Bootstrapping Scattering Amplitudes in Effective Field Theories

by

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ABSTRACT

The modern approach to scattering amplitudes has provided a plethora of techniques that allow us to circumvent Lagrangians and Feynman rules. The bootstrap of amplitudes in quantum field theories allows us to study the landscape of effective field theories. In this dissertation, we apply the amplitudes bootstrap to address a series of questions: 1) We exploit Goldstone soft theorems and supersymmetry to study EFTs that result from spontaneous symmetry-breaking and their supersymmetrizations. 2) Using methods of generalized unitarity, we construct a certain class of all-multiplicity 1-loop amplitudes in Born-Infeld theory. 3) These 1-loop results coupled with an assumption of tree-like factorization allow us to show that electromagnetic duality may be non-anomalous at 1-loop in Born-Infeld theory. 4) We introduce a novel formalism to double-copy scattering amplitudes with massive states along with spectral conditions that ensure that the resulting double-copy is local.

CHAPTER 1

Introduction

A model describing the fundamental physics of our universe must consist of two basic components: a spectrum of particles of various charges, masses and spins and a set of interactions between them. Interactions between fundamental particles in nature can be characterized as electroweak, strong or gravitational. At present, the Standard Model of particle physics describes electroweak and strong interactions, while Einstein's theory of general relativity describes classical gravity at weak fields. In this dissertation, we study quantum field theory (QFT), the framework within which particle physics and perturbative (around flat space) gravity are formulated.

Quantum field theory was first introduced as a formalism that describes quantum mechanical particles and is consistent with Einstein's special theory of relativity. It has been very successful in predicting the outcomes of experiments. The quintessential experimental probe is particle collisions. These are examples of scattering processes whose probabilities or 'cross-sections' are the primary observables in QFT scattering. Cross-sections contain two parts: integrals over kinematic space and the scattering amplitude.

Traditionally, scattering amplitudes in QFT have been calculated via a sum over Feynman diagrams. Each diagram can be interpreted via the Feynman rules associated to the theory at hand. Though this method is efficient in the simplest cases, as the number of interacting particles or the order of perturbation theory grows, so does its complexity. The number of diagrams itself grows combinatorially with the number of external states. Thus there is a need for sleeker and more efficient methods.

The 'modern approach' to scattering amplitudes is a broad term that encompasses a class of non-traditional, often symmetry-driven, techniques to calculate scattering amplitudes. Some modern methods circumvent Lagrangians entirely. This provides a tool to build possible scattering amplitudes without knowledge of the underlying Lagrangian, and this has been particularly effective in surveying the landscape of field theories perturbatively. The idea of the bootstrap is to allow theories that give rise to consistent amplitudes[1] and rule out ones that do not. This dissertation employs modern methods and uses a bottom-up approach to bootstrap scattering amplitudes in effective field theories.

Let us understand why the landscape of field theories is worth studying. One strategy to answer open questions in fundamental physics such as the cosmological constant problem, the field content of dark matter, grand unification etc. is to build models and check how their predictions hold up against experimental results. Model building involves the construction of theories with a wide range of particles and interactions, which populate different parts of the field theory landscape. In condensed matter physics, QFT is employed to describe quantum quench physics, BCS superconductivity and the fractional quantum Hall effect. Systems in condensed matter physics too vary greatly in terms of what kinds of (possibly composite) particles and interactions they contain. In cosmology, building quantum field theories that could possibly describe baryogenesis or inflation are topics of active research. Thus the landscape of field theories is far-reaching and a better understanding of it is relevant to many areas of physics.

Many theories are associated with a certain regime of validity. For example a theory of baryons or mesons is necessarily an effective description, valid only below the QCD confinement scale. In gravitational physics, despite not knowing how gravity behaves at high energy scales, one can still use the Einstein-Hilbert Lagrangian expanded around Minkowski space at low-energies. These are examples of effective field theory (EFT), the language in which much of physics is viewed today. Effective field theories have an associated scale of validity, above which new physics enters. Below this scale, the EFT has multiple (possibly infinitely many) Lagrangian operators, each suppressed by the scale.

Since we are only interested in on-shell observables, Lagrangian operators carry some unnecessary redundancies. First, some operators may vanish on-shell. Second, operators may be related to one another by partial integration. Finally, off-shell actions are subject to field re-definitions. Hence it is useful to focus on on-shell amplitudes as a tool to survey the landscape of EFTs.

There are two main ways to deal with effective field theories. Let us better understand them via an example. One of the biggest unanswered questions in theoretical physics is the quantization of gravity. Gravity at macroscopic scales is well-described by Einstein's theory of general relativity and one possible description at high energies is provided by string theory. Thus one strategy to understand scattering of fundamental particles is to begin with a string theory (that typically lives in a higher-dimensional space) and study it in the low-energy regime. This is the so-called 'top-down' approach. On the other hand one can remain agnostic about the ultra-violet regime and instead begin with known properties of a theory at low energies such as its symmetries and

[1] We discuss more about what consistency conditions an amplitude must satisfy a little later.

use these to build the most general model compatible with the given set of properties. This is called the 'bottom-up' approach. The traditional strategy is to identify all Lagrangian operators compatible with low-energy symmetries and study the resulting observables. This dissertation attacks EFTs bottom-up via their on-shell amplitudes and in particular, bootstraps EFT amplitudes compatible with the principles of unitarity, locality and Lorentz invariance.

Indeed to understand what it means to 'bootstrap' scattering amplitudes, we must start by studying the consistency conditions on the amplitude. Consider Lorentz invariance for example. All particles in relativistic QFTs have mass and spin, i.e. they transform under a particular representation of the Lorentz group. Since amplitudes describe the scattering of these particles, they must be Lorentz-covariant. In the case of scattering of scalar particles for example, the amplitude must be built out of Lorentz invariant contractions of the external momenta.

Scattering amplitudes in QFTs involve the time evolution of a specified 'incoming' state in the far past to an 'outgoing' state in the far future. These are elements of a scattering- or S-matrix. For our purposes, unitarity of a theory refers to the unitarity of its S-matrix. Unitarity of the S-matrix manifests on scattering amplitudes as factorization properties on their singularities. Tree-level amplitudes must factorize on poles into lower-point amplitudes. Unitarity also relates higher-loop integrands to lower-loop or possibly tree-level amplitudes.

Now let us discuss locality. All of the four fundamental forces in nature are propagated by gauge bosons that interact locally. On a lattice, the idea of local interactions is simple- it boils down to having a scale associated with the interaction that dictates how many neighbors a lattice site is allowed to interact with. Low energy effective theories of our interest live in continuum space. Here locality manifests in two ways: 1) When dealing with Lagrangian theories, an operator that is evaluated at more than one space-time point is disallowed by locality. 2) Derivatives are a bit more tricky because an increase in the number of derivatives in an operator is analogous to increasing the interaction radius on a lattice. Thus an interaction that contains the inverse of a derivative operator (which may be interpreted as an infinite number of derivatives) is non-local.

It should be noted that the number of derivatives in a given term is ambiguously defined in off-shell actions due to field re-definitions. On the other hand, the S-matrix is unaffected by these. Since we can always move a finite number of extra derivatives from the kinetic terms into the interaction terms via field re-definition, we see that when studying the S-matrix of a local QFT, we need only consider a 2-derivative kinetic term i.e. a propagator that goes as $1/p^2$. This heavily constrains the analytic properties of S-matrix elements in momentum space.

The ramifications of locality or analyticity of the S-matrix are broad and have been the subject of a lot of study. For example at the leading order in perturbation theory i.e. at tree-level, the amplitude is only allowed to have simple poles in the Mandelstam variables at locations where intermediate momenta go on-shell. Coupled with unitarity, this tells us that the amplitude is a rational function of momenta with simple poles in Mandelstam variables and the residue of the amplitude on these simple poles is the product of two lower-point amplitudes. Beyond the leading order, locality and unitarity dictate that the amplitude must develop branch cuts at specific locations, the discontinuities across which are given by lower-order amplitudes.

Equipped with the understanding of how Lorentz invariance, locality and unitarity constrain scattering amplitudes, this dissertation embarks on a journey through the space of possible S-matrices. Each viable S-matrix corresponds to a field theory and thus we can study the landscape of field theories via the space of possible scattering amplitudes. This technique of constructing amplitudes compatible with unitarity, locality, Lorentz invariance and often other symmetries is dubbed the 'amplitudes bootstrap'[2].

Chapter 2 An important step in the amplitude bootstrap is to identify the on-shell action of the low-energy symmetries. For example if our effective action had an unbroken global $U(1)$ symmetry, all amplitudes that violate conservation of the associated additive charge would vanish. Indeed this is how the conservation of electric charge arises in the Standard Model. Using charge conservation as input for the amplitude bootstrap produces all possible amplitudes compatible with the specified $U(1)$ symmetry.

A more non-trivial example, relevant for this thesis, is that of spontaneous symmetry-breaking. There are two important theorems that are useful in this case. The first, Goldstone's theorem states that the breaking of an internal symmetry leads to a massless scalar mode in the spectrum, referred to as the Goldstone boson. This has since been extended to space-time symmetry-breaking, and supersymmetry-breaking which results in a Goldstone fermion, or Goldstino. Whatever the broken symmetry, the resulting action is known to realize it via its non-linear action on the Goldstone mode [3]. This leads us to the second important theorem: Adler's soft pion theorem and its extensions. These soft theorems inform how non-linear symmetries in the action cause scattering amplitudes to vanish in the limit of vanishing momentum of an external Goldstone mode[3]. Stated differently, spontaneous symmetry-breaking results in so-called Adler zeros in the soft limit of an external Goldstone mode.

Chapter 2 is based on the paper *Soft Bootstrap and Supersymmetry* published in Journal of High Energy Physics (JHEP) in 2019, that I wrote in collaboration with Henriette Elvang, Marios Hadjiantonis and Callum Jones [5]. Here we use soft theorems as input to bootstrap scattering amplitudes in effective field theories that result from some symmetry-breaking pattern. We ex-

[2]This is not to be confused with the conformal bootstrap, where crossing symmetry is used to survey the landscape of possible conformal field theories (CFTs). The philosophy of both these approaches is quite similar as is explored in [2].

[3]This is true only when symmetry-breaking results in a so-called symmetric coset space and the EFT lacks cubic interactions. A counterexample is provided by broken conformal invariance that leads the non-symmetric coset space [4].

tend a soft recursion technique introduced for scalars in [6] to particles with spin and survey the landscape of Goldstone effective field theories.

Supersymmetry is a symmetry that relates bosons and fermions. It is a necessary ingredient to construct a healthy string theory and it is very useful in other models of physics beyond the Standard Model. Combined with supersymmetry, the soft bootstrap proves to be a powerful technique. An important example is that of supersymmetric Born-Infeld theory. This is a theory of a massless vector supermultiplet that arises in the open string effective action. Though the photons have no soft properties of their own, the other fields in the supermultiplet are Goldstone modes. A combination of the soft bootstrap and supersymmetry Ward identities are able to fix the Born-Infeld photon S-matrix at leading order as we show in [5].

Chapter 3 In its own right, Born-Infeld theory exists outside of this supersymmetric definition and has many interesting properties. First introduced as a solution to the electron self-energy problem [7], it has been noted to have no vacuum birefringence [8] and multi-chiral soft limits at tree-level [1]. Another fascinating property of Born-Infeld theory is electromagnetic duality [9]. This is a purely on-shell symmetry i.e. it is a symmetry of the equations of motion. In fact, the duality is a peculiar symmetry in that it has no known off-shell Lorentz-covariant formulation [10]. This makes the scattering amplitude approach singularly useful in the study of this duality symmetry. On-shell, the existence of such a duality implies optical helicity conservation. This conserved $U(1)$ charge can actually be seen as a result of R-symmetry in supersymmetric Born-Infeld theory[4]. In non-supersymmetric Born-Infeld, other methods such as the novel recursion relations introduced in our paper [11] can be used to prove tree-level EM duality.

Chapter 3 focuses on understanding Born-Infeld theory beyond tree-level. In particular, it focuses on two duality-violating sectors: the all-plus and all-but-one-plus. It is based on my paper *All-Multiplicity One-Loop Amplitudes in Born-Infeld Electrodynamics from Generalized Unitarity* published in JHEP in 2020, written in collaboration with Henriette Elvang, Marios Hadjiantonis and Callum Jones [12]. Using the constraints of unitarity, we construct all-multiplicity 1-loop amplitudes in the self-dual and next-to-self-dual helicity sectors. This is a first step towards understanding EM duality at 1-loop.

Chapter 4 A full investigation of loop-level EM duality is carried out in my paper *Electromagnetic Duality and D3-Brane Scattering Amplitudes Beyond Leading Order* accepted for publication in JHEP in 2021, written in collaboration with Henriette Elvang, Marios Hadjiantonis and Callum Jones [11]. Using the fact that the 1-loop all-plus and all-but-one-plus amplitudes are finite and local [12], we show that assuming tree-like factorization, any duality-violating 1-loop

[4]R-symmetry generators commute with all Poincaré generators but act non-trivially on the supercharges that generate supersymmetry transformations.

amplitude can be removed by the addition of finite local counterterms to the action. Thus we see that EM duality can indeed be preserved by Born-Infeld at 1-loop. The details are presented in Chapter 4.

An interesting aspect of the preservation of EM duality in Born-Infeld is its interplay with the so-called 'double copy' construction. The double-copy was first introduced as a map that relates gravity amplitudes to products of gluon amplitudes [13]. It has since grown into a web of relations between a variety of theories, one of which is Born-Infeld theory [14]. We discuss this more in Chapter 4.

Chapter 5 Most earlier work on the double copy has been confined to massless gluons and gravitons, even though physics experiments and observations are consistent with some models of light but massive gravitons [15]. In my paper *Constraints on a Massive Double-Copy and Applications to Massive Gravity* published in JHEP in 2021, written in collaboration with Laura Johnson and Callum Jones [16], we explore the possibility of a massive version of the gluon-graviton double copy consistent with locality. We introduce a novel formalism for a massive double-copy consistent with tree-level unitarity and locality. We derive a set of consistency conditions that must be satisfied by theories that admit a double-copy. While a naive massive extension of Yang-Mills does not double copy to massive gravity, we find that any theory with a Kaluza-Klein tower of states admits a local double copy. Following our work, models satisfying these consistency conditions have been found and the massive double-copy has been used to calculate amplitudes in these theories [17]. Chapter 5 contains further details.

During my time in graduate school, I also had the pleasure of working on a couple of other papers that are not included in this dissertation:

- *On the Supersymmetrization of Galileon Theories in Four Dimensions* published in Physical Letters B in 2018, written in collaboration with Henriette Elvang, Marios Hadjiantonis and Callum Jones [18].
- *Thermodynamics of Near BPS Black Holes in AdS_4 and AdS_7* under review at JHEP, written in collaboration with Finn Larsen [19].

In the following sections, I introduce in more detail various aspects of the scattering amplitudes bootstrap that are relevant background material for the chapters that follow.

1.1 On-Shell Methods in Quantum Field Theory

Many methods that fall under the umbrella of the modern amplitudes program rely on the so-called spinor-helicity variables in 4 dimensions, so let us begin our discussion there.

Since scattering amplitudes involve asymptotic particles in a relativistic field theory, each external state is fully specified by its Lorentz representation. In other words, the input data of a scattering amplitude is simply the masses, spins and momenta of the external particles, and the amplitude itself must transform covariantly under a Lorentz transformation. For example consider a massless particle in 4 dimensions. Here the amplitude must transform appropriately under the little group $SO(2)$ according to the specified helicity of the particle.

In the case of massless on-shell spin-$1/2$ fermions in 4 dimensions, the external states are represented by fermion wavefunctions in the amplitude. These satisfy the equations of motion, also known as the massless Dirac equation,

$$\not{p}\, v_{\pm}(p) = 0\,, \qquad \bar{u}_{\pm}(p)\, \not{p} = 0\,, \tag{1.1.1}$$

where $\not{p} = p^{\mu}\gamma_{\mu}$ and the $\pm$ subscripts index the helicity of the external fermion. The solutions to Dirac equation are

$$v_{+}(p) = \begin{pmatrix} |p]_a \\ 0 \end{pmatrix}\,, \qquad v_{-}(p) = \begin{pmatrix} 0 \\ |p\rangle^{\dot{a}} \end{pmatrix}\,, \tag{1.1.2}$$

$$\bar{u}_{+}(p) = \begin{pmatrix} |p]_a & 0 \end{pmatrix}\,, \qquad \bar{u}_{-}(p) = \begin{pmatrix} 0 & |p\rangle^{\dot{a}} \end{pmatrix}\,. \tag{1.1.3}$$

Here a and $\dot{b}$ are spinor indices that take values 1, 2 and we use the conventions of [20]. $|p\rangle^{\dot{b}}$ and $|p]_b$ are called spinor helicity variables and their indices are raised and lowered by the $SU(2)$-invariant Levi-Civita tensors, ϵ_{ab} for the undotted indices and $\epsilon_{\dot{a}\dot{b}}$ for the dotted ones. In this notation, (1.1.1) can be written as Weyl equations,

$$p_{a\dot{b}}|p\rangle^{\dot{b}} = 0\,, \qquad p^{\dot{a}b}|p]_b = 0\,. \tag{1.1.4}$$

where

$$p_{a\dot{b}} = p_{\mu}(\sigma^{\mu})_{a\dot{b}} = \begin{pmatrix} -p^0 + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & -p^0 - p^3 \end{pmatrix}\,, \tag{1.1.5}$$

with a similar expression for $p^{\dot{a}b}$.

In the case of massless vector fields, we do not have external wavefunctions but polarization vectors. Using spinor-helicity variables, we may express the external polarizations as,

$$\epsilon_-^\mu = -\frac{\langle p|\bar{\sigma}^\mu|q]}{\sqrt{2}[qp]}\,, \qquad\qquad \epsilon_+^\mu = -\frac{\langle q|\bar{\sigma}^\mu|p]}{\sqrt{2}\langle qp\rangle}\,. \tag{1.1.6}$$

Here q is a reference spinor and the polarization vectors above are transverse $p_\mu \epsilon_\pm^\mu = 0$ for any value of $q \neq p$ due to (1.1.4). Indeed the final amplitude should not depend on q since it must be gauge-invariant. Finally, graviton polarizations can be built out of massless vector polarizations (1.1.6), and so we can represent all particles of physical interest in these variables.

Tree-level amplitudes with only massless external states in 4 dimensions can be built from spinor helicity variables and their Lorentz-invariant contractions. For example, Mandelstam variables and Levi-Civita contractions,

$$s_{ij} = (p_i + p_j)^2 = \langle ij\rangle[ij] \tag{1.1.7}$$

$$\epsilon(ijkl) = \epsilon_{\mu\nu\rho\sigma} p_i^\mu p_j^\nu p_k^\rho p_l^\sigma = \langle ij\rangle[jk]\langle kl\rangle[li] - [ij]\langle jk\rangle[kl]\langle[li]\rangle\,. \tag{1.1.8}$$

Under a little group $SO(2)$ transformation, the helicity spinors transform as

$$|p\rangle \to t|p\rangle \qquad\qquad |p] \to t^{-1}|p]\,. \tag{1.1.9}$$

This informs how the fermion wavefunctions transform. Under such little group scaling, the polarization vectors undergo the following transformation:

$$\epsilon_-^\mu \to t^2 \epsilon_-^\mu \qquad\qquad \epsilon_+^\mu \to t^{-2}\epsilon_+^\mu\,. \tag{1.1.10}$$

In fact, it is generically true that particles of definite helicity scale as t^{-2h}. So we find that an amplitude written in spinor helicity variables scales as

$$\mathcal{A}_n\left(|1], |1\rangle, \cdots, t^{-1}|i], t|i\rangle, \cdots, |n], |n\rangle\right) = t^{-2h_i}\mathcal{A}_n\left(|1], |1\rangle, \cdots, |i], |i\rangle, \cdots, |n], |n\rangle\right)\,. \tag{1.1.11}$$

This leads us into one advantage of spinor helicity variables: the form of all 3-point amplitudes of massless particles (even beyond tree-level) is fixed by little group scaling to be

$$\mathcal{A}(1^{h_1}2^{h_2}3^{h_3}) = c_{123}\langle 12\rangle^{h_3-h_1-h_2}\langle 23\rangle^{h_1-h_2-h_3}\langle 13\rangle^{h_2-h_1-h_3} \tag{1.1.12}$$

for mostly negative helicity particles and an analogous formula for mostly positive helicity ones.

Another advantage, and one of the best examples of the benefits of using spinor helicity variables over Feynman diagrams, is encompassed by the Parke-Taylor formula [21] for n-gluon scattering in the so-called Maximally Helicity-Violating (MHV) sector[5],

$$\mathcal{A}_n[1^+ \cdots i^- \cdots j^- \cdots n^+] = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle \cdots \langle n1\rangle} \,. \tag{1.1.13}$$

Here the square brackets in the amplitude on the LHS denote color-ordering, meaning it is only a part of the amplitude proportional to a particular color tensor. Partial amplitudes can be put together to get the full color-dressed gluon amplitude using

$$\mathcal{A}_n\left(1^+ \cdots i^- \cdots j^- \cdots n^+\right) = \sum_\sigma \mathcal{A}_n[1^+\sigma(\cdots i^- \cdots j^- \cdots n^+)]\mathrm{Tr}\left(T^{a_1}T^{\sigma(a_2}\cdots T^{a_n)}\right) \,, \tag{1.1.14}$$

where σ runs over all permutations of the labels $2, \ldots, n$. Compared to the Feynman diagram calculation, which involves summing over numerous diagrams (e.g. 220 at 6-point), the Parke-Taylor formula (1.1.13) may be derived via an elegant on-shell recursion relation introduced by Britto, Cachazo, Feng and Witten (BCFW) [22].

So far we have only talked about massless particles. Massive particles also admit a spinor helicity formalism, a comprehensive introduction to which may be found in [23]. Still an active area of current research, there have been recent advances in extending massless techniques such as on-shell supersymmetry [24], on-shell recursion relations, and the double copy [16], to amplitudes of massive particles. For the most part, we focus here on theories of massless particles and study their properties using on-shell methods such as those listed above. We begin with massless Goldstone bosons that arise in effective field theories with spontaneously broken symmetries.

1.2 Effective Field Theory of Goldstone Modes

Effective field theories are associated with a particular regime of validity. EFT Lagrangians may contain higher-derivative interactions, each with its own coupling suppressed by an appropriate power of the cut-off scale. At energies comparable to the cut-off scale, the perturbative quantum field theory is invalid and new physics, in the form of new states and interactions must be introduced.

First, let us begin with an illustrative toy model of two types of scalar fields, a massless ϕ field and a massive π field that interact via a cubic $\phi\phi\pi$ vertex with coupling g. At energies comparable

[5] In supersymmetric theories, the supersymmetry Ward identity forces amplitudes in the all-plus and all-but-one-plus helicity sectors to vanish. Thus the all-but-two-plus helicity sector came to be known as the Maximally Helicity-Violating (MHV) sector. Similarly, the all-but-two-minus helicity sector is referred to as anti-MHV.

to m_π, the 4-point amplitude of $\phi\phi \to \phi\phi$ reads

$$\mathcal{A}_4(1_\phi 2_\phi 3_\phi 4_\phi) = \frac{g^2}{s+m_\pi^2} + \frac{g^2}{t+m_\pi^2} + \frac{g^2}{u+m_\pi^2} \,. \tag{1.2.1}$$

At energies much lower than m_π, the Mandelstam invariants s, t and u are much smaller than m_π^2. In the limit of $m_\pi^2 \gg s,\ t,\ u$, the 4-point EFT amplitude is given by Taylor expanding (1.2.1),

$$\mathcal{A}_4(1_\phi 2_\phi 3_\phi 4_\phi) = 3f_\pi + \frac{f_\pi}{m_\pi^4}\left(s^2+t^2+u^2\right) - \frac{f_\pi}{m_\pi^6}\left(s^3+t^3+u^3\right) + \mathcal{O}\left(\frac{1}{m_\pi^8}\right) , \tag{1.2.2}$$

where $f_\pi = g^2/m_\pi^2$ is dimensionless. Every possible Bose-symmetric contribution appears with a fixed coefficient at each order. Note that at $\mathcal{O}\left(m_\pi^{-2}\right)$, the only possible Bose-symmetric contribution is $(s+t+u)$ which vanishes due to momentum conservation. Up to field redefinitions, the dynamics of the massless scalar field is given by an EFT Lagrangian,

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 + \frac{f_\pi}{8}\phi^4 + \frac{f_\pi}{2m_\pi^4}\phi^2\partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi - \frac{f_\pi}{2m_\pi^6}\phi^2\partial_\mu\partial_\nu\partial_\rho\phi\partial^\mu\partial^\nu\partial^\rho\phi + \mathcal{O}\left(\frac{1}{m_\pi^8}\right) . \tag{1.2.3}$$

This Lagrangian reproduces the amplitude (1.2.2). We cut it off at $\mathcal{O}(m_\pi^{-8})$. Notice that there are an infinite number of higher-derivative interactions, each suppressed by the cut-off scale m_π, near which the dynamics of the π field has to be taken into account.

For effective field theory to be useful, it is important to understand what kinds of interactions can and cannot be included in the effective Lagrangian without full knowledge of the ultra-violet completion. For example, it is important to understand why the $\phi^2\partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi$ coefficient was fixed in the example above. Important contributions towards understanding this problem have been made by imposing the constraints of causality [25]. More recently, EFT coefficients have been constrained to lie within a cyclic polytope dubbed the EFThedron, assuming that the high-energy theory is unitary and local [26].

We are interested in cases where a symmetry present at high energies is spontaneously broken below a cut-off scale. Such spontaneous symmetry-breaking is ubiquitous in physics, ranging from magnetization of ferromagnets to the Higgs mechanism in particle physics. A symmetry is spontaneously broken when a symmetry of the system is not respected by its ground state or the vacuum. Goldstone's theorem predicts the appearance of a massless mode in the spectrum of the low-energy effective field theory. This mode is created by the action of the Noether current J^μ associated with the broken continuous symmetry on the vacuum.

Interestingly, version of this phenomenon is realized in the Standard Model via the breaking of chiral symmetry. At energies much higher than the masses of the up and down quarks, the quark field content of the QCD Lagrangian respects an approximate chiral symmetry. The QCD vacuum has zero (angular) momentum. Thus any quark that appears in the ground state must be accompanied by an anti-quark of opposite chirality and the vacuum necessarily breaks chiral symmetry. Hence we expect the existence of an almost-Goldstone boson in the spectrum. This is realized by the neutral and charged pions. Because chiral symmetry is only approximate, the pions are not exactly massless but have small masses; indeed they are far lighter than the other mesons.

An important feature of scattering amplitudes of Goldstone modes resulting from a spontaneously broken internal symmetry is given by Alder's soft pion theorem. As we previously mentioned, the broken current J^μ creates a Goldstone mode π when it acts on the vacuum[6]. This allows us to relate the matrix elements

$$\langle\beta|J^\mu(0)|\alpha\rangle = N^\mu_{\beta\alpha} + \frac{iFq^\mu}{q^2}M_{\beta\alpha}\,, \tag{1.2.4}$$

where β and α are some (possibly multi-particle) initial and final states, q is the momentum of the Goldstone boson π, N^μ represents contributions to the matrix elements that do not have a q^2 pole while $M_{\beta\alpha}$ is the matrix element of $\alpha \to \beta + \pi$.

The final soft theorem follows from two facts. First, note that the conservation of the current $\partial_\mu J^\mu$ does not imply the existence of a conserved charge due to the spontaneous breaking of the symmetry associated to J^μ. Second, as long as the theory is free from 3-point interactions, N^μ is guaranteed to be regular when $q \to 0$, the so-called soft limit. In [27], the author showed that

$$\lim_{p_\pi \to 0} \mathcal{A}_n(\beta,\alpha,\pi) = \mathcal{O}(p_\pi)\,. \tag{1.2.5}$$

$\mathcal{A}_n$ is said to have an Adler zero when p_π is taken soft.

Another consequence of the broken symmetry is its non-linear action on the Goldstone mode π as discussed by Callan, Coleman, Wess and Zumino (CCWZ) [3]. In the simplest case the non-field-dependent part is a shift symmetry,

$$\pi \to \pi + c\,. \tag{1.2.6}$$

So far, we have only discussed spontaneous internal symmetry-breaking. Goldstone's theorem also applies to the breaking of spacetime symmetries, though the number of Goldstone modes that result no longer matches the number of broken generators [28]. In the more general setup of spacetime symmetry-breaking, the non-field-dependent part of the non-linear realization takes

[6]Despite sharing the same notation, note that this Goldstone mode is distinct from the massive scalar field in the toy model (1.2.1) discussed earlier.

the form,

$$\pi \to \pi + c_0 + c_1^{\mu} x_{\mu} + c_2^{\mu\nu} x_{\mu} x_{\nu} + \cdots , \tag{1.2.7}$$

where x is the space-time coordinate. The degree in x of the shift polynomial σ determines the degree of the Adler zero [29],

$$\lim_{p_\pi \to 0} \mathcal{A}_n(\beta, \alpha, \pi) = \mathcal{O}(p_\pi^{\sigma+1}) . \tag{1.2.8}$$

Thus σ denotes the degree of softness of the Goldstone mode resulting from a shift symmetry of the type (1.2.7).

As per the philosophy of the modern amplitudes program, we can now reverse the logic of Adler's soft pion theorem, studying soft amplitudes directly to get a handle on symmetry-breaking. We can systematically examine the space of theories with spontaneously broken symmetries by constructing S-matrices with the required soft properties. To do this, we make use of on-shell soft subtracted recursion relations first introduced in [6]. This recursive construction provides a litmus test of whether or not one can construct a theory with a given set of lower-point interactions g_i with particular degrees of softness σ_i. In [29], soft recursion was used to survey the landscape of EFTs to determine which values of (g_i, σ_i) can result in a Goldstone scalar EFT.

Extending this technique to non-zero spins and formulating a concrete validity criterion for this *soft bootstrap*, we were able to survey theories with supersymmetry-breaking, with both linearly and non-linearly realized supersymmetries in our paper [5]. The details are presented in Chapter 2. Many of these theories had higher degree softness and resulted from spacetime symmetry-breaking. In the next section, we discuss an important example that demonstrates both these features.

1.3 D-Brane Effective Actions: Galileons and Born-Infeld Photons

A big driver of research in modern theoretical physics is the question of quantizing a theory of gravity. One solution is string theory. Since string theories live in higher-dimensional spaces, one must study them on compactified sub-spaces in order to understand possible lower-dimensional spectra and interactions. The low-energy theory has perturbative features like particles derived from the excited modes on strings and non-perturbative ones like D-branes. In this section, we discuss the low-energy effective action on D-branes.

Let us start with the simplest example: a D3-brane with a 4-dimensional worldvolume embedded at x_5 in a 5-dimensional ambient space. The pull-back of the transverse coordinate x^5 is a scalar field $\phi(x)$ that lives in the worldvolume of the D3-brane. This induces the following Dirac-Born-Infeld (DBI) action on the brane,

$$S_{\text{DBI}} = \Lambda^4 \int d^4x \left(\sqrt{\det \left(\eta_{\mu\nu} + \tfrac{1}{\Lambda^4} \partial_\mu \phi \partial_\nu \phi \right)} - 1 \right) , \tag{1.3.1}$$

where Λ^4 is the brane tension. The 5-dimensional Poincaré symmetries are non-linearly realized by the D-brane scalars ϕ. The x_5 translations act as

$$\phi \to \phi + c , \tag{1.3.2}$$

while the Lorentz transformations between transverse and longitudinal directions $M_{\mu,5}$ act as,

$$\phi \to \phi + v_\mu x^\mu + \text{ field-dependent terms} . \tag{1.3.3}$$

Both (1.3.2) and (1.3.3) are symmetries of the DBI action (1.3.1). The extension of Alder's soft pion theorem [29] now tells us that the scalar mode must enjoy degree 2 soft limits,

$$\lim_{p_\phi \to 0} \mathcal{A}_n = \mathcal{O}(p_\phi^2) . \tag{1.3.4}$$

Interestingly, broken translation and Lorentz invariance give rise to the same Goldstone mode [28] and the number of Goldstone modes equals the number of broken translations. This means that for a 4-brane embedded in d-dimensional space there are $(d-4)$ Goldstone scalars. Let us now investigate what other modes exist on the brane.

10-dimensional string theory is supersymmetric and lives in superspace, a space-time with added Grassmann coordinates. As a result, it is useful to study the effective action of D-branes embedded in superspace. Consider for example a D3-brane in 6d. We know that this has 2 scalar modes with softness $\sigma = 2$, but in this case there are additional symmetries that are broken. The supersymmetries are non-linearly realized on the resulting Goldstino modes ψ as shift symmetries,

$$\psi \to \psi + c \;\Rightarrow\; \sigma_\psi = 1 . \tag{1.3.5}$$

The dynamics of these modes are governed by the Akulov-Volkov action [30]. Finally, the low-energy dynamics of the open strings encoded in massless vector fields A^μ, also induce an action on the D-brane,

$$S_{\text{BI}} = -\Lambda^4 \int \mathrm{d}^4x \left[\sqrt{-\det \left(g_{\mu\nu} + \frac{1}{\Lambda^2} F_{\mu\nu} \right)} - 1 \right] . \tag{1.3.6}$$

This is known as the Born-Infeld action after it was first proposed in the context of the electron self-energy problem [7]. At first glance it may seem as if these photons do not have any special soft limits since they have $\sigma = 0$, but note that such non-divergent soft limits are very different from the usual divergent soft limits of minimally-coupled photons [31].

The leftover (unbroken) supersymmetries are realized linearly on $(A_\mu, \psi, \bar{\psi})$ and $(\psi, \bar{\psi}, Z, \bar{Z})$ which transform as $\mathcal{N} = 1$ supermultiplets. Here $Z = \phi_1 + i\phi_2$ is a complex scalar built out of the two real Goldstone scalars. This is enhanced to one $\mathcal{N} = 4$ vector supermultiplet for a D3-brane in 10d. In addition, each model can be studied in its own right. They are discussed in Chapter 2.

The action induced on the D3-brane (1.3.1) necessarily has sub-leading contributions. The first cubic interactions stem from the extrinsic curvature K when considering an end-of-the-world brane, quartic coupling Λ_4 originates from the Ricci scalar R and the quintic interaction results from the boundary Gibbons-Hawking-York term. Together these are called the 4d Galileon action,

$$S = \int \mathrm{d}^4x\sqrt{-G}\left[\Lambda^4 + \Lambda_3^3 K[G] + \Lambda_4^2 R[G] + \Lambda_5 \mathcal{K}_{\text{GHY}}[G]\right] . \tag{1.3.7}$$

In a particular double-scaling limit, the Galileons can be decoupled from the leading DBI term:

$$T^4 \to \infty \tag{1.3.8}$$

$$g_3 = \frac{\Lambda_3^3}{\Lambda^6}, \quad g_4 = \frac{\Lambda_4^2}{\Lambda^8}, \quad g_5 = \frac{\Lambda_5}{\Lambda^{10}} . \tag{1.3.9}$$

This scaling limit results in an İnönü-Wigner contraction of the symmetry algebra. The resulting cubic and quintic Galileon actions have soft degree $\sigma = 2$, similar to the DBI scalars. In the case of the quartic Galileon, this leads to an enhancement of the shift symmetry on the scalar to

$$\phi \to \phi + c + v_\mu x^\mu + s_{\mu\nu} x^\mu x^\nu \quad \Rightarrow \quad \sigma_\phi = 3 , \tag{1.3.10}$$

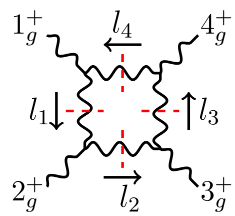
where $s_{\mu\nu}$ is a symmetric traceless tensor. The quartic theory is called the special Galileon due to this special enhancement in softness. In fact Galileons have the highest soft degree of any exceptional EFT [5, 29].

In my paper [18], we used the on-shell amplitude approach to study the possible supersymmetrizations of the cubic, quartic and quintic Galileons. This lead to a novel construction of the supersymmetric quartic and quintic Galileon theories, while the cubic Galileon could be related to these via a field redefinition. These results are summarized in Section 2.9.

1.4 Loop Techniques and Anomalous Symmetries

In our discussion so far, we have focused on calculating tree amplitudes or studying properties of theories at tree-level. In this section, we consider loop-level techniques: primarily generalized unitarity and integrand cuts.

Unitarity of the S-matrix relates the imaginary part of 1-loop integrals given by 'cuts' of loop integrands, to a product of tree-level amplitudes. Here a cut refers to when one or more propagators inside the loop are put on-shell. Evaluating the 1-loop integrand on the cut conditions results in the product of two or more tree-level amplitudes in the case of a 2- or higher-line cut. For example, consider the 4-line cut of a 1-loop 4-point self-dual Yang-Mills integrand that breaks up into four on-shell 3-point tree amplitudes:



(1.4.1)

In 4 dimensions any 1-loop integral can be rewritten using the following basis of integrals using Passarino-Veltman reduction [32],

$$\mathcal{A}_n^{\text{1-loop}} = c_{\text{bubble}} I_{\text{bubble}} + c_{\text{triangle}} I_{\text{triangle}} + c_{\text{box}} I_{\text{box}} + \text{rational terms}\,, \tag{1.4.2}$$

where the box, triangle and bubble integrals are scalar loops with the respective topologies. In 4 dimensions, only the bubble integral carries UV divergences while the triangle integral is IR divergent. In the case of a D-dimensional amplitude, the topologies extend from bubbles to D-gons.

The coefficients c_i can be calculated via unitarity. Cuts like the one in (1.4.1) relate the box coefficients c_{box} to a 4-line cut of the integrand. Once the box coefficients are found, the triangle coefficients follow from 3-line cuts, and finally the bubble coefficients from 2-line cuts. Since it is impossible to cut more than 4 lines in 4 dimensions, we do not need to consider box topologies at 1-loop in 4 dimensions.

Since unitarity cuts give the necessary bubble, triangle and box coefficients, all that is left to be determined is the rational terms. These arise from the part of the integrand that is not captured by the 4d unitarity cuts. To calculate the rational contributions, we use a technique called 'generalized unitarity' [33], where the cut loop propagators are d-dimensional rather than 4d. It is precisely the $(d-4)$-dimensional part of the loop momentum that gives rise to rational terms.

Thus generalized unitarity methods can be used to circumvent Feynman rules and calculate amplitudes at 1-loop.

Modern loop techniques can also be used to better understand the symmetries of EFTs. For example, consider electromagnetic duality in Born-Infeld theory. The equations of motion of BI are invariant under electromagnetic duality transformations. The symmetry manifests itself on tree-level scattering amplitudes as the conservation of optical helicity,

$$\mathcal{A}_n\big(\underbrace{\gamma^+\ldots\gamma^+}_{n_+}\underbrace{\gamma^-\ldots\gamma^-}_{n_-}\big) = 0 \;\text{ for }\; n_+ \neq n_-\,. \tag{1.4.3}$$

Classical, i.e. tree-level, symmetries of a field theory need not be preserved in the quantum theory. Thus a natural question is whether or not electromagnetic duality is preserved in Born-Infeld at 1-loop. The answer lies in the structure of loop amplitudes with external states that violate conservation of optical helicity. If helicity-violating 1-loop amplitudes can be removed by adding finite local counterterms to the Born-Infeld action, then EM duality is a 1-loop symmetry.

The first step towards answering this question is presented in my paper [12]. Here we present the calculation of all-multiplicity 1-loop amplitudes in the self-dual and next-to-self-dual helicity sectors of Born-Infeld theory. This is discussed further in Chapter 3. Building on these results, in our subsequent paper [11] we found that assuming tree-like factorization properties hold at 1-loop, electromagnetic duality can be restored via addition of local counterterms. The results are presented in Chapter 4.

1.5 The Double Copy

The classic example of the double copy is that of gluonic tree amplitudes in Yang-Mills theory double-copying to gravitational amplitudes in Einstein gravity coupled to a dilaton and an antisymmetric 2-form. Concretely,

$$\mathcal{M}_n^{\text{grav}} = \sum_{\alpha\in A,\beta\in B} \mathcal{A}_n^{\text{YM}}[\alpha] S_n[\alpha|\beta] \mathcal{A}_n^{\text{YM}}[\beta]\,, \tag{1.5.1}$$

where A and B are two bases of color-orderings and S_n is the so-called KLT kernel which defines the double-copy map. This representation of the double copy results from the field theory limit of the KLT formula [34] in string theory that expresses closed string amplitudes in terms of a sum of products of open string amplitudes. In the field theory limit, the double copy map is from gluons to gravitons coupled to a dilaton and an anti-symmetric 2-form.

Another formulation of the double copy is the color-kinematics duality, or the BCJ double copy,

[13]. Here we start with an amplitude,

$$\mathcal{A}_n^{\text{A}} = \sum_i \frac{c_i n_i}{d_i}\,, \tag{1.5.2}$$

where c_i are color factors built out of f^{abc} structure constants of the group in question, d_i are products of propagators in n-point trivalent graphs and n_i are the leftover kinematic factors in the amplitude of theory A. In order to write down the double-copy of this amplitude, it must obey color-kinematics duality, i.e.

$$c_i + c_j + c_k = 0 \qquad\qquad \Rightarrow \qquad\qquad n_i + n_j + n_k = 0\,. \tag{1.5.3}$$

The double copy of such an amplitude is then given by

$$\mathcal{M}_n^{\text{C=A}\otimes\text{B}} = \sum_i \frac{\tilde{n}_i n_i}{d_i}\,, \tag{1.5.4}$$

where $\tilde{n}_i$ are numerators of theory B and n_i are those of A. For massless particles in the adjoint representation, we can easily commute between the BCJ and KLT double copy formulae and each has different strengths. While it can be shown that the KLT formula ensures the double-copy amplitude has the correct factorization properties as we see in Chapter 5, the BCJ representation has a natural generalization to loop-level [35]. At 1-loop this reads,

$$\mathcal{A}_n^{\text{1-loop}} = \int \frac{d^D l}{(2\pi)^D} \sum_i S_i \frac{c_i n_i(l)}{d_i(l)} \qquad \Rightarrow \qquad \mathcal{M}_n^{\text{1-loop}} = \int \frac{d^D l}{(2\pi)^D} \sum_i S_i \frac{\tilde{n}_i(l) n_i(l)}{d_i(l)}\,, \tag{1.5.5}$$

where d_i are propagators containing loop momentum l in various loop configurations.

Finally, let us look at other examples of the double copy. Many of these stem from a double-copy formula proposed by Cachazo, He and Yuan (CHY) [36] which takes the schematic form,

$$\mathcal{M}_n^{\text{C}} = \int d^n\sigma \mathcal{I}_n^{\text{A}} \mathcal{I}_n^{\text{B}}\,, \tag{1.5.6}$$

at tree-level, where the integral is over a worldsheet in kinematic space. Varying the integrands $\mathcal{I}_n$ lead to the discovery of a whole slew of new double copy maps [14], some of which are given in Table 1.1.

The models in Table 1.1 contain only massless modes. Some extensions of the double copy map broaden these examples by considering massive matter couplings, but the case of scattering with all massive particles had been left unstudied until my paper [16] with Johnson and Jones. We study the constraints of locality on a possible extension of the KLT double copy to massive external states. In particular, we consider the example of a massive Yang-Mills theory. We show

R / L	BAS	χPT	YM	$\mathcal{N}=4$ SYM
BAS	BAS	χPT	YM	$\mathcal{N}=4$ SYM
χPT	χPT	sGal	BI	$\mathcal{N}=4$ sDBI
YM	YM	BI	gravity+	$\mathcal{N}=4$ SG
$\mathcal{N}=4$ SYM	$\mathcal{N}=4$ SYM	$\mathcal{N}=4$ sDBI	$\mathcal{N}=4$ SG	$\mathcal{N}=8$ SG

Table 1.1: The table shows the tree-level double-copy A $\otimes$ B of a selection of different choices for the A and B single-color models. BAS is the cubic bi-adjoint scalar model. The other single-color models are χPT = chiral perturbation theory (NLSM), YM = Yang-Mills theory, and $\mathcal{N}=4$ super Yang Mills theory (sYM). For the results of the double-copy, sGal stands for the special Galileon, BI is Born-Infeld theory, gravity+ is Einstein gravity with a dilaton and antisymmetric 2-form, and SG stands for supergravity.

that while a naive single-mass spectrum fails to give a local double copy, when equipped with a Kaluza-Klein tower of states, the double copy of massive Yang-Mills will give a local theory of massive gravity. The details are presented in Chapter 5.

Another aspect of the models in Table 1.1 is that they are leading order in the effective field theory derivative expansion. Extending the double copy beyond the leading order is the subject of our current work [37]. This question is intricately linked with the stringy origins of the KLT formula whose string theory version provides one example of a higher-derivative double-copy. To construct the most general effective field theory double copy we can take a locality-driven bottom-up approach similar to that presented in Chapter 5, except instead of considering massive bi-adjoint scalar amplitudes we consider higher-derivative deformations of the bi-colored theory.

CHAPTER 2

Soft Bootstrap and Supersymmetry

2.1 Introducing Exceptional EFTs

Constructing effective actions one by one is not an efficient approach to the problem of classifying Goldstone EFTs and studying the properties of the associated scattering amplitudes. The effective actions for Goldstone modes typically have the unusual property that while there may be an *infinite* number of gauge invariant local operators at a fixed order in the derivative expansion, the associated infinite set of Wilson coefficients is determined in terms of a *finite* number of independent parameters. How can this be understood in purely on-shell terms?

The traditional explanation is that the spontaneously broken symmetries are nonlinearly realized on the fundamental fields and therefore mix operators in the effective action of different valence. From a more physical perspective, the spontaneously broken symmetries manifest themselves on the physical observables via low-energy or soft theorems. The non-independence of the Wilson coefficients is required to produce a cancellation between Feynman diagrams that ensures the low-energy theorem to hold. This is a redundant statement: while the number of independent parameters required to specify the effective action at a given order is reparametrization invariant, the actual Wilson coefficients are not. As we will see, from a purely on-shell perspective the collapse from an infinite number of free parameters to a finite number is a symptom of the underlying *recursive constructiblility* of the S-matrix, which itself can be understood as a consequence of the low-energy theorems.

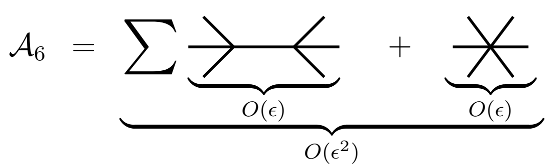
It is instructive to consider an explicit example that illustrates these ideas. Consider a flat 3-brane in 5d Minkowski space. There is a Goldstone mode ϕ associated with the spontaneous breaking of translational symmetry in the direction transverse to the brane, and it is well-known that the leading low-energy dynamics is governed by the Dirac-Born-Infeld (DBI) action. In static gauge, it takes the form

$$S_{\text{DBI}} = \Lambda^4 \int d^4x \left(\sqrt{\det\left(\eta_{\mu\nu} + \tfrac{1}{\Lambda^4}\partial_\mu\phi\partial_\nu\phi\right)} - 1 \right), \tag{2.1.1}$$

where Λ^4 is the brane tension. The action trivially has a constant shift symmetry $\phi \to \phi + c$ which implies that the DBI amplitudes have vanishing single-soft limits. In particular, when one of its momentum lines is taken soft,

$$p^\mu_{\text{soft}} \to \epsilon\, p^\mu_{\text{soft}} \quad \text{with} \quad \epsilon \to 0\,, \tag{2.1.2}$$

the Feynman vertex it sits on goes to zero as $O(\epsilon)$. There are no cubic interactions, so propagators remain finite. Hence, every tree-level Feynman diagram goes to zero as $O(\epsilon)$. What may be surprising is that a cancellation occurs between Feynman diagrams such that the soft behavior of any tree-level DBI n-point amplitude is enhanced to $O(\epsilon^2)$. For example for the 6-point amplitude, the $O(\epsilon)$-contributions of the pole diagrams cancel against those of the 6-point contact term, leaving an overall $O(\epsilon^2)$ soft behavior:



(2.1.3)

The cancellation of the $O(\epsilon)$-contributions requires the coefficients of the 4- and 6-particle interactions $(\partial\phi)^4$ and $(\partial\phi)^6$ to be uniquely related. Interestingly we can invert the logic of this argument. Begin with the most general effective action constructed from the operators present in the DBI action, but now with *a priori* independent Wilson coefficients c_i, schematically

$$S_{\text{eff}} \sim \int \mathrm{d}^4x \; \left[(\partial\phi)^2 + \frac{c_1}{\Lambda^4}\partial^4\phi^4 + \frac{c_2}{\Lambda^8}\partial^6\phi^6 + \ldots\right]. \tag{2.1.4}$$

Imposing that the amplitudes of this model satsify $O(\epsilon^2)$ low-energy theorems generates an infinite set of relations among the c_i. Up to non-physical ambiguities related to field redefinitions, the unique solution to these constraints is the DBI action. In that sense, DBI is the unique leading-order 4d real single-scalar theory with $O(\epsilon^2)$ low-energy theorems [6].

The cancellation of the $O(\epsilon)$-terms in the DBI amplitudes is a manifestation of a less obvious symmetry of the action. The broken Lorentz transformations transverse to the brane induce an enhanced shift symmetry on the brane action of the form $\phi \to \phi + c_\mu x^\mu + \ldots$, where the "$+\ldots$" stand for field-dependent terms. A theory with interaction terms built from scalar fields with at least two derivatives on every field would trivially have the enhanced shift symmetry that leads to the $O(\epsilon^2)$ soft behavior, but this is not the case for DBI. Therefore DBI is in a class of EFTs that have been described in previous work as *exceptional* [6]. This example illustrates the Lagrangian-based description of what is meant by an exceptional EFT: a local field theory of massless particles with shift symmetries that lead to an enhanced soft behavior of the scattering amplitudes beyond what is obvious from simple counting of derivatives on the fields.[1]

The on-shell significance of the exceptional EFTs was first described in [29, 38]. It was shown, for the case of scalar effective field theories, that the class of exceptional EFTs as defined above coincides precisely with the class of EFTs for which there exists a valid method of on-shell recursion. On-shell recursion for scattering amplitudes in the form of BCFW [22,39] or those based on various types of multi-line shifts [40–43] have been around for several years now, but they are often not valid in EFTs. Technically, this is because higher-derivative interactions tend to give 'bad' large-z behavior of the amplitudes under the complex momentum shifts and as a result there are non-factorizable contributions from a pole at $z = \infty$. A more physical reason is that in order for a recursive approach to have a chance, it has to be given information about how higher-point terms are possibly connected to the lower-point interactions. Standard recursion relations basically only 'know' gauge-invariance, so in the DBI example they have no opportunity to know about any relation between the couplings of $(\partial\phi)^4$ and $(\partial\phi)^6$. So, naturally, a recursive approach to calculate amplitudes in exceptional EFTs needs to know about the low-energy theorems, since — as illustrated for DBI — this is what ties the higher-point interactions to the lower-point ones. This is the additional input used in the early work of [44–46] in which vanishing soft limit arguments were used to construct the amplitudes of pion scattering without an explicit action. The idea was more recently formalized in the form of the *soft subtracted recursion relations* presented in [38]; they provide a tool to calculate the leading (and possibly next-to-leading) order contribution to the S-matrix of an exceptional EFT.[2].

The existence of valid recursion relations gives us our sought-after on-shell characterization of the relation among the Wilson coefficients of Goldstone EFTs. The infinite set of *a priori* independent local operators at leading order in the derivative expansion determine the leading-order part of the S-matrix. For a generic EFT, the presence of independent operators of valence n corresponds to the appearance of independent coefficients on contact contributions for amplitudes with n external particles. If the scattering amplitudes are recursively constructible at a given order, then no such *independent* coefficients can appear since the entire amplitude must be determined by factorization into amplitudes with fewer external particles. Furthermore, the recursion must take as its input a finite set of seed amplitudes that depend on only a finite number of parameters.

[1]This definition is a little imprecise. In standard usage, an EFT is defined by some physical data including the spectrum of particles and associated symmetries and corresponds to an effective action with operators at all orders in the derivative expansion. The defining property of an exceptional EFT however is typically only valid at leading or next-to-leading order. The equivalent on-shell statement is that the scattering amplitudes of the EFT are only recursively constructible at the same order in the expansion.

[2]A recent paper [47] derives Ward identities from the enhanced shift symmetry in expectional EFTs and discusses a different type of recursion relations.

Beyond being an efficient method for calculating explicit scattering amplitudes in known models, the subtracted recursion relations can be implemented as a numerical algorithm to explore and classify the landscape of possible EFTs. We term this program the *soft bootstrap* due to the structural similarity of the method with the conformal bootstrap [48,49]. The method is described in detail in Section 2.4.5, here we give a simplified description. We consider EFTs as defined by a set of on-shell *soft data*: a spectrum of massless states, linearly realized symmetries and low-energy theorems. We use general ansätze for scattering amplitudes of low valence and low mass dimension, consistent with the assumed spectrum and linear symmetries, as input for subtracted recursion. If the ansätze satisfy a certain criterion guaranteeing the validity of the subtracted recursion relations and if the assumed soft data corresponds to a valid EFT, then the output of the recursion should correspond to a physical scattering amplitude. Here *valid EFT* means the existence of the assumed EFT as a local, unitary, Poincaré invariant quantum field theory.

For tree-level scattering amplitudes this includes the requirement that the only singularities of the amplitude correspond to factorization on a momentum channel. Conversely if no such valid EFT exists, or equivalently if the assumed soft data is inconsistent, then the output of the recursion generically will not correspond to a physical scattering amplitude and this may be detected through the presence of non-physical or spurious singularities. In practice, the ansätze are parametrized by a finite number of coefficients, and the removal of spurious singularities often places constraints on these coefficients.

The soft bootstrap program was initiated in [29], where it was used to explore the landscape of real scalar EFTs with vanishing low-energy theorems. The results are reviewed and extended in Section 2.5. This paper should be understood as a continuation and generalization of this program, incorporating richer soft data including spinning particles and linearly realized supersymmetry. In Section 2.2 we provide a brief overview of exceptional EFTs studied in this paper.

2.2 Exceptional EFTs Studied

In this paper, we extend the application of the soft bootstrap from real scalars to any massless helicity-h particle and we derive a precise criterion for the validity of the soft subtracted recursion relations. By the new validity criterion, the on-shell characterization of an exceptional EFT will precisely be that its amplitudes are constructible using soft recursion.

Our work requires a precise definition of the degree of softness of the amplitude. This is given in Section 2.4.1. For now, let us simply introduce the *soft weight* σ as

$$\mathcal{A}_n(\epsilon p_1, p_2, \ldots) = \epsilon^\sigma \, \mathcal{S}_n^{(0)} + O(\epsilon^{\sigma+1}) \quad \text{as} \quad \epsilon \to 0\,, \tag{2.2.1}$$

Soft degree σ	Spin s	Type of symmetry breaking
1	0	Internal symmetry (symmetric coset)
0	0	Internal symmetry (non-symmetric coset)
1	1/2	Supersymmetry
0	0	Conformal symmetry
0	1/2	Superconformal symmetry
2	0	Higher-dimensional Poincaré symmetry
0	0	Higher-dimensional AdS symmetry
3	0	Special Galileon symmetry

Table 2.1: The table lists soft weights σ associated with the soft theorems $\mathcal{A}_n \to O(\epsilon^\sigma)$ as $\epsilon \to 0$ for several known cases. The soft limit is taken holomorphically in 4d spinor helicity, see Section 2.4.1 for a precise definition. Conformal and superconformal breaking is discussed in Section 2.6.3.

where $\mathcal{S}_n^{(0)} \neq 0$. Table 2.1 summarizes the soft weights for various known cases of spontaneous symmetry breaking. The earlier example of DBI corresponds to the case of spontaneously broken higher-dimensional Poincaré symmetry; only the breaking of the translational symmetry actually gives rise to a Goldstone mode [50] and it will have $\sigma = 2$.

Here follows a brief overview of exceptional EFTs that appear in this paper. We include the connection between their soft behavior and Lagrangian shift symmetries:

- **DBI** can be extended to a complex scalar Dirac-Born-Infeld theory and coupled supersymmetrically to a fermion sector described by the **Akulov-Volkov** action of Goldstinos from spontaneous breaking of supersymmetry. In extended supersymmetric DBI, the vector sector is **Born-Infeld (BI)** theory. The soft weights are $\sigma_Z = 2$ for the complex scalars Z of DBI, $\sigma_\psi = 1$ for the fermions of Akulov-Volkov, and $\sigma_\gamma = 0$ for the BI photon. The soft behaviors can be associated with shift symmetries $Z \to Z + c + v_\mu x^\mu$ and $\psi \to \psi + \xi$, where ξ is a constant Grassmann-number.[3] $\mathcal{N} = 1$ **supersymmetric Born-Infeld** couples the BI vector to the Goldstino of Akulov-Volkov.

- **Nonlinear sigma models (NLSM)** describe the Goldstone modes of sponteneously broken internal symmetries and have scalars with constant shift symmetries that give $\sigma = 1$ soft weights in the low-energy theorems. A common example of an NLSM is **chiral perturbation theory** in which the scalars live in a coset space $U(N) \times U(N)/U(N)$.

  The complex scalar $\mathbb{CP}^1$ NLSM can be supersymmetrized with a fermion sector that is **Nambu-Jona-Lasinio (NJL)** model. The complex scalars have shift symmetry $Z \to Z + c$ and $\sigma_Z = 1$ while the fermions have no shift symmetry and $\sigma_\psi = 0$. We study both the $\mathcal{N} = 1$ and 2 supersymmetric $\mathbb{CP}^1$ NLSM.[4]

[3] We leave out field-dependent terms for simplicity when stating the shift symmetries.

[4] In Section 2.7.2 we show that the $\mathcal{N} = 2$ $\mathbb{CP}^1$ NLSM requires the presence of 3-point interactions and the soft

- A NLSM can have a non-trivial subleading operator that respects the shift symmetry and hence also the low-energy theorems with $\sigma = 1$. This operator is known as the **Wess-Zumino-Witten (WZW)** term and has a leading 5-point interaction.

- **Galileon** scalar EFTs arise in various contexts and have the extended shift symmetry $\phi \to \phi + c + v_\mu x^\mu$ that gives low-energy theorems with $\sigma = 2$. As such they can be thought of as subleading operators of the DBI action, and are called **DBI-Galileons**. They can also be decoupled from DBI (at the cost of having no UV completion).

  In 4d there are two independent Galileon operators: the quartic and quintic Galileon. (By a field redefinition, the cubic Galileon is not independent from the quartic and quintic.) When decoupled from DBI, the quartic Galileon has an even further enhanced shift symmetry $\phi \to \phi + c + v_\mu x^\mu + s_{\mu\nu} x^\mu x^\nu$ that gives low-energy theorems with soft weight $\sigma = 3$ and is then called the **Special Galileon** [29, 51].

- The quartic Galileon has a complex scalar version with $\sigma_Z = 2$ (but it cannot have $\sigma_Z = 3$). It has an $\mathcal{N} = 1$ supersymmetrization [18, 52] in which the fermion sector trivially realizes a constant shift symmetry that gives $\sigma_\psi = 1$.

- There is evidence [18] that the quintic Galileon may have an $\mathcal{N} = 1$ supersymmetrization. This involves a complex scalar whose real part is a Galileon with $\sigma = 2$ and imaginary part is an R-axion with $\sigma = 1$.

2.3 Structure of the Effective Action

The low-energy dynamics of a physical system can be described by a Wilsonian effective action containing a set of local quantum fields for each of the on-shell asymptotic states with *all possible* local interactions allowed by the assumed symmetries:

$$S_{\text{effective}} = S_0 + \sum_{\mathcal{O}} \frac{c_{\mathcal{O}}}{\Lambda^{\Delta[\mathcal{O}]-4}} \int \mathrm{d}^4 x\, \mathcal{O}(x)\,. \tag{2.3.1}$$

Here S_0 denotes the free theory, i.e. the kinetic terms, Λ is a characteristic scale of the problem, and $c_{\mathcal{O}}$ are dimensionless constants. The sum is over all local Lorentz invariant operators $\mathcal{O}(x)$ of the schematic form

$$\mathcal{O}(x) \sim \partial^A \phi(x)^B \psi(x)^C F(x)^D\,, \tag{2.3.2}$$

weight of the scalar is reduced to $\sigma_Z = 0$.

where $A, \ldots, D$ are integer exponents. In this paper we focus on EFTs in which the operators $\mathcal{O}$ are manifestly gauge invariant.[5]

We assign the following quantities to a local operator

- *Dimension:* $\Delta[\mathcal{O}]$ defined as the engineering dimension with bosonic fields of dimension 1 and fermionic fields of dimension 3/2.
- *Valence:* $N[\mathcal{O}]$ defined as the sum of the total number of field operators appearing. Equivalently, this is the valence of the Feynman vertex derived from such an interaction.

The schematic operator in (2.3.2) has $\Delta[\mathcal{O}] = A + B + \frac{3}{2}C + 2D$ and $N[\mathcal{O}] = B + C + D$.

In standard EFT lore, operators of lowest dimension dominate in the IR. In many cases this means the marginal and relevant interactions dominate and the irrelevant interactions are sub-dominant and suppressed by powers of the UV scale Λ. In other cases, such as effective field theories describing the dynamics of Goldstone modes, there are only irrelevant interactions and it may be less clear which operators dominate. It is therefore useful to introduce the *reduced* dimension

$$\tilde{\Delta}[\mathcal{O}] = \frac{\Delta[\mathcal{O}] - 4}{N[\mathcal{O}] - 2} \tag{2.3.3}$$

for the operator basis (2.3.1). Operators that minimize $\tilde{\Delta}$ dominate in the IR.

The authors of [6, 29, 38] consider only scalar EFTs and therefore operators of the form $\mathcal{O} \sim \partial^m \phi^n$. They define a quantity

$$\rho \equiv \frac{m-2}{n-2} = \tilde{\Delta}[\mathcal{O}] - 1\,, \tag{2.3.4}$$

to determine when two operators of this form produce tree-level diagrams with couplings of the same mass dimension. Morally ρ is the same as the reduced dimension $\tilde{\Delta}[\mathcal{O}]$. The latter is the natural generalization of ρ to operators containing particles of all spins.

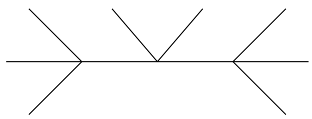
The quantity $\tilde{\Delta}$ is useful for clarifying the notion of what it means for an interaction to be leading order in an EFT with only irrelevant interactions. In the deep IR, the relative size of the dimensionless Wilson coefficients in the effective action is unimportant since lower dimension operators will *always* dominate over higher dimension operators. It is therefore only necessary to isolate the contributions that are leading in a power series expansion of the amplitudes in the inverse UV cutoff scale Λ^{-1}. The dominant interactions in the deep IR are generated by operators that *minimize* this quantity. As an illustrative example, consider an effective action for scalars

[5]This need not be the case in more general scenarios (though of course we insist on overall gauge invariance). For example in Yang-Mills theory, the gauge invariant operator $\mathrm{tr}F^2$ has a quadratic term which we group into the free part S_0 of the action while the interaction terms would be accounted for in the sum of all operators $\mathcal{O}$ in (2.3.1). Similarly, for massless spin-2 fields when $\sqrt{-g}R$ is expanded around flat space.

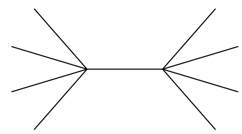
with interaction terms of the form

$$S_{\text{effective}} \supset \int \mathrm{d}^4x \left[\frac{c_4}{\Lambda^4} \partial^4 \phi^4 + \frac{c_5}{\Lambda^5} \partial^4 \phi^5 \right]. \tag{2.3.5}$$

The reduced dimensions $\tilde{\Delta}$ are 2 and $5/3$ for the quartic and quintic interactions respectively. The quintic interaction should therefore dominate over the quartic in the deep IR. To see this explicitly we have to compare amplitudes with the *same* number of external states, so we compare the contributions from tree-level Feynman diagrams to the 8-point amplitude:



$$\sim \quad \frac{1}{\Lambda^{12}}$$



$$\sim \quad \frac{1}{\Lambda^{10}} \tag{2.3.6}$$

This confirms that the diagrams arising from the quintic interaction dominate the 8-point amplitude.

It is useful to introduce the notion of *fundamental interactions* (or *fundamental operators*) in an EFT. These are the lowest dimension operator(s) whose on-shell matrix elements can be recursed to define all matrix elements of the theory at leading order in the low-energy expansion.

Consider the DBI action. The leading interaction comes from an operator of the form $\frac{1}{\Lambda^4}\partial^4\phi^4$ and as discussed in the introduction, with the associated 4-point amplitude as input, all other n-point amplitudes in DBI can be constructed with soft subtracted recursion relations. If the action had contained an interaction term of the form $\frac{c_5}{\Lambda^5}\partial^5\phi^4$, then $\frac{1}{\Lambda^4}\partial^4\phi^4$ would not be sufficient to determine dominating contributions at n-point order, i.e. both interactions would need to be considered fundamental for soft recursion.

The operators immediately subleading to DBI in the brane-effective action are encoded in the DBI-Galileon. In 4d, there are two such independent couplings,[6] namely for a quartic interaction of the schematic form $\frac{b_4}{\Lambda^6}\partial^6\phi^4$ and a quintic interaction of the form $\frac{b_5}{\Lambda^9}\partial^8\phi^5$; these both have $\tilde{\Delta} = 3$ whereas DBI has $\tilde{\Delta} = 2$. Thus the DBI-Galileon has a total of three fundamental operators: the 4-point DBI interaction and the 4- and 5-point Galileon interactions.

2.4 Subtracted Recursion Relations

We review on-shell subtracted recursion relations for scattering amplitudes of Goldstone modes [6, 29, 38, 53, 54] and derive a new precise criterion for their validity.

[6]The cubic Galileon interaction is equivalent to a particular linear combination of the quartic and quintic Galileon after a field redefinition.

2.4.1 Holomorphic Soft Limits and Low-Energy Theorems

We rely on the 4d spinor helicity formalism (for reviews, see [20,31,55,56]) in which a massless on-shell momentum is written $p = -|p\rangle[p|$. This presents an ambiguity in how to take the soft limit (2.1.2): it could for example be taken democratically as $\{|p\rangle, |p]\} \to \{\epsilon^{1/2}|p\rangle, \epsilon^{1/2}|p]\}$, holomorphically $\{|p\rangle, |p]\} \to \{\epsilon|p\rangle, |p]\}$, or anti-holomorphically $\{|p\rangle, |p]\} \to \{|p\rangle, \epsilon|p]\}$. These are all equivalent choices, because the momentum p is invariant under *little group scaling* $\{|p\rangle, |p]\} \to \{t|p\rangle, t^{-1}|p]\}$. Amplitudes scale homogeneously under the little group,

$$\mathcal{A}_n\big(\{|1\rangle, |1]\} \ldots \{t|i\rangle, t^{-1}|i]\}_+ \ldots\big) = t^{-2h_i}\mathcal{A}_n\big(\{|1\rangle, |1]\} \ldots \{|i\rangle, |i]\}_+ \ldots\big)\,,. \tag{2.4.1}$$

so the choice of soft limit is simply reflected in a helicity-dependent overall scaling factor. We choose to minimize the power of ϵ in the soft limit by letting the choice depend on the sign of the helicity of the particle: specfically, we take $p_{\text{soft}} \to \epsilon\, p_{\text{soft}} = -\epsilon|s\rangle[s|$ holomorphically for any state with non-negative helicity:[7]

$$|s\rangle \to \epsilon|s\rangle \quad \text{ for } \; h_s \geq 0\,. \tag{2.4.2}$$

For a negative-helicity particle, we use the anti-holomorphic prescription $|s] \to \epsilon|s]$. For scalars, it makes no difference which choice is made.

We characterize the soft behavior of amplitudes of massless particles in terms of a *holomorphic soft weight* σ (or, for brevity, just *soft weight*). It is defined in terms of the holomorphic soft limit (2.4.2) as

$$\mathcal{A}_n\big(\{|1\rangle, |1]\} \ldots \{\epsilon|s\rangle, |s]\}_+ \ldots\big) = \epsilon^{\sigma}\, \mathcal{S}_n^{(0)} + O(\epsilon^{\sigma+1}) \quad \text{as } \; \epsilon \to 0\,, \tag{2.4.3}$$

where $\mathcal{S}_n^{(0)} \neq 0$. This way of taking the soft limit is closely correlated with the shifts introduced for the soft subtracted recursion relations in the following.

2.4.2 Review of Soft Subtracted Recursion Relations

We consider complex momentum deformations of the form

$$p_i \to \hat{p}_i = (1 - a_i z)p_i \quad \text{ with } \quad \sum_{i=1}^{n} a_i p_i = 0\,. \tag{2.4.4}$$

[7]Taking the soft limit as simply as in (2.4.2) is not compatible with overall momentum conservation. To stay on the algebraic locus of momentum conservation in momentum space, we take the limit with appropriate shifts in a subset of the $n-1$ other momentum variables. The precise prescription can be found in equation (6) of [57]. The details will not affect the main line of the discussion in this paper, but we note that all calculations are done manifestly on-shell, including the soft limits.

The label $i = 1, 2, \ldots, n$ runs over the n massless particles in the scattering amplitude. The shifted momenta $\hat{p}_i$ are on-shell by virtue of $p_i^2 = 0$ and satisfy momentum conservation when the shift coefficients a_i satisfy the condition in (2.4.4). (We discuss the solutions to this condition in Section 2.4.5.) When evaluated on the shifted momenta $\hat{p}_i$, an n-point amplitude becomes a function of z and we write it as $\hat{\mathcal{A}}_n(z)$.

The *subtracted recursion relations* for an n-point tree-level amplitude $\mathcal{A}_n$ are derived from the Cauchy integral

$$\oint \frac{dz}{z} \frac{\hat{\mathcal{A}}_n(z)}{F(z)} \;=\; 0\,, \tag{2.4.5}$$

where the contour surrounds all the poles at finite z and the function F is defined as

$$F(z) = \prod_{i=1}^{n} (1 - a_i z)^{\sigma_i}\,. \tag{2.4.6}$$

The vanishing of the integral in (2.4.5) requires absence of a simple pole at $z = \infty$. We derive a sufficient criterion for this behavior in Section 2.4.3.

The shift (2.4.4) is implemented on the spinor helicity variables according to the sign of the helicity h_i of particle i as

$$\begin{aligned} h_i \geq 0: &\qquad |i\rangle \to (1 - a_i z)|i\rangle\,, &\qquad |i] \to |i]\,, \\ h_i < 0: &\qquad |i\rangle \to |i\rangle\,, &\qquad |i] \to (1 - a_i z)|i]\,. \end{aligned} \tag{2.4.7}$$

The limit $z \to 1/a_i$ is then precisely the soft limit $\hat{p}_i \to 0$ of the ith particle in the deformed amplitude. Hence, *if the amplitude satisfies low-energy theorems of the form (2.4.3) with weights σ_i for each particle* i, the integral (2.4.5) will not pick up any non-zero residues from poles arising from the function F when it is chosen as in (2.4.6). Therefore the only simple poles in (2.4.5) arise from $z = 0$ and factorization channels in the deformed tree amplitude. They occur where internal momenta go on-shell, $\hat{P}_I^2 = 0$. The residue theorem then states that the residue at $z = 0$ equals minus the sum of all such residues, and factorization on these poles gives

$$\mathcal{A}_n = \hat{\mathcal{A}}_n(z=0) = \sum_I \sum_{|\psi^{(I)}\rangle} \sum_{\pm} \frac{\hat{\mathcal{A}}_L^{(I)}(z_I^\pm)\hat{\mathcal{A}}_R^{(I)}(z_I^\pm)}{F(z_I^\pm) P_I^2 (1 - z_I^\pm / z_I^\mp)}\,. \tag{2.4.8}$$

The sums are over all factorization channels I, the two solutions $z_I^\pm$ to $\hat{P}_I^2 = 0$, and all possible particle types $|\psi^{(I)}\rangle$ that can be exchanged in channel I. These recursion relations are called *soft subtracted recursion relations*. When $F = 1$, the recursion is called *unsubtracted*.

The expression for the solutions $z_I^\pm$ to the quadratic equation $\hat{P}_I^2 = 0$ involves square roots, but those must cancel since the tree amplitude is a rational function of the kinematic variables.

On channels where the amplitude factorizes into two *local* lower-point amplitudes (meaning that they have no poles), the cancellations of the square roots can be made manifest. This is done by a second application of Cauchy's theorem, which for each channel I converts the sum of residues at $z = z_I^{\pm}$ to the sum of the residues at $z = 0$ and $z = 1/a_i$ for all i. Details are provided in Appendix A, here we simply state the result: *if $\mathcal{A}_L^{(I)}$ and $\mathcal{A}_R^{(I)}$ are local for all factorization channels*, the soft recursion relations take the form

$$\mathcal{A}_n = \sum_I \sum_{|\psi^{(I)}\rangle} \left(\frac{\hat{\mathcal{A}}_L^{(I)}(0)\hat{\mathcal{A}}_R^{(I)}(0)}{P_I^2} + \sum_{i=1}^{n} \text{Res}_{z=\frac{1}{a_i}} \frac{\hat{\mathcal{A}}_L^{(I)}(z)\hat{\mathcal{A}}_R^{(I)}(z)}{z\,F(z)\,\hat{P}_I^2} \right) . \tag{2.4.9}$$

Note that this form of the recursion relation is typically only valid at low points since it requires that the amplitude factorizes into a form where all subamplitudes are local.

The recursion relation in the form (2.4.9) is manifestly rational in the kinematic variables, and we will be using (2.4.9) for the applications in this paper. Note that only the first term in (2.4.9) has poles. Therefore the sum of the $1/a_i$ residues over all channels must be a local polynomial in the momenta.

2.4.3 Validity Criterion

The purpose of including $F(z)$ in (2.4.5) is to improve the large-z behavior of the integrand so that one can avoid a pole at $z = \infty$. This is necessary in EFTs, where the large-z behavior of the amplitude typically does not allow for unsubtracted recursion relations with $F(z) = 1$ to be valid without a boundary term from $z = \infty$. A sufficient condition for absence of a simple pole at infinity is that the deformed amplitude vanishes as $z \to \infty$. Below we show that for a theory with a single fundamental interaction (see Section 2.3) of valence v and coupling of mass-dimension $[g_v]$ the criterion for validity of the subtracted recursion relations is

$$\boxed{4 - n - \frac{n-2}{v-2}\,[g_v] - \sum_{i=1}^{n} s_i - \sum_{i=1}^{n} \sigma_i < 0\,.} \tag{2.4.10}$$

Here s_i is the spin (not helicity) of particle i and σ_i is its soft behavior (2.4.3). Alternatively, one can write the constructibility criterion in terms of the reduced dimension $\tilde{\Delta}$, introduced in (2.3.3), as

$$\boxed{4 - n + (n-2)\tilde{\Delta} - \sum_{i=1}^{n} s_i - \sum_{i=1}^{n} \sigma_i < 0\,.} \tag{2.4.11}$$

The criterion generalizes to theories with more than one fundamental coupling by replacing $\frac{n-2}{v-2}\,[g_v]$ in (2.4.10) by the sum over all couplings contributing to the diagrammatic expansion of the amplitude in question; the precise criterion is given in (2.4.19).

Proof of the criterion (2.4.10)

To avoid a pole at infinity in the Cauchy integral (2.4.5), it is sufficient to require $\hat{\mathcal{A}}_n(z)/F(z) \to 0$ as $z \to \infty$. To start with, we determine the large-z behavior of the deformed amplitude $\hat{\mathcal{A}}_n(z)$.

Generically, in a theory of massless particles with couplings g_k, a tree-level amplitude takes the form

$$\mathcal{A}_n = \sum_j \left(\prod_k g_k^{n_{jk}} \right) M_j \,, \tag{2.4.12}$$

where $\prod_k g_k^{n_{jk}}$ is a product of coupling constants and M_j is a function of spinor brackets only. Since there can be no other dimensionful quantities entering M_j, the mass dimension $[M_j]$ can be determined via a homogenous scaling of all spinors:

$$|i\rangle \to \lambda^{1/2}|i\rangle \quad \text{and} \quad |i] \to \lambda^{1/2}|i] \qquad \Longrightarrow \qquad M_j \to \lambda^{[M_j]} M_j \,. \tag{2.4.13}$$

The mass dimension is also fixed by simple dimensional analysis to be

$$[M_j] = 4 - n - \sum_k n_{jk}[g_k] \,, \tag{2.4.14}$$

since an n-point scattering amplitude in 4d has to have mass-dimension $4 - n$.

It is useful to consider a modified scale transformation defined as

$$\begin{aligned} h_i \geq 0: &\qquad |i\rangle \to \lambda|i\rangle \,, \qquad |i] \to |i] \,, \\ h_i < 0: &\qquad |i\rangle \to |i\rangle \,, \qquad |i] \to \lambda|i] \,. \end{aligned} \tag{2.4.15}$$

The effect of this scaling can be obtained from the uniform scaling (2.4.13) via a little group transformation (2.4.1) on all momenta with $t = \lambda^{1/2}$. Therefore under (2.4.15), M_j scales as $M_j \to \lambda^{[M_j] - \sum_i s_i} M_j$, where s_i is the spin (*not* helicity) of particle i.

For the case of a theory with a single fundamental interaction of valence v with coupling g_v, the number of couplings appearing in an n-point amplitude is $\frac{n-2}{v-2}$, and therefore we have

$$\mathcal{A}_n \to \lambda^D \mathcal{A}_n \,, \quad D = 4 - n - \frac{n-2}{v-2}[g_v] - \sum_i s_i \tag{2.4.16}$$

under the modified scale transformation (2.4.15).

Under the momentum shift (2.4.7), the deformed tree amplitude $\hat{\mathcal{A}}_n(z)$ can be written

$$\begin{aligned}\hat{\mathcal{A}}_n(z) &= \hat{\mathcal{A}}_n\big(\dots\{(1-a_iz)|i\rangle, |i]\}_+ \dots \{|j\rangle, (1-a_jz)|j]\}_-\big)\\ &= \hat{\mathcal{A}}_n\big(\dots\{z(1/z-a_i)|i\rangle, |i]\}_+ \dots \{|j\rangle, z(1/z-a_j)|j]\}_-\big)\\ &= z^D\,\hat{\mathcal{A}}_n\big(\dots\{(1/z-a_i)|i\rangle, |i]\}_+ \dots \{|j\rangle, (1/z-a_j)|j]\}_-\big)\,,\end{aligned} \tag{2.4.17}$$

where the subscripts $\pm$ refer to the sign of the helicity of each particle. In the last line we used the behavior (2.4.16) under the modified scaling (2.4.15).

At large z, the amplitude in the last line of (2.4.17) is the original unshifted amplitude evaluated at a momentum configuration with $q_i = -a_ip_i$. These momenta are all on-shell and satisfy, via (2.4.4), momentum conservation. The only way the tree amplitude could have a singularity at this momentum configuration would be if an internal line went on-shell. This can always be avoided for generic momenta.[8] Thus we conclude from (2.4.17) that for large z, the deformed amplitude behaves as

$$\hat{\mathcal{A}}_n(z) \to z^N \quad \text{with} \quad N \le D\,, \tag{2.4.18}$$

where D is given in (2.4.16). The inequality allows for the possibility that $\mathcal{A}_n$ could have a zero at $q_i = -a_ip_i$.

Our mission was to find a criterion for $\hat{\mathcal{A}}_n(z)/F(z) \to 0$ as $z \to \infty$. By the definition (2.4.6), we have $F(z) \to z^{\sum_i \sigma_i}$ for large z. From our analysis of the large-z behavior of $\hat{\mathcal{A}}_n(z)$, we can therefore conclude that, at worst, $\hat{\mathcal{A}}_n(z)/F(z) \to z^{D-\sum_i \sigma_i}$. The sufficient criterion for absence of a pole at infinity, and hence for validity of the subtracted recursion relation, is then $D-\sum_i \sigma_i < 0$. This is precisely the condition (2.4.10). This concludes the proof.

It is straightforward to generalize the constructibility criterion to EFTs with more than one fundamental interaction,

$$4 - n - \min_j\Big(\sum_k n_{jk}[g_k]\Big) - \sum_{i=1}^n s_i - \sum_{i=1}^n \sigma_i < 0\,. \tag{2.4.19}$$

Recall that in effective field theories, the couplings have negative mass-dimension. This means that the constructibility criterion tends to be dominated by the fundamental interactions associated with operators of the highest mass-dimension that can contribute to the n-point amplitude.

Example 1

Let us once again return to the example of DBI. The action has a fundamental quartic vertex $g_4(\partial\phi)^4$ with a coupling of mass-dimension $[g_4] = -4$. The constructibility criterion (2.4.10)

[8] The condition (2.4.4) has a trivial solution with all a_i equal. Therefore any solution to (2.4.4) can be shifted uniformly $a_i \to a_i + a$ for any real number a. Hence, we can always avoid the discrete set of momentum configurations for which an internal line in $\mathcal{A}_n$ goes on-shell.

for the n-scalar amplitude is $n(1 - \sigma_S) < 0$, where σ_S is the soft behavior of the scalar ϕ. Since $\sigma_S = 2$ in DBI, all DBI tree amplitudes are constructible via the subtracted soft recursion relations, as claimed in the introduction.

The failure of the constructibility criterion for $\sigma_S = 1$ is simply the statement that an EFT whose interactions are built from powers of $(\partial\phi)^2$ trivially has a constant shift symmetry and hence $\sigma_S = 1$, so there are no constraints from shift symmetry on the coefficients of $(\partial\phi)^{2k}$ in terms of that of $(\partial\phi)^4$ and then one has no chance of recursing $\mathcal{A}_4$ to get all-point amplitudes.

Example 2
Consider a theory of massless fermions with quartic coupling of mass-dimension $[g_4] = -2$. The criterion (2.4.10) says that the n-fermion amplitudes are constructible when $4 < n(1 + 2\sigma_\psi)$. Thus all $n > 4$ point tree-amplitudes are constructible by (2.4.8) for any soft weight $\sigma_\psi \geq 0$. No such theory exists for $\sigma_\psi > 0$ (as we prove in Section 2.5.2), but for $\sigma_\psi = 0$ this is exactly the Nambu–Jona-Lasinio (NJL) model, which consists of the simple 4-fermion interaction $\psi^2\bar{\psi}^2$ [58].

2.4.4 Non-Constructibility = Triviality

We have derived a constructibility criterion, but what does it mean? The answer is quite simple: if an n-point amplitude can be constructed recursively from lower-point on-shell amplitudes, there cannot exist a local gauge-invariant n-field operator that contributes to the amplitude without modifying its soft behavior. We define a *trivial operator* to be one with at least 4 fields whose matrix elements manifestly have a given soft weight σ. Let us now assess what it takes to make an operator of scalar, fermion, and vector fields trivial.

Triviality.

Scalars. Operators with at least m derivatives on each scalar field will trivially have single-soft scalar limits with $\sigma_S = m$.

Fermions. We have chosen the soft limit (2.4.2) according to the helicity such that the fermion wavefunctions do not generate any soft factors of ϵ. Thus a trivial soft behavior must come from derivatives on each fermion field in the Lagrangian. We conclude that the trivial soft behavior $\sigma_F =$ smallest number of derivatives on each fermion field.

Photons. Gauge invariance tells us that we should construct the interaction terms using the field strength $F_{\mu\nu}$.[9] When associated with an external photon, the Feynman rule for $F_{\mu\nu}$ gives $p_\mu\epsilon_\nu - p_\nu\epsilon_\mu$. Naively, it may seem to be linear in the soft momentum, but under the holomorphic soft shift (2.4.7) it is actually $O(\epsilon^0)$. Recall that in spinor helicity formalism, a positive helicity vector

[9]Or covariant derivatives $D_\mu = \partial_\mu + igA_\mu$. In this paper, we focus on scalars and fermions that do not transform under any gauge-$U(1)$, therefore photons must couple via $F_{\mu\nu}$.

polarization takes the form $\epsilon_+^\mu \bar{\sigma}_\mu^{\dot{a}b} = \epsilon_+^{\dot{a}b} = |q\rangle^{\dot{a}}[p|^b/\langle pq\rangle$, where q is a reference spinor. Hence, for a positive helicity photon we have

$$(F_+)_a{}^b \equiv (\sigma^{\mu\nu})_a{}^b F_{\mu\nu} \longrightarrow (\sigma^{\mu\nu})_a{}^b(p_\mu \epsilon_{+\nu} - p_\nu \epsilon_{+\mu}) \sim |p]_a \langle p|_{\dot{c}} \frac{|q\rangle^{\dot{c}}[p|^b}{\langle pq\rangle} = |p]_a [p|^b \,. \qquad (2.4.20)$$

This is explicitly independent of the reference spinor q because $F_{\mu\nu}$ is gauge invariant. For a positive helicity particle, we take the soft limit holomorphically as $|p\rangle \to \epsilon|p\rangle$ (while $|p] \to |p]$), so we explicitly see that $F_{\mu\nu} \longrightarrow |p][p|$ is $O(\epsilon^0)$ when p is taken soft. Likewise, for a negative helicity photon, $(F_-)^{\dot{a}}{}_{\dot{b}} \longrightarrow |p\rangle\langle p|$. We conclude that an operator with photons has trivial soft behavior that is determined by the smallest number of derivatives on each field strength $F_{\mu\nu}$.

In an EFT where photon interactions are built only from the field strengths, the matrix elements are $O(1)$ when a photon is taken soft. This, for example, is exactly the case for Born-Infeld theory in which the photons have $\sigma = 0$.

Constructibility. Suppose we study an n-particle amplitude with n_s scalars, n_f fermions, and n_γ photons in an EFT whose fundamental v-particle interactions all have couplings of the same mass-dimension $[g_v]$. The criterion (2.4.10) for constructibility via subtracted soft recursion relations can be written as

$$4 - n - n_v[g_v] - \frac{1}{2}n_f - n_\gamma - n_s\sigma_s - n_f\sigma_f - n_\gamma\sigma_\gamma < 0 \,, \qquad (2.4.21)$$

where $n_v = (n-2)/(v-2)$ is the number of vertices needed at n-point.

Non-constructibility = Triviality. Let us assess if there can be a local contact term for an n-particle amplitude with n_s scalars, n_f fermions, and n_γ photons and soft behaviors σ_s, σ_f, and σ_γ, respectively. As discussed above, a contact term that has such trivial soft behavior takes the form

$$g_n \underbrace{(\partial^{\sigma_s}\phi)\cdots(\partial^{\sigma_s}\phi)}_{n_s}\underbrace{(\partial^{\sigma_f}\psi)\cdots(\partial^{\sigma_f}\psi)}_{n_f}\underbrace{(\partial^{\sigma_\gamma}F)\cdots(\partial^{\sigma_\gamma}F)}_{n_\gamma} \qquad (2.4.22)$$

(for brevity we have not distinguished ψ and $\bar{\psi}$). In 4d, the mass-dimension of the coupling g_n is easily computed as

$$[g_n] = 4 - \big(n_s + n_s\sigma_s\big) - \big(\tfrac{3}{2}n_f + n_f\sigma_f\big) - \big(2n_\gamma + n_\gamma\sigma_\gamma\big) \,. \qquad (2.4.23)$$

Using $n = n_s + n_f + n_\gamma$, we can rewrite this as

$$4 - n - [g_n] - \frac{1}{2}n_f - n_\gamma - n_s\sigma_s - n_f\sigma_f - n_\gamma\sigma_\gamma = 0 \,. \qquad (2.4.24)$$

Compare this with (2.4.21); we note that the constructibility criterion is simply that $n_v[g_v] > [g_n]$, or maybe more intuitively, that g_n has more negative mass-dimension than n_v g_v-vertices. So, when constructibility holds, the n-particle amplitude constructed from the n_v v-valent vertices *cannot* be influenced by a contact term that trivially has the soft behavior: such a contact term would be too high order in the EFT due to all the derivatives needed to trivialize the soft behavior. That of course makes sense; were there such an independent local contact term, it could be added to the result of recursion with any coefficient without changing any of the properties of the amplitude. Hence recursion cannot possibly work in that case. (This is analogous to the example in [20, 31] for constructibility in scalar-QED via BCFW; the difference here is that the subtracted soft recursion relations "know" about the soft behavior in addition to gauge-invariance.)

The argument is easily extended to the case where the theory has fundamental vertices of different valences and mass-dimensions. We conclude that the constructibility criterion (2.4.10) is equivalent to the non-existence of local n-particle operators with couplings of the same mass-dimension and trivial soft behavior: Non-constructibility = Triviality.

2.4.5 Implementation of the Subtracted Recursion Relations

Here we present details relevant for the practical implementation of the soft subtracted recursion relations.

Solving the shift constraints. Conservation of the momentum for the shifted momenta $\hat{p}_i$ (2.4.4) requires the shift variables a_i to satisfy

$$\sum_i a_i p_i^\mu = 0. \tag{2.4.25}$$

In 4d, the LHS can be viewed as a $4 \times n$ matrix p_i^μ of rank 4 (if $n \geq 5$) multiplying a n-component vector a_i. Hence the valid choices of parameters a_i form a vector space given by the kernel of the matrix p_i^μ. For $n \geq 5$ any subset of four momenta are generically linearly independent, so the p_i^μ-matrix has full rank. By the rank-nullity theorem, the dimension of the kernel is therefore $n-4$. However, there is always a trivial solution which consists of all a_i's equal, hence non-trivial solutions to (2.4.25) exist only when $n \geq 6$.

Practically, the linear system of equations is solved by dotting in p_j, i.e. we have

$$\sum_i s_{ji}\, a_i = 0 \quad \text{for} \quad j = 1, 2, \ldots, n\,. \tag{2.4.26}$$

The symmetric $n \times n$-matrix with entries s_{ji} has rank 4, so the linear system (2.4.26) can be solved for say a_1, a_2, a_3, and a_4 in terms of the $n-4$ other a_i's.

Soft bootstrap. Subtracted recursion relations can be used to calculate tree amplitudes in EFTs of

Goldstone modes in theories we already know well, such as DBI, Akulov-Volkov etc. However, the soft subtracted recursion relations can also be used as a tool to *classify* and assess the existence of exceptional EFTs with a given spectrum of massless particles and low-energy theorems with given weights σ.

The approach to the classification of special EFTs is as follows:

(1) Model input: the spectrum of massless particles and the coupling dimensions of the fundamental interactions in the model.

(2) Symmetry assumptions: the n-particle amplitudes have soft behavior with weight σ_i for the ith particle.

If the constructibility criterion (2.4.10) is *not* satisfied, the assumptions (1) and (2) are trivially satisfied and we cannot constrain the couplings in the EFTs; it is not exceptional.

If the constructibility criterion (2.4.10) *is* satisfied for input (1) and (2), one can use the soft subtracted recursion relations to test whether a theory can exist with the above assumptions. One proceeds as follows.

The fundamental vertices give rise to local amplitudes which must be polynomials[10] in the spinor helicity brackets, and it is simple to construct the most general such ansatz for the local input amplitudes. One can further restrict this ansatz by imposing on it the soft behaviors associated with the assumed symmetries. The result of recursing this input from the fundamental vertices is supposed to be a physical amplitude and therefore it must necessarily be independent of the $n-4$ parameters a_i that are unfixed by (2.4.25). If that is *not* the case for any ansatz of the fundamental input amplitudes (vertices), we learn that there cannot exist a theory with the properties (1) and (2) above. On the other hand, an a_i-independent result is evidence (but not proof) of the existence of such a theory. It may well be that a_i-independence requires some of the free parameters in the input amplitudes to be fixed in certain ways and this can teach us important lessons about the underlying theory. The test of a_i-independence can be done efficiently numerically, and this way one can scan through theory-space to test which symmetries are compatible with a given model input.

Additionally, one can impose further constraints from unbroken global symmetries, for example, one can restrict the input from the fundamental amplitudes by imposing the supersymmetry Ward identities. We shall see examples of this in later sections.

4d and 3d consistency checks. There is a subtlety that must be addressed for $n = 6$. In that case, the solution space is 2-dimensional, but one solution is the trivial one with all a_i equal.

[10]This is true at 4-point and higher; for 3-point, massless particle amplitudes are uniquely fixed by the little group scaling.

Furthermore, one can rescale all a_i. This means that if the recursed result for the amplitude depends on the a_i only through ratios of the form

$$\frac{(a_i - a_j)}{(a_k - a_l)}\,, \tag{2.4.27}$$

it will appear to be a_i-independent numerically, but the result will nonetheless have spurious poles. To detect this problem numerically, we dimensionally reduce the recursed result to 3d.[11]. Then the space of solutions to (2.4.25) is $(n-3)$-dimensional, so there are non-trivial solutions and a numerical 3d test will reveal dependence on ratios such as (2.4.27) for $n = 6$.

We refer to the consistency checks of a_i-independence as *4d and 3d consistency checks*, respectively, or simply as *n-point tests* when applied to construction of n-point amplitudes. In this paper, we use 6-, 7- and 8-point tests. In Section 2.5, we present an overview of the resulting space of exceptional pure real and complex scalar, fermion, and vector EFTs.

Special requirements for non-trivial 5-point interactions. Consider 5-particle interactions which are non-trivial with respect to a given soft behavior. This could for example be the Wess-Zumino-Witten (WZW) term, which with 4 derivatives on 5 scalars has a non-trivial $\sigma = 1$ soft behavior. Or the 5-point Galileon, which with 8 derivatives on 5 scalars has a non-trivial $\sigma = 2$. Constructibility tells us that one must be able to calculate such 5-point amplitudes from soft recursion relations via factorization, i.e.

$$\mathcal{A}_5 = \sum_I \frac{\hat{\mathcal{A}}_3 \hat{\mathcal{A}}_4}{P_I^2}\,. \tag{2.4.28}$$

However, there are no 3-point amplitudes available that could possibly make this work. The reason is that the only 3-scalar interaction with a non-zero on-shell amplitude is ϕ^3, which gives rise to amplitudes with $\sigma = -1$ [57]. So we appear to have a contradiction: the constructibility criterion tells us that these 5-particle amplitudes are recursively constructible, but it is obviously impossible to construct them from lower-point input.

What goes wrong is that at 5-points, there are no non-trivial choices of the a_i parameters that give valid recursion relations in 4d. So we have to go to 3d kinematics to resolve this issue. The above contradiction persists in 3d, so the only resolution is that these non-trivial constructible 5-point amplitudes must vanish in 3d kinematics.

Indeed they do: for WZW term and the quintic Galileon, the 5-point matrix elements are

$$A_5^{\text{WZW}} = g_5\, \epsilon_{\mu\nu\rho\sigma} p_1^\mu p_2^\nu p_3^\rho p_4^\sigma\,, \qquad A_5^{\text{Gal}} = g_5'\, (\epsilon_{\mu\nu\rho\sigma} p_1^\mu p_2^\nu p_3^\rho p_4^\sigma)^2\,. \tag{2.4.29}$$

[11] The dimensional reduction from 4d to 3d is carried out by simply replacing all square spinors by angle spinors.

The Levi-Civita contraction makes it manifest that these amplitudes vanish in 3d.

We conclude that *any* non-trivial (in the sense of soft behavior) 5-particle interaction must vanish in 3d. Thus, it is no coincidence that the WZW and quintic Galileon 5-point amplitudes are proportional to Levi-Civita contractions.

2.5 Soft Bootstrap

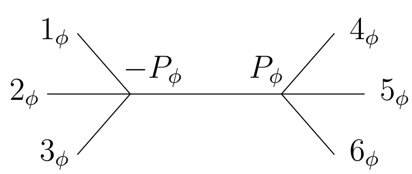
We now turn to examples of how the soft recursion relations can be used to examine the existence of exceptional EFTs. The landscape of real scalar theories was previously studied in [6,29,38,43]. We outline it briefly below for completeness, but otherwise focus on new results, in particular for complex scalars, fermions, and vectors. This section considers only theories with one kind of massless particle. One can of course also couple scalars, fermions, and vectors in EFTs, and this is discussed in Sections 2.7, 2.8, and 2.9.

2.5.1 Pure Scalar EFTs

Consider an EFT with a single real scalar field ϕ. There can only be non-vanishing 3-point amplitudes in ϕ^3-theory and this gives amplitudes with soft weight $\sigma = -1$. Focusing on EFTs with soft weights $\sigma \geq 0$, the lowest-point amplitude is 4-point.

The on-shell factorization diagrams that contribute in the recursion relations (2.4.9) for $\mathcal{A}_6(1_\phi\, 2_\phi\, 3_\phi\, 4_\phi\, 5_\phi\, 6_\phi)$ are composed of a product of two 4-point amplitudes, for example the 123-channel diagram is

$$\mathcal{A}_6^{(123)} =$$



$$= \frac{\hat{\mathcal{A}}_L(0)\hat{\mathcal{A}}_R(0)}{P_{123}^2} + \sum_{i=1}^{6} \text{Res}_{z=\frac{1}{a_i}} \frac{\hat{\mathcal{A}}_L(z)\hat{\mathcal{A}}_R(z)}{z\, F(z)\, \hat{P}_{123}^2}\,, \tag{2.5.1}$$

where $\hat{\mathcal{A}}_L = \hat{\mathcal{A}}_4(1_\phi\, 2_\phi\, 3_\phi\, -P_\phi)$ and $\hat{\mathcal{A}}_R = \hat{\mathcal{A}}_4(P_\phi\, 4_\phi\, 5_\phi\, 6_\phi)$.[12] One sums over the 10 independent permutations corresponding to the 10 distinct factorization channels.[13]

For complex scalars, we assume that the input 4-point amplitudes are of the form $\mathcal{A}_4(1_Z\, 2_{\bar{Z}}\, 3_Z\, 4_{\bar{Z}})$;[14] one can also consider more general input but it would not be compatible

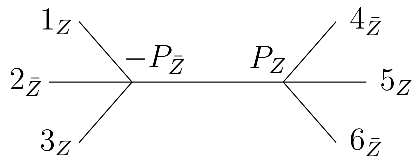
[12] The momenta in the hatted amplitudes are shifted; for simplicity, we do not write the hats on the momentum variables explicitly. Note that in particular P_ϕ should really be understood as $\hat{P}_\phi$ with $\hat{P}_\phi^2 = 0$.

[13] We do not consider color-ordering in this section. With color-ordering, one only includes the factorization diagrams from cyclic permutations of the external lines.

[14] There is no color-ordering implied in any of the amplitudes here. We simply alternate Z and $\bar{Z}$ states as odd/even numbered momentum lines. In later sections, other helicity states are grouped similarly, in particular for supersym-

with supersymmetry, so in the present paper we do not discuss such options. At 6-point, there is only one type of amplitude that can arise from such 4-point input via recursion, and that is $\mathcal{A}_6(1_Z\, 2_{\bar{Z}}\, 3_Z\, 4_{\bar{Z}}\, 5_Z\, 6_{\bar{Z}})$. The 123-channel diagram is

$$\mathcal{A}_6^{(123)} = \qquad (2.5.2)$$



To get the full amplitude, one must sum over all factorization channels:

$$\mathcal{A}_6(1_Z\, 2_{\bar{Z}}\, 3_Z\, 4_{\bar{Z}}\, 5_Z\, 6_{\bar{Z}}) = \Big(\mathcal{A}_6^{(123)} + (2 \leftrightarrow 4) + (2 \leftrightarrow 6)\Big) + (1 \leftrightarrow 5) + (3 \leftrightarrow 5)\,. \qquad (2.5.3)$$

In the following we consider real and complex scalar theories with 4- and 5-point fundamental vertices.

2.5.1.1 Fundamental 4-point Interactions

Consider a theory of a single real scalar with fundamental 4-point interactions. We parameterize $\mathcal{A}_4^{\text{ansatz}}$ as the most general polynomial in the Mandelstam variables s, t, u (with $s + t + u = 0$) and full Bose symmetry. We subject the recursed result for $\mathcal{A}_6$ to the test of a_i-independence, as described in Section 2.4.5. The result is

$$\partial^{2m}\phi^4 \qquad (2.5.4)$$

- [g]	m	$\mathcal{A}_4^{\text{ansatz}}(1_\phi\, 2_\phi\, 3_\phi\, 4_\phi)$	$\sigma = 0$	1	2	3	4
0	0	g	ϕ^4-theory	F	F	F	F
2	1	0	–	F	F	F	F
4	2	$g(s^2+t^2+u^2)$	–	–	DBI	F	F
6	3	$g\, stu$	–	–	Gal$_4$	Spec Gal$_4$	F
8	4	$g(s^4+t^4+u^4)$	–	–	–	F	F

In the table, we list the coupling dimension $[g]$ of the fundamental quartic couplings along with the most general ansatz for the corresponding 4-point amplitude. The dash, $-$, indicates that the constructibility criterion (2.4.10) fails; this means "triviality" in the sense described in Section 2.4.4). "F" indicates that the soft recursion fails to give an a_i-independent result, and hence no such theory can exist with the given assumptions. When a case passes the 6-point test, we are able to uniquely identify which theory it is. In the above table, the non-trivial theories that pass

metric cases, states that belong to the positive helicity sector sit on odd-numbered lines and negative helicity sector states on even-numbered lines. This is convenient for the practical implementation but should not be misunderstood as an indication of color-ordering.

the 6-point test are: ϕ^4-theory, DBI, and the quartic Galileon. The latter automatically has $\sigma = 3$ (which is called the Special Galileon) and passes 6-point test for both $\sigma = 2$ and $\sigma = 3$.

The analysis for complex scalars proceeds similarly and the results are

$$\partial^{2m} Z^2 \bar{Z}^2 \tag{2.5.5}$$

- [g]	m	$\mathcal{A}_4^{\text{ansatz}}(1_Z, 2_{\bar{Z}}, 3_Z, 4_{\bar{Z}})$	$\sigma = 0$	1	2	3
0	0	g	$\lvert Z\rvert^4$-theory	F	F	F
2	1	gt	–	$\mathbb{CP}^1$ NLSM	F	F
4	2	$gt^2 + g'su$	–	–	$g' = 0$ cmplx DBI	F
6	3	$gt^3 + g'stu$	–	–	$g = 0$ cmplx Gal_4	F
8	4	$gt^4 + g't^2su + g''s^2u^2$	–	–	–	F

The non-trivial theories are $|Z|^4$-theory, the $\mathbb{CP}^1$ NLSM (which is studied in further detail in Section 2.7), and the complex scalar versions of DBI and the quartic Galileon. Note that there does not exist a complex scalar version of the Special Galileon with $\sigma = 3$. The results for the 6-point amplitudes of each of the theories with $\sigma > 0$ can be found in Appendix B.

2.5.1.2 Fundamental 5-point Interactions

At 5-point, the input amplitudes are constructed as polynomials of Mandelstam variables s_{ij} and Levi-Civita contractions of momenta. They must obey (1) momentum conservation, (2) Bose symmetry, and (3) assumed soft behavior σ. In many cases, these constraints on the 5-point input amplitudes are sufficient to rule out such theories (assuming no other interactions) without even applying soft recursion.

As discussed at the end of Section 2.4.5, non-trivial 5-point amplitudes must vanish in 3d kinematics, so they are naturally written using the Levi-Civita tensor, as in the two cases of WZW and the quintic Galileon (2.4.29).

We can summarize the results in the following:

- **1 real scalar.** There are only two non-trivial theories based on a fundamental 5-point interaction, namely ϕ^5-theory, which has $[g_5] = -1$ and $\sigma = 0$, and the quintic Galileon, which has $[g_5] = -9$ and $\sigma = 2$.

- **1 complex scalar.** We assume input amplitudes of the form $\mathcal{A}_5(1_Z 2_{\bar{Z}} 3_Z 4_{\bar{Z}} 5_Z)$. Two cases pass the 8-point test:

  The quintic $g_5(Z^3\bar{Z}^2 + Z^2\bar{Z}^3)$-theory with $[g_5] = -1$ has $\sigma_Z = 0$.

  The complex-scalar version of the quintic Galileon with $[g_5] = -9$ and $\sigma_Z = 2$. The 5-point amplitude is

$$\mathcal{A}_5(1_Z 2_{\bar{Z}} 3_Z 4_{\bar{Z}} 5_Z) = g_5(\epsilon_{\mu\nu\rho\sigma} p_1^\mu p_2^\nu p_3^\rho p_4^\sigma)^2\,, \tag{2.5.6}$$

same as for the real-scalar quintic Galileon. The fact that it passes the 8-point test is somewhat trivial: because of the two explicit factors of momentum for 4 out of 5 particles, the residues at $1/a_i$ vanish identically for each factorization channel. The same is true for the real Galileon, so the 8-point test is not really effective as an indicator of whether such a theory may exist.

Suppose the putative complex-scalar quintic Galileon is coupled to the complex scalar DBI. Then we can conduct a 7-point test based on factorization into a quantic Galileon and a quartic DBI subamplitude. The test of a_i-independence requires the coupling constant g_5 to vanish. This means that the DBI-Galileon with a complex scalar cannot have a 5-point interaction.

At $[g_5] = -9$, there is a 6-parameter family of 5-point amplitudes with $\sigma_Z = 1$. The EFT with such amplitudes is generally non-constructible. However, a 1-parameter sub-family is compatible with the constraints of supersymmetry. As discussed in [18] and further in Section 2.9.1 this may be a candidate for a supersymmetric quintic Galileon with a limited sector of constructible amplitudes.

2.5.2 Pure Fermion EFTs

Let us now consider EFTs with only fermions and fundamental interactions of the form $\partial^{2m}\psi^2\bar{\psi}^2$. This is not the only choice, but it is the option compatible with supersymmetry. Moreover, we have found that couplings of "helicity violating" 4-point interactions in the fermion sector must vanish by the 6-point test in all pure-fermion cases we tested. The calculations proceed much the same way as for scalars, except that one must be more careful with signs when inserting fermionic states on the internal line. The diagrams needed for the recursive calculation of the 6-fermion amplitude $A_6(1^+_\psi\, 2^-_\psi\, 3^+_\psi\, 4^-_\psi\, 5^+_\psi\, 6^-_\psi)$ are just like those in the scalar case (2.5.2), but now the permutations have to be taken with a sign:

$$\mathcal{A}_6(1^+_\psi\, 2^-_\psi\, 3^+_\psi\, 4^-_\psi\, 5^+_\psi\, 6^-_\psi) = \Big(\mathcal{A}_6^{(123)} - (1 \leftrightarrow 5) - (3 \leftrightarrow 5)\Big) - (2 \leftrightarrow 4) - (2 \leftrightarrow 6). \tag{2.5.7}$$

The input 4-point amplitudes $\mathcal{A}_4(1^+_\psi\, 2^-_\psi\, 3^+_\psi\, 4^-_\psi)$ are fixed by little group scaling to be $\langle 24\rangle[13]$ times a Mandelstam polynomial of degree $m-1$ that must be symmetric under $s \leftrightarrow u$ to ensure Fermi antisymmetry for identical fermions. The most general input amplitudes for low values of

m are summarized in the table below that also shows the result of the recursive 6-point test:

$$\partial^{2m}\psi^2\bar{\psi}^2 \tag{2.5.8}$$

- [g]	m	$\mathcal{A}_4(1^+_\psi\, 2^-_\psi\, 3^+_\psi\, 4^-_\psi) = \langle 24\rangle[13]\times$	$\sigma = 0$	1	2	3
2	0	g	NJL	F	F	F
4	1	gt	–	A-V	F	F
6	2	$gt^2 + g'su$	–	–	F	F
8	3	$gt^3 + g'stu$	–	–	$g = 0$ new	F

We comment briefly on these results:

- The **NJL model** has the fundamental 4-fermion interaction $\bar{\psi}^2\psi^2$ and the result of recursing it to 6-point is given in Appendix B.0.1. The relevance of this model will for our purposes be as part of the supersymmetrization of the NLSM (see Section 2.7).
- **Akulov-Volkov theory** of Goldstinos is the only non-trivial EFT with coupling of mass-dimension -4. The Goldstinos in this theory have low-energy theorems with $\sigma = 1$. The 6-fermion amplitude is given in (B.0.14) in Appendix B.0.2.
- There are no constructible purely fermionic EFTs with fundamental quartic coupling $[g_4] = -6$. Nonetheless, as was shown in [18], the quartic Galileon has a supersymmetrization with a 4-fermion fundamental interaction, however, the fermion has $\sigma = 1$, so the all-fermion amplitudes in that theory are not constructible by soft recursion: one needs additional input from supersymmetry. We refer the reader to [18] and present some further details in Section 2.9.1.
- For $[g] = -8$ and $\sigma = 2$, the 6-point numerical test is passed in 4d kinematics without constraints on g and g'; that is because the recursed result depends only on ratios (2.4.27). When the 3d consistency check is employed, we learn that we must set $g = 0$ to ensure a_i-independence. (This is not a strong test since the particular form of the interaction, stu, ensures that all $1/a_i$-poles cancel in each factorization individual diagram.) Hence, the theory that passes the 6-point test with $\sigma = 2$ has $\mathcal{A}_4(1_\psi, 2_{\bar{\psi}}, 3_\psi, 4_{\bar{\psi}}) = g'\langle 24\rangle[13]stu$. The subtracted recursion relations fail at $n > 6$, which means that at 8-point and higher, this model is not uniquely determined by its symmetries. The Lagrangian construction of this theory has been studied as a fermionic generalization of the scalar Galileon [59].

2.5.3 Pure Vector EFTs

Pure abelian vector EFTs consist of interaction terms built from $F_{\mu\nu}$-contractions, possibly dressed with extra derivatives. In 4d, the Cayley-Hamilton relations imply that theories built from just field strengths $F_{\mu\nu}$ can be constructed from two types of index-contractions, namely (see for example [1])

$$f = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad \text{and} \quad g = -\frac{1}{4}F_{\mu\nu}\tilde{F}^{\mu\nu}\,, \tag{2.5.9}$$

where $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$. If one assumes parity, the Lagrangian can only contain even powers of g. One can then write an ansatz for the Lagrangian as

$$\mathcal{L} = f + \frac{b_1}{\Lambda^4}f^2 + \frac{b_2}{\Lambda^4}g^2 + \frac{b_3}{\Lambda^8}f^3 + \frac{b_4}{\Lambda^8}fg^2 + \dots \tag{2.5.10}$$

As established in Section 2.4.4, a model with photon interactions built of $F_{\mu\nu}$-contractions only have soft behavior $\sigma = 0$. The simplest 4-photon interactions may naively look like the vector equivalent of the constructible ϕ^4 scalar EFT. However, that is not the case. For the scalar, the 6-particle operator $\frac{1}{\Lambda^2}\phi^6$ is subleading to the pole contributions with two ϕ^4-vertices. However, for photons the pole terms with two $\frac{1}{\Lambda^4}F^4$-vertices are exactly the same order as $\frac{1}{\Lambda^8}F^6$. Therefore amplitudes in a theory with F^n interactions and $\sigma = 0$ are non-constructible, in other words it is trivial to have $\sigma = 0$ for any choice of coefficients b_i. One may ask if it is possible to choose the parameters b_i in (2.5.10) such that the amplitudes have enhanced soft behavior $\sigma > 0$. The 6-point soft recursive test shows that this is impossible, i.e. no models exist with Lagrangians of the form (2.5.10) and $\sigma > 0$.

Nonetheless, the class of theories with pure F^n-interactions do include one particularly interesting case, namely Born-Infeld (BI) theory. The BI Lagrangian can be written in 4d as

$$\mathcal{L}_{\mathrm{BI}} = \Lambda^4\left(1 - \sqrt{-\det\left(\eta_{\mu\nu} + F_{\mu\nu}/\Lambda^2\right)}\right). \tag{2.5.11}$$

Upon expansion, the Lagrangian will take the form (2.5.10) with some particular coefficients b_i. As noted, those particular coefficients do not change the single-soft behavior of amplitudes, the BI photon also has $\sigma = 0$. Nonetheless, BI theory does have the distinguishing feature of being the vector part of a supersymmetric EFT. In particular, $\mathcal{N} = 1$ supersymmetric Born-Infeld theory couples the BI vector to a Goldstino mode whose self-interactions are described by the Akulov-Volkov action. One can also view Born-Infeld as the vector part of the $\mathcal{N} = 2$ or $\mathcal{N} = 4$ supersymmetrization of DBI. It was argued recently [1] that supersymmetry ensures BI amplitudes to vanish in certain multi-soft limits. Based on that, the BI amplitudes can be calculated unambiguously using on-shell techniques [1]. Alternatively, one can show that the $\mathcal{N} = 1$ supersymmetry Ward identities uniquely fix the BI amplitudes in terms of amplitudes with Goldstinos; we discuss this briefly in Section 2.8 and in further detail in the context of partial breaking of supersymmetry in a forthcoming paper.

Next, one can consider EFTs in which the field strengths are dressed with derivatives, for example

$$\mathcal{L} = -\frac{1}{4}F^2 + \frac{c_1}{\Lambda^6}\partial^2 F^4 + \frac{c_1}{\Lambda^{12}}\partial^4 F^6 + \ldots \tag{2.5.12}$$

Theories with fundamental 4-point interactions are non-constructible for $\sigma = 0$ and fail the soft recursion a_i-independence 6-point test for $\sigma > 0$. One implication of this is that there can be no vector Goldstone bosons with vanishing low-energy theorems. This conclusion was also reached in [60], but from a very different algebraically-based analysis. A second implication is that the pure vector sector of an $\mathcal{N} \geq 2$ Galileon model is non-constructible with the basic soft recursion, and other properties (such as supersymmetry) have to be specified in order to determine those amplitudes recursively.

There are other interesting vector EFTs: we study in detail the $\mathcal{N} = 2$ supersymmetric NLSM in Section 2.7. Furthermore, massive gravity [61–63] motivates the existence of a vector-scalar theory coupling Galileons to a vector field; we explore this in Section 2.9.2.

2.6 Soft Limits and Supersymmetry

For models with unbroken supersymmetry, the on-shell amplitudes satisfy a set of linear relations known as the *supersymmetry Ward identities* [64, 65]. (For recent reviews and results, see [20, 31, 66].) In this section, we use $\mathcal{N} = 1$ supersymmetry to derive general consequences for the soft behavior for massless particles in the same supermultiplet. It is not assumed that these particles are Goldstone or quasi-Goldstone modes; the results apply to all $\mathcal{N} = 1$ supermultiplets of massless particles. The consequences for extended supersymmetry are directly inferred from the $\mathcal{N} = 1$ constraints.

g

2.6.1 $\mathcal{N} = 1$ Supersymmetry Ward Identities

We consider $\mathcal{N} = 1$ *chiral* and *vector* supermultiplets. We use the following shorthand for the action of the supercharges on individual particles with momentum label i: for chiral multiplets

state i	$\mathcal{Q} \cdot i$	$\mathcal{A}_n$ prefactor	$\mathcal{Q}^\dagger \cdot i$	$\mathcal{A}_n$ prefactor
ψ^+	Z	$\vert i]$	0	0
Z	0	0	ψ^+	$-\vert i\rangle$
$\overline{Z}$	ψ^-	$\vert i]$	0	0
ψ^-	0	0	$\overline{Z}$	$-\vert i\rangle$

(2.6.1)

where Z is a complex scalar and ψ is a Weyl fermion. The superscripts $\pm$ refer to the helicity of the particle. $\mathcal{Q}^\dagger$ raises helicity by $1/2$ while $\mathcal{Q}$ lowers it by $1/2$. The prefactor is what goes outside the amplitude when the supercharge acts on it, e.g.

$$\begin{aligned}\mathcal{Q}\cdot\mathcal{A}_n\big(1_Z\,2^+_\psi\,3^+_\psi\,4_{\overline{Z}}\ldots\big) = \ 0\ +\ |2]\mathcal{A}_n\big(1_Z\,2_Z\,3^+_\psi\,4_{\overline{Z}}\ldots\big)\ -\ |3]\mathcal{A}_n\big(1_Z\,2^+_\psi\,3_Z\,4_{\overline{Z}}\ldots\big)\\ +\ |4]\mathcal{A}_n\big(1_Z\,2^+_\psi\,3^+_\psi\,4^-_\psi\ldots\big)\ +\ldots\end{aligned} \tag{2.6.2}$$

Due to the Grassmann nature of the supercharges, there is a minus sign for each fermion that the supercharge has to move past to get to the ith state.

Similarly for a vector multiplet:

state i	$\mathcal{Q}\cdot i$	$\mathcal{A}_n$ prefactor	$\mathcal{Q}^\dagger\cdot i$	$\mathcal{A}_n$ prefactor
γ^+	ψ^+	$\lvert i]$	0	0
ψ^+	0	0	γ^+	$-\lvert i\rangle$
ψ^-	γ^-	$-\lvert i]$	0	0
γ^-	0	0	ψ^-	$\lvert i\rangle$

(2.6.3)

where ψ is a Weyl fermion and γ is a vector boson.

In this notation, the supersymmetry Ward identities are equivalent to the statement that the following action of the supercharges annihilates the amplitude [20, 31, 66]

$$\begin{aligned}0 = \mathcal{Q}\cdot\mathcal{A}_n\left(1,\ldots,n\right) &= \sum_{i=1}^{n}(-1)^{L_i+P_i}|i]\mathcal{A}_n\left(1,\ldots,\mathcal{Q}\cdot i,\ldots,n\right),\\ 0 = \mathcal{Q}^\dagger\cdot\mathcal{A}_n\left(1,\ldots,n\right) &= \sum_{i=1}^{n}(-1)^{L_i+P_i}|i\rangle\mathcal{A}_n\left(1,\ldots,\mathcal{Q}^\dagger\cdot i,\ldots,n\right),\end{aligned} \tag{2.6.4}$$

where L_i is equal to the number of fermions to the *left* of $\mathcal{Q}^{(\dagger)}\cdot i$ and the factors $P_i = 0$ or 1 correspond to the additional minus signs associated with the spinor prefactors as described in Tables 2.6.1 and 2.6.3. Note that the action of the supercharges always changes the number of fermions by ±1, but that amplitudes are non-vanishing only if the number of fermions is even. So to get an interesting relation among amplitudes on the right-hand-side, the amplitude on the left-hand-side must vanish identically.

2.6.2 Soft Limits and Supermultiplets

We consider the chiral multiplet and vector multiplet separately and then extend the results to enhanced supersymmetry.

Chiral multiplet. Define the *soft factors* $\mathcal{S}_n^{(i)}$ as the momentum dependent coefficients in the holomorphic soft expansion taken here for simplicity on the first particle

$$\begin{aligned}
\mathcal{A}_n\left(\{\epsilon|1\rangle,|1]\}_Z,\ldots\right) &\rightarrow \mathcal{S}_n^{(0)}(1_Z,\ldots)\,\epsilon^{\sigma_Z}+\mathcal{S}_n^{(1)}(1_Z,\ldots)\,\epsilon^{\sigma_Z+1}+\mathcal{O}\left(\epsilon^{\sigma_Z+2}\right)\,,\\
\mathcal{A}_n\left(\{\epsilon|1\rangle,|1]\}_\psi^+,\ldots\right) &\rightarrow \mathcal{S}_n^{(0)}(1_\psi^+,\ldots)\,\epsilon^{\sigma_\psi}+\mathcal{S}_n^{(1)}(1_\psi^+,\ldots)\,\epsilon^{\sigma_\psi+1}+\mathcal{O}\left(\epsilon^{\sigma_\psi+2}\right).
\end{aligned}\tag{2.6.5}$$

The soft weights are σ_Z and σ_ψ for the scalar and fermion, respectively. To see how supersymmetry forces relations among the soft weights and soft factors we use (2.6.4) to write

$$\begin{aligned}
\mathcal{A}_n\left(1_Z,\ldots,n\right) &= \sum_{i=2}^{n}(-1)^{L_i+P_i+1}\frac{[Xi]}{[X1]}\,\mathcal{A}_n\left(1_\psi^+,\ldots,\mathcal{Q}\cdot i,\ldots,n\right),\\
\mathcal{A}_n\left(1_\psi^+,\ldots,n\right) &= \sum_{i=2}^{n}(-1)^{L_i+P_i+1}\frac{\langle Xi\rangle}{\langle X1\rangle}\,\mathcal{A}_n\left(1_Z,\ldots,\mathcal{Q}^\dagger\cdot i,\ldots,n\right),
\end{aligned}\tag{2.6.6}$$

where the arbitrary X-spinor cannot be proportional to $|1\rangle$ or $|1]$.

Taking the holomorphic soft expansion on the right-hand-side of these expressions, in the second line only, an extra power of ϵ appears in the denominator and we find

$$\begin{aligned}
\mathcal{S}_n^{(0)}(1_Z,\ldots)\,\epsilon^{\sigma_Z}+\mathcal{O}\left(\epsilon^{\sigma_Z+1}\right) &= \sum_{i=2}^{n}(-1)^{L_i+P_i+1}\frac{[Xi]}{[X1]}\mathcal{S}_n^{(0)}(1_\psi^+,\ldots,\mathcal{Q}\cdot i,\ldots)\,\epsilon^{\sigma_\psi}+\mathcal{O}\left(\epsilon^{\sigma_\psi+1}\right),\\
\mathcal{S}_n^{(0)}(1_\psi^+,\ldots)\,\epsilon^{\sigma_\psi}+\mathcal{O}\left(\epsilon^{\sigma_\psi+1}\right) &= \sum_{i=2}^{n}(-1)^{L_i+P_i+1}\frac{\langle Xi\rangle}{\langle X1\rangle}\mathcal{S}_n^{(0)}(1_Z,\ldots,\mathcal{Q}^\dagger\cdot i,\ldots)\,\epsilon^{\sigma_Z-1}+\mathcal{O}\left(\epsilon^{\sigma_Z}\right).
\end{aligned}$$

The leading power of ϵ on the right-hand-side must match the leading power on the left. It is possible that cancellations among the terms on the right-hand-side may effectively increase the leading power but never decrease it. This then gives the following inequalities

$$\sigma_Z \geq \sigma_\psi \quad \text{and} \quad \sigma_\psi \geq \sigma_Z - 1\,,\tag{2.6.7}$$

for which there are only two solutions

$$\boxed{\sigma_Z = \sigma_\psi + 1 \quad \text{or} \quad \sigma_Z = \sigma_\psi\,.}\tag{2.6.8}$$

These two options have different consequences for the soft factors. For $\sigma_Z = \sigma_\psi + 1$, we have

$$\begin{aligned}
0 &= \sum_{i=2}^{n}(-1)^{L_i+P_i}[Xi]\,\mathcal{S}_n^{(0)}\left(1_\psi^+,\ldots,\mathcal{Q}\cdot i,\ldots\right),\\
\mathcal{S}_n^{(0)}\left(1_\psi^+,\ldots\right) &= \sum_{i=2}^{n}(-1)^{L_i+P_i+1}\frac{\langle Xi\rangle}{\langle X1\rangle}\,\mathcal{S}_n^{(0)}\left(1_Z,\ldots,\mathcal{Q}^\dagger\cdot i,\ldots\right),
\end{aligned}\tag{2.6.9}$$

while for $\sigma_\phi = \sigma_\psi$, we have

$$\begin{aligned} 0 &= \sum_{i=2}^{n} (-1)^{L_i+P_i} \langle Xi \rangle \mathcal{S}_n^{(0)} \left(1_Z, \ldots, \mathcal{Q}^\dagger \cdot i, \ldots\right), \\ \mathcal{S}_n^{(0)} \left(1_Z, \ldots\right) &= \sum_{i=2}^{n} (-1)^{L_i+P_i+1} \frac{[Xi]}{[X1]} \mathcal{S}_n^{(0)} \left(1_\psi^+, \ldots, \mathcal{Q} \cdot i, \ldots\right). \end{aligned} \tag{2.6.10}$$

In addition there will be an infinite number of similar relations which come from matching higher powers in ϵ.

Vector multiplet. We define the soft factors as

$$\mathcal{A}_n \left(\{\epsilon|1\rangle, |1]\}_\gamma^+, \ldots\right) \rightarrow \mathcal{S}_n^{(0)}(1_\gamma^+, \ldots)\, \epsilon^{\sigma_\gamma} + \mathcal{S}_n^{(1)}(1_\gamma^+, \ldots)\, \epsilon^{\sigma_\gamma+1} + \mathcal{O}\left(\epsilon^{\sigma_\gamma+2}\right). \tag{2.6.11}$$

The analysis of the supersymmetry Ward identities proceeds similarly to that of the chiral multiplet and results in only two options for the soft weights:

$$\boxed{\sigma_\psi = \sigma_\gamma + 1, \quad \text{or} \quad \sigma_\psi = \sigma_\gamma\,.} \tag{2.6.12}$$

The consequences for the soft factors are for $\sigma_\psi = \sigma_\gamma + 1$

$$\begin{aligned} 0 &= \sum_{i=2}^{n} (-1)^{L_i+P_i} [Xi] \mathcal{S}_n^{(0)} \left(1_\gamma^+, \ldots, \mathcal{Q} \cdot i, \ldots\right), \\ \mathcal{S}_n^{(0)} \left(1_\gamma^+, \ldots\right) &= \sum_{i=2}^{n} (-1)^{L_i+P_i+1} \frac{\langle Xi \rangle}{\langle X1 \rangle} \mathcal{S}_n^{(0)} \left(1_\psi^+, \ldots, \mathcal{Q}^\dagger \cdot i, \ldots\right), \end{aligned} \tag{2.6.13}$$

and for $\sigma_\gamma = \sigma_\psi$

$$\begin{aligned} 0 &= \sum_{i=2}^{n} (-1)^{L_i+P_i} \langle Xi \rangle \mathcal{S}_n^{(0)} \left(1_\psi^+, \ldots, \mathcal{Q}^\dagger \cdot i, \ldots\right), \\ \mathcal{S}_n^{(0)} \left(1_\psi^+, \ldots\right) &= \sum_{i=2}^{n} (-1)^{L_i+P_i+1} \frac{[Xi]}{[X1]} \mathcal{S}_n^{(0)} \left(1_\gamma^+, \ldots, \mathcal{Q} \cdot i, \ldots\right). \end{aligned} \tag{2.6.14}$$

Note that we have made no assumptions about the sign of σ, so the relations derived here are totally general. Also, the supersymmetry Ward identities hold at all orders in perturbation theory, so the relations among the soft behaviors remain true at loop-level.

Extended supersymmetry. Relations between the soft weights of particles in the same massless supermultiplets in extended supersymmetry follow directly from the $\mathcal{N} = 1$ results above, since the supersymmetry Ward identities take the same form for each pair of $(s, s+\frac{1}{2})$-multiplets. In particular, the soft weights of the boson (σ_B) and fermion (σ_F) in a $(s, s+\frac{1}{2})$-multiplet are related as

$$\begin{cases} \sigma_B = \sigma_F + 1 \quad \text{or} \quad \sigma_B = \sigma_F \quad \text{for } s \text{ integer}\,, \\ \sigma_B = \sigma_F - 1 \quad \text{or} \quad \sigma_B = \sigma_F \quad \text{for } s \text{ half-integer}\,. \end{cases} \tag{2.6.15}$$

These relations will be useful in later applications in this paper. For now, we make a small aside and demonstrate the application of (2.6.15) to the case of spontaneously broken superconformal symmetry and for unbroken extended supergravity.

2.6.3 Application to Superconformal Symmetry Breaking

The breaking of conformal symmetry gives rise to a single Goldstone mode [50], often called the dilaton. It has been established in the literature [4,67,68] that this dilaton obeys low-energy theorems with $\sigma = 0$. In a superconformal theory, breaking of conformal invariance must be accompanied by breaking of the superconformal symmetries. This follows from the algebra: $\{\mathcal{S}, \mathcal{S}^\dagger\} = \mathcal{K}$, $[\mathcal{Q}, \mathcal{K}] = \mathcal{S}^\dagger$ and $[\mathcal{Q}^\dagger, \mathcal{K}] = \mathcal{S}$, where $\mathcal{K}$ are the generators of conformal boosts, $\mathcal{S}$ and $\mathcal{S}^\dagger$ are the superconformal fermionic generators, and $\mathcal{Q}$ and $\mathcal{Q}^\dagger$ are the regular supercharges with $\{\mathcal{Q}, \mathcal{Q}^\dagger\} = \mathcal{P}$.

Assuming $\mathcal{Q}$-supersymmetry to be unbroken, the dilaton will be joined by a Goldstone mode from the broken R-symmetry to form a complex scalar Z with $\sigma_Z = 0$.[15] It follows from our general analysis that the fermionic partner of Z will have $\sigma = 0$ or $\sigma = -1$. For the latter, Yukawa-interactions are necessary [57] and supersymmetry then requires cubic scalar interactions $Z|Z|^2 + \text{h.c.}$ which would imply $\sigma = -1$ for the dilaton. Since $\sigma_Z = 0$, $\sigma = -1$ is not possible for the dilaton and we conclude that the Goldstino mode associated with the breaking of the superconformal fermionic symmetries generated by $\mathcal{S}$ and $\mathcal{S}^\dagger$ must have low-energy theorems with soft weight $\sigma = 0$.

An example is $\mathcal{N} = 4$ SYM on the Coulomb branch with the simplest breaking pattern.[16] The R-symmetry is broken from $SO(6)$ to $SO(5)$ and the five broken generators give rise to five Goldstone modes which join the dilaton of the conformal breaking to be the 6 real scalars of an $\mathcal{N} = 4$ massless multiplet. The supermultiplet also contains the 4 Goldstinos associated with the four broken superconformal generators. The supermultiplet is capped off by a $U(1)$ vector whose soft weight, by the above analysis, must be either $\sigma = 0$ or -1. The states that are charged under this $U(1)$ are the massive W-multiplets and in their presence, one can have $\sigma = -1$, otherwise $\sigma = 0$ for the vector.

[15] An example of the bosonic part of an $\mathcal{N} = 1$ effective action of the dilaton and a $U(1)_R$ Goldstone boson can be found in [69].

[16] See [68,70] for explicit amplitudes on the Coulomb branch of $\mathcal{N} = 4$ SYM.

2.6.4 Application to Supergravity

It is well-known that gravitons have a universal soft behavior [71]: when the soft limit (2.4.2) is applied to a single graviton, the amplitude diverges as $1/\epsilon^3$, i.e. the soft weight is $\sigma_2 = -3$. (In this section, we use a subscript on the soft weight to indicate the spin of the particle.) Applying (2.6.15) shows that the gravitino can have $\sigma_{3/2} = -2$ or -3. However, unitarity and locality constraints show [57] that amplitudes cannot be more singular than $1/\epsilon^2$ for a single soft gravitino, so it must be that $\sigma_{3/2} = -2$. This must be true in any supergravity theory.

Consider now a graviphoton in $\mathcal{N} \geq 2$ supergravity. Its supersymmetry Ward identities with the gravitino imply $\sigma_1 = -2$ or $\sigma_1 = -1$. The $\sigma_1 = -2$ behavior requires the graviphoton, and by supersymmetry also the gravitino, to interact with a pair of electrically charged particles via a dimensionless coupling; however, for the gravitino such a coupling is inconsistent with unitarity and locality [57]. So there is only one option, namely $\sigma_1 = -1$.

In pure $\mathcal{N} \geq 3$ supergravity, we also have spin-$\frac{1}{2}$ fermions in the graviton supermultiplet. By (2.6.15) and the previous results, they can have either $\sigma_{1/2} = -1$ or 0. The analysis in [57] shows that $\sigma_{1/2} = -1$ requires a dimensionless coupling of the spin-$\frac{1}{2}$ particle with two other particles, for example via a Yukawa coupling. Since there are no dimensionless couplings in pure supergravity, it follows from [57] that the amplitude has to be $O(\epsilon^0)$ or softer. This leaves only one option, namely that $\sigma_{1/2} = 0$ in pure supergravity.

In pure $\mathcal{N} \geq 4$ supergravity, the scalars in the supermultiplet can have $\sigma_0 = 0$ or $\sigma_0 = 1$. If we focus on the MHV sector, the supersymmetry Ward identities give

$$\mathcal{A}_n(1_Z\, 2_{\bar{Z}}\, 3_h^-\, 4_h^+ \ldots n_h^+) = \frac{\langle 13\rangle^4}{\langle 23\rangle^4}\, \mathcal{A}_n(1_h^+\, 2_h^-\, 3_h^-\, 4_h^+ \ldots n_h^+)\,, \tag{2.6.16}$$

where Z and $\bar{Z}$ denote any pair of conjugate scalars and h are gravitons. Taking line 1 soft holomorphically, $|1\rangle \to \epsilon|1\rangle$, the graviton amplitude on the RHS diverges as $1/\epsilon^3$ but the prefactor vanishes as ϵ^4. It follows that the MHV amplitude vanishes as $O(\epsilon)$ in the single soft-scalar limit. In other words, for MHV amplitudes $\sigma_0 = 1$. It is tempting to conclude that one must have $\sigma_0 = 1$ for all amplitudes, but that is too glib, as we now explain.

It is known that the scalar cosets of $\mathcal{N} \geq 4$ pure supergravity theories in 4d are symmetric, and therefore lead to $\sigma_0 = 1$ vanishing low-energy theorems. But at the level of the on-shell amplitudes, this conclusion does not follow from the supersymmetry Ward identities alone: as we have seen, they give $\sigma_0 = 1$ or $\sigma_0 = 0$. That analysis has to remain true at all loop-orders. In $\mathcal{N} = 4$ supergravity, for example, the anomaly of the $U(1)$ R-symmetry can be expected to affect the soft behavior at some order. Our arguments show that it cannot happen in the MHV sector, but does not rule it out beyond MHV; this is what the $\sigma_0 = 0$ accounts for. Furthermore, one can add

helicity state	σ
+2 graviton	-3
+3/2 gravitino	-2
+1 graviphoton	-1
+1/2 fermion	0
0 scalar	0 or $+1$
-1/2 fermion	$+1$
-1 graviphoton	$+1$
-3/2 gravitino	$+1$
-2 graviton	$+1$

Table 2.2: Holomorphic soft weights σ for the $\mathcal{N}=8$ supermultiplet. Note that the soft weights in this table follow from taking the soft limit holomorphically, $|i\rangle \to \epsilon|i\rangle$ for all states, independently of the sign of their helicity. At each step in the spectrum, the soft weight either changes by 1 or not at all. Note that one could also have used the anti-holomorphic definition $|i] \to \epsilon|i]$ of taking the soft limit; in that case the soft weights would just have reversed, to start with $\sigma=-3$ for the negative helicity graviton, but no new constraints would have been obtained on the scalar soft weights. In $\mathcal{N}=8$ supergravity, the 70 scalars are Goldstone bosons of the coset $E_{7(7)}/SU(8)$ and hence $\sigma=1$. Including higher-derivative corrections may change this behavior to $\sigma=0$ depending on whether the added terms are compatible with the coset structure.

higher-derivative operators to the supergravity action such that supersymmetry is preserved but the low-energy theorems are not. Indeed, string theory does this in the α'-expansion by adding to the $\mathcal{N}=8$ tree-level action a supersymmetrizable operator $\alpha'^3 e^{-6\phi} R^4$. This operator does not affect the soft behavior of MHV amplitudes, but it is known that it does result in non-vanishing single soft scalar limits for 6-particle NMHV amplitudes at order α'^3 [72, 73].

The results for $\mathcal{N}=8$ supersymmetry are summarized in Table 2.2.

2.6.5 MHV Classification and Examples of Supersymmetry Ward Identities

For later convenience, we state here the explicit form of the supersymmetry Ward identities (2.6.4) for a few particularly useful cases. We focus on the chiral multiplet, but similar results apply to the vector multiplet.

First we make the simple observation that amplitudes with all Z's or only one $\bar{Z}$ and rest Z's vanish:

$$\mathcal{A}_n\big(1_Z\, 2_Z\, 3_Z\, 4_Z \ldots n_Z\big) = 0 \quad \text{and} \quad \mathcal{A}_n\big(1_Z\, 2_{\bar{Z}}\, 3_Z\, 4_Z \ldots\big) = 0\,. \tag{2.6.17}$$

This follows from the supersymmetry Ward identities such as

$$0 = \mathcal{Q} \cdot \mathcal{A}_n\big(1^+_\psi\, 2_Z\, 3_Z\, 4_Z \ldots n_Z\big) = |1]\, \mathcal{A}_n\big(1_Z\, 2_Z\, 3_Z\, 4_Z \ldots n_Z\big)\,,$$
$$0 = \mathcal{Q} \cdot \mathcal{A}_n\big(1^+_\psi\, 2_{\bar{Z}}\, 3_Z\, 4_Z \ldots n_Z\big) = |1]\, \mathcal{A}_n\big(1_Z\, 2_{\bar{Z}}\, 3_Z\, 4_Z \ldots n_Z\big) - |2]\, \mathcal{A}_n(1^+_\psi\, 2^-_\psi\, 3_Z\, 4_Z \ldots n_Z\big)\,.$$

Dotting in $[2|$ gives (2.6.17). Similarly $\mathcal{A}_n(1^+_\psi\, 2^-_\psi\, 3_Z\, 4_Z \ldots n_Z\big) = 0$ and so on. In the context of gluon scattering, the equivalent statements are that amplitudes with helicity structure $+++\ldots+$ or $-++\ldots+$ vanish. These helicity configurations are often called "helicity violating".

The simplest non-vanishing amplitudes are often denoted MHV (Maximally Helicity Violating) in the context of gluon scattering and we adapt the same nomenclature here. MHV amplitudes obey the simplest supersymmetry Ward identities in that they are just linear proportionality relations. For example, it follows from

$$\begin{aligned} 0 &= \mathcal{Q} \cdot \mathcal{A}_n\big(1^+_\psi\, 2_{\bar{Z}}\, 3_Z\, 4_{\bar{Z}}\, 5_Z \ldots n_Z\big) \\ &= |1]\, \mathcal{A}_n\big(1_Z\, 2_{\bar{Z}}\, 3_Z\, 4_{\bar{Z}} \ldots\big) - |2]\, \mathcal{A}_n(1^+_\psi\, 2^-_\psi\, 3_Z\, 4_{\bar{Z}} \ldots\big) - |4]\, \mathcal{A}_n(1^+_\psi\, 2_{\bar{Z}}\, 3_Z\, 4^-_\psi \ldots\big) \end{aligned} \tag{2.6.18}$$

upon dotting in $[4|$ that

$$\mathcal{A}_n(1^+_\psi\, 2^-_\psi\, 3_Z\, 4_{\bar{Z}}\, 5_Z \ldots n_Z) = \frac{[14]}{[24]}\, \mathcal{A}_n(1_Z\, 2_{\bar{Z}}\, 3_Z\, 4_{\bar{Z}}\, 5_Z \ldots n_Z). \tag{2.6.19}$$

Similarly, one finds that the MHV amplitude with four fermions is proportional to the one with two fermions. To summarize, MHV amplitudes satisfy

$$\begin{aligned} \mathcal{A}_n(1^+_\psi\, 2^-_\psi\, 3^+_\psi\, 4^-_\psi\, 5_Z \ldots n_Z) &= \frac{[13]}{[14]}\, \mathcal{A}_n(1^+_\psi\, 2^-_\psi\, 3_Z\, 4_{\bar{Z}}\, 5_Z \ldots n_Z) \\ &= \frac{[13]}{[24]}\, \mathcal{A}_n(1_Z\, 2_{\bar{Z}}\, 3_Z\, 4_{\bar{Z}}\, 5_Z \ldots n_Z)\,. \end{aligned} \tag{2.6.20}$$

The second-simplest class of supersymmetric Ward identities relate amplitudes in the NMHV class. In this paper, the 6-particle amplitudes play a central role, so we write down the 6-point NMHV supersymmetry Ward identities explicitly:

$$\begin{aligned} &|1]\mathcal{A}_6(1_Z\, 2_{\bar{Z}}\, 3_Z\, 4_{\bar{Z}}\, 5_Z\, 6_{\bar{Z}}) - |2]\mathcal{A}_6(1^+_\psi\, 2^-_\psi\, 3_Z\, 4_{\bar{Z}}\, 5_Z\, 6_{\bar{Z}}) \\ &\qquad - |4]\mathcal{A}_6(1^+_\psi\, 2_{\bar{Z}}\, 3_Z\, 4^-_\psi\, 5_Z\, 6_{\bar{Z}}) - |6]\mathcal{A}_6(1^+_\psi\, 2_{\bar{Z}}\, 3_Z\, 4_{\bar{Z}}\, 5_Z\, 6^-_\psi) = 0\,, \end{aligned} \tag{2.6.21}$$

$$\begin{aligned} &|1]\mathcal{A}_6(1_Z\, 2^-_\psi\, 3^+_\psi\, 4_{\bar{Z}}\, 5_Z\, 6_{\bar{Z}}) + |3]\mathcal{A}_6(1^+_\psi\, 2^-_\psi\, 3_Z\, 4_{\bar{Z}}\, 5_Z\, 6_{\bar{Z}}) \\ &\qquad - |4]\mathcal{A}_6(1^+_\psi\, 2^-_\psi\, 3^+_\psi\, 4^-_\psi\, 5_Z\, 6_{\bar{Z}}) - |6]\mathcal{A}_6(1^+_\psi\, 2^-_\psi\, 3^+_\psi\, 4_{\bar{Z}}\, 5_Z\, 6^-_\psi) = 0\,, \end{aligned} \tag{2.6.22}$$

$$\begin{aligned} &|1]\mathcal{A}_6(1_Z\, 2^-_\psi\, 3^+_\psi\, 4^-_\psi\, 5^+_\psi\, 6_{\bar{Z}}) + |3]\mathcal{A}_6(1^+_\psi\, 2^-_\psi\, 3_Z\, 4^-_\psi\, 5^+_\psi\, 6_{\bar{Z}}) \\ &\qquad + |5]\mathcal{A}_6(1^+_\psi\, 2^-_\psi\, 3^+_\psi\, 4^-_\psi\, 5_Z\, 6_{\bar{Z}}) - |6]\mathcal{A}_6(1^+_\psi\, 2^-_\psi\, 3^+_\psi\, 4^-_\psi\, 5^+_\psi\, 6^-_\psi) = 0\,. \end{aligned} \tag{2.6.23}$$

We now turn to applications of these results.

2.7 Supersymmetric Non-linear Sigma Model

Perhaps the simplest and most familiar class of models that exhibit both linearly realized supersymmetry and interesting low-energy theorems are the *supersymmetric non-linear sigma models*. Of particular interest are the *coset* sigma models for which the target manifold is a homogeneous space G/H. At lowest order, the coset sigma model captures the universal low-energy behavior of the scalar Goldstone modes of a spontaneous symmetry breaking pattern $G \to H$, where G and H are the *isometry* and *isotropy* groups of the target manifold respectively. If the target manifold is additionally a *symmetric* space and there are no 3-point interactions, then the off-shell Ward-Takahashi identities for the spontaneously broken currents imply $\sigma = 1$ vanishing low-energy theorems for the Goldstone scalars. An interesting recent perspective on coset sigma models can be found in [74].

At leading order it is fairly straightforward to calculate the on-shell scattering amplitudes for such a model from the (two-derivative) non-linear sigma model effective action. Using the methods of on-shell recursion, the use of an effective action is unnecessary. Instead, we may assume low-energy theorems and on-shell Ward identities of the isotropy group H as the on-shell data that defines the model. Using the procedure of the soft bootstrap described in Section 2.4.5, we may apply subtracted recursion to construct the contributions to the S-matrix at leading order.

A particularly simple and well-studied example of such a construction has previously been given for the $\frac{U(N)\times U(N)}{U(N)}$ coset sigma model [38, 53]. There are several nice features of this model which make it an appealing toy-model to study on-shell. As will be discussed in Section 2.9.4, at leading order ($\tilde{\Delta} = 1$ or equivalently two-derivative) the isotropy $U(N)$ symmetry allows for the construction of *flavor*-ordered partial amplitudes with only $(n-3)!$ independent amplitudes for the scattering of n Goldstone scalars.

The situation is somewhat less straightforward for models describing the low-energy dynamics of the Goldstone modes of internal symmetry breaking with some amount of linearly realized supersymmetry.[17] There are several interesting consequences of this combination of symmetries. The states must form mass degenerate multiplets of the supersymmetry algebra, which in this case means that the Goldstone scalars must always transform together with additional massless spinning states. As discussed in Section 2.6.2, the low-energy theorems of each of the particles in these *Goldstone* multiplets are not independent.

It is well-known in the literature of supersymmetric field theories that to construct a supersym-

[17] In this more general context *internal* symmetry includes R-symmetry. For our purposes the relevant property is that the conserved charges are Lorentz scalars and so correspond to a spectrum of spin-0 Goldstone modes.

metric action, the massless scalar modes must parametrize a target space manifold with *Kähler* structure for $\mathcal{N}=1$ supersymmetry [75]. For $\mathcal{N}=2$ supersymmetry the target space manifold must have the structure

$$\mathcal{M}_{\mathcal{N}=2}=\mathcal{M}_{\mathrm{V}}\times\mathcal{M}_{\mathrm{H}}, \tag{2.7.1}$$

where the scalars of the *vector* multiplets parametrize the *special-Kähler* manifold $\mathcal{M}_{\mathrm{V}}$ while the scalars belonging to *hyper* multiplets parametrize the *hyper-Kähler* manifold $\mathcal{M}_{\mathrm{H}}$ [76]. As a consequence, despite the obvious virtues of a flavor ordered representation, this makes studying the supersymmetrization of the $\frac{U(N)\times U(N)}{U(N)}$ coset sigma model using subtracted recursion more difficult, since even in the $\mathcal{N}=1$ case the target manifold is not Kähler. This does not mean that the internal symmetry breaking pattern $U(N)\times U(N)\to U(N)$ is impossible in an $\mathcal{N}=1$ supersymmetric model. Rather it means that the target space contains $\frac{U(N)\times U(N)}{U(N)}$ as a non-Kähler submanifold and includes additional directions in field space or equivalently includes additional massless *quasi-Goldstone* scalars [77]. In general there is no unique way to extend the symmetry breaking coset to a Kähler manifold, because in any given example the spectrum of quasi-Goldstone modes depends on the details of the UV physics. Correspondingly, the quasi-Goldstone scalars do not satisfy the kind of universal low-energy theorems necessary for us to construct the scattering amplitudes recursively.

Instead, in this section we will study the interplay of low-energy theorems and supersymmetry by considering the simplest symmetric coset that is both Kähler and special-Kähler

$$\frac{SU(2)}{U(1)}\cong\mathbb{CP}^1\,, \tag{2.7.2}$$

and therefore should admit both an $\mathcal{N}=1$ and $\mathcal{N}=2$ supersymmetrization. Our assumption here is that the target manifold *is* the coset manifold and therefore the massless spectrum should contain only two real scalar degrees of freedom, both Goldstone modes. They form a single complex scalar field $Z,\overline{Z}$ which carries a conserved charge associated with the isotropy $U(1)$. These properties uniquely determine the Goldstone multiplets as an $\mathcal{N}=1$ chiral and $\mathcal{N}=2$ vector multiplet respectively.

The main results of this section are (1) the demonstration that both the $\mathcal{N}=1$ and $\mathcal{N}=2$ $\mathbb{CP}^1$ non-linear sigma models are constructible on-shell using recursion without the need to explicitly construct an effective action. And (2) this construction gives a new *on-shell* perspective on the relationship between the linearly realized target space isotropies of $\mathcal{M}_{\mathrm{V}}$ and electric-magnetic duality transformations of the associated vector bosons.

2.7.1 $\mathcal{N}=1$ $\mathbb{CP}^1$ NLSM

The $\mathcal{N}=1$ $\mathbb{CP}^1$ non-linear sigma model is defined by the following on-shell data:

- A spectrum consisting of a massless $\mathcal{N}=1$ chiral multiplet $(Z, \bar{Z}, \psi^+, \psi^-)$.

- Scattering amplitudes satisfy $\mathcal{N}=1$ supersymmetry Ward identities.

- Scattering amplitudes satisfy isotropy $U(1)$ Ward identities under which $Z, \bar{Z}$ are charged.

- $\sigma_Z = \sigma_{\bar{Z}} = 1$ soft weight for the scalars.

Using the approach of the soft bootstrap, we begin by constructing the most general on-shell amplitudes at lowest valence that are consistent with the above data and minimize $\tilde{\Delta}$. There are no possible 3-point amplitudes consistent with the assumptions and so we must begin at 4-point. A $|Z|^4$ interaction, corresponding to $\tilde{\Delta}=0$, is consistent with $U(1)$ conservation but violates the assumed low-energy theorem. The next-to-lowest reduced dimension interactions correspond to $\tilde{\Delta}=1$ and have a unique 4-point amplitude consistent with the assumptions

$$\mathcal{A}_4(1_Z\, 2_{\bar{Z}}\, 3_Z\, 4_{\bar{Z}}) = \frac{1}{\Lambda^2} s_{13}. \tag{2.7.3}$$

Note that at 4-point, the conservation of the $U(1)$-charge for the complex scalar is automatically enforced as a consequence of the supersymmetry Ward identitites. We will see that this implies the conservation of the $U(1)$ charge for amplitudes with arbitrary number of external particles corresponding to $\tilde{\Delta}=1$. Note that this is not automatic for higher order ($\tilde{\Delta}>1$) corrections and must be imposed as a separate constraint. Using (2.6.20) the remaining 4-point amplitudes are completely determined by supersymmetry; it is convenient to summarize the component amplitudes in a single *superamplitude* [78]

$$\mathcal{A}_4(1_{\Phi^+}2_{\Phi^-}3_{\Phi^+}4_{\Phi^-}) = \frac{1}{\Lambda^2}\,[13]\,\delta^{(2)}(\tilde{Q}) = \frac{1}{2\Lambda^2}\,[13]\sum_{i,j=1}^{4}\langle ij\rangle\eta_i\eta_j\,. \tag{2.7.4}$$

Here we have introduced two chiral superfields Φ^+ and Φ^- that contain the positive and negative helicity fields of the $\mathcal{N}=1$ chiral multiplet as

$$\Phi^+ = \psi^+ + \eta Z\ , \qquad \Phi^- = \bar{Z} - \eta\,\psi^-. \tag{2.7.5}$$

η is the Grassmann coordinate of $\mathcal{N}=1$ on-shell superspace and η_i denotes the η-coordinate of the i^{th} superfield. We can obtain all the component amplitudes by projecting out components of the superfield. For example, the all-fermion amplitude can be derived as follows

$$\mathcal{A}_4(1^+_\psi 2^-_\psi 3^+_\psi 4^-_\psi) = \frac{\partial}{\partial\eta_2}\frac{\partial}{\partial\eta_4}\mathcal{A}_4(1_{\Phi^+}2_{\Phi^-}3_{\Phi^+}4_{\Phi^-}) = -\frac{1}{\Lambda^2}\,[13]\,\langle 24\rangle. \tag{2.7.6}$$

It is useful to note that the expression (2.7.4) is manifestly local. It follows that all component amplitudes are free of factorization singularities, indicating the absence of 3-point interactions in this theory. Note also that the pure fermion sector is exactly the NJL model detected by the soft bootstrap in Section 2.5.2.

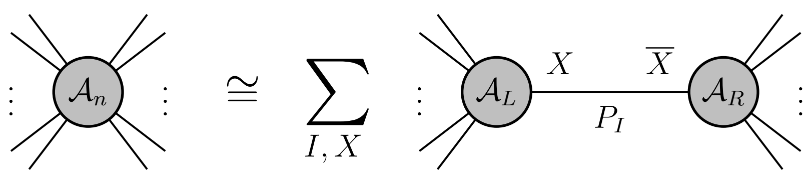
Next, we use these 4-point amplitudes to recursively construct n-point amplitudes. Following the discussion in Section 2.6, we note that the soft weight of the fermion must be either $\sigma_\psi = 0$ or $\sigma_\psi = 1$. Making the conservative choice $\sigma_\psi = 0$, we evaluate the constructibility criterion on the above on-shell data,

$$4 < 2n_s + n_f, \tag{2.7.7}$$

where n_f is the number of external fermion states of the n-point amplitude and $n_s = n - n_f$ is the number of external scalar states. For $n > 4$, this condition is satisfied for all n-point amplitudes. We find that recursively constructing the 6-point amplitudes yields an a_i-independent expression. All the 6-point amplitudes can be found in Appendix B.0.1. Since our input 4-point amplitudes are MHV, the only non-zero constructible amplitudes at 6-point are NMHV and can be verified to satisfy the NMHV 6-point Ward identities (2.6.21), (2.6.22), (2.6.23).

If however we make the stronger assumption $\sigma_\psi = 1$, the recursively constructed 6-point amplitude is a_i-dependent and therefore fails the consistency checks. As a result we conclude that the true soft weight of the fermion of our theory is $\sigma_\psi = 0$ and this is sufficient to construct the S-matrix at leading order from the 4-point seed amplitudes (2.7.4).

The recursive constructibility of the S-matrix has non-trivial consequences for the possible conserved additive quantum numbers. In a recursive model the only non-zero amplitudes are those which can be constructed by gluing together lower-point on-shell amplitudes



(2.7.8)

where the states $X, \bar{X}$ on either side of the factorization channel I have CP conjugate quantum numbers. As discussed further in Appendix C, if an additive quantum number is conserved by all seed amplitudes then it must be conserved by all recursively constructible amplitudes.

For example, in the present context the seed amplitudes conserve two *independent* $U(1)$ charges:

	$U(1)_A$	$U(1)_B$
Z	q_A	0
$\bar{Z}$	$-q_A$	0
ψ^+	0	q_B
ψ^-	0	$-q_B$
η	$-q_A$	q_B
Φ^+	0	q_B
Φ^-	$-q_A$	0

We know to expect the existence of an isotropy $U(1)$ under which the scalars are charged, but from our on-shell construction it is unclear whether this should be $U(1)_A$ or a combination of $U(1)_A$ and $U(1)_B$. We have presented the charges as two independent R-symmetries but more correctly we should consider them as a single *global* $U(1)$ and a $U(1)_R$. The presence of a second conserved quantum number is not part of the definition of the $\mathbb{CP}^1$ non-linear sigma model but is instead an *emergent* or *accidental* symmetry at lowest order in the EFT. In general one would expect $U(1)_A \times U(1)_B$ to be explicitly broken to the isotropy $U(1)$ by higher dimension operators.

2.7.2 $\mathcal{N} = 2$ $\mathbb{CP}^1$ NLSM

The $\mathcal{N} = 2$ $\mathbb{CP}^1$ NLSM is defined by the following on-shell data:

- A spectrum consisting of a massless $\mathcal{N} = 2$ vector multiplet $(Z, \bar{Z}, \psi^{a+}, \psi_a^-, \gamma^+, \gamma^-)$, where $a = 1, 2$.
- Scattering amplitudes satisfy $\mathcal{N} = 2$ supersymmetry Ward identities.
- Scattering amplitudes satisfy isotropy $U(1)$ Ward identities under which $Z, \bar{Z}$ are charged.

Note that, importantly, we do *not* impose the the soft weight of the scalars $\sigma_Z = \sigma_{\bar{Z}} = 1$. As we will explain further below, no model with the above properties *and* vanishing scalar soft limits exists.

To proceed, interactions with reduced dimension $\tilde{\Delta} = 0$ (such as Yukawa interactions) are incompatible with $\mathcal{N} = 2$ supersymmetry for a single vector multiplet. Thus, the minimal value is $\tilde{\Delta} = 1$; that is of course also the value for the $\mathcal{N} = 1$ model. It is curious to note that $\mathcal{N} = 2$ supersymmetry is sufficient to uniquely construct the S-matrix at this order in $\tilde{\Delta}$. As we show in the following, *without* assuming vanishing scalar soft limits, the restriction of the external states to a single chiral multiplet $(Z, \bar{Z}, \psi^{1+}, \psi_1^-)$ reproduces the $\mathcal{N} = 1$ $\mathbb{CP}^1$ sigma model.

As in the previous section, for $\tilde{\Delta} = 1$ the 4-point scalar amplitude takes the form (2.7.3). All 4-point component amplitudes are uniquely fixed by the 4-scalar amplitudes by the $\mathcal{N} = 2$ supersymmetry Ward identities and they can be encoded compactly into superamplitudes using two chiral superfields [78]

$$\begin{aligned}\Phi^+ &= \gamma^+ + \eta_1\psi^{1+} + \eta_2\psi^{2+} - \eta_1\eta_2 Z,\\ \Phi^- &= \bar{Z} + \eta_1\psi_2^- - \eta_2\psi_1^- - \eta_1\eta_2\gamma^- \,.\end{aligned} \tag{2.7.9}$$

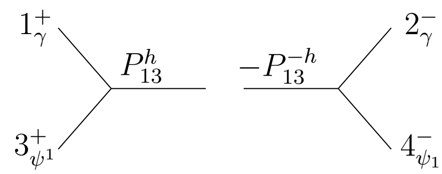
Here η_1 and η_2 are the Grassmann coordinates of $\mathcal{N}=2$ on-shell superspace. The R-indices on ψ^a are raised and lowered using ϵ_{ab}, so $\psi_2^- = \epsilon_{21}\psi^{1-} = \psi^{1-}$ and $\psi_1^- = \epsilon_{12}\psi^{2-} = -\psi^{2-}$. In terms of the superfields, the 4-point superamplitude can be expressed as

$$\mathcal{A}_4(1_{\Phi^+}2_{\Phi^-}3_{\Phi^+}4_{\Phi^-}) = \frac{1}{\Lambda^2}\frac{[13]}{\langle 13\rangle}\delta^{(4)}(\tilde{Q}) = \frac{1}{4\Lambda^2}\frac{[13]}{\langle 13\rangle}\prod_{a=1}^{2}\sum_{i,j=1}^{4}\langle ij\rangle\eta_{ia}\eta_{ja}. \tag{2.7.10}$$

We use η_{ia} to denote the a^{th} Grassmann coordinate of the i^{th} external superfield. In contrast to (2.7.4), the superamplitude (2.7.10) generates component amplitudes that are not local due to the factorization singularity at $P_{13}^2 \to 0$. For example, consider the following component amplitude

$$\mathcal{A}_4(1_\gamma^+ 2_\gamma^- 3_{\psi^1}^+ 4_{\psi_1}^-) = -\frac{\partial}{\partial\eta_{21}}\frac{\partial}{\partial\eta_{22}}\frac{\partial}{\partial\eta_{31}}\frac{\partial}{\partial\eta_{42}}\mathcal{A}_4(1_{\Phi^+}2_{\Phi^-}3_{\Phi^+}4_{\Phi^-}) = -\frac{1}{\Lambda^2}\frac{[13]\,[14]\,\langle 24\rangle}{[24]}. \tag{2.7.11}$$

Locality and unitarity imply that this 4-point amplitude must factorize into 3-point amplitudes on the singularity at $P_{13}^2 \to 0$. Denoting the helicity of the exchanged particle h, the amplitude factorizes as



(2.7.12)

The contribution to the residue on the singularity takes the form

$$\begin{aligned} P_{13}^2\mathcal{A}_4(1_\gamma^+ 2_\gamma^- 3_{\psi^1}^+ 4_{\psi_1}^-)\Big|_{P_{13}^2=0} &= \mathcal{A}_3\big(1_\gamma^+ 3_{\psi^1}^+ (P_{13})_h\big)\mathcal{A}_3\big((-P_{13})_{-h} 2_\gamma^- 4_{\psi_1}^-\big)\\ &= \left(\frac{g_1}{\Lambda}[13]^{3/2-h}[1P_{13}]^{1/2+h}[3P_{13}]^{-1/2+h}\right)\left(\frac{g_2}{\Lambda}\langle 24\rangle^{3/2-h}\langle 2P_{13}\rangle^{1/2+h}\langle 4P_{13}\rangle^{-1/2+h}\right)\\ &= \frac{g_1 g_2}{\Lambda^2}(-1)^{2h}\,[13]^{3/2-h}\,\langle 24\rangle^{3/2+h}\,[23]^{1/2-h}\,[14]^{1/2+h}\,, \end{aligned} \tag{2.7.13}$$

with the 3-point amplitudes completely determined by Poincaré invariance and little group scaling. Comparing with the explicit form of the residue calculated from (2.7.11)

$$P_{13}^2\mathcal{A}_4(1_\gamma^+ 2_\gamma^- 3_{\psi^1}^+ 4_{\psi_1}^-)\Big|_{P_{13}^2=0} = \frac{1}{\Lambda^2}\,[13]\,[14]\,\langle 24\rangle^2, \tag{2.7.14}$$

we find that $h = 1/2$ and $g_1 g_2 = -1$. The exchanged particle of helicity $h = 1/2$ can be either ψ^{1+} or ψ^{2+}. The locality of the $\mathcal{A}_4(1^+_{\psi^1} 2^-_{\psi_1} 3^+_{\psi^1} 4^-_{\psi_1})$ and $\mathcal{A}_4(1^+_{\psi^2} 2^-_{\psi_2} 3^+_{\psi^2} 4^-_{\psi_2})$ tells us that they do not factorize on the $(P_{13})^2 \to 0$ pole. We conclude that $\mathcal{A}_3(1^+_\gamma 2^+_{\psi_1} 3^+_{\psi_1}) = \mathcal{A}_3(1^+_\gamma 2^+_{\psi_2} 3^+_{\psi_2}) = 0$, while

$$\mathcal{A}_3(1^+_\gamma 2^+_{\psi_1} 3^+_{\psi_2}) = \frac{g_1}{\Lambda}[12][13] \,, \quad \mathcal{A}_3(1^-_\gamma 2^-_{\psi_1} 3^-_{\psi_2}) = \frac{g_2}{\Lambda}\langle 12\rangle\langle 13\rangle \,. \tag{2.7.15}$$

We carry out a similar exercise with $\mathcal{A}_4(1^+_\gamma 2^-_\gamma 3^+_\gamma 4^-_\gamma)$ for a particle of helicity h in the $P^2_{13} \to 0$ factorization channel. Comparing with the 4-point amplitude (2.7.10) fixes $h = 0$. This could correspond to either Z or $\bar{Z}$ exchange. The absence of a $P^2_{14} \to 0$ pole in $\mathcal{A}_4(1^+_\gamma 2^-_\gamma 3_Z 4_{\bar{Z}})$ shows that $\mathcal{A}_3(1^+_\gamma 2^+_\gamma 3_{\bar{Z}}) = 0$ and

$$\mathcal{A}_3(1^+_\gamma 2^+_\gamma 3_Z) = \frac{g_3}{\Lambda}[12]^2 \,, \quad \mathcal{A}_3(1^-_\gamma 2^-_\gamma 3_{\bar{Z}}) = \frac{g_4}{\Lambda}\langle 12\rangle^2 \,, \tag{2.7.16}$$

where $g_3 g_4 = 1$. Demanding that all non-local 4-point amplitudes factorize correctly fixes $-g_1 = g_2 = g_3 = g_4 = -1$. The 3-point superamplitudes are

$$\begin{aligned}
\mathcal{A}_3(1_{\Phi^-} 2_{\Phi^-} 3_{\Phi^-}) &= \delta^{(4)}(\tilde{Q}) = \frac{1}{4\Lambda}\prod_{a=1}^{2}\sum_{i,j=1}^{3}\langle ij\rangle \eta_{ia}\eta_{ja} \,, \qquad (2.7.17)\\
\mathcal{A}_3(1_{\Phi^+} 2_{\Phi^+} 3_{\Phi^+}) &= \frac{1}{\Lambda}\delta^{(2)}(\eta_1[23] + \eta_2[31] + \eta_3[12]) = \frac{1}{\Lambda}\prod_{a=1}^{2}(\eta_{1a}[23] + \eta_{2a}[31] + \eta_{3a}[12]) \,,
\end{aligned}$$

where $\prod_{a=1}^{2} f_a$ is defined as $f_1 f_2$.

It is interesting to observe that even though the $\mathcal{N} = 0, 1$ and 2 $\mathbb{CP}^1$ NLSM have the pure scalar 4-point amplitude in common, in the latter case the extended supersymmetry together with locality require the presence 3-point interactions.

We are now in a position to address the constructibility of general n-point amplitudes. Since we are not assuming vanishing soft limits as part of our on-shell data, we are not able to make use of subtracted recursion. This is only problematic for a subset of the amplitudes in this model, at least at leading order. The unsubtracted constructibility criterion for this model reads

$$4 < n_f + 2n_v, \tag{2.7.18}$$

where n_f and n_v are the number of fermions and vector bosons respectively. It turns out that the amplitudes that do not satisfy this criterion can be determined from the $\mathcal{N} = 2$ supersymmetry Ward identities in terms of those that do; explicit formulae are given in Appendix D. Remarkably, without making any strong assumptions about the structure of low-energy theorems for the scalars, which usually characterize the sigma model coset structure, the $\mathcal{N} = 2$ supersymmetry is sufficient at leading order to both construct the entire S-matrix and reproduce the amplitudes of the $\mathcal{N}=1$ and $\mathcal{N}=0$ models as special cases.

This same statement can be made in the perhaps more familiar language of local field theory. At this order in the EFT expansion, the S-matrix elements should be calculable from some effective action, the bosonic sector of which should be described by a two-derivative Lagrangian of the general form

$$\mathcal{L}_{\text{eff}} = P\left(|Z|^2\right)|\partial_\mu Z|^2 + Q\left(|Z|^2\right) Z\, F_+^2 + \text{h.c.} \tag{2.7.19}$$

where $P(|Z|^2)$ and $Q(|Z|^2)$ are some functions analytic around $Z \sim 0$. Insisting that the S-matrix elements satisfy the on-shell $\mathcal{N}=2$ supersymmetry Ward identities is equivalent to requiring the existence of off-shell $\mathcal{N}=2$ supersymmetry transformations under which the effective action is invariant. The on-shell uniqueness result is equivalent to the statement that the off-shell $\mathcal{N}=2$ supersymmetry uniquely (up to field redefinitions) determines the form of the two-derivative effective action. In particular, the function $P(|Z|^2)$ is uniquely determined to be

$$P\left(|Z|^2\right) = \left(\frac{1}{1+|Z|^2}\right)^2, \tag{2.7.20}$$

corresponding to the Fubini-Study metric on $\mathbb{CP}^1$.

Since the entire S-matrix is determined, we can explicitly demonstrate how the presence of the vector bosons modifies the structure of the low-energy theorems from the naive vanishing soft limits suggested by the coset structure. Consider the following relation among 5-point amplitudes given by the $\mathcal{N}=2$ supersymmetry Ward identities

$$\mathcal{A}_5\left(1_\gamma^+, 2_\gamma^+, 3_Z, 4_Z, 5_{\bar{Z}}\right) = \frac{\langle 34\rangle^2}{\langle 45\rangle^2}\mathcal{A}_5\left(1_\gamma^+, 2_\gamma^+, 3_\gamma^+, 4_Z, 5_\gamma^-\right). \tag{2.7.21}$$

The amplitude on the right-hand-side satisfies (2.7.18) and therefore is constructible using unsubtracted recursion. This gives the non-constructible amplitude on the left-hand-side as

$$\mathcal{A}_5\left(1_\gamma^+, 2_\gamma^+, 3_Z, 4_Z, 5_{\bar{Z}}\right) = \frac{1}{\Lambda^3}\langle 34\rangle^2\left(\frac{[12][34]}{\langle 12\rangle\langle 34\rangle} + \frac{[23][14]}{\langle 23\rangle\langle 14\rangle} + \frac{[31][24]}{\langle 31\rangle\langle 24\rangle}\right). \tag{2.7.22}$$

The soft limits on particles 1, 2, 3 and 4 vanish, as expected. The soft limit on particle 5, however, is $\mathcal{O}(1)$, contrary to the expected soft behavior for a Goldstone mode of a symmetric coset. Explicitly

$$\mathcal{A}_5\left(1_\gamma^+, 2_\gamma^+, 3_Z, 4_Z, 5_{\bar{Z}}\right) \xrightarrow{|5]\to\epsilon|5]} \frac{1}{\Lambda^3}[12]^2 + \mathcal{O}(\epsilon). \tag{2.7.23}$$

It is interesting that the coupling to the photons, required by $\mathcal{N}=2$ supersymmetry, results in non-vanishing soft scalar limits for a theory with a symmetric coset. In principle, this amplitude could have had a contact contribution of the form $\propto [12]^2$, but our calculation shows that such a term would be incompatible with $\mathcal{N}=2$ supersymmetry.

The maximal R-symmetry group that this model can realize is $U(2)_R = U(1)_R \times SU(2)_R$. We will now verify that the $SU(2)_R$ symmetry Ward identities hold for the seed amplitudes, the $U(1)_R$ we will address separately. To do this we choose a basis for the generators of $SU(2)_R$. The scalars and vectors both transform as $SU(2)$ singlets. The positive helicity fermion species $\psi^{1,2+}$ will transform in the fundamental representation under

$$\mathcal{T}_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathcal{T}_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{T}_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{2.7.24}$$

The negative helicity fermions transform in the anti-fundamental with $\bar{\mathcal{T}}_i = -\mathcal{T}_i^\dagger$. This tells us that the $\mathcal{T}_0$-Ward identity is satisfied as long as the fermion species appear in pairs of (a) different helicity, same species or (b) same helicity, different species. This is true of all the non-zero amplitudes in this model. The action of $\mathcal{T}_+$ and $\mathcal{T}_-$ are

state i	$\mathcal{T}_+ \cdot i$	$\mathcal{A}_n$ prefactor	$\mathcal{T}_- \cdot i$	$\mathcal{A}_n$ prefactor	$\mathcal{T}_0 \cdot i$	$\mathcal{A}_n$ prefactor
ψ^{1+}	0	0	ψ^{2+}	1	ψ^{1+}	1
ψ^{2+}	ψ^{1+}	1	0	0	ψ^{2+}	-1
ψ_1^-	ψ_2^-	-1	0	0	ψ_1^-	-1
ψ_2^-	0	0	ψ_1^-	-1	ψ_2^-	1

(2.7.25)

We find that all 3-point and 4-point amplitudes in this model satisfy the $SU(2)_R$ Ward identities, for example

$$\begin{aligned} \mathcal{T}_- \cdot \mathcal{A}_4(1^+_{\psi^1} 2^-_{\psi_2} 3^+_{\psi^1} 4^-_{\psi_1}) &= \mathcal{A}_4(1^+_{\psi^2} 2^-_{\psi_2} 3^+_{\psi^1} 4^-_{\psi_1}) - \mathcal{A}_4(1^+_{\psi^1} 2^-_{\psi_1} 3^+_{\psi^1} 4^-_{\psi_1}) + \mathcal{A}_4(1^+_{\psi^1} 2^-_{\psi_2} 3^+_{\psi^2} 4^-_{\psi_1}) \\ &= -\frac{[13]}{[24]}\,(s+t+u) = 0\,. \end{aligned} \tag{2.7.26}$$

As discussed above, we conclude that at leading order the $SU(2)_R$ Ward identities are satisfied by all amplitudes in the $\mathcal{N}=2$ model.

Following the same approach as described for the $\mathcal{N}=1$ model, conservation laws satisfied by the seed amplitudes imply that the same quantities are conserved by all leading-order amplitudes if they are recursively constructible (see Appendix C). This result extends to non-Abelian symmetries, which in the on-shell language correspond to Ward identities for non-diagonal generators; this is shown for $SU(2)$ in Appendix C. The amplitudes that are not constructible using recursion are fixed by supersymmetry in terms of those that are. Therefore, they will also respect the conservation laws and non-Abelian symmetries of the seed amplitudes.

This model also conserves a separate $U(1)_R$ charge. We know to expect the conservation of the charge associated with the $U(1)$ isotropy group. In the $\mathcal{N}=1$ case we found that the scattering amplitudes conserve an R-charge $U(1)_A$ assigned only to the complex scalar but it was consistent with the existence of $U(1)_B$ that the isotropy $U(1)$ might also assign a charge to the fermion or even to assign equal charges in the form of a global symmetry. In the present context we also have two independent $U(1)$ symmetries. The first is the $U(1) \subset SU(2)_R$ which assigns opposite charges to the fermions ψ^{1+} and ψ^{2+}. The second assigns charges to each of the states which, up to overall normalization can be deduced from the 3- and 4-point seed amplitudes and are summarized in the following table:

	$U(1)_R$	$SU(2)_R$
Z	-4	**1**
$\bar{Z}$	4	**1**
ψ^{a+}	-1	**2**
ψ_a^-	1	**2**
γ^+	2	**1**
γ^-	–2	**1**
η_a	3	**2**
Φ^+	2	**1**
Φ^-	4	**1**

These are the *only* linear symmetries compatible with the seed amplitudes. The isotropy $U(1)$ must therefore be identified with some linear combination of $U(1)_R$ and $U(1) \subset SU(2)_R$. This is perhaps surprising, it tells us that the massless vector boson must also be charged under the isotropy $U(1)$. Just as for the fermions, the vector charges are *chiral* meaning that the positive and negative helicity states have opposite charges. Such charges for vectors are associated with electric-magnetic duality symmetries.

Such an extra $U(1)_R$ symmetry is possible because the maximal outer-automorphism group of the $\mathcal{N}=2$ supersymmetry algebra is $U(2)_R$. The assignment of the associated charges is, up to normalization, fixed by the charge of the highest helicity state in the multiplet. It is interesting to observe that in the present context, knowledge of the non-vanishing 4-point amplitudes is *insufficient* to determine the $U(1)_R$ charge assignments. It is only from considering the 3-point amplitudes that we find the assignment of a non-zero chiral charge for the vector bosons unavoidable. Consider for example the amplitudes (2.7.16). Since the scalar is required to be charged under the isotropy $U(1)$, which in this case must be the $U(1)_R$ since there are no other symmetries under which the scalar is charged, we see that the vector must also be charged and satisfy $2q[\gamma^+] = -q[Z]$. The existence of fundamental 3-point interactions in this model was deduced by demanding that the singularities of the 4-point amplitudes be identified with physical factorization channels. From an on-shell point of view, it is therefore an *unavoidable* consequence of locality, unitarity and supersymmetry that the $\mathcal{M}_V$ isotropy group of an $\mathcal{N}=2$ non-linear sigma model acts on the vector bosons as an electric-magnetic duality transformation.

The necessary existence of the fundamental 3-point amplitudes (2.7.15) and (2.7.16) has a further interesting consequence for the low-energy behavior of the vector boson. In [57] it was shown that *singular* low-energy theorems arise from the presence of certain 3-point amplitudes. In the notation used in [57] the 3-point amplitudes (2.7.15) and (2.7.16) are classified as $\mathtt{a}=1$ in the soft limit of a positive helicity vector boson. Therefore a vector boson present in amplitudes which contain at least one of the following other particles: Z, ψ^{a+} or γ^+ has soft weight $\sigma_\gamma=-1$. Using the general formalism developed in [57], we can write down the low-energy theorem of the vector bosons in this subclass of amplitudes

$$\mathcal{A}_{n+1}\left(s_\gamma^+,1,2,...,n\right)\xrightarrow{p_s\to\epsilon p_s \text{ as } \epsilon\to 0}\sum_{k=1}^{n}\frac{[sk]}{\epsilon\langle sk\rangle}\mathcal{A}_n\left(1,2,...,\mathcal{F}_+\cdot k,...,n\right)+\mathcal{O}\left(\epsilon^0\right). \quad (2.7.27)$$

Here we are using a notation similar to [79] with the introduction of an operator $\mathcal{F}_+$ which acts on the one-particle states as

state i	$\mathcal{F}_+\cdot i$	$\mathcal{A}_n$ prefactor
Z	γ^-	1
ψ^{1+}	ψ_2^-	-1
ψ^{2+}	ψ_1^-	-1
γ^+	$\overline{Z}$	-1

(2.7.28)

and annihilates the states of the negative helicity multiplet. A similar operator $\mathcal{F}_-$ can be defined for the soft limit of a negative helicity vector. Using equation (2.6.13) in conjunction with the soft behavior (2.7.27) of the $n+1$-point amplitude results in the following identity for the residual n-point amplitudes

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(-1)^{L_i+P_i}\frac{[Xi][Yj]}{\langle Yj\rangle}\mathcal{A}_n\left(1,2,...,\mathcal{Q}_1\cdot i,...,\mathcal{F}_+\cdot j,...,n\right)=0\,, \quad (2.7.29)$$

where here $P_i=0$ or 1 corresponds to the additional signs associated with the prefactors of both the supersymmetry Ward identities and the operator $\mathcal{F}_+$ given in Table 2.7.28. Note that the action of $\mathcal{Q}_1$ and $\mathcal{F}_+$ commute on all physical states, so there is no ambiguity when $i=j$ in the sums. Moreover, rearranging the order of the sums, it becomes clear that for each fixed j, the sum over i expresses a supersymmetry Ward identity for the n-point amplitudes. As such, the identity (2.7.29) does not impose further constraints beyond supersymmetry.

2.8 Super Dirac-Born-Infeld and Super Born-Infeld

In the soft bootstrap analysis of Section 2.5, we encountered three theories with a fundamental quartic interaction whose couplings are of mass-dimension -4: DBI, Akulov-Volkov, and Born-Infeld. These EFTs can all be related by supersymmetry. We will discuss them in further detail in future work, so for now we simply note the following:

- The $\mathcal{N} = 1$ supersymmetric Dirac-Born-Infeld model has as its pure scalar sector the complex scalar DBI theory with $\sigma_Z = 2$ and as its pure fermion sector Akulov-Volkov theory with $\sigma_\psi = 1$. All amplitudes are constructible with soft subtracted recursion. We present the expressions for the 4- and 6-point amplitudes in Appendix B.0.2.
- The $\mathcal{N} = 1$ supersymmetric Born-Infeld model combines Akulov-Volkov theory with Born-Infeld theory with $\sigma_\gamma = 0$. All amplitudes are constructible with the soft subtracted recursion relations of Section 2.4, except the pure vector ones, but they are uniquely fixed by the supersymmetry Ward identities. The 4- and 6-point amplitudes are given in Appendix B.0.3.
- Extended supersymmetry binds BI, Akulov-Volkov, and DBI into one supersymmetric exceptional EFT. In particular, $\mathcal{N} = 4$ supersymmetry binds together DBI, Akulov-Volkov, and Born-Infeld theory and will be discussed further in forthcoming work. The $\mathcal{N} = 4$ super-DBI amplitudes can be constructed using the CHY approach [80].

2.9 Galileons

Galileons are scalar effective field theories that arise in a multitude of contexts and as a result can be defined in different ways. In 4d, Galileons are

1. Higher-derivative scalar field theories with second-order equations of motion and absence of Ostrogradski ghosts. These theories have three free parameters: the cubic, quartic and quintic interaction coupling constants. A field redefinition removes the cubic interaction in favor of a linear combination of the quartic and quintic. The scattering amplitudes are of course invariant under the field redefinition, so for the purpose of studying perturbative scattering amplitudes, we consider only the quartic and quintic Galileons.
2. The non-linear realization of the algebra $\mathfrak{Gal}(4, 1)$ which is an İnönü-Wigner contraction of the ISO$(4, 1)$ symmetry algebra [81]. Truncated to leading order in the reduced dimension $\tilde{\Delta}$, this gives an effective field theory of a real massless scalar ϕ with $\sigma = 2$ vanishing

soft limits and coupling dimensions $[g_4] = -6$ and $[g_5] = -9$ for the quartic and quintic interactions respectively.

3. Subleading contributions to the low-energy effective action on a 3-brane embedded in a $5d$ Minkowski space. The leading contribution to this EFT is the DBI action and including the Galileon terms, the model is often called the DBI-Galileon. In the limit of infinite brane tension, the Galileons decouple from DBI. The non-$\mathbb{Z}_2$-symmetric cubic and quintic interactions arise from considering the effective action on an end-of-the-world brane.

4. Scalar effective field theories that arise from the massless decoupling limit of Fierz-Pauli-type massive gravity [61, 62] and from the decoupling limit of Proca theories.

It is not obvious if these definitions are equivalent. The equivalence between Definitions 2 and 3 is straightforward since ISO$(4,1)$ is the Poincaré symmetry of the 5d embedding space. In the brane picture of Definition 3, the DBI-Galileon scalar is a Goldstone boson that arises from the spontaneous breaking of translational symmetry transverse to the brane, with the contraction of the 5d Poincaré algebra equivalent to the non-relativistic limit of the fluctuations of the brane into the extra dimension [82].

In an approach based on scattering amplitudes, it is natural to use the second definition of Galileon theories, based on their soft weight $\sigma = 2$ and fundamental coupling dimension. This is what we do in the following, however, we do comment on the connections to the other definitions. In Section 2.9.1, we briefly review our recent results about the supersymmetrization of (DBI-)Galileon theories in 4d and cover some details that were left out in [18]. Motivated by Definition 4, we investigate the possibility of a scalar-vector Galileon theory in Section 2.9.2. In Sections 2.9.3 and 2.9.4, we focus our attention on the Special Galileon. In Section 2.9.3 we address the question of subleading operators respecting the enhanced $\sigma = 3$ soft behavior. In Section 2.9.4, we approach the same question from a double-copy construction.

2.9.1 Galileons and Supersymmetry

This section reviews and expands on the results of [18] for $\mathcal{N} = 1$ supersymmetrization of Galileon models. Two approaches to forming a complex scalar $Z = \phi + i\chi$ are considered:

(a) Both ϕ and χ are Galileons so that the complex scalar Z has soft weight $\sigma_Z = 2$, or

(b) ϕ is a Galileon but χ only has constant shift symmetry; then $\sigma_\phi = 2$ and $\sigma_\chi = 1$, and hence $\sigma_Z = 1$. A natural interpretation of χ is as an R-axion.

Both options were considered in [18].

Option (a): $\sigma_Z = 2$

Consider first the *quartic* Galileon. As discussed in Section 2.6.5, to be compatible with supersymmetry, the 4-point complex scalar amplitudes must have two Z's and two $\bar{Z}$'s; such an amplitude is in the MHV class. It is also clear from the table of "soft bootstrap" results in (2.5.5) that there is a unique complex scalar quartic Galileon theory[18] with $\sigma_Z = 2$ based on the 4-point interaction with $\mathcal{A}_4(1_Z\, 2_{\bar{Z}}\, 3_Z\, 4_{\bar{Z}}) = g_4 stu$. The other 4-point amplitudes in a supersymmetric theory are fixed by $\mathcal{A}_4(1_Z\, 2_{\bar{Z}}\, 3_Z\, 4_{\bar{Z}})$ using the supersymmetry Ward identity (2.6.20).

By (2.6.8), the soft behavior of the fermion must be either $\sigma_\psi = 1$ or 2. The all-fermion amplitudes are constructible when $\sigma_\psi = 2$, and our soft bootstrap results for fermion theories (2.5.8) show that no such theory exists. Therefore, the fermions in a supersymmetric Galileon theory with $\sigma_Z = 2$ must have $\sigma_\psi = 1$.

In a supersymmetric quartic Galileon theory with $\sigma_Z = 2$ and $\sigma_\psi = 1$, the constructibility criterion (2.4.10) for n-point amplitudes with n_s scalars and n_f fermions is $n_f < 4$. Thus at 6-point, we can only use soft subtracted recursion to compute the amplitudes with at most two fermions. However, as discussed in [18], two of the six supersymmetry Ward identities (2.6.21)-(2.6.23) uniquely determine the 4- and 6-fermion amplitudes. The remaining four identities in (2.6.21)-(2.6.23) are used as consistency checks. The expressions for the 6-point amplitudes of the supersymmetric quartic Galileon can be found in Appendix B.0.4. We have checked that the recursively constructed 4- and 6-point amplitudes match those that we calculate from the Lagrangian superspace construction of the quartic Galileon in [52].

The supersymmetry Ward identities at 8-point and higher do not uniquely determine the non-constructible amplitudes of the supersymmetric quartic Galileon. We therefore suspect that the quartic Galileon fails to be unique at 8-point and higher [18].

The *quintic* Galileon does not admit a supersymmetrization with $\sigma_Z = 2$ for the complex scalar. As discussed at the end of Section 2.5.1.2, there are no obvious obstructions from the soft-recursion tests to a complex scalar decoupled quintic Galileon with $\mathcal{A}_5(1_Z\, 2_{\bar{Z}}\, 3_Z\, 4_{\bar{Z}}\, 5_Z) = (\epsilon_{\mu\nu\rho\sigma} p_1^\mu p_2^\nu p_3^\rho p_4^\sigma)^2$. However, it is not compatible with the 5-point supersymmetry Ward identities. It follows that the cubic Galileon also cannot be supersymmetrized with $\sigma_Z = 2$.

Option (b): $\sigma_Z = 1$**.**

Consider a *quartic* complex scalar theory where the real part of the complex scalar Z is the Galileon ϕ and the imaginary part is an R-axion χ. The constructibility criterion with $\sigma_\phi = 2$ and $\sigma_\chi = \sigma_\psi = 1$ is $2n_\chi + n_f < 4$, so there are only two mixed amplitudes to check; they do not restrict the 2-parameter family of input amplitudes [18]. We have checked that the constructible

[18]That analysis also shows that it is impossible for this kind of model to have special Galileon symmetry with $\sigma_Z = 3$.

6-point amplitudes are compatible with DBI.

For a *quintic* Galileon with $\sigma_Z = 1$, we found a 3-parameter solution [18] to the supersymmetry Ward identities. Requiring that it is compatible with DBI restricts to a unique solution,

$$\mathcal{A}_5(1_Z\, 2_{\bar{Z}}\, 3_Z\, 4_{\bar{Z}}\, 5_Z) = -\frac{[24]}{[25]}\mathcal{A}_5(1_Z\, 2_{\bar{Z}}\, 3_Z\, 4_{\bar{\psi}}\, 5_\psi) = \frac{[24]}{[35]}\mathcal{A}_5(1_Z\, 2_{\bar{\psi}}\, 3_\psi\, 4_{\bar{\psi}}\, 5_\psi)\,, \tag{2.9.1}$$

namely

$$\begin{aligned}\mathcal{A}_5(1_Z\, 2_{\bar{Z}}\, 3_Z\, 4_{\bar{Z}}\, 5_Z) = s_{24}\,\Big(6s_{24}s_{25}s_{45} + \big(&4s_{12}s_{23}s_{45} + 2s_{12}s_{24}s_{34} + 2s_{25}^2 s_{45} + s_{24}s_{25}^2 + (2\leftrightarrow 4)\big)\\ &+(1\leftrightarrow 5) + (3\leftrightarrow 5)\Big) - 4s_{24}^2\,.\end{aligned}$$

The amplitudes $\mathcal{A}_5(1_{\bar{Z}}\, 2_Z\, 3_{\bar{Z}}\, 4_Z\, 5_{\bar{Z}})$, $\mathcal{A}_5(1_{\bar{Z}}\, 2_Z\, 3_{\bar{Z}}\, 4_\psi 5_{\bar{\psi}})$, and $\mathcal{A}_5(1_{\bar{Z}}\, 2_\psi\, 3_{\bar{\psi}}\, 4_\psi\, 5_{\bar{\psi}})$ follow from conjugation of the above.[19] It is interesting that the fermions in these 5-point amplitudes automatically have $\sigma_\psi = 1$.

To test consistency of a supersymmetric quintic Galileon with $\sigma_\phi = 2$, $\sigma_\chi = 1$, and $\sigma_\psi = 1$, we consider the 7-point and 8-point amplitudes in the decoupled Galileon theory. In both cases, the constructibility criterion is $2n_\chi + n_f < 4$. The (few) non-trivial constructible amplitudes pass the soft subtraction recursive tests of a_i-independence. We have also tested compatibility with the supersymmetric DBI interactions: at 7-point the constructibility criterion is $2n_\chi + n_f < 8$ and again the constructible 7-point amplitudes pass the test. This indicates that there may indeed be a supersymmetric brane-theory with both quartic and quintic terms subleading to DBI. The scalar ϕ is the Goldstone mode of the broken transverse translational symmetry whereas the scalar χ is an R-axion. The fermion ψ is a genuine Goldstino of partial broken supersymmetry. We discuss such scenarios further in forthcoming work.

2.9.2 Vector-Scalar Special Galileon

It is known that scalar Galileon theories arise in certain limits of massive gravity [61, 62] (for a review, see [63]). An on-shell massive graviton in 4d has 5 polarization states and the decoupling limit gives one real massless scalar (the Galileon) and a massless photon in addition to the massless graviton. So we expect there to be an EFT of a real Galileon scalar coupled to vector.[20] The vector couples quadratically to the scalar and was consistently truncated off in [62]. Some subsequent studies have discussed the photon-scalar coupling of Galileons, see for example [83]. Here, we use soft recursion to give some definitive results about the possible scattering amplitudes in such a theory.

[19] These 5-point amplitudes are not required to vanish in $3d$ kinematics (and they do not) because they do not satisfy the constructibility criterion.

[20] The decoupling of these interactions from the graviton is not clear [63].

If the scalar has $\sigma_\phi = 2$, only the scalar amplitudes are constructible, and we are not able to say anything about the vector sector and its couplings to the scalar. If however the couplings are tuned in such a way that the cubic and quintic Galileon interactions are set to zero then in the scalar sector the soft weight of the scalar is enhanced to $\sigma_\phi = 3$, the *special Galileon* scenario. At present it is unknown whether this enhancement of symmetry can be understood in some natural way from the decoupling limit of some model of massive gravity. Moreover, it is not a priori clear if the $\sigma_\phi = 3$ enhancement can survive coupling to other particles.

We use the power of the soft bootstrap to construct the most general amplitudes consistent with the special Galileon low-energy theorem. We use the 6-point test to exclude EFTs with a special Galileon coupled non-trivially to a photon with $\sigma_\gamma > 0$. For the model with $\sigma_\phi = 3$ and $\sigma_\gamma = 0$, we find that the soft recursion 6-point test reduces the most general 6 real-parameter ansatz for the scalar and scalar-vector interactions to a 3 real-parameter family:

$$\begin{aligned}
\mathcal{A}_4(1_\phi\, 2_\phi\, 3_\phi\, 4_\phi) &= g_1 stu\,, \\
\mathcal{A}_4(1_\phi\, 2_\phi\, 2_\gamma^+\, 4_\gamma^+) &= g_2[34]^2\left(t^2+u^2+3tu\right)\,, \\
\mathcal{A}_4(1_\gamma^-\, 2_\phi\, 3_\phi\, 4_\gamma^+) &= g_1\langle 12\rangle[24]\langle 13\rangle[34]u\,, \\
\mathcal{A}_4(1_\phi\, 2_\phi\, 3_\gamma^-\, 4_\gamma^-) &= g_2^*\langle 34\rangle^2\left(t^2+u^2+3tu\right)\,.
\end{aligned} \tag{2.9.2}$$

The couplings of the pure vector sector are unconstrained; the most general ansatz is

$$\begin{aligned}
\mathcal{A}_4(1_\gamma^+\, 2_\gamma^+\, 3_\gamma^+\, 4_\gamma^+) &= g_3\Big([12]^2[34]^2 s + [13]^2[24]^2 t + [14]^2[23]^2 u\Big)\,, \\
\mathcal{A}_4(1_\gamma^-\, 2_\gamma^-\, 3_\gamma^+\, 4_\gamma^+) &= g_4\langle 12\rangle^2[34]^2 s\,, \\
\mathcal{A}_4(1_\gamma^-\, 2_\gamma^-\, 3_\gamma^-\, 4_\gamma^-) &= g_3^*\Big(\langle 12\rangle^2\langle 34\rangle^2 s + \langle 13\rangle^2\langle 24\rangle^2 t + \langle 14\rangle^2\langle 23\rangle^2 u\Big)\,.
\end{aligned} \tag{2.9.3}$$

The most interesting feature of the above result is the relation between the coefficients of the amplitudes $\mathcal{A}_4(1_\phi\, 2_\phi\, 3_\phi\, 4_\phi)$ and $\mathcal{A}_4(1_\gamma^-\, 2_\phi\, 3_\phi\, 4_\gamma^+)$. The former is the familiar quartic Galileon, while the latter would arise from an operator of the form

$$\mathcal{O} \sim g_1(\partial_\mu F_+^{\alpha\beta})(\partial^\mu F_-^{\dot\alpha\dot\beta})(\sigma^\nu_{\alpha\dot\alpha}\partial_\nu\phi)(\sigma^\rho_{\beta\dot\beta}\partial_\rho\phi), \tag{2.9.4}$$

where $F_\pm$ are as defined in and below (2.4.20)

The relation between the couplings strongly indicates the existence of a non-linear symmetry which mixes the scalar and vector modes. Describing the action of this symmetry and its consequences is left for future work.

2.9.3 Higher Derivative Corrections to the Special Galileon

The real quartic Galileon has low-energy theorems with $\sigma = 3$ soft weight. Being agnostic about the origin of the special Galileon, from an EFT perspective, one should write a Lagrangian with all possible operators that respect the symmetries of the theory in a derivative expansion. The authors of [84] found that among a specific subclass of Lagrangian operators, namely those with the schematic form $\partial^4\phi^4$, $\partial^6\phi^4$ and $\partial^8\phi^5$, the special Galileon is the unique choice that can give enhanced soft limits with $\sigma = 3$ soft weight. In this section, we investigate much more exhaustively the possible higher-derivative quartic and quintic operators compatible with $\sigma = 3$ soft behavior. This is done using soft-subtracted recursion relations to calculate the 6- and 7-point scattering amplitudes of the model.

Let us start our discussion with the 6-point case. The constructibility criterion (2.4.21) implies that recursion relations are valid if the coupling constant g_6 of the 6-point amplitude satisfies

$$[g_6] > -20\,. \tag{2.9.5}$$

Given that this coupling is the product of two quartic couplings and that the leading order quartic coupling has mass dimension -6 recursion relations can probe contributions to the 4-point amplitude with mass dimension in the range

$$-14 < [g_4] \leq -6\,. \tag{2.9.6}$$

Taking into account Bose symmetry, the most general ansatz one can write down for the 4-point matrix element of local operators is

$$\begin{aligned}\mathcal{A}_4(1_\phi 2_\phi 3_\phi 4_\phi) =& \frac{c_0}{\Lambda^6} stu \\ &+ \frac{c_1}{\Lambda^8}\left(s^4+t^4+u^4\right) \\ &+ \frac{c_2}{\Lambda^{10}}\left(s^5+t^5+u^5\right) \\ &+ \frac{1}{\Lambda^{12}}\left(c_3\left(s^6+t^6+u^6\right)+c_3' s^2t^2u^2\right)+\mathcal{O}(\Lambda^{-14})\,.\end{aligned} \tag{2.9.7}$$

The leading term with coupling c_0/Λ^6 is the usual quartic Galileon. The terms suppressed by higher powers of the the UV cutoff Λ encode all possible higher-derivative quartic operators of the scalar field up to order Λ^{-14}.

We apply the 6-point test with $\sigma = 3$ and find that consistency requires $c_1 = c_3 = 0$ in the ansatz (2.9.7). The 4-point amplitude then becomes

$$\mathcal{A}_4(1_\phi\, 2_\phi\, 3_\phi\, 4_\phi) = \frac{c_0}{\Lambda^6} stu + \frac{c_2}{\Lambda^{10}}\left(s^5+t^5+u^5\right) + \frac{c_3'}{\Lambda^{12}} s^2t^2u^2 + \mathcal{O}(\Lambda^{-14})\,. \tag{2.9.8}$$

From this, we understand that there cannot exist an 8-derivative Lagrangian operator that preserves the special Galileon symmetry. Additionally, at 6-, 10- and 12-derivative order there exist unique quartic operators compatible with $\sigma = 3$. In Section 2.9.4, we show explicitly that the result (2.9.8) can also be obtained from an application of the BCJ double-copy.

Next we examine the possible existence of quintic operators compatible with $\sigma = 3$. We combine input from the quartic Galileon with the most general possible ansatz for the 5-point matrix elements and use the 7-point test to assess compatibility with $\sigma = 3$. The soft subtracted recursion relations at 7 points are valid if

$$[g_7] > -24\,. \tag{2.9.9}$$

Since the 7-point coupling constant is the product of a quartic (with mass dimension -6 or lower) and a quintic coupling, the latter must then satisfy

$$[g_5] > -18\,. \tag{2.9.10}$$

With Bose symmetry and the requirement that the ansatz for the 5-point amplitude must have soft weight $\sigma = 3$, we are left with

$$\begin{aligned}
\mathcal{A}_5(1_\phi\, 2_\phi\, 3_\phi\, 4_\phi\, 5_\phi) &= \frac{d_1}{\Lambda^{15}}\epsilon(1234)\sum_P (-1)^{|P|} s_{P_1P_2}s_{P_2P_3}s_{P_3P_4}s_{P_4P_5}s_{P_5P_1} \\
&+\frac{1}{\Lambda^{17}}\Bigg[d_2\,\epsilon(1234)^4 + d_3\epsilon(1234)\sum_P(-1)^{|P|}s_{P_1P_2}s^2_{P_2P_3}\left(s^2_{P_2P_3}s_{P_3P_4}-s^2_{P_1P_2}s_{P_2P_4}\right) \\
&+d_4\left(\frac{4}{5}\sum_{i<j}s^3_{ij}\sum_{i<j}s^5_{ij}+\sum_{i<j}\sum_{k\neq i,j}\left(20s^2_{ij}s^3_{ik}s^3_{jk}+9s^4_{ij}s^2_{ik}s^2_{jk}-2s^6_{ij}s_{ik}s_{jk}\right)\right)\Bigg]+\mathcal{O}(\Lambda^{-19})\,.
\end{aligned} \tag{2.9.11}$$

In the above, $\epsilon(1234) = \epsilon_{\mu\nu\rho\sigma}p_1^\mu p_2^\nu p_3^\rho p_4^\sigma$, the sum $\sum_{i<j}$ means $\sum_{i=1}^4\sum_{j=i+1}^5$, while the sum $\sum_P$ is over all permutations of $\{1,2,3,4,5\}$, $(-1)^{|P|}$ is the signature of the permutation and P_i is its ith element. There are no contributions to the amplitude that have less than 14 derivatives. The $1/\Lambda^{14}$-term satisfies the constructibility criterion and vanishes in 3d kinematics, in agreement with the discussion of Section 2.4.5. Two of the $1/\Lambda^{17}$-terms also vanish in 3d kinematics, but this was not a priori expected since they are too high order to satisfy constructibility.

The 7-point test implies no constraints on the coefficients d_1, d_2, d_3 and d_4. This is evidence in favor of the existence of four 5-point operators that preserve the special Galileon symmetry. Next, in Section 2.9.4, we investigate whether this result can be obtained from a double-copy prescription, similar to the 4-point case.

2.9.4 Comparison with the Field Theory KLT Relations

The significance of the special Galileon extends well beyond the contraction limit of the 3-brane effective field theory and the decoupling limit of massive gravity. The enhancement of the soft behavior to $\sigma = 3$ (which degenerates to $\sigma = 2$ when the DBI interactions are re-introduced) or correspondingly the extension of the non-linearly realized symmetry algebra suggests that this model has a fundamental significance of its own that is at present only partially understood. Perhaps one of the deepest and least understood aspects of the special Galileon is its role in the (field theory) *KLT algebra* as the product of two copies of the $\frac{U(N)\times U(N)}{U(N)}$ non-linear sigma model. For $N = 2, 3$ this coset sigma model has been intensively studied as a phenomenological model of the lightest mesons under the name *Chiral Perturbation Theory* (χPT). Henceforth we will use this name to avoid confusion with the $\mathbb{CP}^1$ non-linear sigma model discussed in Section 2.7.

The *double-copy* relation between χPT and the special Galileon was first understood in the CHY auxilliary world-sheet formalism [85]. Specifically, it was shown in the CHY formalism that the *leading order* contribution to scattering in the special Galileon model can be obtained from the KLT product

$$\mathcal{A}_n^{\text{sGal}} = \sum_{\alpha,\beta} \mathcal{A}_n^{\chi\text{PT}}[\alpha] S_{\text{KLT}}[\alpha|\beta] \mathcal{A}_n^{\chi\text{PT}}[\beta] \,, \tag{2.9.12}$$

where α, β index the $(n-3)!$ independent color(flavor)-orderings.[21] The KLT kernel $S_{\text{KLT}}[\alpha|\beta]$ is universal in the sense that the explicit form of the relations (2.9.12) are identical to the perhaps more familiar field theory KLT relations giving a double-copy construction of Einstein-dilaton-$B_{\mu\nu}$ gravity from two copies of Yang-Mills theory. Concretely, the first few relations have the form

$$\begin{aligned}
\mathcal{A}_4^{\text{sGal}}(1,2,3,4) &= -s_{12}\mathcal{A}_4^{\chi\text{PT}}[1,2,3,4]\,\mathcal{A}_4^{\chi\text{PT}}[1,2,4,3] \,, \\
\mathcal{A}_5^{\text{sGal}}(1,2,3,4,5) &= s_{23}s_{45}\mathcal{A}_5^{\chi\text{PT}}[1,2,3,4,5]\,\mathcal{A}_5^{\chi\text{PT}}[1,3,2,5,4] + (3 \leftrightarrow 4) \,, \\
\mathcal{A}_6^{\text{sGal}}(1,2,3,4,5,6) &= -s_{12}s_{45}\mathcal{A}_6^{\chi\text{PT}}[1,2,3,4,5,6]\Big(s_{35}\mathcal{A}_6^{\chi\text{PT}}[1,5,3,4,6,2] \\
&\qquad + (s_{34}+s_{35})\mathcal{A}_6^{\chi\text{PT}}[1,5,4,3,6,2]\Big) + \mathcal{P}(2,3,4) \,,
\end{aligned} \tag{2.9.13}$$

where $\mathcal{P}(2,3,4)$ denotes the sum of all permutations of legs 2, 3 and 4.

For the formulae (2.9.12) and (2.9.13) to even be well-defined, the color-ordered amplitudes on the right-hand-side must satisfy a number of non-trivial relations to reduce the number of independent partial amplitudes to $(n-3)!$ for the scattering of n particles. The existence of a color-ordered representation is itself non-trivial and not guaranteed to be satisfied in all models with color structure [86]. In all known cases where the double-copy relations (2.9.12) give a

[21] We use square brackets for the arguments of a color-ordered amplitude.

sensible, physical output, the reduction to a reduced basis of size $(n-3)!$ is accomplished by two sets of identities among the partial amplitudes, namely the *Kleiss-Kuijf* and *fundamental Bern-Carrasco-Johansson* relations. That these identites obtain for amplitudes calculated in the leading two-derivative action of χPT was first established in [87] using semi-on-shell recursion techniques developed in [88].

Our goal in this section is to connect two (possibly discrepant) definitions of the special Galileon model:

1. The special Galileon is the most general effective field theory of a real massless scalar with $\sigma = 3$ vanishing soft limits.

2. The special Galileon is the double-copy of two copies of χPT.

What we have described above is the known fact that these definitions agree at the lowest non-trivial order. In the previous section we used soft subtracted recursion to construct the most general 4- and 5-point amplitudes consistent with the first definition up to order Λ^{-12} and Λ^{-17} respectively. To determine if these results agree with the second definition we must first construct the most general 4- and 5-point amplitudes in χPT compatible with the requirements of the double-copy. Here we are following the approach of [86] and making the most conservative possible assumptions. Specifically we assume that both the explicit form of the double-copy (2.9.13) *and* the relations the amplitudes must satisfy to reduce the basis of partial amplitudes to size $(n-3)!$ are *identical* to what is required at leading order.

Let us begin with the 4-point amplitudes. The relations we impose are cyclicity (C)

$$\mathcal{A}_4^{\chi\text{PT}}[1,2,3,4] = \mathcal{A}_4^{\chi\text{PT}}[2,3,4,1]\,, \tag{2.9.14}$$

Kleiss-Kuijf (KK) or $U(1)$-decoupling

$$\mathcal{A}_4^{\chi\text{PT}}[1,2,3,4] + \mathcal{A}_4^{\chi\text{PT}}[2,1,3,4] + \mathcal{A}_4^{\chi\text{PT}}[2,3,1,4] = 0\,, \tag{2.9.15}$$

and the fundamental BCJ relation

$$(-s-t)\mathcal{A}_4^{\chi\text{PT}}[1,2,3,4] - t\mathcal{A}_4^{\chi\text{PT}}[1,2,4,3] = 0\,. \tag{2.9.16}$$

Since there are no additional quantum number labels in the partial amplitudes, at each order the 4-point amplitude is determined by a single polynomial function of the available Lorentz singlets

$$\mathcal{A}_4^{\chi\text{PT}}[1,2,3,4] = F^{(0)}(s,t) + \frac{1}{\Lambda^2}F^{(2)}(s,t) + \frac{1}{\Lambda^4}F^{(4)}(s,t) + \dots \tag{2.9.17}$$

The superscript k counts both the mass dimension of the function and the number of derivatives in the underlying effective operator. In this language, the double-copy-compatibility conditions take the form

$$\begin{aligned}
&\text{C:} && F^{(k)}(s,t) = F^{(k)}(-s-t,t)\,, \\
&\text{KK:} && F^{(k)}(s,t) + F^{(k)}(s,-s-t) + F^{(k)}(-s-t,s) = 0\,, \\
&\text{BCJ:} && (-s-t)F^{(k)}(s,t) - tF^{(k)}(s,-s-t) = 0\,.
\end{aligned} \tag{2.9.18}$$

We make a general parametrization of the polynomial functions as

$$\begin{aligned}
F^{(0)}(s,t) &= c_1^{(0)}, \\
F^{(2)}(s,t) &= c_1^{(2)}s + c_2^{(2)}t, \\
F^{(4)}(s,t) &= c_1^{(4)}s^2 + c_2^{(4)}st + c_3^{(4)}t^2, \\
F^{(6)}(s,t) &= c_1^{(6)}s^3 + c_2^{(6)}s^2t + c_3^{(6)}st^2 + c_4^{(6)}t^3, \\
F^{(8)}(s,t) &= c_1^{(8)}s^4 + c_2^{(8)}s^3t + c_3^{(8)}s^2t^2 + c_4^{(8)}st^3 + c_5^{(8)}t^4,
\end{aligned} \tag{2.9.19}$$

and so on. Imposing the conditions (2.9.18) gives a system of linear relations among the coefficients $c_i^{(k)}$. These are straightforward to solve and give

$$\mathcal{A}_4^{\chi\text{PT}}[1,2,3,4] = \frac{g_2}{\Lambda^2}t + \frac{g_6}{\Lambda^6}t(s^2+t^2+u^2) + \frac{g_8}{\Lambda^8}t(stu) + \ldots \tag{2.9.20}$$

A few comments about this result. As expected, the leading 2-derivative contribution is compatible with the conditions (2.9.18). Surprisingly, there are no compatible contributions from 4-derivative operators, but there are unique contributions at 6- and 8-derivative order. Moreover, the structure of the result here agrees with the 4-point amplitude of *Abelian Z-theory* [89]. The Z-theory model is a top-down construction which gives open string scattering amplitudes as the field theory double-copy of Yang-Mills and a higher-derivative extension of χPT. The Z-amplitudes are by construction guaranteed to satisfy the double-copy-compatibility conditions but with Wilson coefficients g_i having precise values calculated from the known string amplitudes. The method of this section can be understood as the bottom-up converse of the Z-theory construction, and at 4-point we find agreement.

To summarize, we have shown that up to 8-derivative order there is a 3-parameter family of operators that generate 4-point matrix elements compatible with the conditions required for the double-copy to be well-defined. We could continue this to higher order, but our ability to compare with the methods of Section 2.9.3 are bounded above at this order by the constructibility criterion.

To construct the associated amplitudes in the special Galileon model (according to the second definition described above) we use the first relation in (2.9.13). The result is

$$\mathcal{A}_4^{\text{sGal}}(1,2,3,4) = \frac{c_1}{\Lambda^6}stu + \frac{c_2}{\Lambda^{10}}\left(s^5+t^5+u^5\right) + \frac{c_3}{\Lambda^{12}}s^2t^2u^2 + \ldots\,, \tag{2.9.21}$$

in precise agreement with the special Galileon amplitude (2.9.8).

As an additional check to the results obtained above, we calculate the 6-point amplitudes of both χPT and the special Galileon. Up to order $\mathcal{O}(\Lambda^{-6})$ the χPT amplitude can be calculated using soft subtracted recursion with (2.9.20) as input. Note that only three factorization channels contribute to this calculation because the rest do not preserve color ordering. The resulting amplitude,

$$\mathcal{A}_6^{\chi\text{PT}}[1,2,3,4,5,6] = \frac{g_2^2}{\Lambda^4}\left[\frac{s_{13}s_{46}}{p_{123}^2} + \frac{s_{24}s_{15}}{p_{234}^2} + \frac{s_{35}s_{26}}{p_{345}^2} - s_{246}\right] + \mathcal{O}(\Lambda^{-8})\,, \tag{2.9.22}$$

satisfies all C, KK and BCJ constraints. Contributions subleading to the ones listed above do not satisfy the constructibility criterion (2.4.21) and cannot be calculated using soft subtracted recursion. However, we were able to uniquely determine them up to order $\mathcal{O}(\Lambda^{-10})$, by demanding that they have the correct pole structure, consistent with unitarity and locality, have $\sigma = 1$ soft weight and satisfy C, KK and BCJ conditions. The result of this calculation is listed in (B.0.30).

We are now in position to calculate the 6-point special Galileon amplitude with two different methods. We can either use the 6-point KLT relation in (2.9.13) or use soft subtracted recursion with (2.9.21) as input. The results of these calculations match perfectly up to order $\mathcal{O}(\Lambda^{-18})$, which is the furthest the recursive calculation can go.

Shifting our focus to 5-point amplitudes, we find that it is *not* possible to reproduce (2.9.11) as a double-copy of two (identical or non-identical) color-ordered scalar amplitudes, despite the perfect agreement at 4- and 6-points. Starting from a general ansatz for the scalar color-ordered amplitude, we find that the leading contribution that satisfies all C, KK and BCJ constraints is $\mathcal{O}(\Lambda^{-15})$ corresponding to a valence 5 scalar-field operator with 14 derivatives. The existence of such an operator at all is interesting since there are apparently *no odd point amplitudes in Z-theory* [89]! At this order we find that the kinematic structure of Z-theory does not coincide with the most general possible double-copy-compatible higher-derivative extension of χPT. Or perhaps said differently, just like string theory fixes the Wilson coefficients in the 4-point result (2.9.20) to take particular (non-zero) values, it appears to fix the Wilson coefficients of the odd-point amplitudes to be zero.

When we use the second relation of (2.9.13) with this result, we obtain a 5-point scalar amplitude of order $\mathcal{O}(\Lambda^{-33})$, which is significantly subleading to the amplitude (2.9.11) we calculated in the previous section for the special Galileon.

CHAPTER 3

All-Multiplicity One-Loop Amplitudes in Born-Infeld Electrodynamics from Generalized Unitarity

3.1 Special Features of Born-Infeld Theory

The Born-Infeld model of non-linear electrodynamics is a low-energy effective field theory of central importance in theoretical physics. Introduced long ago as an (ultimately misguided) proposed classical solution to the electron self-energy problem [7], it subsequently reappeared as the low-energy effective description of world-volume gauge fields on D-branes [90–92]. Independently of this *stringy* characterization, the Born-Infeld model has proven to be a truly exceptional example of a low-energy effective theory of non-linear electrodynamics, though perhaps at times a mysterious one.

As a classical field theory in $d = 4$ the Born-Infeld model can be described by the effective action

$$S_{\text{BI}} = -\Lambda^4 \int \mathrm{d}^4 x \left[\sqrt{-\det\left(g_{\mu\nu} + \frac{1}{\Lambda^2} F_{\mu\nu} \right)} - 1 \right], \tag{3.1.1}$$

where Λ is the characteristic scale in the problem. In the D-brane picture, Λ is related to the brane tension.

Low-energy scattering of light-by-light in the Born-Infeld model can be calculated as a perturbative expansion in $1/\Lambda$. The tree-approximation to these scattering amplitudes has been a subject of interest recently in the context of modern on-shell approaches to quantum field theory. For example, in [1] two novel on-shell approaches for calculating 4d tree-level Born-Infeld amplitudes were given: by imposing *multi-chiral* low-energy theorems derived from supersymmetric relations with Goldstone fermions, and from T-duality constraints under dimensional reduction. Also very striking is the discovery in [85], in the context of the CHY formulation of the tree-level S-matrix, that the KLT formula relating Yang-Mills (YM) and gravity amplitudes also gives Born-Infeld tree amplitudes if one of the gauge theory factors is replaced with the flavor-ordered amplitudes of Chiral Perturbation Theory (χPT):

$$\mathrm{BI}_d = \mathrm{YM}_d \otimes_{\mathrm{KLT}} \chi\mathrm{PT}_d. \tag{3.1.2}$$

The subscript d indicates the spacetime dimensions of these theories. What all of these discoveries make clear is that there is an enormous amount of structure hidden behind the action (3.1.1) which may be leveraged to make possible previously unattainable calculations. It should also be noted that Born-Infeld plays a central role in the ever growing web of mysterious connections between gauge theories, gravity theories, and EFTs in diverse dimensions [93, 94]. Also of great relevance in this paper, pure Born-Infeld can be defined as a consistent truncation of $\mathcal{N} > 1$ supersymmetrizations of Dirac-Born-Infeld theory.

The tree amplitudes in 4d Born-Infeld theory exhibit an important and interesting feature: they vanish unless the external states have an equal number of positive and negative helicity states. This is the on-shell manifestation of electromagnetic duality of the classical theory in 4d. In particular, the 4-particle tree amplitude[1] is

$$\mathcal{A}_4^{(\mathrm{tree})\ \mathrm{BI}_4}(1_\gamma^+, 2_\gamma^+, 3_\gamma^-, 4_\gamma^-) = \frac{1}{\Lambda^4}[12]^2\langle 34\rangle^2\,, \tag{3.1.3}$$

while all other helicity configurations vanish. Note that the emergence of electromagnetic duality is highly non-trivial in the double-copy construction (3.1.2). Some of the key properties of the BI tree amplitudes are summarized in Figure 3.1.

The recent progress in Born-Infeld scattering has so far been restricted to tree-level amplitudes. Given the development of powerful unitarity based methods for recycling trees into loops [33], there is every reason to believe that interesting structures are waiting for us in the loop amplitudes. In this context almost nothing is known.[2] There are good reasons for this; the calculations in Born-Infeld electrodynamics at one-loop are challenging, in ways that are importantly different from superficially similar calculations in perturbative quantum gravity. Similar to calculations at one-loop using Feynman rules derived from expanding the linearized Einstein-Hilbert action, the first computational bottleneck in Born-Infeld is given by the problem of determining the off-shell

[1] Compared to the action (3.1.1), we have rescaled $\Lambda^4 \to \Lambda^4/2$, such that the 4-point amplitude has coupling $1/\Lambda^4$.

[2] One of the few explicit calculations is the determination of the cut-constructible part of the 4-point MHV amplitude in $\mathcal{N} = 4$ DBI_4 in [95].

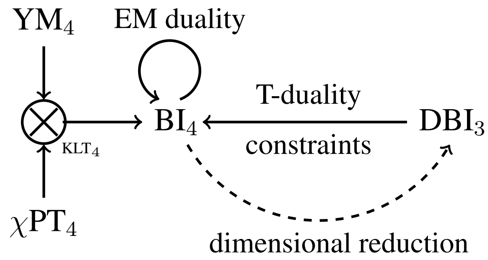


Figure 3.1: Some key-properties of BI amplitudes at tree-level, in particular the double-copy construction and 4d electromagnetic duality. The idea behind the T-duality constraint [1] is that when dimensionally reduced along one direction, a linear combination of the photon polarizations become a scalar modulus of the compactified direction,. i.e. it is the Goldstone mode of the spontaneously broken translational symmetry and as such it must have enhanced $\mathcal{O}(p^2)$ soft behavior.

vertex factors for the interaction terms given by expanding (3.1.1)

$$S_{\mathrm{BI}} \sim \int \mathrm{d}^4 x \left[F^2 + \frac{c_1}{\Lambda^4} F^4 + \frac{c_2}{\Lambda^8} F^6 + \ldots \right]. \tag{3.1.4}$$

As the multiplicity of external states increases, more and more terms in this expansion must be kept, and so an ever growing list of increasingly long vertex factors must be calculated. At multiplicity n, operators of the form F^n will contribute; with vertex factors given as sums over permutations growing exponentially in n. Beyond the lowest multiplicity, calculating such an amplitude by hand is almost unthinkable, and even with state-of-the art computing power one soon hits a hard wall when performing such a brute force calculation. The situation here is a little different from perturbative gravity. In gravity, the vertex factors are not independent since they are not separately gauge invariant; the higher-point interactions are in principle completely determined by locality and Lorentz invariance by the three-particle ones. This can have dramatic consequences, for example in [96] all-multiplicity, rational one-loop results are obtained from the lowest multiplicity results by enforcing the correct collinear and soft limits. In Born-Infeld, however, these higher-valence operators are genuinely gauge invariant physical operators, the associated Wilson coefficients are not related by any inviolable field theory principle and must instead be fixed by imposing additional physical constraints. No analysis of soft or collinear limits could possibly determine the all-multiplicity one-loop amplitudes in Born-Infeld, unless it incorporated additional physical information beyond Lorentz invariance and locality.

The second computational bottleneck occurs when evaluating the required loop integrals. Even if the required loop integrands can be constructed, we still have to integrate the resulting expressions. Operators of the form F^n are n-derivative operators and the associated vertex factors have n powers of momentum. The resulting loop-integrands therefore involve tensors with ranks that grow larger and larger with the multiplicity. This is unlike gravity that only has two-derivative interactions. Attempting to apply traditional Passarino-Veltman reduction algorithms to such high-rank tensor expressions again quickly leads to a confrontation with the limits of computing power. Such a direct calculation is primarily limited by the fact that the method of Feynman diagrams is completely general. It therefore makes no use of any of the aforementioned properties that make Born-Infeld electrodynamics exceptional. For example, such an approach would be equally well-suited to calculating loop corrections in the Euler-Heisenberg effective theory [97], another well-studied example of a model of non-linear electrodynamics.

In this paper, we initiate a study of 4d non-supersymmetric Born-Infeld theory at the loop-level. We use modern on-shell methods (supersymmetric decomposition, double-copy, T-duality…) that are specialized to the particular properties of Born-Infeld and to the objects we compute. We derive results that would be impossible to obtain with traditional methods. Specifically, we derive all-multiplicity results for the one-loop amplitudes in the *self-dual* (SD) and *next-to-self-dual* (NSD) sectors of 4d non-supersymmetric Born-Infeld:

$$\mathcal{A}_n^{\text{SD}}\left(1_\gamma^+,\, 2_\gamma^+,\, \ldots\, (n-1)_\gamma^+,\, n_\gamma^+\right) \quad \text{and} \quad \mathcal{A}_n^{\text{NSD}}\left(1_\gamma^+,\, 2_\gamma^+,\, \ldots\, (n-1)_\gamma^+,\, n_\gamma^-\right). \tag{3.1.5}$$

Any 4d cuts of these amplitudes vanish, hence to obtain them d-dimensional unitarity is used and the results are necessarily rational functions of the external momenta.

One motivation for these calculations is to examine the fate of electromagnetic duality at loop-level in pure Born-Infeld theory. We make some observations at the end of the paper, but otherwise this will be the subject of a forthcoming paper.

3.2 Overview of Method

Our goal in this paper is to calculate SD and NSD one-loop amplitudes in non-supersymmetric Born-Infeld in $d = 4$. As discussed in Section 3.1 instead of traditional Feynman diagrammatics we make extensive use of modern on-shell methods to construct the amplitudes. In particular, we use d-dimensional generalized unitarity methods [98] to construct the complete loop-integrand in a physically motivated dimensional scheme. We begin with a brief overview of unitarity methods and then describe in detail the approach taken in this paper. In Section 3.2.1, we introduce the techniques in the familiar context of Yang-Mills theory, then adapt the methods to Born-Infeld in Section 3.2.2.

3.2.1 Generalized Unitarity and Supersymmetric Decomposition

The main idea of unitarity based methods [99] is to exploit that the loop integrand is a complex rational function of the loop momentum with singularity structure constrained by factorization into on-shell tree amplitudes. Here we focus specifically on one-loop order and all calculations are made in a given dimensional regularization scheme. This means that while the external momenta and polarizations are strictly $d = 4$-dimensional, the loop momentum is formally regarded as $d = (4 - 2\epsilon)$-dimensional.

4-Dimensional Unitarity Methods

Expanding the loop-*integrand* around $\epsilon = 0$, the leading $\mathcal{O}(\epsilon^0)$ component has an unambiguous physical meaning related to unitarity of the S-matrix. Via the Cutkosky theorem [100], the factorization of the integrand into on-shell tree amplitudes on 4d cuts

$$l_1^2 \;\cdots\; l_k^2 \, \mathcal{I}_n[l]\bigg|_{l_1^2=\cdots=l_k^2=0} = \sum_{\text{states}} \mathcal{A}^{\text{tree}_4}_{(1)} \ldots \mathcal{A}^{\text{tree}_4}_{(k)}, \qquad k \leq 4, \tag{3.2.1}$$

where l_i^μ, for $i = 1, \cdots k$ are 4d momenta, ensures that the integrated amplitude has the correct branch cut discontinuities required by the optical theorem. A rational function with all the correct 4d cuts (and no spurious cuts) then yields the correct amplitude at $\mathcal{O}(\epsilon^0)$ after integration, up to a function with no branch cuts, i.e. a rational function. This is the idea of the *4-dimensional unitarity* approach: the cut-constructible part of the amplitude is completely fixed by the physical tree amplitudes. Due to a complete understanding of integrand reduction to a basis of master scalar integrals at one-loop this procedure can be completely automated [101]. The remaining rational function ambiguity must then be determined by imposing additional physical constraints, such as cancellation of spurious singularities in the cut-constructible part or by imposing known behavior in soft or collinear limits [96,102]. One advantage of calculating the 4d-cut-constructible part and the rational part separately in this way is that at all stages of the calculation we make use of regularization scheme-independent, physical objects (on-shell 4d tree-amplitudes). The primary disadvantage to this approach is the relative difficulty in calculating the rational terms separately.

d-Dimensional Unitarity Methods

In certain cases, the cut-constructible part vanishes and the integrated loop-amplitude is purely rational. In that case, the method outlined above for determining the rational part is not applicable. This, in particular, will be the situation for the amplitudes (3.1.5) of interest in this paper.

A more familiar example is the SD and NSD sectors of pure Yang-Mills theory (i.e. the all-plus and all-plus-one-minus gluon amplitudes): at one-loop, any 4d cut has factors of tree amplitudes of the SD and NSD helicity configurations and those vanish [65], hence all the 4d cuts vanish. According to the discussion above, the absence of 4d cuts implies that the resulting integrand is zero at $\mathcal{O}(\epsilon^0)$ (vanishes in $d=4$), but may have non-zero contributions at $\mathcal{O}(\epsilon)$. As a result, SD and NSD one-loop amplitudes have no branch cut discontinuities and are instead purely rational functions. These rational contributions arise from subtle ϵ/ϵ cancellations after integration; the same mechanism gives rise to the chiral anomaly in dimensional regularization [103]. Since the SD and NSD sectors of YM and BI theory are very similar, we introduce the method here for YM , then adapt it to BI theory in the Section 3.2.2.

The method of *d-dimensional unitarity* [98] does not separate the 4d-cuts and rational terms. In the d-dimensional unitarity approach, we must first define a suitable dimensional regularization scheme in which d-dimensional integrand cuts have the form

$$l_1^2 l_2^2 \dots l_k^2 \mathcal{I}_n[l]\bigg|_{l_1^2=\dots=l_k^2=0} = \sum_{\text{states}} \mathcal{A}_{(1)}^{\text{tree}_d} \dots \mathcal{A}_{(k)}^{\text{tree}_d}\,, \tag{3.2.2}$$

where the on-shell cut momenta l_i are d-dimensional. The additional constraint of correct cuts in d-dimensions is sufficient to construct the integrand to all orders in ϵ, allowing us to determine both the 4d cut-constructible and rational parts at the same time. This approach is therefore well-suited to the purely rational SD and NSD one-loop amplitudes of Yang-Mills. The difficulty of this approach is that we are forced to work with regularization scheme-dependent quantities, which are therefore non-unique, and furthermore since the cuts are in d-dimensions, we lose the simplicity of spinor-helicity variables.

In certain special cases, such as pure Yang-Mills and pure Born-Infeld in $d=4$, we can maneuver around these difficulties and define a regularization scheme in which both the d-dimensional-cut structure is quite simple *and* we can still make use of spinor-helicity variables. This simplified implementation of d-dimensional unitarity is sometimes referred to as *supersymmetric decomposition* and this is what we describe next.

Consistent Truncation and Supersymmetric Decomposition

It is instructive to first review the concept of supersymmetric consistent truncation at tree-level. In general we say that model A is a *consistent truncation* of model B if the on-shell states of A form a subset of the on-shell states of B and (when restricted to the A-states) the S-matrices are identical at tree-level.[3] This occurs in any model in which the states of B/A (B-states that are not

[3] This is equivalent to the statement that solutions to the classical equations of motion for model A are also solutions to the equations of motion of model B with the fields in B/A turned off.

A-states) carry an independent charge or parity; such states can give no contribution to state-sums on factorization singularities and hence no contribution to the tree-level S-matrix elements with all external A-states. A simple example of this occurs in any model containing both Bosonic and Fermionic states; since the quantity $(-1)^F$ is conserved we can always construct a consistent truncation by restricting to the Bosonic sector. If there are additional conserved quantities in the Bosonic sector, then it may be possible to give a further truncation.

As a relevant example, consider $\mathcal{N} = 2$ super Yang-Mills (without matter hypermultiplets) in $d = 4$. The spectrum consists of a massless vector multiplet containing a gauge boson $g^{\pm}$, two Weyl fermions $\psi^{\pm}_{1,2}$ and a complex scalar ϕ, $\overline{\phi}$. Restricting to the Bosonic sector gives a consistent truncation, the resulting model is non-supersymmetric and describes Yang-Mills coupled to a massless (adjoint) complex scalar. In this model there is an additional *global* symmetry, descended from R-symmetry, under which the states are charged as

$$Q[g^{\pm}] = 0, \qquad Q[\phi] = 1, \qquad Q[\overline{\phi}] = -1. \tag{3.2.3}$$

Consequently, we can define a further truncation to the purely *gluonic sector*, the resulting model is precisely pure non-supersymmetric Yang-Mills. The statement of consistent truncation in this example is then

$$\mathcal{A}_n^{\text{(tree) }\mathcal{N}=2\text{ SYM}}\left[1_g, \ldots n_g\right] = \mathcal{A}_n^{\text{(tree) YM+Adj}}\left[1_g, \ldots n_g\right] = \mathcal{A}_n^{\text{(tree) YM}}\left[1_g, \ldots n_g\right]. \tag{3.2.4}$$

Since gluonic amplitudes in $\mathcal{N} = 2$ SYM in the SD and NSD helicity sectors vanish at all orders of perturbation theory, these same helicity sectors must likewise vanish in tree-level non-supersymmetric Yang-Mills.

The notion of consistent truncation in the form of equalities such as (3.2.4) does not continue to hold at loop-level. We can, however, make use of supersymmetric truncations at one-loop to form a *supersymmetric decomposition*. Let us illustrate this in the context of Yang-Mills. At one-loop, all states in the model generically run in every loop, for $\mathcal{N} = 0, 1$ and 2 SYM we can schematically represent the contributions to purely gluonic amplitudes as

$$\begin{aligned}
\mathcal{A}_n^{\text{(1-loop) YM}}\left[1_g \ldots n_g\right] &= \mathcal{A}_n^{[V]}\left[1_g \ldots n_g\right] \\
\mathcal{A}_n^{\text{(1-loop) }\mathcal{N}=1\text{ SYM}}\left[1_g \ldots n_g\right] &= \mathcal{A}_n^{[V]}\left[1_g \ldots n_g\right] + \mathcal{A}_n^{[F]}\left[1_g \ldots, n_g\right] \\
\mathcal{A}_n^{\text{(1-loop) }\mathcal{N}=2\text{ SYM}}\left[1_g \ldots n_g\right] &= \mathcal{A}_n^{[V]}\left[1_g \ldots n_g\right] + 2\mathcal{A}_n^{[F]}\left[1_g \ldots n_g\right] + \mathcal{A}_n^{[S]}\left[1_g \ldots n_g\right],
\end{aligned} \tag{3.2.5}$$

where V, F, and S represent contributions from vector bosons, Weyl fermions, and complex scalars, respectively. The contributions on the right-hand-side have no invariant physical meaning, even in the context of a Feynman diagram expansion, as a grouping of terms they depend on the choice of regularization scheme. One can, however, give invariant physical meaning to these expressions on 4d-unitarity cuts: the decomposition reflects the contributions to the state sums. Note that it is the existence of the same conservation laws that allowed us to construct consistent truncations at tree-level that make this decomposition sensible. In particular, due to (3.2.3), there are no mixed scalar/gluon contributions to 4d cuts. If the amplitudes are calculated in the Four Dimensional Helicity (FDH) or similar schemes, in which the one-to-one correspondence between the (external) 4-dimensional helicity states and the (internal) d-dimensional states is preserved [104] then the relations (3.2.5) are well-defined on d-dimensional cuts.

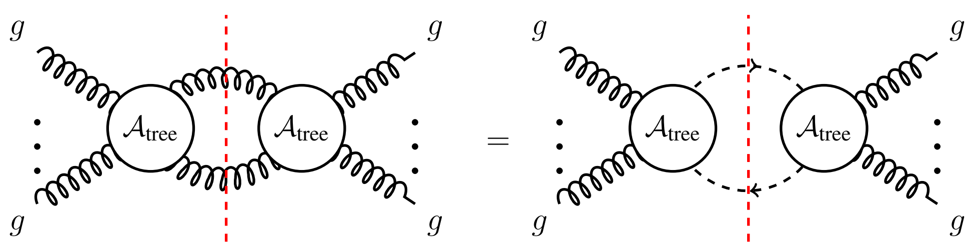
The notion of a *supersymmetric decomposition* is a rearrangement of (3.2.5) such that one-loop amplitudes in non-supersymmetric Yang-Mills can be given as sums over contributions from $\mathcal{N} = 1, 2$ vector multiplets and adjoint scalars

$$\begin{aligned} &\mathcal{A}_n^{\text{(1-loop) YM}}[1_g \dots n_g] \\ &\quad = -\mathcal{A}_n^{\text{(1-loop) }\mathcal{N}=2\text{ SYM}}[1_g \dots n_g] + 2\mathcal{A}_n^{\text{(1-loop) }\mathcal{N}=1\text{ SYM}}[1_g \dots n_g] + \mathcal{A}_n^{[S]}[1_g \dots n_g]\,. \end{aligned} \tag{3.2.6}$$

Next, we assume that our regularization scheme is supersymmetric (for example FDH [105]), and therefore the one-loop amplitudes satisfy the same supersymmetry Ward identities as the tree-level amplitudes.[4] This dramatically simplifies in the SD and NSD sectors, since the contributions from the $\mathcal{N} > 0$ components vanish. In these sectors the supersymmetric decomposition simplifies to

$$\mathcal{A}_n^{\text{(1-loop) YM}}\left[1_g^+ \dots (n-1)_g^+,\, n_g^\pm\right] = \mathcal{A}_n^{[S]}\left[1_g^+ \dots (n-1)_g^+,\, n_g^\pm\right]. \tag{3.2.7}$$

We refer to this as the *scalar-loop representation* of the one-loop amplitude. Again, in the context of d-dimensional unitarity we can interpret this statement unambiguously as a statement about the d-dimensional unitarity cuts of the loop-integrand.



(3.2.8)

As a consequence, the complete one-loop integrand can be reconstructed by requiring the correct

[4] In a non-supersymmetric scheme such as conventional dimensional regularization (CDR) the result of the loop integrals will typically not satisfy the supersymmetry Ward identities. Supersymmetry must be restored by adding finite local counterterms which modify the rational part of the one-loop amplitudes.

d-dimensional unitarity cuts into d-dimensional tree-amplitudes of the form

$$\mathcal{A}_n^{(\text{tree})}\left[1_\phi, 2_g \ldots (n-1)_g, n_{\bar{\phi}}\right]. \tag{3.2.9}$$

Here only the momenta of the scalars are d-dimensional, while the momenta and polarizations of the gluons are 4-dimensional.

We rewrite the d-dimensional momenta in terms of 4-dimensional momenta as

$$l^\mu = l^\mu_{[4]} + l^\mu_{[-2\epsilon]}. \tag{3.2.10}$$

Due to the orthogonality of 4-dimensional and (-2ϵ)-dimensional subspaces, we can rewrite the various Lorentz singlets that appear in the amplitude as

$$q \cdot l = q \cdot l_{[4]}, \qquad l^2 = l^2_{[4]} + l^2_{[-2\epsilon]} \equiv l^2_{[4]} + \mu^2, \tag{3.2.11}$$

where q^μ is any 4-dimensional vector and $\mu^2 \equiv l^2_{[-2\epsilon]}$. Using these relations we find that we can rewrite all d-dimensional amplitudes (3.2.9) as 4-dimensional amplitudes with a massive scalar of mass μ^2.

Up to this point we have not explicitly defined the regularization scheme, we have only made use of some assumed general properties. It is important to emphasize that physical observables are independent of the choice of regularization scheme. In this paper, the calculation we describe is made in a particular version of dimensional regularization that has certain convenient properties, but the physical conclusions should be independent of this choice, we discuss this further in the Discussion section.

We shall define the massive scalar amplitudes directly in 4d, requiring all of the standard tree-level properties of Lorentz invariance, locality and unitarity, in addition to the requirement

$$\mathcal{A}_n^{\text{tree}}\left[1_\phi, 2_g, \ldots, (n-1)_g, n_{\bar{\phi}}\right] \xrightarrow{\mu^2 \to 0} \mathcal{A}_n^{\text{tree }(\mathcal{N}=2)}\left[1_\phi, 2_g, \ldots, (n-1)_g, n_{\bar{\phi}}\right]. \tag{3.2.12}$$

Even though the 4d cuts vanish in the SD and NSD amplitudes of consideration, the relations (3.2.5) make sense for all helicity amplitudes, and for those with non-vanishing 4d cuts the $\mathcal{A}_n^{[S]}$ cuts must be equal to products of tree-amplitudes of $\mathcal{N} = 2$ SYM. The problem of constructing the integrand in the scalar loop representation then has two parts:

1. Define a model of a massive adjoint scalar coupled to Yang-Mills which reduces to the Bosonic sector of $\mathcal{N} = 2$ SYM in the massless limit.

2. Construct a complex rational function of 4d momenta with correct cuts into the massive scalar tree amplitudes and no spurious cuts.

The required massless limit (3.2.12) is not sufficient to determine the massive scalar model described in Step 1. In addition to the *minimal* coupling,[5] we could also add generic terms to the scalar potential or higher-derivative couplings, for example we might consider a model described by the action

$$S[A_\mu, \phi, \overline{\phi}] = S_{\text{minimal}}[A_\mu, \phi, \overline{\phi}] + \int \mathrm{d}^4x \left[\frac{\mu^2}{\Lambda_1^4} |\phi|^6 + \frac{\mu^2}{\Lambda_2^4} |\phi|^2 \mathrm{Tr}[F^2] \right], \tag{3.2.13}$$

where Λ_1 and Λ_2 are independent mass scales. Such a model clearly satisfies the correct massless limit. The presence of independent dimensionful parameters however makes this physically unacceptable, these would appear in the integrand we construct according to Step 2, and consequently the integrated amplitude. To ensure the absence of such spurious parameters we impose:

3. The result we calculate should agree with the parametric dependence on couplings expected from a full Feynman diagram calculation, therefore an acceptable massive scalar extension of Yang-Mills theory should depend only on the dimensionless Yang-Mills coupling g_{YM}.

By this simple argument all such higher dimension couplings must be absent, and we find that the conditions (1-3) uniquely pick out the minimally coupled massive adjoint scalar with the supersymmetric scalar potential. Such tree amplitudes can be generated efficiently by using massive BCFW recursion, which is reviewed in Section E.0.1.

The strategy described above has been used successfully to calculate all-multiplicity one-loop amplitudes in the SD and NSD sectors of pure Yang-Mills [33]. It has also been implemented in pure Einstein gravity [96] and also recently Einstein Yang-Mills [106]. The purpose of this paper is to implement this approach in non-supersymmetric Born-Infeld electrodynamics in $d = 4$. In the following subsection we will describe the novelties that appear in this model compared to Yang-Mills.

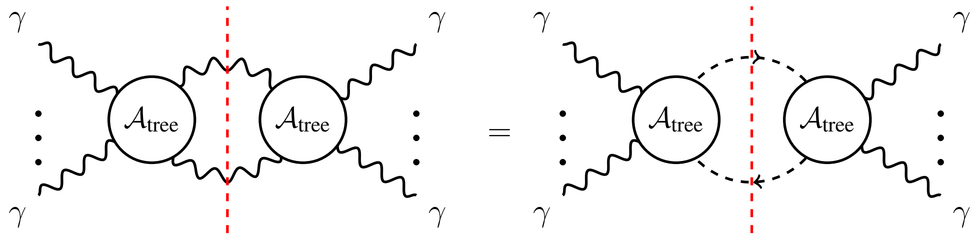
3.2.2 Massive Scalar Extension of Born-Infeld

Almost everything we described in Section 3.2.1 for pure Yang-Mills in $d = 4$ applies to pure Born-Infeld in $d = 4$. At tree-level, non-supersymmetric Born-Infeld is a consistent truncation of $\mathcal{N} = 2$ super Born-Infeld. Consequently, the SD and NSD amplitudes vanish at tree-level. Moreover, in a supersymmetric regularization scheme, the SD and NSD one-loop amplitudes have a scalar-loop representation

$$\mathcal{A}_n^{(\text{1-loop})\ \text{BI}_4}\left(1_\gamma^+, \ldots, (n-1)_\gamma^+, n_\gamma^\pm\right) = \mathcal{A}_n^{[S]}\left(1_\gamma^+, \ldots, (n-1)_\gamma^+, n_\gamma^\pm\right). \tag{3.2.14}$$

[5] This includes the $|\phi|^4$ term in the scalar potential required to satisfy the requirement of $\mathcal{N} = 2$ supersymmetry in the massless limit.

These one-loop amplitudes have no $d = 4$ cuts, so are purely rational. We compute the integrand using d-dimensional unitarity in which the cuts factor into tree amplitudes with two massive scalars coupled to the Born-Infeld photons.



(3.2.15)

Here the massive scalar model should reduce to $\mathcal{N} = 2$ super Born-Infeld in the massless limit, analogously to (3.2.12). Since there are independent gauge-invariant local operators coupling the Born-Infeld photon and a massive scalar which vanish in the massless limit, this is not sufficient to determine the massive model. Unlike Yang-Mills, we can construct an infinite number of such operators *without* introducing spurious dimensionful parameters. In other words, the analogue of conditions (1)-(3) above are *not sufficient* to uniquely pick out a specific model.

To proceed, additional physical constraints must be applied to uniquely define the massive scalar extension of Born-Infeld. In the remainder of this section, we describe the model, which we call mDBI$_4$ (massive DBI in 4d), and argue from two points of view why it is an appropriate definition. In Section 3.3 we then calculate the mDBI$_4$ tree amplitudes

$$\mathcal{A}_n^{\text{mDBI}_4}\big(1_\phi, 2_\gamma^+, \ldots, (n-2)_\gamma^+, (n-1)_\gamma^\pm, n_{\bar{\phi}}\big)\,, \tag{3.2.16}$$

needed for the unitarity cuts, where the complex scalar has mass $\mu^2 \equiv l^2_{[-2\epsilon]}$ in $d = 4$. As stated, these tree amplitudes must satisfy

$$\mathcal{A}_n^{\text{mDBI}_4}\big(1_\phi, 2_\gamma, \ldots, (n-1)_\gamma, n_{\bar{\phi}}\big) \xrightarrow{\mu^2 \to 0} \mathcal{A}_n^{\mathcal{N}=2\,\text{BI}_4}\big(1_\phi, 2_\gamma, \ldots, (n-1)_\gamma, n_{\bar{\phi}}\big). \tag{3.2.17}$$

The two approaches to define mDBI$_4$ are dimensional reduction and the double-copy; we now describe each in turn.

Dimensional Reduction and Supersymmetry

We define mDBI$_4$ as the dimensional reduction of pure Born-Infeld from $d = 6$ (BI$_6$). Specifically we take 6d tree-amplitudes with momenta and polarizations in the configuration described in Table 3.1, i.e. the photon momenta and polarizations lie in a 4d subspace for lines $2, 3, \ldots, n-1$ while lines 1 and n have genuinely 6d momenta but polarizations orthogonal to the 4d subspace, so in the 4d setting they are scalars. This is an appropriate definition because the amplitudes

	1	2	3	4	5
$\vec{p}_{1,n}$	x	x	x	x	x
$\vec{\epsilon}_{1,n}$				x	x
$\vec{p}_{2,3,\ldots,n-1}$	x	x	x		
$\vec{\epsilon}_{2,3,\ldots,n-1}$	x	x	x		

Table 3.1: Kinematic configuration of momenta and polarizations of BI_6 defining mDBI_4 and for YM_6 defining $(\mathrm{YM}+\mathrm{mAdj})_4$.

(3.2.16) arise from d-dimensional cuts of a loop-integrand in a supersymmetric regularization scheme.

As in the previous subsection, it is instructive to first describe the case of pure Yang-Mills. In any scheme, on 4d cuts the integrand factors into tree-amplitudes of YM_4, which by virtue of being a consistent truncation of $\mathcal{N}=2$ SYM_4 satisfy the supersymmetry Ward identities for 8 supercharges. On d-dimensional cuts, however, we would generically expect the action of the supersymmetry algebra to be explicitly broken. To construct a supersymmetric regularization scheme, we want to define a dimensional continuation from $d=4$ in which the action of the 8 supercharges of $\mathcal{N}=2$ is unbroken.

A natural way to do this is to recognize that the Yang-Mills-scalar tree amplitudes (3.2.9) can be obtained from pure Yang-Mills in $d=6$ (YM_6) with momenta and polarizations in the configuration given in Table 3.1. Since YM_6 is a consistent truncation of $\mathcal{N}=(1,0)$ SYM_6, the YM_6 tree amplitudes must satisfy the full set of $\mathcal{N}=(1,0)$ supersymmetry Ward identities. It therefore follows that in the configuration given in Table 3.1, the 6d amplitudes written in a 4d language, must satisfy (some version of) the supersymmetry Ward identities for 8 supercharges. We should therefore expect a regularization scheme with a scalar-loop representation (3.2.7), with massive scalar amplitudes defined by this dimensional reduction from 6d, to preserve (some version of) the full $\mathcal{N}=2$ supersymmetry on *d-dimensional cuts*, and it is therefore a supersymmetric scheme. This definition of the Yang-Mills-scalar amplitudes satisfies the criteria we gave in the previous subsection of absence of spurious parametric dependence. The massive scalar extension of 4d Yang-Mills theory defined this way will be denoted $(\mathrm{YM}+\mathrm{mAdj})_4$; as it turns out, it will be useful in our amplitude constructions.

The same argument applies essentially verbatim to Born-Infeld. BI_6 is a consistent truncation of $\mathcal{N}=(1,0)$ super Born-Infeld (SBI_6), so the tree-amplitudes of mDBI_4 defined by the configuration given in Table 3.1 must preserve the action of 8 supercharges. Hence the SD and NSD one-loop integrands of BI_4 in the scalar loop representation (3.2.1) preserve the action of $\mathcal{N}=2$ supersymmetry on d-dimensional cuts, and therefore define a scheme that we expect to be supersymmetric. We do not have a formal proof of this statement.

BCJ Double-Copy

A complimentary argument, with the same conclusion, is given by considering the BCJ double copy. It was shown in [85], in the context of the CHY formalism [14, 107], that the field theory KLT formulae which give gravity tree amplitudes as the double-copy of gauge theory tree amplitudes also give Born-Infeld if one of the gauge theory factors is replaced by Chiral Perturbation Theory (χPT). χPT is a non-linear sigma model with target space $\frac{SU(N)\times SU(N)}{SU(N)}$. This double-copy statement applies at tree-level in d-dimensions

$$\mathrm{BI}_d = \mathrm{YM}_d \otimes_{\mathrm{KLT}} \chi\mathrm{PT}_d\,. \tag{3.2.18}$$

It has been conjectured by BCJ that the double-copy could be extended to loop integrands [35]. This remains a conjecture, though it has been successfully applied in many examples and represents the current state of the art for high loop order calculations in maximal supergravity [108]. In this spirit we conjecture that the tree-level double-copy construction of Born-Infeld extends to a complete loop-level double copy following BCJ.

In this paper we do not make use of explicit color-kinematics dual BCJ integrands. Rather, we proceed by assuming that such a representation of the BI_4 integrand exists in a supersymmetric regularization scheme which admits a scalar-loop representation (3.2.14). Then on d-dimensional cuts, the integrand factors into tree amplitudes in a model coupling Born-Infeld photons to a massive scalar. Furthermore, these tree amplitudes should be given by the tree-level double-copy of YM_4 coupled to a massive scalar and $\chi\mathrm{PT}_4$ coupled to a massive scalar. The existence of such double-copy compatible massive scalar models is quite non-trivial.

We now want to show that the proposed definition of mDBI_4 is indeed generated by the tree-level double copy. The key to this is that the KLT product is valid in d-dimensions, it therefore commutes with dimensional reduction[6] in the sense described by the configuration in Table 3.1. This is summarized in Figure 3.2.

Since both Yang-Mills and χPT satisfy the conditions necessary for the double-copy to be well-defined in d-dimensions, we can begin with these models in $d = 6$. As illustrated in the diagram above we have two choices, either take the 6d double-copy first and then dimensionally reduce to 4d, or dimensionally reduce to the 4d massive scalar models first and then take the 4d double-copy; it is clear these choices will agree. In the first case, the validity of the d-dimensional double copy gives precisely the definition of mDBI_4 given above, the second case gives us exactly the massive scalar double-copy we expect on d-dimensional cuts if the loop BCJ conjecture is correct. The advantage of working in the 4d formulation is that we can take advantage of the 4d spinor helicity formalism.

[6] The dimensional reduction of $\chi\mathrm{PT}_6$ to $d = 4$ is defined by the momentum configuration in Table 3.1.

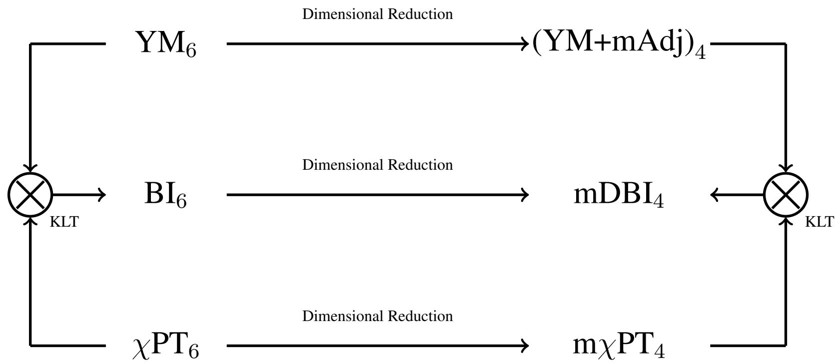


Figure 3.2: Dimensional reduction commutes with the KLT product. This provides us with two alternative approaches to mDBI$_4$.

3.3 Calculating mDBI$_4$ Tree Amplitudes

3.3.1 General Structure

As described in the previous section, the input required for constructing the (N)SD loop integrands using d-dimensional unitarity are tree amplitudes in some model (which we call mDBI$_4$) describing a massless Born-Infeld photon coupled to a massive complex scalar. We need two types of tree amplitudes:

- mDBI$_4$ NSD amplitudes: These are of the form $\mathcal{A}_n^{\text{mDBI}_4}\left(1_\phi, 2_\gamma^+, \ldots, (n-1)_\gamma^+, n_{\overline{\phi}}\right)$ and will be used to calculate BI$_4$ SD and NSD amplitudes in Sections 3.4.2 and 3.4.3 respectively.
- mDBI$_4$ MHV amplitudes: These are of the form $\mathcal{A}_n^{\text{mDBI}_4}\left(1_\phi, 2_\gamma^+, \ldots, (n-1)_\gamma^-, n_{\overline{\phi}}\right)$ and will be used to calculate BI$_4$ NSD amplitudes in Section 3.4.3.

First we will give a general parametrization of such tree amplitudes, then in the following section we will fix all ambiguities using two complimentary approaches.

The analytic structure of the mDBI$_4$ amplitudes have the general form of a rational function of external kinematic data and can be split into contributions

$$\begin{aligned}
&\mathcal{A}_n^{\text{mDBI}_4}\left(1_\phi, 2_\gamma^+, \ldots, (n-1)_\gamma^\pm, n_{\overline{\phi}}\right) \\
&= \mathcal{A}_n^{\text{mDBI}_4}\left(1_\phi, 2_\gamma^+, \ldots, (n-1)_\gamma^\pm, n_{\overline{\phi}}\right)\Big|_{\text{factoring}} + \mathcal{A}_n^{\text{mDBI}_4}\left(1_\phi, 2_\gamma^+, \ldots, (n-1)_\gamma^\pm, n_{\overline{\phi}}\right)\Big|_{\text{contact}}.
\end{aligned} \tag{3.3.1}$$

The *factoring* terms contain all kinematic singularities, which are required to be simple poles on invariant masses of subsets of external momenta, and have residues given by sums of products of lower point amplitudes. In this sense the factoring terms are recursively determined by amplitudes at lower multiplicity. In EFTs (such as mDBI$_4$) the resulting rational function is incompletely determined by factorization, and there is some remaining polynomial ambiguity. These ambiguities are contained in the *contact* contribution, which encodes all independent local operators compatible with the assumed properties of the model. We can give a general parametrization of these contact contributions for mDBI$_4$ through a combination of dimensional analysis, little group scaling and analysis of the massless limit.

In $d = 4$ the amplitudes have mass dimension $[\mathcal{A}_n] = 4 - n$, this includes both dimensionful coupling constants and kinematic dependence. The contact contribution is then a sum over terms of the schematic form

$$\mathcal{A}_n^{\text{mDBI}_4}\left(1_\phi, 2_\gamma^+, \ldots, (n-1)_\gamma^\pm, n_{\overline{\phi}}\right)\Big|_{\text{contact}} \sim \frac{1}{\Lambda^m} F_n^\pm\left(\{|i\rangle, |i]\}, p_{[4]}^{1,n}, \mu^2\right), \tag{3.3.2}$$

where $[\Lambda] = 1$ is the dimensionful scale appearing in the Born-Infeld action (3.1.1) and $[F_n] - m = 4 - n$. Since this is a contact contribution F_n must be a polynomial in the Lorentz invariant spinor contractions and the mass of the scalar μ^2. These polynomials must have the correct little group scaling dictated by their helicity configurations. This sets a lower-bound on the mass dimension of $F_n^\pm$ since we must have

$$\begin{aligned} F_n^+\left(\{|i\rangle, |i]\}, p_{[4]}^{1,n}, \mu^2\right) &\sim |2]^2|3]^2 \ldots |n-1]^2 G_n^+\left(\{|i\rangle, |i]\}, p_{[4]}^{1,n}, \mu^2\right) \\ F_n^-\left(\{|i\rangle, |i]\}, p_{[4]}^{1,n}, \mu^2\right) &\sim |2]^2|3]^2 \ldots |n-1\rangle^2 G_n^-\left(\{|i\rangle, |i]\}, p_{[4]}^{1,n}, \mu^2\right). \end{aligned} \tag{3.3.3}$$

Here $G^\pm$ are again polynomials in helicity spinors, but with zero little group weight. Since $[G^\pm] \geq 0$ we must have $[F_n^\pm] \geq n - 2$.

Next we impose that the complete mDBI$_4$ amplitudes should agree with $\mathcal{N} = 2$ BI$_4$ in the limit $\mu^2 \to 0$. This constraint is quite powerful due to the conservation of a $U(1)_R$ duality charge in $\mathcal{N} = 2$ BI$_4$. Up to an arbitrary normalization, the states of the $\mathcal{N} = 2$ massless vector multiplet can be assigned the following additive quantum numbers

$$Q[\gamma^\pm] = \pm 1, \quad Q[\psi_{1,2}^\pm] = \pm 1/2, \quad Q[\phi] = Q[\overline{\phi}] = 0. \tag{3.3.4}$$

It is straightforward to show that these charges are conserved at tree-level since they are conserved by the leading $n = 4$ interactions and the entire tree-level S-matrix is constructible by on-shell subtracted recursion [5]. Note that this $U(1)_R$ is *not* a subgroup of the $SU(2)_R$ symmetry group under which the fermions ψ_A transform as a doublet. It is an independent symmetry which enhances the full R-symmetry group of $\mathcal{N} = 2$ BI$_4$ to $U(2)_R$. The analogous enhancement of R-symmetry in maximally supersymmetric Born-Infeld was first discussed in [80]. As a consequence of the conservation of the duality charges (3.3.4), in the NSD and MHV sectors of mDBI_4 the massless limits are given by

$$\begin{aligned}
&\mathcal{A}_4^{\text{mDBI}_4}\left(1_\phi, 2_\gamma^+, 3_\gamma^+, 4_{\overline{\phi}}\right) \xrightarrow{\mu^2\to 0} 0, \\
&\mathcal{A}_4^{\text{mDBI}_4}\left(1_\phi, 2_\gamma^+, 3_\gamma^-, 4_{\overline{\phi}}\right) \xrightarrow{\mu^2\to 0} -\langle 3|p_1|2]^2, \\
&\mathcal{A}_n^{\text{mDBI}_4}\left(1_\phi, 2_\gamma^+, \ldots, (n-1)_\gamma^\pm, n_{\overline{\phi}}\right) \xrightarrow{\mu^2\to 0} 0, \quad n>4.
\end{aligned} \tag{3.3.5}$$

Due to the different singularity structure, the factoring and contact terms cannot interfere in this limit, and so the contact terms must vanish independently. For this to happen the contact terms must be proportional to some positive power of μ^2, which further increases the minimal dimension to $[F_n^\pm] \geq n$. The contact terms must then have the schematic form

$$\begin{aligned}
\mathcal{A}_n^{\text{mDBI}_4}\left(1_\phi, 2_\gamma^+, \ldots, (n-1)_\gamma^+, n_{\overline{\phi}}\right)\Big|_{\text{contact}} &\sim \frac{\mu^2}{\Lambda^{2n-4}}|2]^2|3]^2\ldots|n-1]^2 + \mathcal{O}\left(\frac{1}{\Lambda^{2n-3}}\right) \\
\mathcal{A}_n^{\text{mDBI}_4}\left(1_\phi, 2_\gamma^+, \ldots, (n-1)_\gamma^-, n_{\overline{\phi}}\right)\Big|_{\text{contact}} &\sim \frac{\mu^2}{\Lambda^{2n-4}}|2]^2|3]^2\ldots|n-1\rangle^2 + \mathcal{O}\left(\frac{1}{\Lambda^{2n-3}}\right).
\end{aligned} \tag{3.3.6}$$

It is easy to see that in the $(n-1)^-$ (MHV) case no contact term of this leading mass dimension can exist since there is no non-vanishing way to contract the angle spinors.

Next we recall our discussion from Section 3.2, such contact contributions should not introduce any spurious dimensionful parameters which might appear in the final integrated amplitude. We should not consider contributions with more inverse powers of Λ at a fixed multiplicity n. In Appendix F we give a short proof that at each multiplicity n there is a unique contact term, the final result can be parametrized as

$$\mathcal{A}_n^{\text{mDBI}_4}\left(1_\phi, 2_\gamma^+, \ldots, (n-1)_\gamma^+, n_{\overline{\phi}}\right)\Big|_{\text{contact}} = \frac{c_n\mu^2}{\Lambda^{2n-4}}\left([23]^2[45]^2\ldots[n-2,n-1]^2 + \ldots\right), \tag{3.3.7}$$

where $+\ldots$ denotes the sum over all ways of partitioning the set $\{2,\ldots,n-1\}$ into subsets of length 2. Such local matrix elements can be generated from local operators of the form

$$\mathcal{L}_{\text{mDBI}_4} \supset \frac{c_{2n}\mu^2}{\Lambda^{4n-4}}|\phi|^2\left(F_{\alpha\beta}^+F^{+\alpha\beta}\right)^{n-1} \quad + \quad \text{h.c.} \tag{3.3.8}$$

In subsequent sections the Λ dependence of the scattering amplitudes will be suppressed, they can trivially be restored by dimensional analysis.

The remarkable result (which we will verify using two complimentary approaches in the following sections, the first presented below and the second described in Appendix E) is that if we define mDBI_4 as the dimensional reduction of BI_6 as described above, then $c_n = 0$ for $n > 4$. The com-

	1	2	3	4	5
$\vec{p}_{1,n}$	x	x		x	x
$\vec{\epsilon}_{1,n}$				x	x
$\vec{p}_{2,3,\ldots,n-2}$	x	x			
$\vec{\epsilon}_{2,3,\ldots,n-2}$	x	x			
$\vec{p}_{n-1}$	x	x			
$\vec{\epsilon}_{n-1}$			x		

Table 3.2: Kinematic configuration of momenta and polarizations of BI_6 defining the 3d dimensional reduction of mDBI_4. The 3-direction will be T-dualized, mapping the polarization of the photon labeled $n-1$ to a brane modulus.

plete tree amplitudes are then completely fixed by recursive factorization into the fundamental 4-point mDBI_4 amplitudes.

3.3.2 T-Duality and Low-Energy Theorems

One of the most important and remarkable properties of D-branes (of which Born-Infeld and related models provide the low-energy effective description) is their behaviour under T-duality [109]. Though this is a non-perturbative stringy property, a useful remnant remains even in the tree-level scattering amplitudes of pure Born-Infeld. We will consider the configuration of momenta and polarizations described in Table 3.2.

At tree-level all internal momenta are linear combinations of external momenta, and so in this configuration the amplitudes are independent of the 3-direction in momentum space. This means that the tree-amplitudes are invariant under compactification of the spatial 3-direction on S^1. T-duality in this context is the statement that a space-filling D5-brane on $\mathbb{R}^{4+1} \times S^1$ with the radius of S^1 given by R, is equivalent to a codimension-1 D4-brane on $\mathbb{R}^{4+1} \times S^1$, where S^1 is the transverse dimension with radius $\sim 1/R$. In the full string theory, T-duality relates infinite towers of KK and winding modes. In this low-energy EFT containing only the massless states as on-shell degrees of freedom, the only non-trivial mapping is between photons polarized in the compact direction on the D5-brane and the brane modulus of the D4-brane

$$|\gamma^{\top}(\vec{p})\rangle \leftrightarrow |\Phi(\vec{p})\rangle. \tag{3.3.9}$$

Since the tree-level amplitudes in Table 3.2 are independent of the compactification, they must remain invariant in the limit $R \to 0$. In the T-dual configuration this corresponds to the decompactification limit in which we have a D4 brane embedded in $\mathbb{R}^{5+1}$. In this limit, the spontaneous symmetry breaking pattern in the T-dual frame jumps discontinuously

$$\frac{\text{ISO}(4,1)\times\text{SO}(2)}{\text{ISO}(4,1)} \xrightarrow{R\to 0} \frac{\text{ISO}(5,1)}{\text{ISO}(4,1)}. \tag{3.3.10}$$

The brane modulus is then identified as the Goldstone mode of both the translation symmetry in the 3-direction *and* the Lorentz transformations mixing the 3- and world-volume directions. In the physical scattering amplitudes this manifests as *enhanced* soft theorems for the brane modulus

$$\mathcal{A}_n^{\text{mDBI}_4}\left(1_\phi, 2_\gamma^+, \ldots, (n-2)_\gamma^+, (n-1)_\Phi, n_{\overline{\phi}}\right) \sim \mathcal{O}\left(p_{n-1}^2\right), \quad \text{as} \quad p_{n-1}\to 0, \tag{3.3.11}$$

where the momenta and polarizations are as given in Table 3.2. In this section we will use this result to fix the contact term ambiguities of the mDBI$_4$ amplitudes. This momentum configuration is an effective further dimensional reduction from 4d to 3d and so we will write the explicit form of the amplitudes in 3d language. In our conventions, the dimensional reduction map takes an especially simple form

$$4d \to 3d: \qquad \langle ij\rangle \to \langle ij\rangle, \qquad [ij] \to \langle ij\rangle, \tag{3.3.12}$$

which we will then further simplify (for purely Bosonic amplitudes this means rewriting all helicity spinor contractions as Mandelstam invariants). To apply these results to the Ansatz form of the mDBI$_4$ amplitudes described above, which are in the helicity basis, we must relate the transverse polarization $\gamma^\top$ to a linear combination of helicity states. In our conventions the correct linear combination is found to be

$$|\gamma^\top(\vec{p})\rangle = |\gamma^+(\vec{p})\rangle - |\gamma^-(\vec{p})\rangle, \tag{3.3.13}$$

which for the helicity amplitudes means

$$\begin{aligned}&\mathcal{A}_n^{\text{mDBI}_4}\left(1_\phi, 2_\gamma^+, \ldots, (n-2)_\gamma^+, (n-1)_\gamma^\top, n_{\overline{\phi}}\right) = \\ &\mathcal{A}_n^{\text{mDBI}_4}\left(1_\phi, 2_\gamma^+, \ldots, (n-2)_\gamma^+, (n-1)_\gamma^+, n_{\overline{\phi}}\right) - \mathcal{A}_n^{\text{mDBI}_4}\left(1_\phi, 2_\gamma^+, \ldots, (n-2)_\gamma^+, (n-1)_\gamma^-, n_{\overline{\phi}}\right).\end{aligned} \tag{3.3.14}$$

The method used in this section will be to form this linear combination of Ansatze, apply the dimensional reduction map and then take the soft limit $p_{n-1}\to 0$. Compatibility with T-duality then requires that the $\mathcal{O}(p_{n-1})$ terms cancel amongst themselves, this requirement uniquely fixes the c_n coefficients.

3.3.2.1 Explicit Examples of T-duality Constraints

We will begin with the 4-point amplitudes in mDBI$_4$. As described above the MHV amplitude is uniquely fixed by the $\mu^2 \to 0$ limit, while the NSD amplitudes are fixed up to an overall coefficient

$$\mathcal{A}_4^{\mathrm{mDBI}_4}\left(1_\phi, 2_\gamma^+, 3_\gamma^+, 4_{\overline{\phi}}\right) = c_4 \mu^2 [23]^2. \tag{3.3.15}$$

By taking the appropriate linear combination according to (3.3.13) we can form an amplitude for which particle 3 is polarized in the direction *transverse* to a particular 2d subspace

$$\begin{aligned} &\mathcal{A}_4^{\mathrm{mDBI}_4}\left(1_\phi, 2_\gamma^+, 3_\gamma^\top, 4_{\overline{\phi}}\right) \\ &\quad = \mathcal{A}_4^{\mathrm{mDBI}_4}\left(1_\phi, 2_\gamma^+, 3_\gamma^+, 4_{\overline{\phi}}\right) - \mathcal{A}_4^{\mathrm{mDBI}_4}\left(1_\phi, 2_\gamma^+, 3_\gamma^-, 4_{\overline{\phi}}\right) \\ &\quad = c_4 \mu^2 [23]^2 + \langle 3|p_1|2]^2. \end{aligned} \tag{3.3.16}$$

We then apply the dimensional reduction map, after reduction to 3d the various spinor contractions reduce to

$$\begin{aligned} &[23]^2 \to s_{23} \\ &\langle 3|p_1|2]^2 \to \mathrm{Tr}\left[p_3 \cdot p_1 \cdot p_2 \cdot p_1\right] = 2\left(2(p_1 \cdot p_3)(p_1 \cdot p_2) - p_1^2 (p_2 \cdot p_3)\right). \end{aligned} \tag{3.3.17}$$

Applying this gives

$$\mathcal{A}_4^{\mathrm{mDBI}_4}\left(1_\phi, 2_\gamma^+, 3_\gamma^\top, 4_{\overline{\phi}}\right) \xrightarrow{3d} 2(c_4+1)\mu^2 (p_2 \cdot p_3) + 4(p_1 \cdot p_3)(p_4 \cdot p_3). \tag{3.3.18}$$

In the limit where $p_3 \to 0$ we can see that the first term vanishes at $\mathcal{O}(p_3)$ while the second term vanishes at $\mathcal{O}(p_3^2)$, the T-duality constraint then forces us to choose $c_4 = -1$. This result can also be obtained in a completely different way by using a massive version of the KLT relations (E.0.30).

At 6-point and higher it is necessary to define the soft degree more precisely. Let's quickly review the rigorous definition of a soft limit (see [57] for more details). We evaluate our amplitude on a one-parameter family of momenta of the form

$$\hat{p}_5(\epsilon) = \epsilon p_5, \quad \hat{p}_i(\epsilon) = p_i + \epsilon q_i, \quad i \neq 5. \tag{3.3.19}$$

The deformed momenta should satisfy momentum conservation and the on-shell conditions for all values of $\epsilon \in \mathbb{C}$, which requires

$$p_5^2 = 0, \quad p_i \cdot q_i = 0, \quad q_i^2 = 0, \quad \sum_{i \neq 5} p_i = 0, \quad p_5 + \sum_{i \neq 5} q_i = 0. \tag{3.3.20}$$

At leading order in the ϵ-expansion the q_i momenta do not appear. After dimensional reduction our amplitudes are trivially at least $\mathcal{O}(\epsilon)$, our goal is then to show that these leading terms are actually zero and that therefore the leading term in the expansion is $\mathcal{O}(\epsilon^2)$. For this purpose, taking the soft limit is equivalent to taking $p_i,\ i \neq 5$ to satisfy 5-particle momentum conservation, and p_5 as an unrelated null vector. We should bare this in mind when making algebraic manipulations involving conservation of momentum.

Let's now proceed with the calculation of the 6-point soft limit. We begin with a general Ansatze which has the correct factorization properties and generally parametrized contact terms, and make a dimensional reduction

$$\begin{aligned}&\mathcal{A}_6^{\text{mDBI}_4}\left(1_\phi, 2_\gamma^+, 3_\gamma^+, 4_\gamma^+, 5_\gamma^+, 6_{\overline{\phi}}\right)\\ &\xrightarrow{3d+\text{soft}} \frac{(\mu^2)^2 s_{23}s_{45}}{s_{123}+\mu^2} + \frac{(\mu^2)^2 s_{24}s_{35}}{s_{124}+\mu^2} + \frac{(\mu^2)^2 s_{25}s_{34}}{s_{12}+\mu^2} + c_6\mu^2\left(s_{23}s_{45}+s_{24}s_{35}+s_{25}s_{34}\right),\end{aligned} \tag{3.3.21}$$

also,

$$\begin{aligned}&\mathcal{A}_6^{\text{mDBI}_4}\left(1_\phi, 2_\gamma^+, 3_\gamma^+, 4_\gamma^+, 5_\gamma^-, 6_{\overline{\phi}}\right)\\ &\xrightarrow{3d+\text{soft}} \frac{\mu^2}{2}\left[\frac{s_{23}\left(2(p_5\cdot p_6)(s_{46}+\mu^2)+\mu^2 s_{45}\right)}{s_{123}+\mu^2} + \frac{s_{34}\left(2(p_5\cdot p_1)(s_{12}+\mu^2)+\mu^2 s_{25}\right)}{s_{12}+\mu^2}\right.\\ &\qquad\left. + \frac{s_{34}\left(4(p_5\cdot p_{34})(p_2\cdot p_{34})+2\mu^2(p_2\cdot p_5)\right)}{s_{126}}\right] + \mathcal{P}(2,3,4)\,.\end{aligned} \tag{3.3.22}$$

Taking the difference we find that the $(\mu^2)^2$ terms cancel and the remaining terms are purely local

$$\begin{aligned}&\mathcal{A}_6^{\text{mDBI}_4}\left(1_\phi, 2_\gamma^+, 3_\gamma^+, 4_\gamma^+, 5_\gamma^\top, 6_{\overline{\phi}}\right)\\ &= \mathcal{A}_6^{\text{mDBI}_4}\left(1_\phi, 2_\gamma^+, 3_\gamma^+, 4_\gamma^+, 5_\gamma^+, 6_{\overline{\phi}}\right) - \mathcal{A}_6^{\text{mDBI}_4}\left(1_\phi, 2_\gamma^+, 3_\gamma^+, 4_\gamma^+, 5_\gamma^-, 6_{\overline{\phi}}\right)\\ &\xrightarrow{3d+\text{soft}} \frac{1}{2}c_6\mu^2 s_{23}s_{45} - \mu^2 s_{23}(p_5\cdot p_6) - \mu^2 s_{34}(p_1\cdot p_5) - 2\mu^2(p_5\cdot p_{16})(p_2\cdot p_{16})\\ &\qquad + \mu^2 s_{16}(p_2\cdot p_5) + \mathcal{P}(2,3,4)\\ &\qquad = c_6\mu^2\left(s_{23}s_{45}+s_{24}s_{35}+s_{25}s_{34}\right) - 2\mu^2 s_{12}(p_5\cdot p_{16}) + 4\mu^2 s_{12}(p_5\cdot p_{16})\\ &\qquad\quad - 2\mu^2 s_{12}(p_5\cdot p_{16})\\ &\qquad = c_6\mu^2\left(s_{23}s_{45}+s_{24}s_{35}+s_{25}s_{34}\right).\end{aligned} \tag{3.3.23}$$

Somewhat miraculously all of the terms cancel except for the unknown contact term. Since this is manifestly $\mathcal{O}(p_5)$, we must choose $c_6 = 0$ to satisfy the constraint of T-duality. Again, this same conclusion can also be reached after a rather lengthy numerical calculation involving the massive KLT relations (E.0.35). In Appendix G we give the explicit calculation of c_8, again we confirm the result of the numerical KLT calculation. In the next subsection we will give an explicit all-multiplicity proof that the T-duality constraints require $c_n = 0$ for $n > 4$.

3.3.2.2 Small Mass Expansion and the Absence of Contact Terms

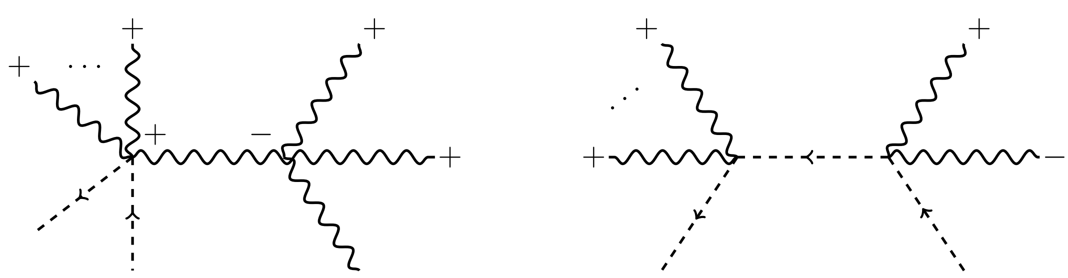
That the 6-point dimensional reduction and soft limit calculation gave $c_6 = 0$ is somewhat remarkable, and could not easily have been anticipated without a detailed calculation. For $n \geq 8$ the conclusion that $c_n = 0$ is less mysterious and can be argued on general grounds by considering the structure of the mDBI$_4$ amplitudes *as an expansion around* the $\mu^2 \to 0$ limit. In Appendix F we show that there is a unique contact term at each multiplicity of the form

$$\mathcal{A}_n^{\text{mDBI}_4}\left(1_\phi, 2_\gamma^+, \ldots, (n-1)_\gamma^+, n_{\overline{\phi}}^-\right)\Big|_{\text{contact}} = c_n \mu^2 \left([23]^2[45]^2 \ldots [n-2, n-1]^2 + \ldots\right). \quad (3.3.24)$$

Dimensionally reducing to 3d this becomes

$$\xrightarrow{3d} c_n \mu^2 \left(s_{23} s_{45} \ldots s_{n-2,n-1} + \ldots\right), \quad (3.3.25)$$

which is manifestly $\mathcal{O}(p_{n-1})$ in the soft limit of particle $n-1$. If $c_n \neq 0$ then this term must cancel against some term in the factoring part of the Ansatz to give the correct $\mathcal{O}(p_{n-1}^2)$ soft limit. To show that this can never happen we expand in the limit $\mu^2 \to 0$. The contact terms clearly always contribute at $\mathcal{O}(\mu^2)$. Since μ^2 is a free parameter (corresponding to our choice of momenta in the 4 and 5 directions from the 6d perspective), the T-duality constraints should apply order-by-order in the expansion. For a non-trivial cancellation between the contact and factoring terms to occur, the factoring terms must give a contribution at $\mathcal{O}(\mu^2)$. If such a contribution exists then we must be able to identify a factorization channel for which the product of the leading small mass behavior on both sides is $\mathcal{O}(\mu^2)$. Since negative and odd powers of μ do not appear, one half of the factorization diagram must be $\mathcal{O}(\mu^0)$. At each multiplicity there are only two possible factorization channels which can give such a contribution:



(3.3.26)

both of which have the form of a lower-point NSD amplitude glued to an $\mathcal{O}(\mu^0)$ 4-point amplitude. For $n = 8$, the $\mathcal{O}(\mu^2)$ contribution to the NSD amplitude arises solely from the contact term which we explicitly verified (by two different methods) was absent. So we conclude there cannot be an $\mathcal{O}(\mu^2)$ contribution to the $n = 8$ MHV amplitude and hence no contact term. We can continue in this way and make an inductive argument that the absence of contact terms at $n-2$-point implies the absence of contact terms at n-point. Together with the explicit $n = 6$ case, we find that all higher point contact terms are zero in mDBI$_4$, the amplitudes are (almost) as simple as possible. We will leverage this simplicity in the following section to construct all-multiplicity one-loop integrands for the SD and NSD sectors of BI$_4$.

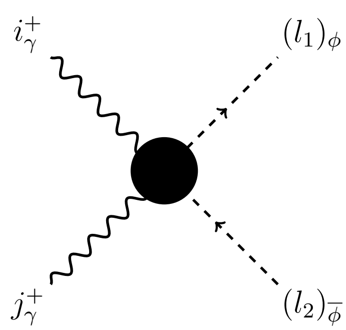
3.4 All Multiplicity Rational One-Loop Amplitudes

3.4.1 Diagrammatic Rules for Constructing Loop Integrands

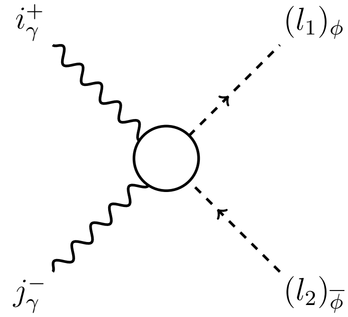
With the results in the previous section, and the discussion in Section 3.2, we have in principle obtained a complete understanding of the structure of the d-dimensional unitarity cuts of SD and NSD BI$_4$ one-loop integrands. Our goal is now to use this to engineer the explicit form of the integrands and then integrate them to obtain the full amplitudes. Ordinarily, gluing together on-shell tree-amplitudes into full loop integrands is a delicate business. Constructing expressions with the correct cuts in one channel may give *polluting* contributions to another channel. Separating these contributions and building up loop integrands in a systematic way has been a subject of intense study over the past several decades [99].

Fortunately for us, the mDBI$_4$ tree amplitudes are of sufficiently simple form that it is straightforward to construct integrands with all of the correct cuts using a set of *diagrammatic rules*. There are two properties that allow us to do this; first, locality is manifest in the mDBI$_4$ amplitudes, and second, due to the absence of contact terms above $n = 4$ the number of *elementary* vertex rules is strictly finite. Notice how much simpler this is than calculating loop diagrams directly from ordinary Feynman rules! If we were calculating loop amplitudes in Born-Infeld the old-fashioned way we would need to calculate new (and increasingly complicated) Feynman vertex rules at each multiplicity.

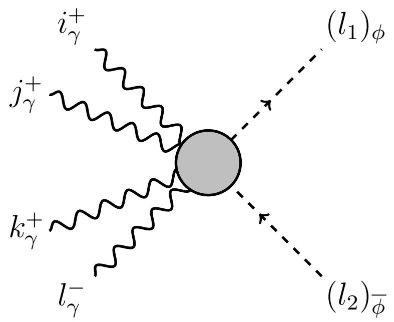
Since we are constructing loop integrands in the scalar loop representation (3.2.1) we will construct a diagrammatic representation in which each diagram consists of a scalar loop *decorated* with any of the following vertex factors:



$$= \quad -\mu^2[ij]^2 \tag{3.4.1}$$

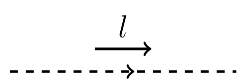


$$= \quad -\langle j|l_1|i]^2 \tag{3.4.2}$$



$$= \quad \frac{\mu^2[k|p_{ij}|l\rangle^2[ij]^2}{s_{ijl}} + \mathcal{C}(i,j,k) \tag{3.4.3}$$

Here $+\,\mathcal{C}(i,j,k)$ denotes the sum over cyclic permutations, all of the momenta are defined to be out-going with photon lines on-shell, while the scalar lines are off-shell. These vertex rules can be glued together on scalar lines in the usual way with the standard massive scalar propagator



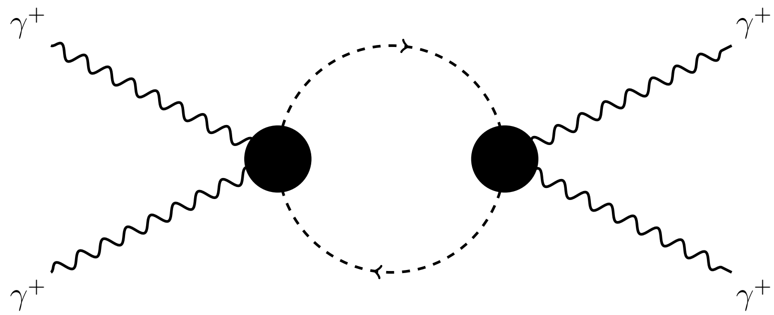
$$= \quad \frac{1}{l^2+\mu^2} \tag{3.4.4}$$

These diagrammatic rules can be justified *post hoc*, by verifying that the resulting loop integrands have the correct massive scalar cuts. These are not Feynman rules in the usual sense, and have not been derived from a Lagrangian. This is especially clear in the 6-point vertex rule (denoted with a gray blob), which is a non-local expression; the poles encode factorization singularities into Born-Infeld photons. Due to the helicity selection rules of BI_4 at tree-level arising from supersymmetric truncation, no further photonic singularities can appear in amplitudes with at most a single negative helicity external state.

In the following sections we will give explicit examples of the applications of these diagrammatic rules to 4- and 6-point SD and NSD loop integrands, and then present explicit expressions for the all-multiplicity results together with the integrated expressions at $\mathcal{O}(\epsilon^0)$.

3.4.2 Self-Dual Sector

In the self-dual sector, since there are only positive helicity external states, at each multiplicity there is only a single topologically distinct diagram and it is constructed solely from black vertices. Beginning with $n = 4$, the diagram has the form:



(3.4.5)

There are three non-trivial permutations of the external labels. The integrand is then

$$\mathcal{I}_4^{\text{SD}}[l;\mu^2] = \frac{1}{2}\left[\frac{(\mu^2)^2[12]^2[34]^2}{[l^2+\mu^2]\left[(l-p_{12})^2+\mu^2\right]} + \mathcal{P}\left(2,3,4\right)\right], \tag{3.4.6}$$

where the factor of $\frac{1}{2}$ compensates for the equivalent permutations in $\mathcal{P}\left(2,3,4\right)$ that are summed over.

We now explicitly verify that the diagrammatic rules of Section 3.4.1 yield an integrand that satisfies the cut conditions. Since the integrand has only one distinct two-particle cut (all others are related by label permutations), we choose to consider the p_{12}-cut. When the on-shell conditions $l^2 = -\mu^2$ and $(l-p_{12})^2 = -\mu^2$ are imposed, the integrand yields

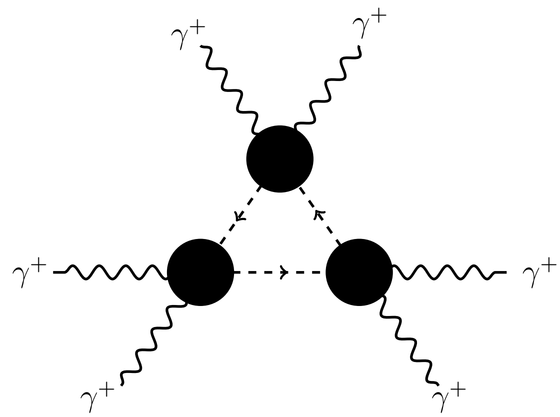
$$\begin{aligned}\left[l^2+\mu^2\right]\left[(l-p_{12})^2+\mu^2\right]\mathcal{I}_4^{\text{SD}}[l;\mu^2]\Big|_{p_{12}\text{-cut}} \\ &= \mathcal{A}_4\left(1_\gamma^+,2_\gamma^+,-l_\phi,(l-p_{12})_{\bar{\phi}}\right)\mathcal{A}_4\left(l_{\bar{\phi}},(p_{12}-l)_\phi,3_\gamma^+,4_\gamma^+\right) \\ &= (\mu^2)^2[12]^2[34]^2 \end{aligned} \tag{3.4.7}$$

as expected. The NSD amplitudes above are given in (E.0.29).

Using the general result for rational loop integrals (H.0.17) gives

$$\begin{aligned}&\mathcal{A}_4^{\text{BI}_4\ \text{1-loop}}\left(1_\gamma^+,2_\gamma^+,3_\gamma^+,4_\gamma^+\right) \\ &\quad= \frac{1}{2}\int\frac{\mathrm{d}^4 l}{(2\pi)^4}\int\frac{\mathrm{d}^{-2\epsilon}\mu}{(2\pi)^{-2\epsilon}}\left[\frac{(\mu^2)^2[12]^2[34]^2}{[l^2+\mu^2]\left[(l-p_{12})^2+\mu^2\right]}+\mathcal{P}\left(2,3,4\right)\right] \\ &\quad= [12]^2[34]^2 I_2^{d=4-2\epsilon}[(\mu^2)^2;p_{12}] + [13]^2[24]^2 I_2^{d=4-2\epsilon}[(\mu^2)^2;p_{13}] \\ &\qquad + [14]^2[23]^2 I_2^{d=4-2\epsilon}[(\mu^2)^2;p_{14}] \\ &\quad= -\frac{i}{960\pi^2}\left([12]^2[34]^2 s_{12}^2 + [13]^2[24]^2 s_{13}^2 + [14]^2[23]^2 s_{14}^2\right) + \mathcal{O}(\epsilon). \end{aligned} \tag{3.4.8}$$

Similarly for $n = 6$ there is a unique topologically distinct class of diagram:



(3.4.9)

The integrand is then given by

$$\mathcal{I}_6^{\text{SD}}[l;\mu^2] = -\frac{1}{4}\left[\frac{(\mu^2)^3[12]^2[34]^2[56]^2}{[l^2+\mu^2]\left[(l-p_{34})^2+\mu^2\right]\left[(l+p_{12})^2+\mu^2\right]} + \mathcal{P}\left(2,3,4,5,6\right)\right]. \quad (3.4.10)$$

The integrand has only one distinct cut into tree-level amplitudes. Consider for example the integrand on the p_{12}-cut,

$$\begin{aligned}
&\left[l^2+\mu^2\right]\left[(l+p_{12})^2+\mu^2\right]\mathcal{I}_6^{\text{SD}}[l;\mu^2]\Big|_{p_{12}\text{-cut}} \\
&\quad = \mathcal{A}_4\left(1_\gamma^+,2_\gamma^+,l_\phi,-(l+p_{12})_{\bar{\phi}}\right)\mathcal{A}_6\left(-l_{\bar{\phi}},(l+p_{12})_\phi,3_\gamma^+,4_\gamma^+,5_\gamma^+,6_\gamma^+\right) \\
&\qquad\quad + \mathcal{A}_4\left(1_\gamma^+,2_\gamma^+,l_{\bar{\phi}},-(l+p_{12})_\phi\right)\mathcal{A}_6\left(-l_\phi,(l+p_{12})_{\bar{\phi}},3_\gamma^+,4_\gamma^+,5_\gamma^+,6_\gamma^+\right) \\
&\quad = 2\mathcal{A}_4\left(1_\gamma^+,2_\gamma^+,l_\phi,-(l+p_{12})_{\bar{\phi}}\right)\mathcal{A}_6\left(-l_{\bar{\phi}},(l+p_{12})_\phi,3_\gamma^+,4_\gamma^+,5_\gamma^+,6_\gamma^+\right). \qquad (3.4.11)
\end{aligned}$$

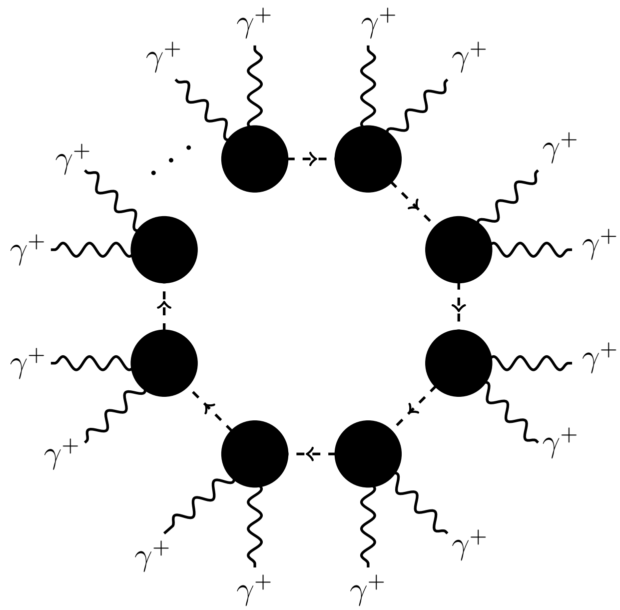
where the amplitudes are given in (E.0.29) and (E.0.33) and the form of the 6-point amplitude (E.0.33) makes it apparent that there are no local contributions to two-scalar cuts.

The factor of 2 in (3.4.11) is multiplied by $\frac{1}{8}$ (which compensates for the equivalent permutations in $\mathcal{P}\left(2,3,4,5,6\right)$ that are summed over). This matches the factor of $\frac{1}{4}$ in the integrand and hence verifies the rules of Section 3.4.1.

Integrating this using the formula (H.0.17) gives

$$\begin{aligned}
&\mathcal{A}_6^{\text{BI}_4\ \text{1-loop}}\left(1_\gamma^+,2_\gamma^+,3_\gamma^+,4_\gamma^+,5_\gamma^+,6_\gamma^+\right) \\
&\quad = \frac{1}{4}\Bigg[\frac{i}{2880\pi^2}[12]^2[34]^2[56]^2\left(s_{12}^2+s_{34}^2+s_{56}^2+s_{12}s_{34}+s_{12}s_{56}+s_{34}s_{56}\right) \\
&\qquad\quad + \mathcal{P}\left(2,3,4,5,6\right)\Bigg] + \mathcal{O}(\epsilon). \qquad (3.4.12)
\end{aligned}$$

The generalization to all multiplicity in the SD sector is now clear. There is always a single topologically distinct diagram with a corresponding scalar rational integral:



(3.4.13)

The complete integrand is then

$$\mathcal{I}_{2n}^{\text{SD}}[l;\mu^2]$$
$$= \left(\frac{1}{2}\right)^{n-1} \left([12]^2[34]^2 \ldots [2n-1,2n]^2 \frac{(-\mu^2)^n}{\prod_{i=1}^{n} \left[\left(l - \sum_{j=1}^{2i} p_j \right)^2 + \mu^2 \right]} + \mathcal{P}(2,3,\ldots,2n) \right) . \tag{3.4.14}$$

Using the result of equation (H.0.17), we find that the integrated amplitude is

$$\boxed{\begin{aligned} &\mathcal{A}_{2n}^{\text{BI}_4\text{ 1-loop}}\left(1_\gamma^+, 2_\gamma^+, \ldots, 2n_\gamma^+\right) \\ &\quad = \frac{i}{32\pi^2}\left(-\frac{1}{2}\right)^{n-1} \frac{1}{n(n+1)(n+2)(n+3)} \\ &\quad \times \left[[12]^2[34]^2 \ldots [2n-1,2n]^2 \left(\sum_{i<j}^{n} \sum_{k<l}^{n} a_{ijkl} \left(\sum_{m=2i+1}^{2j} p_m \right)^2 \left(\sum_{m=2k+1}^{2l} p_m \right)^2 \right) \right. \\ &\quad\quad \left. + \mathcal{P}(2,3,\ldots,2n) \right] + \mathcal{O}(\epsilon)\,, \end{aligned}} \tag{3.4.15}$$

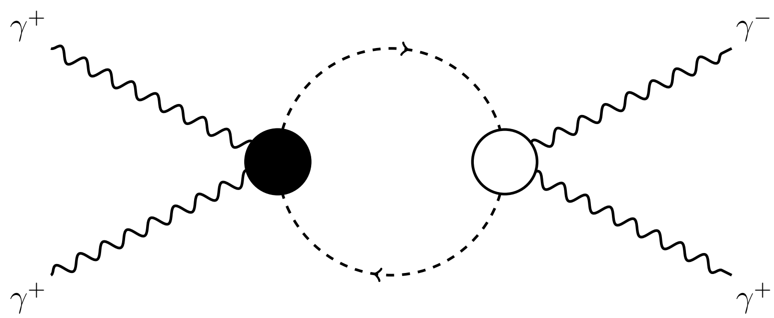
with

$$a_{ijkl} = \begin{cases} 1 & \text{if all } i,j,k,l \text{ are different} \\ 2 & \text{if exactly 2 of } i,j,k,l \text{ are identical} \\ 4 & \text{if } i=k \text{ and } j=l \end{cases} . \tag{3.4.16}$$

It is straightforward to check that this result matches the results of the explicit calculations for the cases of $n = 2$ and $n = 3$, presented above.

3.4.3 Next-to-Self-Dual Sector

In the NSD sector the diagrams have a similar structure, consisting a single scalar loop decorated with the vertex factors. The novelty here is the appearance of a single negative helicity photon, and so each diagram contains either a single white or gray vertex. At 4-point there is only a single topologically distinct class of diagram, and contains both a black and white vertex[7]:



(3.4.17)

There are three non-trivial permutations of the external labels. Consider a single such permuation corresponding to momenta p_1 and p_2 flowing out of the black vertex, the corresponding integrand has the form

$$\mathcal{I}_4^{\text{NSD}}\left[l;\mu^2\right]\Big|_{12} = \frac{\mu^2[12]^2\langle 4|l|3]^2}{[l^2+\mu^2]\left[(l-p_{12})^2+\mu^2\right]}. \tag{3.4.18}$$

We now verify that the diagrammatic rules of Section 3.4.1 give an integrand with the right cuts in the NSD sector. There is only one distinct two-particle cut. As expected, the contribution to the integrand (3.4.18) on the p_{12}-cut is

$$\begin{aligned}\left[l^2+\mu^2\right]\left[(l-p_{12})^2+\mu^2\right]\mathcal{I}_4^{\text{SD}}[l;\mu^2]\big|_{p_{12}\text{-cut}} \\ &= \mathcal{A}_4\left(1_\gamma^+,2_\gamma^+,-l_\phi,(l-p_{12})_{\bar\phi}\right)\mathcal{A}_4\left(l_{\bar\phi},(p_{12}-l)_\phi,3_\gamma^+,4_\gamma^-\right) \\ &= \mu^2[12]^2\langle 4|l|3]^2, \end{aligned} \tag{3.4.19}$$

where the amplitudes are given in (E.0.29) and (E.0.30).

[7]Note that there is no *tadpole* diagram with a single gray vertex since this contributes a scaleless integral which vanishes in dimensional regularization.

Unlike all of the integrals in the SD sector, this is a rational tensor integral. The explicit value of an integral of this form is in (H.0.24), this gives

$$\begin{aligned}
\int \frac{\mathrm{d}^4 l}{(2\pi)^4} \int \frac{\mathrm{d}^{-2\epsilon}\mu}{(2\pi)^{-2\epsilon}} &\left[\frac{\mu^2[12]^2\langle 4|l|3]^2}{[l^2+\mu^2]\,[(l-p_{12})^2+\mu^2]}\right] \\
&= [12]^2 I_2^{d=4-2\epsilon}[\mu^2\langle 4|l|3]^2; p_{12}] \\
&= \frac{-i}{1920\pi^2}[12]^2\langle 4|\sigma_\mu|3]\langle 4|\sigma_\nu|3]\left[g^{\mu\nu}s_{12}^2 - 6p_{12}^\mu p_{12}^\nu s_{12}\right] + \mathcal{O}(\epsilon) \\
&= 0 + \mathcal{O}(\epsilon).
\end{aligned} \tag{3.4.20}$$

Since the remaining channels are simple permutations of this one we conclude

$$\mathcal{A}_4^{\text{BI}_4\ \text{1-loop}}\left(1_\gamma^+, 2_\gamma^+, 3_\gamma^+, 4_\gamma^-\right) = 0 + \mathcal{O}(\epsilon). \tag{3.4.21}$$

Beginning at 6-point there are two distinct classes of diagrams, corresponding to diagrams containing a single white or gray vertex. Note that the 6-point integrand also has two distinct cuts. For instance, take the integrand on the p_{56}-cut,

$$\begin{aligned}
&\left[l^2+\mu^2\right]\left[(l+p_{56})^2+\mu^2\right]\mathcal{I}_6^{\text{SD}}[l;\mu^2]\Big|_{p_{56}\text{-cut}} \\
&\quad= \mathcal{A}_4\left(5_\gamma^+, 6_\gamma^-, l_\phi, -(l+p_{56})_{\bar\phi}\right)\mathcal{A}_6\left(-l_{\bar\phi}, (l+p_{56})_\phi, 1_\gamma^+, 2_\gamma^+, 3_\gamma^+, 4_\gamma^+\right) \\
&\qquad\quad+ \mathcal{A}_4\left(5_\gamma^+, 6_\gamma^-, l_{\bar\phi}, -(l+p_{56})_\phi\right)\mathcal{A}_6\left(-l_\phi, (l+p_{56})_{\bar\phi}, 1_\gamma^+, 2_\gamma^+, 3_\gamma^+, 4_\gamma^+\right) \\
&\quad= 2\mathcal{A}_4\left(5_\gamma^+, 6_\gamma^-, l_\phi, -(l+p_{56})_{\bar\phi}\right)\mathcal{A}_6\left(-l_{\bar\phi}, (l+p_{56})_\phi, 1_\gamma^+, 2_\gamma^+, 3_\gamma^+, 4_\gamma^+\right).
\end{aligned} \tag{3.4.22}$$

where the explicit forms of the amplitudes are given in (E.0.33) and (E.0.30). This generalises to any p_{i6}-cut, where $i \neq 6$.

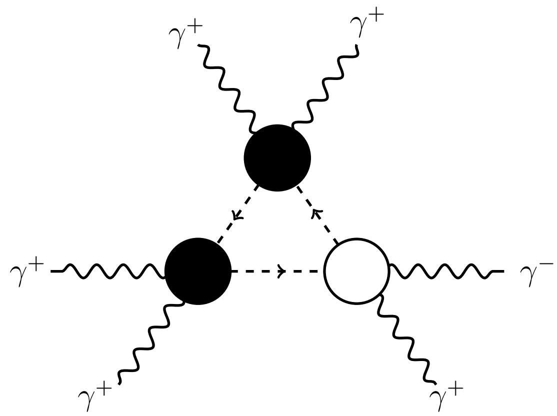
As a representative of the other class of cuts, consider the p_{12}-cut (which generalises to all p_{ij}-cuts where $i, j \neq 6$.),

$$\begin{aligned}
&\left[l^2+\mu^2\right]\left[(l+p_{12})^2+\mu^2\right]\mathcal{I}_6^{\text{SD}}[l;\mu^2]\Big|_{p_{12}\text{-cut}} \\
&\quad= \mathcal{A}_4\left(1_\gamma^+, 2_\gamma^+, l_\phi, -(l+p_{12})_{\bar\phi}\right)\mathcal{A}_6\left(-l_{\bar\phi}, (l+p_{12})_\phi, 3_\gamma^+, 4_\gamma^+, 5_\gamma^+, 6_\gamma^-\right) \\
&\qquad\quad+ \mathcal{A}_4\left(1_\gamma^+, 2_\gamma^+, l_{\bar\phi}, -(l+p_{12})_\phi\right)\mathcal{A}_6\left(-l_\phi, (l+p_{12})_{\bar\phi}, 3_\gamma^+, 4_\gamma^+, 5_\gamma^+, 6_\gamma^-\right) \\
&\quad= 2\mathcal{A}_4\left(1_\gamma^+, 2_\gamma^+, l_\phi, -(l+p_{12})_{\bar\phi}\right)\mathcal{A}_6\left(-l_{\bar\phi}, (l+p_{12})_\phi, 3_\gamma^+, 4_\gamma^+, 5_\gamma^+, 6_\gamma^-\right)
\end{aligned} \tag{3.4.23}$$

where the amplitudes are given in (E.0.29) and (E.0.34). Note that there are two kinds of contributions to $\mathcal{A}_6\left(-l_{\bar\phi}, (l+p_{12})_\phi, 3_\gamma^+, 4_\gamma^+, 5_\gamma^+, 6_\gamma^-\right)$: one factorizes on an internal scalar and the other factorizes on an internal photon,

$$\begin{aligned}\mathcal{A}_6\left(-l_{\bar{\phi}},(l+p_{12})_\phi,3_\gamma^+,4_\gamma^+,5_\gamma^+,6_\gamma^-\right) =&\mathcal{A}_6^{\text{scalar}}\left(-l_{\bar{\phi}},(l+p_{12})_\phi,3_\gamma^+,4_\gamma^+,5_\gamma^+,6_\gamma^-\right)\\ &+\mathcal{A}_6^{\text{photon}}\left(-l_{\bar{\phi}},(l+p_{12})_\phi,3_\gamma^+,4_\gamma^+,5_\gamma^+,6_\gamma^-\right). \end{aligned} \tag{3.4.24}$$

The first class of contributing diagrams is similar to the 4-point calculation and takes the form:



(3.4.25)

Summing over all permutations of the external labels gives the following contribution to the integrand

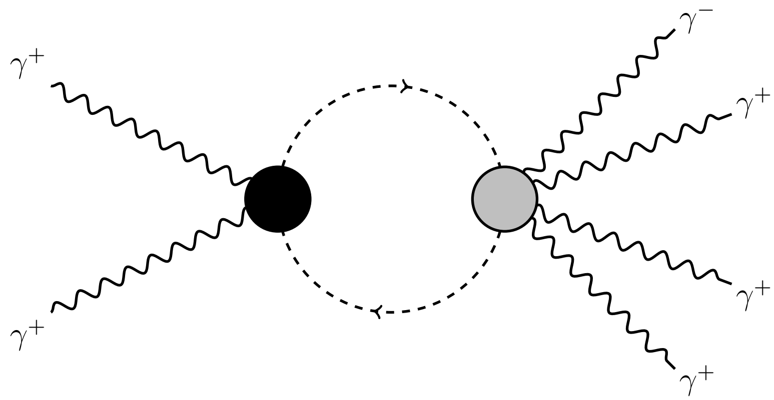
$$\mathcal{I}_6^{\text{NSD}}[l;\mu^2]\Big|_{\text{white}} = \frac{1}{4}\left[\frac{-(\mu^2)^2[12]^2[34]^2\langle 6|l|5]^2}{[l^2+\mu^2]\,[(l-p_{12})^2+\mu^2]\,[(l+p_{56})^2+\mu^2]}+\mathcal{P}\left(1,2,3,4,5\right)\right]. \tag{3.4.26}$$

This contribution has the correct $i6$-cuts (3.4.22). On a p_{12}-cut, (3.4.26) produces

$$\begin{aligned}&\left[l^2+\mu^2\right]\left[(l+p_{12})^2+\mu^2\right]\mathcal{I}_6^{\text{SD}}[l;\mu^2]\Big|_{p_{ij}\text{-cut}}\\ &\quad=2\mathcal{A}_4\left(1_\gamma^+,2_\gamma^+,l_\phi,-(l+p_{12})_{\bar{\phi}}\right)\mathcal{A}_6^{\text{scalar}}\left(-l_{\bar{\phi}},(l+p_{12})_\phi,3_\gamma^+,4_\gamma^+,5_\gamma^+,6_\gamma^-\right).\end{aligned} \tag{3.4.27}$$

The rest of the 6-point MHV amplitude is accounted for by the second class of diagrams.

The contributions from diagrams containing a single gray vertex:



(3.4.28)

which contributes the following to the integrand

$$\mathcal{I}_6^{\rm NSD}[l;\mu^2]\Big|_{\rm gray} = \frac{1}{2}\left[\frac{-(\mu^2)^2[12]^2[34]^2\langle 6|p_{12}|5]^2}{s_{125}\,[l^2+\mu^2]\,[(l-p_{12})^2+\mu^2]} + \mathcal{P}\,(1,2,3,4,5)\right]. \tag{3.4.29}$$

Here the p_{12}-cut yields

$$\begin{aligned}&\left[l^2+\mu^2\right]\left[(l+p_{12})^2+\mu^2\right]\mathcal{I}_6^{\rm SD}[l;\mu^2]\big|_{p_{12}\text{-cut}}\\ &\quad = 2\mathcal{A}_4\left(1_\gamma^+,2_\gamma^+,l_\phi,-(l+p_{12})_{\bar\phi}\right)\mathcal{A}_6^{\rm photon}\left(-l_{\bar\phi},(l+p_{12})_\phi,3_\gamma^+,4_\gamma^+,5_\gamma^+,6_\gamma^-\right).\end{aligned} \tag{3.4.30}$$

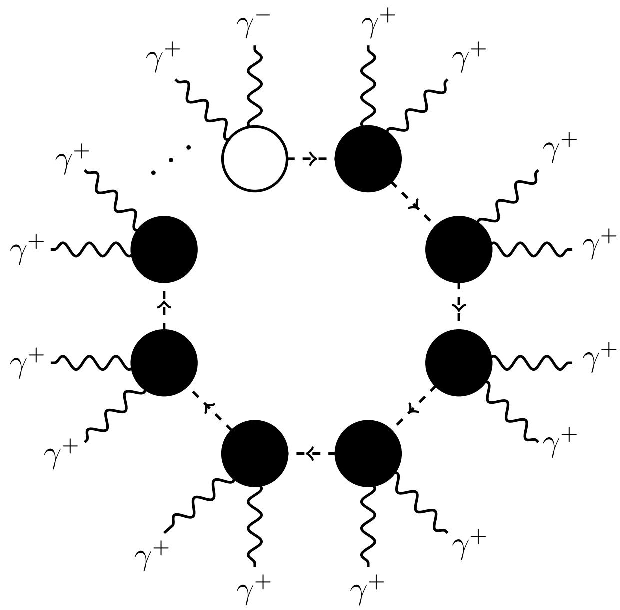
Thus the combined contributions to the integrand from both diagrams (3.4.26) and (3.4.29) is verified to have the correct cuts.

The integration of (3.4.26) and (3.4.29) can be carried out straightforwardly using the general results (H.0.17) and (H.0.24)

$$\begin{aligned}&\mathcal{A}_6^{\rm BI_4\ 1\text{-}loop}\left(1_\gamma^+,2_\gamma^+,3_\gamma^+,4_\gamma^+,5_\gamma^+,6_\gamma^-\right)\\ &\quad = \frac{-i}{23040\pi^2}[12]^2[34]^2\langle 6|p_{125}|5]^2\left(s_{56}+3s_{12}+3s_{34}-6\frac{s_{12}^2}{s_{125}}\right)+\mathcal{P}\,(1,2,3,4,5)+\mathcal{O}(\epsilon).\end{aligned} \tag{3.4.31}$$

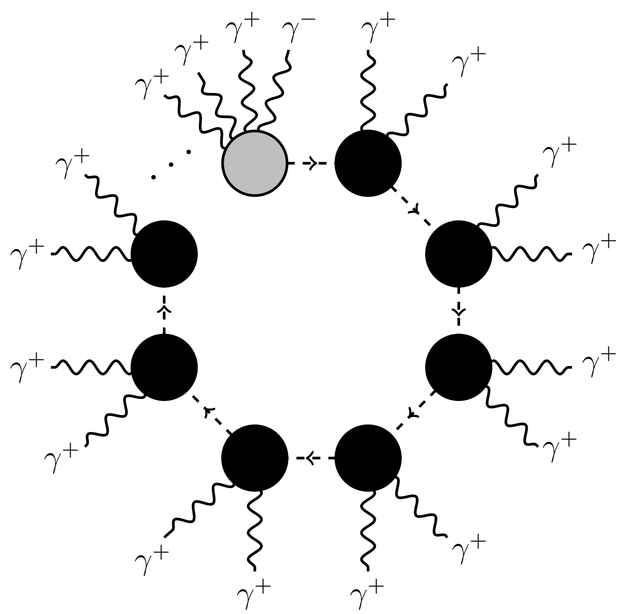
Unlike the cases we have seen so far, this expression is non-local. The factorization poles in the amplitude can be traced back to the non-local gray vertex factor and the associated set of gray loop diagrams. Calculating residues on these poles yields a 4-point SD amplitude times a Born-Infeld tree.

Finally we consider the all-multiplicity result in the NSD sector. Similar to the NSD 6-point example, there will be local contributions from diagrams containing a single white vertex:



(3.4.32)

as well as non-local contributions from diagrams containing a single gray vertex:



(3.4.33)

The explicit contributions to the integrand are, respectively

$$\mathcal{I}_{2n}^{\mathrm{NSD}}[l;\mu^2]\Big|_{\text{white}} = -\left(-\frac{1}{2}\right)^{n-1}[12]^2\ldots[2n-3\;2n-2]^2[2n-1|l|2n\rangle^2 \times \frac{(\mu^2)^{n-1}}{\prod_{i=1}^{n}\left[\left(l-\sum_{j=1}^{2i}p_j\right)^2+\mu^2\right]} + \mathcal{P}(1,2,\ldots,2n-1), \tag{3.4.34}$$

and

$$\mathcal{I}_{2n}^{\text{NSD}}[l;\mu^2]\Big|_{\text{gray}}$$
$$= -\left(-\frac{1}{2}\right)^{n-1} \frac{[12]^2 \ldots [2n-3\;2n-2]^2[2n-1|p_{2n}+p_{2n-2}+p_{2n-3}|2n\rangle^2}{s_{2n,2n-2,2n-3}}$$
$$\times \frac{(\mu^2)^{n-1}}{\prod_{i=1}^{n-2}\left[\left(l-\sum_{j=1}^{2i}p_j\right)^2+\mu^2\right]\left(l-\sum_{j=1}^{2n}p_j\right)^2} + \mathcal{P}(1,2,\ldots,2n-1)\,. \quad (3.4.35)$$

Integrating these contributions separately using (H.0.17) and (H.0.24) gives the result

$$\boxed{\begin{aligned}&\mathcal{A}_{2n}^{\text{BI}_4\text{ 1-loop}}\left(1_\gamma^+,2_\gamma^+,\ldots,(2n-1)_\gamma^+,2n_\gamma^-\right) = \\ &\mathcal{A}_{2n}^{\text{BI}_4\text{ 1-loop}}\left(1_\gamma^+,2_\gamma^+,\ldots,(2n-1)_\gamma^+,2n_\gamma^-\right)\Big|_{\text{white}} + \mathcal{A}_{2n}^{\text{BI}_4\text{ 1-loop}}\left(1_\gamma^+,2_\gamma^+,\ldots,(2n-1)_\gamma^+,2n_\gamma^-\right)\Big|_{\text{gray}},\end{aligned}} \quad (3.4.36)$$

where

$$\boxed{\begin{aligned}&\mathcal{A}_{2n}^{\text{BI}_4\text{ 1-loop}}\left(1_\gamma^+,2_\gamma^+,\ldots,(2n-1)_\gamma^+,2n_\gamma^-\right)\Big|_{\text{white}} \\ &\quad= \frac{-i}{16\pi^2}\left(-\frac{1}{2}\right)^{n-1}\frac{1}{(n-1)n(n+1)(n+2)(n+3)}[12]^2\ldots[2n-3\;2n-2]^2 \\ &\quad\times \sum_{i<j}^{n}\left(\sum_{m=2i+1}^{2j}p_m\right)^2\left[\sum_{k<l}^{n}2\,a_{ijkl}\left(\sum_{m=1}^{2k}[2n-1|p_m|2n\rangle\right)\left(\sum_{m=1}^{2l}[2n-1|p_m|2n\rangle\right)\right. \\ &\qquad\left.+\sum_{k=1}^{n}b_{ijk}\left(\sum_{m=1}^{2k}[2n-1|p_m|2n\rangle\right)^2\right] + \mathcal{P}(1,2,\ldots,2n-1)+\mathcal{O}(\epsilon),\end{aligned}} \quad (3.4.37)$$

with

$$b_{ijk} = \begin{cases} 2 & \text{if } i\neq k \text{ and } j\neq k \\ 6 & \text{if } i=k \text{ or } j=k \end{cases}\,. \quad (3.4.38)$$

And also

$$\boxed{\begin{aligned}
&\mathcal{A}_{2n}^{\text{BI}_4\text{ 1-loop}}\left(1_\gamma^+, 2_\gamma^+, \ldots, (2n-1)_\gamma^+, 2n_\gamma^-\right)\Big|_{\text{gray}} \\
&= \frac{i}{32\pi^2}\frac{(n-2)!}{(n+2)!}\left(-\frac{1}{2}\right)^{n-1}\frac{[12]^2\ldots[2n-3\;2n-2]^2[2n-1|p_{2n-2}+p_{2n-3}|2n\rangle^2}{s_{2n,2n-2,2n-3}} \\
&\times\left[\sum_{i<j}^{n-2}\sum_{k<l}^{n-2}a_{ijkl}\left(\sum_{m=2i+1}^{2j}p_m\right)^2\left(\sum_{m=2k+1}^{2l}p_m\right)^2+4\sum_{i\leq j}^{n-2}\left(\sum_{m=1}^{2i}p_m\right)^2\left(\sum_{m=1}^{2j}p_m\right)^2\right. \\
&\left.+2\sum_{i=1}^{n-2}\sum_{k<l}^{n-2}a_{i(n-1)kl}\left(\sum_{m=1}^{2i}p_m\right)^2\left(\sum_{m=2k+1}^{2l}p_m\right)^2\right]+\mathcal{P}(1,2,\ldots,2n-1)+\mathcal{O}(\epsilon).
\end{aligned}} \tag{3.4.39}$$

It is easy to check that these generic result match the cases of $n=2$ and $n=3$ that were presented above.

As we have already discussed for the 6-particle case, the NSD $(2n)$-particle amplitudes we calculate have poles that can be traced back to the associated poles of the gray vertex factors for $n \geq 3$. These poles are located at $s_{i,j,2n}=0$, for $i<j\leq 2n-1$, and the associated residues are products of the tree 4-particle amplitude and a SD $(2n-2)$-particle amplitude of the form (3.4.15). Let us now demonstrate this factorization explicitly. Consider for example the residue of (3.4.36) at $s_{2n-2,2n-1,2n}=0$,

$$\begin{aligned}
&\operatorname*{Res}_{p_f^2=0}\mathcal{A}_{2n}^{\text{BI}_4\text{ 1-loop}}\left(1_\gamma^+, 2_\gamma^+, \ldots, (2n-1)_\gamma^+, 2n_\gamma^-\right) \\
&= 2\frac{1}{32\pi^2}\frac{(n-2)!}{(n+2)!}\left(-\frac{1}{2}\right)^{n-1}[12]^2\ldots[2n-5\;2n-4]^2[2n-2\;2n-1]^2[2n-3|p_f|2n\rangle^2 \\
&\times\left[\sum_{i<j}^{n-2}\sum_{k<l}^{n-2}a_{ijkl}\left(\sum_{m=2i+1}^{2j}p_m\right)^2\left(\sum_{m=2k+1}^{2l}p_m\right)^2+4\sum_{i\leq j}^{n-2}\left(\sum_{m=1}^{2i}p_m\right)^2\left(\sum_{m=1}^{2j}p_m\right)^2\right. \\
&\left.+2\sum_{i=1}^{n-2}\sum_{k<l}^{n-2}a_{i(n-1)kl}\left(\sum_{m=1}^{2i}p_m\right)^2\left(\sum_{m=2k+1}^{2l}p_m\right)^2\right]+\mathcal{P}(1,2,\ldots,2n-3)+\mathcal{O}(\epsilon)\,,
\end{aligned} \tag{3.4.40}$$

where $p_f = p_{2n-2}+p_{2n-1}+p_{2n}$ is the momentum on the factorization channel. Notice that not all permutations listed in (3.4.39) contribute to the residue while the additional factor of 2 in the right-hand side comes from the trivial permutation $2n-2 \leftrightarrow 2n-1$. Now, on the factorization channel

$$[2n-3|p_f|2n\rangle = -[2n-3,p_f]\langle p_f,2n\rangle = -i[2n-3,p_f]\langle -p_f,2n\rangle\,. \tag{3.4.41}$$

Also, we can use momentum conservation to write

$$\sum_{m=1}^{2i} p_m = -p_f - \sum_{m=2i+1}^{2n-3} p_m = - \sum_{m=2i+1}^{2n-2} \tilde{p}_m \,, \tag{3.4.42}$$

where we have defined

$$\tilde{p}_m = \begin{cases} p_m & \text{if } m \leq 2n-3 \\ p_f & \text{if } m = 2n-2 \end{cases} \tag{3.4.43}$$

With this definition we can write the above residue as

$$\begin{aligned}
&\operatorname*{Res}_{p_f^2=0} \mathcal{A}_{2n}^{\mathrm{BI}_4 \text{ 1-loop}} \left(1_\gamma^+, 2_\gamma^+, \ldots, (2n-1)_\gamma^+, 2n_\gamma^-\right) = \left([2n-2, 2n-1]^2 \langle -p_f, 2n\rangle^2\right) \\
&\times \Bigg[\frac{1}{32\pi^2} \frac{(n-2)!}{(n+2)!} \left(-\frac{1}{2}\right)^{n-2} [12]^2 \ldots [2n-5, 2n-4]^2 [2n-3, p_f]^2 \\
&\qquad \times \sum_{i<j}^{n-1} \sum_{k<l}^{n-1} a_{ijkl} \left(\sum_{m=2i+1}^{2j} \tilde{p}_m\right)^2 \left(\sum_{m=2k+1}^{2l} \tilde{p}_m\right)^2 + \mathcal{P}(1, 2, \ldots, 2n-3) + \mathcal{O}(\epsilon) \Bigg] \,,
\end{aligned} \tag{3.4.44}$$

which clearly shows its factorized form. More precisely, we can write

$$\begin{aligned}
&\operatorname*{Res}_{p_f^2=0} \mathcal{A}_{2n}^{\mathrm{BI}_4 \text{ 1-loop}} \left(1_\gamma^+, \ldots, (2n-1)_\gamma^+, 2n_\gamma^-\right) \\
&= \mathcal{A}_{2n-2}^{\mathrm{BI}_4 \text{ 1-loop}} \left(1_\gamma^+, \ldots, (2n-3)_\gamma^+, (p_f)_\gamma^+\right) \times \mathcal{A}_4^{\mathrm{BI}_4} \left((-p_f)_\gamma^-, (2n-2)_\gamma^+, (2n-1)_\gamma^+, (2n)_\gamma^-\right) .
\end{aligned} \tag{3.4.45}$$

The fact that the pole terms of the NSD 1-loop amplitude factorize to a SD 1-loop and a tree-level MHV amplitude at all multiplicities means that if we choose to remove the SD amplitudes by introducing finite local counter-terms, then the NSD amplitudes become local and can also be set to zero with the introduction of further finite local counter-terms. The consequences will be discussed in the next section.

3.5 Discussion

The main results of this paper are (3.4.15) and (3.4.36), explicit expressions for the SD and NSD amplitudes at one-loop, that would have been impossible to obtain by using traditional Feynman diagrammatics. As expected, they are finite and at $\mathcal{O}(\epsilon^0)$ given by rational functions. For the SD and NSD sectors, these properties follow from the property of BI_4 being a consistent truncation of a supersymmetric model at tree-level. More generally however, we expect both of these properties to obtain in all helicity sectors *except* the duality-conserving sector

$$\mathcal{A}_{2n}^{\mathrm{BI}_4}\left(1_\gamma^+,\ldots,n_\gamma^+,(n+1)_\gamma^-,\ldots,(2n)_\gamma^-\right). \tag{3.5.1}$$

As a consequence of an electromagnetic duality symmetry, these amplitudes which conserve a chiral charge for the photon are the only non-vanishing amplitudes at tree-level [9, 110]. At one-loop, only amplitudes in the duality-conserving sector can have non-vanishing 4d cuts and consequently non-rational functional dependence.

The methods of this paper do not directly extend to calculations at one-loop beyond the SD and NSD sectors. In a sense then we have explored only a small fraction of the structure of Born-Infeld at one-loop. At higher multiplicity the majority of non-duality-conserving sectors, which are expected to be rational, cannot be calculated by constructing integrands from massive scalar cuts. In the duality-conserving sector, the cut-constructible parts can be obtained using the non-vanishing 4d cuts, this will be explored in detail in a separate paper.

It is important to note that the explicit results (3.4.15) and (3.4.36) were obtained in a particular version of dimensional regularization. Specifically, we imposed that the tree-amplitudes appearing in d-dimensional cuts should satisfy the low-energy theorem described in Section 3.3.2. Physically this is equivalent to requiring that the low-energy consequences of T-duality are preserved by the dimensional regulator. While this choice of regularization scheme greatly simplifies the analysis, physical observables must be independent of this choice. It would be an interesting and useful consistency check to re-calculate the one-loop amplitudes in the SD and NSD sectors (and beyond) using a different regularization scheme. For example, one might consider an alternate non-supersymmetric dimensional scheme (such as the 't Hooft-Veltman scheme) or more ambitiously a *non-dimensional* scheme such as Passarino-Veltman. Though potentially much more complicated, the latter case has the virtue of being defined intrinsically in $d = 4$ and may therefore avoid explicitly breaking the electromagnetic duality symmetry. Such questions are important, but are outside of the scope of this paper.

Having explicit forms for two infinite classes of duality-violating one-loop amplitudes, we are in a position to make an interesting observation about the fate of electromagnetic duality at the one-loop quantum-level (see [111] for recent discussion). Recall that electromagnetic duality is not a symmetry in the usual sense. In the standard covariant approach to perturbative quantization, the (effective) quantum theory of Born-Infeld electrodynamics is defined by a path integral

$$e^{i\Gamma[J]} = \int [DA]\, e^{iS[A]+i\int A\wedge J}, \tag{3.5.2}$$

where S is the manifestly Lorentz-invariant effective action (3.1.1), and the path integral measure includes appropriate gauge fixing terms. Curiously, electromagnetic duality is a symmetry only on the *saddle points* of this integral. This is equivalent to the familiar statement that electromagnetic duality is a symmetry of the classical equations of motion, *but not* a symmetry of the off-shell effective action [112]. The practical consequences of this observation is that the standard Feynman rules (for example in R_ξ-gauge) derived from the action (3.1.1) do not manifest the conservation of duality charge at each vertex. At tree-level, duality violating scattering amplitudes are seen to vanish only after summing over all relevant Feynman graphs.

One approach to realizing electromagnetic duality off-shell was given by Deser and Teitelboim who proposed a modified transformation of the covariant action (3.1.1) which acts *non-locally* on the gauge potential [10, 113]. The consequences of such non-local symmetries on perturbative scattering amplitudes are unclear. Another road is to maintain the standard local form of the duality transformation, but replace (3.1.1) with a classically equivalent non-covariant action. This was the approach of Schwarz and Sen [114, 115] and also the first-order (or phase space path integral) approach of Deser and Teitelboim [10,113]. While such non-covariant actions do indeed manifest the conservation of duality charge vertex-by-vertex in the Feynman diagram expansion, we have simply traded one hard problem for another since now it is not clear that the loop-level scattering amplitudes we calculate are Lorentz invariant (see [116] for related discussion). In summary, it would appear to not be possible to define electromagnetic duality as a local, off-shell symmetry of the action while preserving manifest Lorentz covariance. In retrospect we should not expect such a thing to be possible, if it were then the standard Noether procedure would allow us to construct a local Lorentz covariant current operator for the duality charge. But since this charge is carried by massless spin-1 states, such an object is forbidden by the Weinberg-Witten theorem [117]. Given this state of affairs, it is unclear if it is possible to define a quantization of Born-Infeld electrodynamics that preserves electromagnetic duality *in addition to* the standard properties of Lorentz invariance, locality and unitarity. In the specific context considered in this paper we would like to know if it is possible to define an S-matrix at loop-level which respects the helicity selection rules associated with the conservation of duality charge.

The approach we took, constructing local loop integrands consistent with d-dimensional unitarity and 4-dimensional Lorentz invariance, would appear to preserve all of the expected properties manifestly, with the exception of duality invariance. Determining if our explicit results are consistent with the existence of such a duality-respecting quantization is a little subtle. It is too naive to simply observe that the duality-violating one-loop amplitudes (3.4.15) and (3.4.36) are non-zero. Similar to $U(1)$ symmetries acting on chiral fermions, duality rotations act as chiral rotations on states of spin-1, and are therefore only defined in exactly 4-dimensions. Our explicit results however were obtained in a dimensional regularization scheme which explicitly breaks the symmetry. To determine if a genuine anomaly is present, we must first recall that the classical action used to define the full quantum theory as a path integral (3.5.2) is ambiguous up to the addition of *finite local counterterms*. If a consistent set of local, Lorentz-invariant counterterms can be added to the action such that their contribution *cancels* the explicitly calculated rational one-loop amplitudes, then there is no anomaly and the symmetry is preserved. In a related context, recent explicit calculations in $\mathcal{N} = 4$ supergravity in $d = 4$ have revealed that the conventional understanding of the physical consequences of chiral anomalies may be modified in the context of duality symmetries [118, 119].

For the one-loop Born-Infeld amplitudes considered in this paper, in the SD sector the expressions (3.4.15) are manifestly local and Lorentz-invariant, and so can be consistently cancelled by local counterterms. In the NSD sector the expressions (3.4.36) are non-local, here we must sum over both contact contributions from independent local operators and factoring contributions containing both counterterms and tree-level Born-Infeld vertices. The condition that these non-local contributions can be removed with finite local counterterms requires that our explicit results (3.4.36) have the singularity and factorization properties of tree-amplitudes, and we verified this explicitly at the end of Section 3.4.3. The structure of the local counterterms will be discussed further in a separate paper.

These results give an infinite number of non-trivial checks on the preservation of duality under quantization, but do not constitute a proof. Extending the results of this paper to the remaining duality-violating sectors and beyond is therefore essential to understanding the ultimate fate of electromagnetic duality in quantum Born-Infeld.

CHAPTER 4

Electromagnetic Duality and D3-Brane Scattering Amplitudes Beyond Leading Order

4.1 Electromagnetic Duality in Born-Infeld Theory

Electromagnetic (EM) duality is a continuous, global symmetry of certain models containing abelian gauge fields in $d = 4$.[1] On-shell, the charge eigenstates of the symmetry coincide with the helicity eigenstates of a massless spin-1 field (and also potentially additional states), and as such give a concrete realization of *chiral symmetry* for spin $> 1/2$. The primary consequences of such a symmetry for the physically observable S-matrix elements are selection rules, or on-shell Ward identities, of the form

$$\mathcal{A}_n\big(\underbrace{\gamma^+\ldots\gamma^+}_{n_+}\underbrace{\gamma^-\ldots\gamma^-}_{n_-}\big) = 0 \;\text{ for }\; n_+ \neq n_- \,, \tag{4.1.1}$$

where we are using the common convention that helicity states are labelled with all particles outgoing.

Off-shell the meaning of electromagnetic duality is more subtle. In the standard manifestly Lorentz-covariant formulation, the off-shell action for an abelian gauge field is constructed using field operators $A^\mu(x)$ that are not in one-to-one correspondence with physical states. It is

[1]This notion of EM duality is related to, but not exactly the same as, the Montonen-Olive S-duality of $\mathcal{N} = 4$ super Yang-Mills theory [120]; S-duality is necessarily a discrete symmetry since it acts on the quantized charge lattice. In the zero-coupling limit, $g_{\text{YM}} \to 0$, the charged states decouple and the symmetry is enhanced to the continuous $SO(2)$ duality symmetry of free Maxwell theory. The electromagnetic duality considered in this chapter is of the continuous kind, without charged states, but with non-linear self-interactions of the gauge field which preserve the duality symmetry.

perhaps not surprising that as a consequence, there is no off-shell symmetry of the action for which the selection rule (4.1.1) is the conservation of the associated Noether charge. There is however, a corresponding symmetry of the classical equations of motion. In the simplest case of duality-invariant non-linear electrodynamics, the equations of motion consist of a Bianchi identity constraint and an Euler-Lagrange equation,

$$\partial_\mu \tilde{F}^{\mu\nu} = 0, \quad \partial_\mu \tilde{G}^{\mu\nu}[F] = 0, \quad \text{where} \quad \tilde{G}^{\mu\nu}[F] \equiv 2\frac{\partial \mathcal{L}}{\partial F_{\mu\nu}}\,. \tag{4.1.2}$$

Given any solution $F_{\mu\nu}$ we can generate a one-parameter family of solutions by a so-called duality rotation

$$F_{\mu\nu} \to \cos(\theta) F_{\mu\nu} + \sin(\theta) G_{\mu\nu}[F]\,, \tag{4.1.3}$$

if and only if the model satisfies a non-linear, on-shell constraint known as the *Gaillard-Zumino condition* [121]

$$F_{\mu\nu}\tilde{F}^{\mu\nu} + G_{\mu\nu}[F]\tilde{G}^{\mu\nu}[F] = 0\,. \tag{4.1.4}$$

The connection between duality invariance of the equations of motion and the conservation of a chiral charge (4.1.1) in the on-shell scattering amplitudes, is perhaps not widely known. At tree-level, the connection was demonstrated for classically duality invariant models of non-linear electrodynamics in [111], generalizing earlier demonstrations in the context of the non-supersymmetric Born-Infeld (BI) model [9]. More generally, the conservation of a chiral charge is an empirical fact about tree-level scattering amplitudes in a wide variety of classically duality-invariant models, including models of extended supergravity [118].

The goal of this chapter is to improve our understanding of duality invariance, in the form of the selection rule (4.1.1), using modern on-shell methods. This includes a novel understanding of why such a symmetry may, in certain classes of models, be present. Using D-brane world-volume EFTs as theoretical laboratories in which to make explicit calculations, we see from one perspective, using dimensional reduction/oxidation together with subtracted on-shell recursion, that such a symmetry emerges naturally. Contrarily, from the perspective of the BCJ double-copy construction [13, 34, 85],

$$\begin{aligned}&\Big(\text{Born-Infeld with } \mathcal{N} \text{ SUSY}\Big)\\ &= \Big(U(N) \text{ Yang-Mills with } \mathcal{N} \text{ SUSY}\Big) \otimes \left(\frac{U(N)\times U(N)}{U(N)} \text{ Nonlinear Sigma Model}\right),\end{aligned} \tag{4.1.5}$$

at both tree- and loop-level, the appearance of such symmetries is an unexplained miracle. Specifically, amplitudes in YM theory do not satisfy a non-abelian analogue of the selection rule (4.1.1), so for example any odd-point YM amplitude and any MHV amplitude with more than 4 particles have to give zero upon being double-copied with NLSM amplitudes. Such cancellations do indeed happen, as is necessary for the double-construction to work, but it is an emergent rather than a manifest property.

In the non-supersymmetric case of (4.1.5), we analyze the higher-derivative corrections to both Yang-Mills (YM) and the nonlinear sigma model (χPT) and find that the double-copy by itself is not sufficient to guarantee duality invariance. The result is a mysterious subset of higher-derivative corrections, which are neither completely duality-preserving nor completely general.

A further goal of this chapter, building on earlier work by the same authors [12], is to improve our understanding of electromagnetic duality in the context of *quantum* corrections. That the duality rotation (4.1.3) is only a symmetry of the classical equations of motion, and not the off-shell action, might lead us to suspect that the selection rule (4.1.1) is only valid at tree-level. This conclusion is complicated by the fact that there exist alternative, non-manifestly Lorentz-covariant, formulations of duality invariant models which do realize duality rotations off-shell [113,114]. It is an interesting and important problem to determine if classically duality-invariant models can be quantized (at least perturbatively) in a way that preserves both the selection rule (4.1.1) *and* Lorentz invariance, or if there is an anomaly which requires breaking the former to preserve the latter. We address this question by explicit calculation in the context of the worldvolume EFT of a probe D3-brane, described by the abelian $\mathcal{N} = 4$ *Dirac-Born-Infeld (DBI)* model. A central result is that we obtain an explicit, all-multiplicity expression for the 1-loop integrand of the MHV sector of $\mathcal{N} = 4$ DBI in dimensional regularization at all orders in ϵ. For $n > 4$ the resulting manifestly Lorentz-invariant expressions, integrated in $d = 4 - 2\epsilon$ up to $\mathcal{O}(\epsilon^0)$, are shown to be cancelled by $\mathcal{N} = 4$ invariant finite local counterterms, and hence there is no duality anomaly in this sector. From previous results obtained by the present authors in the self-dual (all-plus) and next-to-self-dual (one-minus) sectors of non-supersymmetric Born-Infeld [12], together with the above general analysis of the higher-derivative double-copy, we demonstrate that the requisite finite counterterms cannot always be constructed as a KLT product. This may indicate that there exists no loop-level regularization scheme that respects both electromagnetic duality and color-kinematics duality, suggesting a deep conflict between these two physical principles.

Below we give a brief review of the physics of probe D-brane worldvolume EFTs, with emphasis on the manifestation of physical properties in the S-matrix.

4.1.1 Review of D3-Brane Worldvolume EFTs

The models we consider in this chapter describe the low-energy dynamics of the massless excitations of a probe D3-brane with up to $\mathcal{N} = 4$ linear supersymmetry, embedded in a D-dimensional bulk Minkowski spacetime. The effective field theory on the 4d worldvolume takes the form of a $U(1)$ gauge theory with nonlinear self-interactions, coupled to N_f Weyl fermions and N_s real scalars. The bosonic part of the leading-order (DBI) effective action for a probe Dp-brane has the form

$$S_{Dp}[X^I, F_{\mu\nu}] = T_{Dp} \int \mathrm{d}^{p+1}x \sqrt{-\det\left(\eta_{\mu\nu} + k^2 \partial_\mu X_I \partial_\nu X^I + k F_{\mu\nu}\right)}, \qquad k = 2\pi\alpha', \tag{4.1.6}$$

where $I = 1, 2, ..., N_s$, with $N_s = D - p - 1$ and $T_{Dp} \sim 1/\alpha'^2$. The complete κ-symmetric action including fermionic degrees-of-freedom for $p = 3$ is given explicitly in [122]. In this chapter we take an on-shell perspective, so rather than calculating scattering amplitudes directly from the action (4.1.6), we begin with the physical properties we expect of the model, and use these to *bootstrap* physical S-matrix elements. The massless degrees of freedom and their symmetries have a direct physical interpretation:

- The world-volume *photon* $\gamma^\pm$ describes the motion of the endpoints of open strings ending on the D3-brane [90]. The associated classical equations of motion are invariant under a duality rotation (4.1.3), so as discussed above the photon helicity states carry a conserved chiral charge $Q[\gamma^\pm] = \pm 1$. If the external momenta are restricted to a 3d subspace, then a particular linear polarization is identified with the scalar modulus of a D2-brane in a T-dual frame. The associated enhanced low-energy theorem is sufficient to bootstrap the entire tree-level S-matrix of the non-supersymmetric Born-Infeld model [1].

- The N_f Weyl fermions $\psi_A^\pm$ are Goldstone fermions arising from (potentially partial) spontaneous supersymmetry breaking. All of the fermions are identical and transform together in the fundamental representation of $U(N_f)$. The broken supersymmetry is non-linearly realized in the effective action [122], and consequently the on-shell scattering amplitudes satisfy an Adler-zero type low-energy theorem

$$\mathcal{A}_n^{\mathrm{D3}}\left(\{|p\rangle, |p]\}_\psi^+, ...\right) \xrightarrow{|p\rangle \sim 0} \mathcal{O}\left(|p\rangle\right). \tag{4.1.7}$$

  For the special case $N_f = 1$, $N_s = 0$, the spectrum consists of a single $\mathcal{N} = 1$ vector multiplet and the scattering amplitudes satisfy the associated $\mathcal{N} = 1$ on-shell supersymmetry Ward identities. Together with the low-energy theorem (4.1.7) this can be used to prove that the amplitudes of the bosonic truncation (non-supersymmetric Born-Infeld) satisfy a multi-chiral low-energy theorem [1], sufficient to bootstrap the tree-level S-matrix.

- The N_s scalar fields X_I are spacetime Goldstone bosons, or moduli, arising from the spontaneous breaking of the underlying D-dimensional Poincaré symmetry

$$\mathrm{ISO}(D-1,1) \to \mathrm{ISO}(3,1) \times \mathrm{SO}(D-4). \tag{4.1.8}$$

  Naturally, the scalar fields transform in the vector representation of $SO(D-4)$. The spontaneously broken spacetime symmetries are non-linearly realized in the effective action (4.1.6) with the scattering amplitudes satisfying an *enhanced* Adler-zero low-energy theorem [6]

$$\mathcal{A}_n^{\mathrm{D3}}\left(\{p\}_X, \ldots\right) \xrightarrow{p\sim 0} \mathcal{O}\left(p^2\right). \tag{4.1.9}$$

The problem of constructing a model that satisfies all of the above properties is vastly over-constrained, and it is remarkable that any solution exists. A complete formula for the tree-level S-matrix of the bosonic truncation (4.1.6), valid in d-dimensions, was found in the CHY formalism [85]. Recently, a beautiful CHY formula based on *polarized* scattering equations, appeared for the complete tree-level S-matrix of the $\mathcal{N}=4$ probe D3-brane ($N_f=4$ and $N_s=6$), as well as similar formulae for the 6d parent theories describing probe $\mathcal{N}=(1,1)$ D5- and $\mathcal{N}=(2,0)$ M5-branes [80]. Such higher-dimensional constructions have recently been used to obtain 1-loop integrands for the D3-brane theory in dimensional regularization [123].

The existence of various $U(1)$ charges requires that any non-zero amplitude contains an *even* number of states of a given spin. It is therefore consistent to consider truncations to the purely scalar (p-brane) sector as well as the purely fermionic (Akulov-Volkov or Goldstino) sector. Using the above vanishing low-energy theorems, it was shown in [5, 6, 29, 38, 54] that we can bootstrap the leading-order contribution to the tree-level S-matrix, including in the former case, a finite number of higher-derivative or *Galileon* contributions [82].

The CHY construction [85] has revealed the surprising fact that the well-known field theory KLT formula [34], relating Yang-Mills and Einstein gravity, can also be used to construct Dp-brane world-volume EFTs. The generalization of the double-copy relation (4.1.5) takes the schematic form

$$\Big(\text{BI} \oplus N_f \text{ fermions} \oplus N_s \text{ scalars}\Big) = \Big(\text{YM} \oplus N_f \text{ fermions} \oplus N_s \text{ scalars}\Big) \quad \otimes \quad \chi\text{PT}\,, \tag{4.1.10}$$

where $\oplus$ in the Yang-Mills theory on the right-hand-side indicates whatever couplings between the gluons and massless adjoint fermions and scalars (including Yukawa and scalar potentials) are required to satisfy color-kinematics duality [13]. In this construction, the duality symmetry of the Born-Infeld photon is completely obscured since there is no analogue in the non-abelian Yang-Mills theory. Conceptually the double-copy paints an interesting picture of the massless degrees-of-freedom on the Dp-brane. The *pions* of χPT are Goldstone modes of a spontaneously broken internal symmetry, which combine through the double-copy with the scalar fields of the Yang-Mills theory to form the spacetime Goldstone modes of the Dp-brane. Similarly, the pions combine with the adjoint fermions of the Yang-Mills theory to form Goldstone fermions. The Born-Infeld photon itself has no known interpretation as a Goldstone mode, but through the double-copy we see that it is the combination of a Goldstone scalar pion and a non-abelian gauge boson.

4.2 Dimensional Oxidation: the D3-brane and M2-Brane

We present the worldvolume reduction of the D3-brane action to the M2-brane action and show how the latter has a manifest $U(1)$ symmetry. The oxidation of this symmetry to 4d is very non-trivial, but nonetheless we prove that it is true in the leading order theory using a novel recursion relation. Subsequently we examine higher-derivative corrections and prove that there exist duality-violating operators in 4d that are not ruled out by the 3d $U(1)$ symmetry.

4.2.1 World-Volume Analysis

It is well-known that in D dimensions, the compactification of a Dp-brane on a $p-p'$ torus along its worldvolume directions is T-dual to a flat Dp'-brane, again in D dimensions. This property is inherited by the low-energy effective theories on the branes. The $p+1$-dimensional vector boson states of the Dp-brane, polarized in the compact dimensions, are physically identified with the additional $p-p'$ scalar moduli of the Dp'-brane. Since all dimensions transverse to the world-volume are identical, the scalar moduli of the Dp-brane realize a linear $SO(D-p-1)$ symmetry, while after dimensional reduction this is enhanced to $SO(D-p'-1)$.

A special case of this phenomenon occurs in the dimensional reduction of the D3-brane. For simplicity we restrict the discussion to the truncation to the vector boson sector described by pure Born-Infeld electrodynamics in $\mathbb{R}^4$,

$$S_{\mathrm{D3}}[F_{\mu\nu}] = T_{D3}\int \mathrm{d}^4x\sqrt{-\det\left(\eta_{\mu\nu}+kF_{\mu\nu}\right)}\,. \tag{4.2.1}$$

This action does not manifest any continuous global symmetries, but as described in the introduction, this model has a hidden $U(1)$ electromagnetic duality symmetry that leads to the helicity selection rule (4.1.1). Following the above discussion, dimensional reduction on a circle of vanishing radius gives the action of a D2-brane embedded in $\mathbb{R}^4$,

$$S_{\mathrm{D2}}[F_{\mu\nu},X] = T_{D2}\int \mathrm{d}^3x\sqrt{-\det\left(\eta_{\mu\nu}+k^2\partial_\mu X\partial_\nu X+kF_{\mu\nu}\right)}\,. \tag{4.2.2}$$

Since there is only a single scalar modulus, this action again does not manifest any continuous global symmetries. But, in fact this model has a hidden $U(1)$ symmetry that is revealed off-shell by changing variables via a Legendre transformation. To do so, the implicit Bianchi identity constraint is replaced by an explicit Lagrange multiplier term [124]:

$$S_{\mathrm{D2}}[F_{\mu\nu},X,Y] = T_{D2}\int \mathrm{d}^3x\left[\sqrt{-\det\left(\eta_{\mu\nu}+k^2\partial_\mu X\partial_\nu X+kF_{\mu\nu}\right)}-\frac{k^2}{2}Y\epsilon^{\mu\nu\rho}\partial_\rho F_{\mu\nu}\right]. \tag{4.2.3}$$

By integrating by parts in the final term, the roles of the fields $F_{\mu\nu}$ and Y as dynamical and auxiliary, respectively, are interchanged. Integrating out $F_{\mu\nu}$ generates a *dual* representation of the D2-brane model parametrized in terms of a pair of scalar fields X and Y [124]. Remarkably, this new representation is precisely the (truncated) action of an M2-brane,

$$S_{\mathrm{M2}}[X,Y] = T_{M2} \int \mathrm{d}^3x \sqrt{-\det\left(\eta_{\mu\nu} + k^2 \partial_\mu X \partial_\nu X + k^2 \partial_\mu Y \partial_\nu Y\right)}\,. \tag{4.2.4}$$

The physics of the model (4.2.4) is that it describes the spontaneous breaking of spacetime symmetry $ISO(4,1) \to ISO(2,1) \times SO(2)$, with the scalar brane-moduli X and Y identified with the Goldstone modes of the broken transverse translation symmetries. The linearly realized $SO(2)$ symmetry is manifest in the action (4.2.4) as a rotation of the vector (X,Y) and has the physical interpretation as an isometric rotation between the two non-compact dimensions transverse to the membrane.

The physical significance of this observation is well-known [125,126]. The non-truncated version of this argument is a crucial test of the identification of the 10d embedding of the probe D2-brane with the 11d embedding of the probe M2-brane [124]. In that case, the D3-brane model has a manifest $SO(6)$ symmetry acting as a rotation of the moduli associated with the 6 transverse dimensions. Dimensional reduction and T-dualization produces a D2-brane model with a manifest $SO(7)$ symmetry acting on the 7 moduli. Legendre transforming the 3d gauge boson produces a model with 8 moduli and a manifest $SO(8)$ symmetry, with the additional scalar identified with the spontaneous breaking of translation invariance in the 11th, M-theory dimension.

In the truncated model, for the purposes of calculating scattering amplitudes it is more convenient to form a single complex scalar field $Z = (X + iY)/\sqrt{2}$ and rewrite the action in the form

$$S_{\mathrm{M2}}[Z,\bar{Z}] = T_{M2} \int \mathrm{d}^3x \sqrt{-\det\left(\eta_{\mu\nu} + k^2 \partial_{(\mu} Z \partial_{\nu)} \bar{Z}\right)}\,. \tag{4.2.5}$$

The manifest $U(1)$ symmetry in (4.2.5) implies that the scattering amplitudes of this model satisfy the selection rule

$$\mathcal{A}_n^{\mathrm{M2}}(\underbrace{Z \ldots Z}_{n_Z}\ \underbrace{\bar{Z} \ldots \bar{Z}}_{n_{\bar{Z}}}) = 0 \;\text{ for }\; n_Z \neq n_{\bar{Z}}. \tag{4.2.6}$$

While the existence of the conserved $U(1)$ charge of the M2-brane action (4.2.5) is completely obscured in the D2-brane action (4.2.2), obtained by standard dimensional reduction from the D3-brane action (4.2.1), it is almost trivial at the level of the on-shell scattering amplitudes. Since we are scattering massless Kaluza-Klein modes, the 3d tree-level scattering amplitudes are insensitive to the radius of the compactified dimension and so are formally equivalent to the 4d tree-level scattering amplitudes with the external momenta restricted to an arbitrary 3d subspace. In the commonly used 4d spinor-helicity variables this restriction can be efficiently made by making the replacement

$$|p]_a \longrightarrow \langle p|_{\dot{a}}\,. \tag{4.2.7}$$

Since the models only contain bosonic degrees of freedom, the resulting expressions containing only angle-spinors can always be rewritten in terms of Mandelstam invariants: in 3d, $s_{ij} = (p_i + p_j)^2 = -\langle ij\rangle^2$. All that remains is to relabel the on-shell states in a 3d language as

$$\gamma^+ \longleftrightarrow Z, \qquad \gamma^- \longleftrightarrow \bar{Z}. \tag{4.2.8}$$

Using this dictionary, we find that *the $U(1)$ electromagnetic duality symmetry of the M2-brane amplitudes follows as a necessary consequence of the selection rule (4.1.1) of electromagnetic duality of the D3-brane tree amplitudes.*

At the level of the amplitude selection rules (4.1.1) and (4.2.6), along with the dictionary (4.2.7)-(4.2.8), it is clear that no special properties of Born-Infeld were used. Therefore it implies the following general result:

Theorem 1. *Given any 4d local model of non-linear electrodynamics $S_{(4d)}[F_{\mu\nu}]$, we can construct a 3d model of a complex scalar $S_{(3d)}[Z,\bar{Z}]$ by dimensional reduction followed by a Legendre transformation. If the 4d model has a $U(1)$ electromagnetic duality symmetry of the equations of motion, then the 3d model must have an off-shell $U(1)$ symmetry of the action.*

The above argument includes the possibility of higher-derivative terms; at the level of the amplitude selection rules, this is obvious. At the level of the action, the Legendre transformation becomes the procedure of systematically integrating out the field strength.

The reverse of the above statement would be that the off-shell U(1) symmetry in 3d is oxidized to a duality symmetry in 4d. This would be a more surprising and interesting property. We are going to prove this converse statement for F^n-theories:

Theorem 2. *If the action $S_{(4d)}[F_{\mu\nu}]$ depends only on operators of the form F^n and $S_{(3d)}[Z,\bar{Z}]$ has an off-shell $U(1)$ symmetry, then the 4d model must have a $U(1)$ electromagnetic duality symmetry.*

This converse result is at first sight quite surprising. When a model is dimensionally reduced we typically lose some information about the higher-dimensional physics. One might expect the existence of 4d operators that violate the duality symmetry, but vanish when reduced to 3d and are therefore not ruled out by the 3d symmetry. Indeed this does happen, as we show in Section 4.2.4, but this is delayed to higher orders in the derivative expansion and is absent at the leading orders in the effective field theory. In the following two subsections, we prove this oxidation statement.

In the context of the D3-brane, this argument explains (at least from one point of view) the otherwise mysterious fact that the probe D3-brane preserves the continuous EM duality symmetry of free Maxwell theory. This symmetry is the dimensional oxidation of the linearly realized transverse isometries of the M2-brane. It is interesting to contrast this result with the well-known argument that $\mathcal{N}=4$ super Yang-Mills inherits the modular symmetry $SL(2,\mathbb{Z})$ from the compactification to four-dimensions of the $(2,0)$ SCFT on the world-volume of an M5-brane stack [127]. That argument gives an M-theory based *geometric* explanation of the discrete S-duality form of electromagnetic duality, but gives no indication that it should enhance to a continuous symmetry in the abelian limit.

4.2.2 3d → 4d Oxidation: Contact Terms

The models of non-linear electrodynamics we are considering have the form

$$S_{(4d)}[F_{\mu\nu}] = \int \mathrm{d}^4x \; \mathcal{L}\left(X,Y\right), \qquad X = F_{\mu\nu}F^{\mu\nu}, \qquad Y = F_{\mu\nu}\tilde{F}^{\mu\nu}. \tag{4.2.9}$$

In 4d this form of the Lagrangian is completely general at leading-order in the derivative expansion; using the Cayley-Hamilton theorem it is straightforward to show that general operators of the form Tr$[F^n]$ can be reduced to functions of the invariants X and Y. Furthermore, we assume that in the weak field limit, the Lagrangian admits a low-energy EFT expansion with the leading order term given by the standard Maxwell Lagrangian

$$S_{(4d)}[F_{\mu\nu}] = \int \mathrm{d}^4x \left[-\frac{1}{4}X + \frac{1}{\Lambda^4}\mathcal{L}^{(2)}(X,Y) + ... + \frac{1}{\Lambda^{4k-4}}\mathcal{L}^{(k)}(X,Y) + ...\right], \tag{4.2.10}$$

where $\mathcal{L}^{(k)}$ is a homogeneous polynomial of degree k. Simple dimensional analysis then gives that tree-level scattering amplitudes with n external photons are homogeneous rational functions of spinor-helicity variables $\mathcal{A}_n\left(t|i\rangle, t|i]\right) = t^{2n}\mathcal{A}_n\left(|i\rangle, |i]\right)$. Alternatively, stripping off the overall factor of Λ^{-2n+4} the remaining kinematic part of the amplitude must have mass dimension n.

Consider the dimensional reduction of $S_{(4d)}[F_{\mu\nu}]$ to $S_{(3d)}[Z,\bar{Z}]$, as in Theorem 2 of Section 4.2.1. Suppose the 3d amplitudes conserve a global $U(1)$ charge (4.2.6). We want to prove that the 4d amplitudes must then conserve the duality charge (4.1.1). The dictionary (4.2.8) makes it clear that any $U(1)$-conserving amplitude in 3d lifts to a 4d duality-conserving amplitude. What remains to be shown is that there are no 4d duality-violating amplitudes that vanish upon restriction of the external momenta to a 3d subspace.

There can be no odd-point amplitudes in a theory with only F^n interactions, so the lowest multiplicity we need to consider is 4-point. As the lowest-point amplitudes, they cannot have any kinematic singularities, hence they must be polynomial functions of spinor-helicity variables.

The most general ansatze for these amplitudes consistent with locality, Bose symmetry, mass dimension and little group constraints are

$$
\begin{aligned}
&\mathcal{A}_4(1_\gamma^+\,2_\gamma^+\,3_\gamma^+\,4_\gamma^+) = \alpha_+\left([12]^2\,[34]^2 + [13]^2\,[24]^2 + [14]^2\,[23]^2\right),\\
&\mathcal{A}_4(1_\gamma^-\,2_\gamma^-\,3_\gamma^-\,4_\gamma^-) = \alpha_-\left(\langle 12\rangle^2\langle 34\rangle^2 + \langle 13\rangle^2\langle 24\rangle^2 + \langle 14\rangle^2\langle 23\rangle^2\right),\\
&\mathcal{A}_4(1_\gamma^+\,2_\gamma^+\,3_\gamma^+\,4_\gamma^-) = \mathcal{A}_4(1_\gamma^-\,2_\gamma^-\,3_\gamma^-\,4_\gamma^+) = 0\,.
\end{aligned}
\tag{4.2.11}
$$

Reducing the non-zero amplitudes to 3d gives

$$
\mathcal{A}_4(1_\gamma^\pm\,2_\gamma^\pm\,3_\gamma^\pm\,4_\gamma^\pm) \xrightarrow{3d} \alpha_\pm\left(s^2+t^2+u^2\right),
\tag{4.2.12}
$$

which is non-vanishing unless $\alpha_\pm = 0$. So we establish that for $n = 4$, the existence of a $U(1)$ symmetry after dimensional reduction to 3d requires duality conservation in the 4d model (4.2.9).

At higher multiplicity, we examine the factorization properties of duality-violating 4d amplitudes. Consider an n-point amplitude where the number of positive helicity states n_+ does not equal the number of negative helicity states $n_- \neq n_+$. On a factorization pole, an amplitude splits into $\mathcal{A}_L$ and $\mathcal{A}_R$ such that

$$
\begin{aligned}
n_+^L + n_+^R &= n_+ + 1\,,\\
n_-^L + n_-^R &= n_- + 1\,.
\end{aligned}
\tag{4.2.13}
$$

It is clear that $\mathcal{A}_L$ and $\mathcal{A}_R$ cannot both be duality-preserving amplitudes since $n_+^L + n_+^R \neq n_-^L + n_-^R$ by assumption. Thus, at least one of $\mathcal{A}_L$ and $\mathcal{A}_R$ must be duality-violating.

Since the 4-point duality-violating amplitudes vanish, this then means that any 6-point duality-violating amplitude must be local, i.e. it must arise from a local contact term of the form F^6. By dimensional analysis, little group scaling, and Bose symmetry, the only possible options are (focusing on the mostly-plus sectors)

$$
\begin{aligned}
&\mathcal{A}_6(1_\gamma^+\,2_\gamma^+\,3_\gamma^+\,4_\gamma^+\,5_\gamma^+\,6_\gamma^+) = \beta_+\left([12]^2\,[34]^2\,[56]^2 + \text{perms}\right),\\
&\mathcal{A}_6(1_\gamma^+\,2_\gamma^+\,3_\gamma^+\,4_\gamma^+\,5_\gamma^+\,6_\gamma^-) = 0\,,\\
&\mathcal{A}_6(1_\gamma^+\,2_\gamma^+\,3_\gamma^+\,4_\gamma^+\,5_\gamma^-\,6_\gamma^-) = \beta_{\text{MHV}}\langle 56\rangle^2\left([12]^2\,[34]^2 + \text{perms}\right).
\end{aligned}
$$

In 3d kinematics, the non-zero matrix elements become sums of products of Mandelstam variables *squared* and as such there can be no cancellations; they must be non-vanishing. This means that there are no 4d local, or — by the factorization argument above — non-local, duality-violating 6-point amplitudes that are not ruled out by the 3d $U(1)$ symmetry.

Given that electromagnetic duality is preserved at 6-point order, the 8-point duality-violating amplitudes must be polynomial. The structure observed at 4-point and 6-point for the local contact terms continues at higher point, so the 8-point local duality-violating amplitudes do not vanish in 3d and are therefore also ruled out by the 3d $U(1)$ symmetry, and so on.

A derivation of Theorem 2 naturally lends itself to an inductive argument on the number of particles. In the next section we introduce a new on-shell recursion relation that allows us to prove at all multiplicities that the 3d $U(1)$ symmetry implies 4d electromagnetic duality in F^n-type theories.

4.2.3 3d $\rightarrow$ 4d Oxidation: Subtracted Recursion Relations

We construct an inductive proof of Theorem 2 in Section 4.2.1 based on a new version of subtracted recursion relations that accesses information about how 4d amplitudes behave when the external particles are restricted to a 3d subspace. Consider a set of n on-shell external momenta subject to momentum conservation. Define the shift

$$\hat{p}_i^\mu = (1-z)p_i^\mu + z q_i^\mu \,, \tag{4.2.14}$$

subject to the usual constraints that the shifted momenta $\hat{p}_i^\mu$ are on-shell, $\hat{p}_i^2 = 0$, and satisfy momentum conservation for any value of z, i.e. for each i we require

$$p_i \cdot q_i = q_i^2 = 0 \quad \text{and} \quad \sum_{i=1}^{n} q_i = 0 \,. \tag{4.2.15}$$

We choose the shift vectors q_i to be normal to some unit space-like vector N^μ,

$$q_i \cdot N = 0 \,. \tag{4.2.16}$$

Then, at $z = 1$, the shifted momenta are projected onto the 3d subspace normal to N^μ: $\hat{p}_i^\mu N_\mu \big|_{z=1} = 0$. We implement this shift in terms of spinor-helicity brackets via the holomorphic shift

$$\begin{array}{llll} \langle \hat{i}| = & (1-z)\langle i| + z\, a_i [i|N \,, & |\hat{i}] = |i] \,, & \text{for} \quad h_i \geq 0 \,, \\ \langle \hat{i}| = & \langle i| \,, & |\hat{i}] = (1-z)|i] + z\, a_i N|j\rangle \,, & \text{for} \quad h_i < 0 \,, \end{array} \tag{4.2.17}$$

where $N = N^\mu \sigma_\mu$ and a_i are constants constrained by momentum conservation,

$$\sum_{i:\, h_i \geq 0} a_i |i][i|N + \sum_{i:\, h_i < 0} a_i N|i\rangle\langle i| \, = 0. \tag{4.2.18}$$

The shift (4.2.17) ensures that each shifted momentum is on-shell and the whole set of momenta lie in a 3d subspace when $z = 1$. The latter follows from ($h_i \geq 0$ case)

$$N \cdot \hat{p}_i\big|_{z=1} \propto N^\mu N^\nu \mathrm{Tr}(\bar{\sigma}_\mu |i][i|\sigma_\nu) = N^\mu N^\nu [i|\sigma_\nu \bar{\sigma}_\mu |i] = -N^2[ii] = 0\,. \tag{4.2.19}$$

Naively, equations (4.2.15) and (4.2.16) place $3n + 4$ constraints on the $4n$ components of the q_i^μ's. However, the projection (4.2.16) means that momentum conservation in the N-direction is automatic (i.e. the q_i's are 3d vectors), so the number of constraints is actually $3n + 3$. This gives $n - 3$ free parameters. Alternatively, this can be seen from the fact that (4.2.18) places 3 constraints on the set of n parameters a_i in (4.2.17). Thus the system is under-constrained, and a shift can always be constructed for $n \geq 4$. In Appendix I we present an explicit solution for the a_i.

It was shown in [5] that under a linear holomorphic shift, such as (4.2.17), effective field theory amplitudes scale as the following power of z,

$$4 - n - \frac{n-2}{v-2}[g_v] - \sum_i s_i\,, \tag{4.2.20}$$

when z is large. Here v is the valence of the lowest-point interaction, $[g_v]$ is the mass-dimension of the coupling and s_i are spins of the external states. For models of non-linear electrodynamics (4.2.9), $v = 4$, $[g_v] = -4$ and $s_i = 1$, and so

$$\mathcal{A}_n(z) \overset{z\to\infty}{\sim} z^0\,. \tag{4.2.21}$$

This means that in a standard recursion relation based on a contour integral of $\mathcal{A}_n(z)/z$, there is pole at infinity whose residue cannot be determined by factorization. This simply reflects the fact that the action (4.2.9) contains an infinite number of independent, gauge-invariant, local operators of the form F^n that only begin to contribute to on-shell scattering amplitudes at multiplicity $\geq n$. Since such contributions are completely independent of the lower-valence operators, any attempt to derive on-shell recursion relations must fail unless it incorporates additional physical information sufficient to pick out a unique model. In the present context, the additional constraint we impose is the vanishing of duality-violating amplitudes when external momenta are restricted to a 3d subspace. To do so, we employ *subtracted* recursion relations based on a contour integral of $\mathcal{A}_n(z)/z(1 - z)$. The key is to avoid picking up a pole at $z = 1$.

Consider an n-point duality-violating 4d amplitude $\mathcal{A}_n$. By assumption of the 3d $U(1)$ symmetry, any 4d duality-violating amplitude must vanish when its momenta are restricted to a 3d subspace; hence the shifted amplitude satisfies

$$\mathcal{A}_n(z = 1) = 0\,. \tag{4.2.22}$$

The unshifted amplitude can be retrieved via the contour integral

$$\mathcal{A}_n(0) = \oint_{\mathcal{C}_0} \frac{\mathrm{d}z}{2\pi i} \frac{\mathcal{A}_n(z)}{z(1-z)}\,, \tag{4.2.23}$$

where $\mathcal{C}_0$ is a small circular contour surrounding $z = 0$. The extra factor of $(1-z)$ does not affect the residue at $z = 0$, nor does it introduce a pole in the integrand, precisely due to the property (4.2.22).

Cauchy's theorem is then used to re-express the integral (4.2.23) as a sum of residues on poles away from the origin. Locality ensures that the only poles in a tree amplitude correspond to factorization singularities. All poles occur at finite values of z where an intermediate momentum $\hat{P}_I = (1-z)P_I + zQ_I$ goes on-shell,

$$\hat{P}_I^2 = (1-z)^2 P_I^2 + z^2 Q_I^2 + 2z(1-z) P_I \cdot Q_I = (P_I - Q_I)^2 (z - z_+^I)(z - z_-^I)\,, \tag{4.2.24}$$

and the residue on these poles is the factorized amplitude,

$$\operatorname*{Res}_{P_I^2=0} \mathcal{A}_n = \mathcal{A}_L(z)\mathcal{A}_R(z)\,. \tag{4.2.25}$$

Thus the unshifted amplitude can be expressed as

$$\mathcal{A}_n(0) = \sum_I \operatorname*{Res}_{z=z_\pm^I} \frac{\mathcal{A}_L(z)\mathcal{A}_R(z)}{z(1-z)P_I^2(z)}\,. \tag{4.2.26}$$

Let us now use (4.2.26) to prove Theorem 2 from Section 4.2.1 by induction. By the factorization argument surrounding (4.2.13), any duality-violating amplitude in 4d factorizes into sub-amplitudes $\mathcal{A}_L$ and $\mathcal{A}_R$ of which at least one must also be duality-violating. If we assume that all duality-violating amplitudes with fewer than n external states are zero, then from the recursive formula (4.2.26) it follows that n-point duality-violating amplitudes must likewise vanish. The recursion relations we derived are valid for $n > 3$, and since there are no non-zero 3-point amplitudes for a self-interacting abelian gauge boson consistent with Lorentz invariance and Bose symmetry, this serves as the basis of the induction, and the result is proven.

4.2.4 Higher Derivative Corrections

Theorem 2, proven in the previous subsection, demonstrates that the global $U(1)$ symmetry of a 3d complex scalar obtained by dimensional reduction must oxidize to a duality symmetry in 4d F^n-theories. This was restricted to leading-order in the derivative expansion. Physically this means that we can only expect the oxidation to hold in the deep infrared, where terms with the smallest number of derivatives dominate. A natural question concerns extending this theorem to

Operator \ Helicity Sector	SD	NSD	MHV
F^4	1	0	1
$\partial^2 F^4$	1	1	1
$\partial^4 F^4$	1	0	2
$\partial^6 F^4$	1	1	2
$\partial^8 F^4$	2	1	3
$\partial^{10} F^4$	1	1	3
$\partial^{12} F^4$	2	1	4
F^5	0	0	0
$\partial^2 F^5$	0	0	0
$\partial^4 F^5$	0	1	1
$\partial^6 F^5$	0	2	5
$\partial^8 F^5$	1	13 (1)	11 (2)
F^6	1	0	1
$\partial^2 F^6$	1	2	2
$\partial^4 F^6$	3	4	12
$\partial^6 F^6$	–	15 (2)	30 (5)

Table 4.1: The table shows the number of linearly independent 4d matrix elements of the given operator in the SD (self-dual = all-plus), NSD (next-to-self-dual = one minus), and MHV sectors. For 4-point, the MHV amplitudes (dark gray) conserve duality, but for completeness we include the count. No linear combinations of possible matrix elements vanish in 3d, except for operators in helicity sectors corresponding to the (light gray) shaded cells of the $\partial^8 F^5$ and $\partial^6 F^6$ operators. In each shaded cell, the number in parenthesis is the number of linearly independent matrix elements that do vanish when restricted to 3d kinematics. The "–" indicates that we have not studied the SD sector of the $\partial^6 F^6$ operator.

operators of the form $\partial^{2k} F^n$ with $k \neq 0$.

The recursion relation approach developed in the previous subsection is no longer valid at higher-derivative order, so we proceed by analyzing the matrix elements of operators $\partial^{2k} F^n$ at low-multiplicity. For each operator $\partial^{2k} F^n$ and each helicity assignment, we construct all possible linearly independent matrix elements allowed by mass-dimension, little group scaling, and Bose symmetry; the count is shown in Table 4.1. We then test if any linear combination of the (independent in 4d) matrix elements vanishes in 3d kinematics. This tests if they escape the constraints of the 3d $U(1)$ symmetry. Table 4.1 shows that at multiplicity 4, there are no matrix elements that vanish in 3d, up to and including 16 derivative order. However, at 5-point and 6-point we do find such duality-violating amplitudes that vanish in 3d. The light-gray-shaded cells are those for which such duality-violating matrix elements exist. For example in the MHV sector of $\partial^8 F^5$, there are a total of 11 independent matrix elements and a 2-parameter family of these vanish in 3d.

We now discuss a more systematic way to construct duality-violating, yet 3d $U(1)$-compatible, matrix elements. The simplest construction of a 4d matrix element that vanishes in 3d kinematics involves a Levi-Civita tensor. For example, for $n \geq 5$ consider the scalar matrix element

$$\epsilon\,(1,2,3,4) = \epsilon_{\mu\nu\rho\sigma} p_1^{\mu} p_2^{\nu} p_3^{\sigma} p_4^{\rho}\,. \tag{4.2.27}$$

For $n = 5$, this is the matrix element of the Wess-Zumino-Witten (WZW) term and by momentum conservation it is fully antisymmetric in all five momenta.

Suppose we construct a polynomial in the spinor-helicity brackets that has little-group scaling corresponding to five external photons with helicities $+++--$ and is *antisymmetric* in any exchanges of identical particles. The lowest dimension polynomial with these properties has mass-dimension 9 and there are in fact two such independent polynomials. Upon multiplication of these two polynomials with (4.2.27), we obtain two spinor helicity polynomials of mass-dimension 13, with little-group scaling of photons in the MHV sector, and Bose symmetry in identical particles. They are therefore matrix elements of operators of the schematic form $\partial^8 F^5$ involving contractions with a single Levi-Civita tensor. The Levi-Civita ensures that the matrix elements vanish in 3d. These 2 matrix elements are the ones listed in the gray-shaded box for $\partial^8 F^5$ in the MHV sector and we explicitly match them to the ones found by the general analysis of linear combinations of 11 independent MHV matrix elements possible for any $\partial^8 F^5$ operator (see Table 4.1). Similarly for the NSD (one minus) sector of $\partial^8 F^5$: the matrix element that vanishes in 3d is exactly the product of the WZW polynomial (4.2.27) and the unique spinor bracket polynomial of mass-dimension 9 with little group scaling of photons with helicity $++++-$, and full antisymmetry in identical states.

Since duality-violating amplitudes associated with operators of the form $\partial^m F^4$ are excluded for $m \leq 12$, any 6-point duality-violating amplitude must be polynomial in the spinor brackets and correspond to matrix elements of an operator of the form $\partial^{2k} F^6$. MHV matrix elements that vanish in 3d can then be constructed by multiplying $\epsilon\,(1,2,3,4)$ with a polynomial with MHV $(++++--)$ little group scaling that is antisymmetric in $\{1,2,3,4\}$ and symmetric in $\{5,6\}$. The lowest dimension of such a polynomial is 8 and it is unique. The result is a matrix element of an operator of the form $\partial^6 F^6$. To construct the NSD matrix element that vanishes in 3d, take

$$\epsilon\,(1,2,3,4,5) \equiv \epsilon\,(1,2,3,4) - \epsilon\,(1,2,3,5) + \epsilon\,(1,2,4,5) - \epsilon\,(1,3,4,5) + \epsilon\,(2,3,4,5)\ , \tag{4.2.28}$$

that is antisymmetric in labels $\{1,\cdots,5\}$ and multiply it with the unique dimension-8 NSD $(+++++-)$ polynomial antisymmetric in identical helicity states. This construction is included in the $\partial^6 F^6$ matrix elements that we find to vanish in 3d in Table 4.1, although not all matrix elements reported there can be obtained using this method.

Another construction at 5-point is to multiply 5-point photon matrix elements (with proper Bose symmetry) with the quintic Galileon term,

$$\epsilon\,(1,2,3,4)^2\ . \tag{4.2.29}$$

By the results in Table 4.1, the lowest dimension 5-point matrix element arises from $\partial^4 F^5$. When multiplied by the matrix element of the 8-derivative quintic Galileon term, we get a matrix element of $\partial^{12} F^5$ that is guaranteed to vanish in 3d.

The point here is the existence of 4d duality-violating matrix elements that, since they vanish in 3d, are not excluded by the 3d $U(1)$ symmetry. We note that at low multiplicities, the duality violation is delayed until quite high order; the lowest order example given here is 12-derivative at 6-point (compared with the 6-derivative leading BI term F^6).

In the context of string theory, the Born-Infeld action (4.2.1) describes the leading-order in α'-contribution to the dynamics of the D3-brane. Matching to the world-sheet calculation of the open-string S-matrix, there is an infinite set of sub-leading corrections (the first ones calculated originally in [128]). By dimensional analysis, the sub-leading α'-contributions to a given amplitude correspond to derivative corrections to the effective action of the form $\partial^{2k} F^n$. A reasonable question to ask is whether, when dimensionally reduced to 3d, these sub-leading corrections are consistent with the M2-brane picture and preserve the hidden $U(1)$ symmetry. This is the question we have examined here in the generic context of higher derivative corrections. Whether the particular duality-violating operators we present here are in fact produced in string theory (or if they are compatible with supersymmetry) is a question beyond the scope of this chapter.

4.3 Loop Amplitudes from BCJ Double-Copy

The double-copy of Yang-Mills and chiral perturbation theory (χPT) was shown to result in Born-Infeld theory at tree-level [85], using the CHY formalism [14, 36, 107]. BI tree amplitudes can be constructed either via the KLT relations or the BCJ double-copy. The former is used in the context of higher-derivative corrections in Section 4.7. In this section, we begin with the BCJ construction of the 4-point tree amplitude and then use loop-level BCJ to construct the integrand for the self-dual 1-loop 4-point amplitude.

4.3.1 Tree-level BCJ Double-Copy of BI

At 4-point, the BCJ form of a tree amplitude can be written as

$$\mathcal{A}_4\,(1\,2\,3\,4) = \frac{c_{1342}\;n_{1342}}{t} + \frac{c_{1423}\;n_{1423}}{u}, \tag{4.3.1}$$

where we use the convention for the Mandelstam invariants

$$s = (p_1 + p_2)^2, \quad t = (p_1 + p_3)^2, \quad u = (p_1 + p_4)^2, \tag{4.3.2}$$

and the *BCJ color tensors* are defined as

$$c_{ijkl} = f_{a_i a_j b} f_{a_k a_l b}\,. \tag{4.3.3}$$

The Jacobi identity $c_{1234} + c_{1423} + c_{1342} = 0$ has been used here to eliminate the term with color factor c_{1234}. The numerator factors n_{ijkl} can be written in terms of color-ordered amplitudes as

$$n_{1342} = -t\,\mathcal{A}_4[1\,3\,4\,2] \quad \text{and} \quad n_{1423} = u\,\mathcal{A}_4[1\,2\,3\,4]\,. \tag{4.3.4}$$

For our specific application, we use the ordered tree-level amplitudes for Yang-Mills and χPT

$$\mathcal{A}_4^{\text{YM}}[1^+\,2^+\,3^-\,4^-] = -g_{\text{YM}}^2 \frac{[12]^2\langle 34\rangle^2}{su}, \qquad \mathcal{A}_4^{\chi\text{PT}}[1\,2\,3\,4] = \frac{1}{f_\pi^2} t. \tag{4.3.5}$$

Using (4.3.4), the BCJ numerators take the form

$$\begin{aligned} n_{1342}^{\text{YM}} &= g_{\text{YM}}^2 \frac{[12]^2\langle 34\rangle^2}{s}, & n_{1423}^{\text{YM}} &= -g_{\text{YM}}^2 \frac{[12]^2\langle 34\rangle^2}{s}, \\ n_{1342}^{\chi\text{PT}} &= -\frac{1}{f_\pi^2} tu, & n_{1423}^{\chi\text{PT}} &= \frac{1}{f_\pi^2} tu. \end{aligned} \tag{4.3.6}$$

Together with $n_{1234} \equiv 0$, these sets of BCJ numerators satisfy the *kinematic Jacobi identity* $n_{1234} + n_{1342} + n_{1423} = 0$. The BCJ double-copy, at tree-level and at 4-point takes the form

$$\mathcal{A}_4^{A\otimes B}(1, 2, 3, 4) = \frac{1}{\lambda^2}\left[\frac{n_{1234}^A\, n_{1234}^B}{s} + \frac{n_{1342}^A\, n_{1342}^B}{t} + \frac{n_{1423}^A\, n_{1423}^B}{u}\right], \tag{4.3.7}$$

where λ is some arbitrary constant with $[\lambda] = 1$.[2] The tree-amplitudes of BI are given as the double-copy

$$\begin{aligned}
&\mathcal{A}_4^{\text{BI}}\left(1^+\,2^+\,3^-\,4^-\right)\\
&\quad= \frac{1}{\lambda^2}\left[\frac{n^{\text{YM}}_{1342}\,n^{\chi\text{PT}}_{1342}}{t} + \frac{n^{\text{YM}}_{1423}\,n^{\chi\text{PT}}_{1423}}{u}\right]\\
&\quad= \frac{1}{\lambda^2}\left[\frac{1}{t}\left(g_{\text{YM}}^2\frac{[12]^2\langle 34\rangle^2}{s}\right)\left(-\frac{1}{f_\pi^2}tu\right) + \frac{1}{u}\left(-g_{\text{YM}}^2\frac{[12]^2\langle 34\rangle^2}{s}\right)\left(\frac{1}{f_\pi^2}tu\right)\right]\\
&\quad= \frac{g_{\text{YM}}^2}{\lambda^2 f_\pi^2}[12]^2\langle 34\rangle^2.
\end{aligned} \tag{4.3.8}$$

Comparing this to the well-known result for the 4-point tree amplitude in Born-Infeld

$$\mathcal{A}_4^{\text{BI}}\left(1^+\,2^+\,3^-\,4^-\right) = \frac{1}{\Lambda^4}[12]^2\langle 34\rangle^2, \tag{4.3.9}$$

we see that the BCJ double-copy indeed gives the correct result when the couplings are related as

$$\frac{g_{\text{YM}}^2}{\lambda^2 f_\pi^2} = \frac{1}{\Lambda^4} \sim \frac{1}{T_{D3}}\,. \tag{4.3.10}$$

We use this in the following loop-calculations.

4.3.2 1-loop 4-point Self-Dual Amplitude as a Double-Copy

For loop amplitudes, the BCJ construction is conjectured to be valid at the level of the integrand [35]. In this section we use it to construct the 1-loop 4-point BI integrand of the self-dual sector.

The strategy is as follows: we construct the χPT 4-point 1-loop integrand directly using unitarity to ensure that it satisfies all cuts. In that expression we then replace the 1-loop color-factors by the color-kinematic duality-obeying numerator factors for the self-dual 4-point 1-loop YM amplitude obtained previously in [129] in the FDH scheme [105]. This conjecturally gives the self-dual 1-loop BI integrand. We integrate the expression to show that the integrated result agrees exactly with (4.4.1) obtained by different techniques in [12] and Appendix K.

YM Self-Dual 1-loop 4-point Integrand. The calculation is simplified significantly by the fact that there exists a representation of the self-dual 4-point 1-loop YM numerators where only the

[2] From the equivalence between the tree-level BCJ double-copy and the KLT formula [13], together with the interpretation of the KLT kernel as the inverse of a matrix of bi-adjoint scalar amplitudes [14] we find that λ has a physical interpretation as the ϕ^3 coupling constant.

box-numerators are non-vanishing [129]; specifically these box numerators take the form

$$n_{1234}^{(\text{box})}(l) = 2g_{\text{YM}}^4 \frac{[12][34]}{\langle 12\rangle\langle 34\rangle}\left(\mu^2\right)^2, \tag{4.3.11}$$

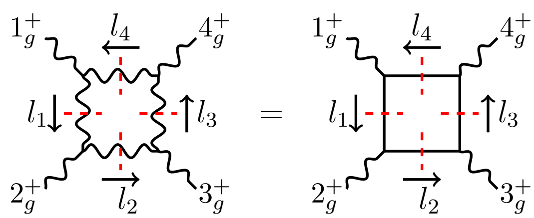
where g_{YM} is the Yang-Mills coupling and $\mu^2 = l_{-2\epsilon}^2$ is the square of the momentum in the -2ϵ extra dimensions, $l^\mu = l_{[4]}^\mu + l_{-2\epsilon}^\mu$. The external state momenta p_i, $i = 1, 2, 3, 4$, are assumed to be on-shell and strictly 4d. The self-dual 1-loop amplitude of YM then takes the form

$$\mathcal{A}_4^{\text{YM}}\left(1_g^+\, 2_g^+\, 3_g^+\, 4_g^+\right) = \int \frac{\mathrm{d}^{4-2\epsilon} l}{(2\pi)^{4-2\epsilon}} \left[\frac{c_{1234}^{(\text{box})} n_{1234}^{(\text{box})}(l)}{l^2(l-p_2)^2(l-p_2-p_3)^2(l+p_1)^2} + (2 \leftrightarrow 3) + (3 \leftrightarrow 4)\right]. \tag{4.3.12}$$

The expression (4.3.11) is symmetric under permutation of the external states, so the box coefficients of all three independent box diagrams of the 1-loop 4-point amplitude are the same: $n_{1324}^{(\text{box})} = n_{1243}^{(\text{box})} = n_{1234}^{(\text{box})}$. The color factors[3] are

$$c_{1234}^{(\text{box})} = f_{a_1 b_1 b_4}\, f_{a_2 b_2 b_1}\, f_{a_3 b_3 b_2}\, f_{a_4 b_4 b_3}\,, \tag{4.3.13}$$

along with the permutations $(2 \leftrightarrow 3)$ and $(3 \leftrightarrow 4)$ (which on the RHS acts on the a-subscripts only). We check the validity of the representation (4.3.12) by computing the maximal cuts, e.g.



(4.3.14)

Here we are using the supersymmetric trick (K.0.9) to replace the gluon in the loop with a complex scalar with 4d mass $\mu^2 = l_{-2\epsilon}^2$. On the cut, the integrand gives

$$\begin{aligned}\text{Cut}_{1234}\left[\mathcal{I}_4^{\text{YM}}\left(1_g^+\, 2_g^+\, 3_g^+\, 4_g^+\right)\right] =& 2\,\mathcal{A}_3^{\text{tree}}\left(1_g^+\,(l_1)_\phi\,(-l_4)_{\overline{\phi}}\right)\mathcal{A}_3^{\text{tree}}\left(2_g^+\,(l_2)_\phi\,(-l_1)_{\overline{\phi}}\right)\\ &\times \mathcal{A}_3^{\text{tree}}\left(3_g^+\,(l_3)_\phi\,(-l_2)_{\overline{\phi}}\right)\mathcal{A}_3^{\text{tree}}\left(4_g^+\,(l_4)_\phi\,(-l_3)_{\overline{\phi}}\right),\end{aligned} \tag{4.3.15}$$

where the factor of 2 accounts for the fact that the complex scalar can run in both directions. The scalar-scalar-gluon amplitude is

$$\mathcal{A}_3^{\text{tree}}\left(1_g^+\,\ell_\phi\,\ell'_{\overline{\phi}}\right) = -i g_{\text{YM}} f_{a_1 a_\ell a_{\ell'}} \frac{[1|\ell|q\rangle}{\langle 1q\rangle}, \tag{4.3.16}$$

where $|q\rangle$ is an arbitrary spinor. The normalization in (4.3.16) is fixed by the 3-gluon amplitude via the SUSY Ward identities in the massless limit $\ell^2 = \ell'^2 \to 0$.

[3]We work in conventions where $[t^a, t^b] = i f^{abc} t^c$ and Tr $\left[t^a t^b\right] = \delta^{ab}$.

It is convenient to choose

$$|q_1\rangle = |2\rangle, \quad |q_2\rangle = |1\rangle, \quad |q_3\rangle = |4\rangle, \quad |q_4\rangle = |3\rangle, \tag{4.3.17}$$

in which case the cut (4.3.15) gives

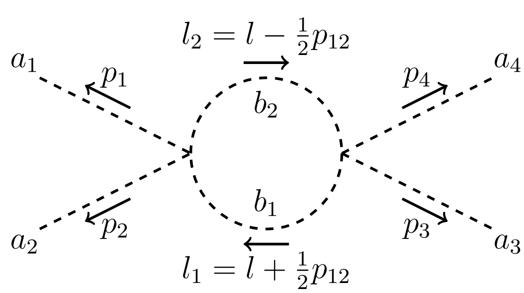
$$\begin{aligned}
&\text{Cut}_{1234}\left[\mathcal{I}_4^{\text{YM}}\left(1_g^+\, 2_g^+\, 3_g^+\, 4_g^+\right)\right] \\
&= c_{1234}^{(\text{box})} 2\, g_{\text{YM}}^4 \frac{[1|l_1 p_2 l_1|1\rangle[3 l_3 p_4 l_3|3\rangle}{\langle 12\rangle^2\langle 34\rangle^2} \\
&= c_{1234}^{(\text{box})}\, 2\, g_{\text{YM}}^4 (\mu^2)^2 \frac{[12][34]}{\langle 12\rangle\langle 34\rangle},
\end{aligned} \tag{4.3.18}$$

where in the second step we use special 3-particle kinematics to show that $[1|l_1 p_2 l_1|1\rangle \times [3|l_3 p_4 l_3|3\rangle = -\langle 12\rangle[12]\mu^2$ and similarly for the other angle-square brackets. Thus we see from (4.3.18) that the product of tree-amplitudes (4.3.15) indeed reproduces the YM numerator factor (4.3.11) of [129].

χ**PT 1-loop 4-point Integrand.** Next, we compute the χPT 1-loop 4-point integrand. We then replace its color factors $c^{(\text{box})}$ by the YM numerators to construct the 1-loop BI self-dual amplitude at 4-point order. The color-dressed χPT tree amplitudes can be written

$$\mathcal{A}_4^{\text{tree}}\left(1_{a_1}\, 2_{a_2}\, 3_{a_3}\, 4_{a_4}\right) = \frac{2}{f_\pi^2}\Big[f_{a_1 a_4 b} f_{a_2 a_3 b}\,(p_1 \cdot p_3) + f_{a_1 a_3 b} f_{a_2 a_4 b}\,(p_1 \cdot p_4)\Big]. \tag{4.3.19}$$

The s-channel cut of the 1-loop amplitude



(4.3.20)

gives

$$
\begin{aligned}
&\mathcal{A}_4^{\text{tree}}\Big(1_{a_1}\, 2_{a_2}\, (-l_1)_{b_1}\, (l_2)_{b_2}\Big)\mathcal{A}_4^{\text{tree}}\left(3_{a_3}\, 4_{a_4}\, (-l_2)_{b_2}\, (l_1)_{b_1}\right)\\
&\quad = \frac{4}{f_\pi^4}\Big[f_{a_1b_2b_3}f_{a_2b_1b_3}\big(-p_1\cdot l_1\big) + f_{a_1b_1b_3}f_{a_2b_2b_3}\big(p_1\cdot l_2\big)\Big]\\
&\qquad\qquad \times\Big[f_{b_2a_4b_4}f_{b_1a_3b_4}\big(-p_3\cdot l_2\big) + f_{b_2a_3b_4}f_{b_1a_4b_4}\big(-p_4\cdot l_2\big)\Big]\\
&\quad = \frac{4}{f_\pi^4}c_{1234}^{(\text{box})}\Big[-(p_1\cdot l)(p_{12}\cdot l) + \tfrac{1}{8}s^2\Big] + \frac{4}{f_\pi^4}c_{1243}^{(\text{box})}\Big[(p_1\cdot l)(p_{12}\cdot l) + \tfrac{1}{8}s^2\Big]\,.
\end{aligned}
\tag{4.3.21}
$$

To obtain the last line we expanded out the product, identified the box color-factors (4.3.13) in terms of the structure constants, and then performed some simplifications. Terms linear in the loop-momentum l were dropped since they integrate to zero.

Including the propagators, this gives the s-channel contribution to the integrand. The t- and u-channels can be obtained by simple permutations of the external momentum labels, so the full integrand is

$$
\begin{aligned}
\mathcal{I}_4^{\chi\text{PT}}\left(1_{a_1}\, 2_{a_2}\, 3_{a_3}\, 4_{a_4}; l, \mu^2\right) = \Bigg(&c_{1234}^{(\text{box})}\frac{2}{f_\pi^4}\frac{-(p_1\cdot l)(p_{12}\cdot l) + \frac{1}{8}s^2}{\left(l-\frac{1}{2}p_{12}\right)^2\left(l+\frac{1}{2}p_{12}\right)^2}\\
&+ c_{1243}^{(\text{box})}\frac{2}{f_\pi^4}\frac{(p_1\cdot l)(p_{12}\cdot l) + \frac{1}{8}s^2}{\left(l-\frac{1}{2}p_{12}\right)^2\left(l+\frac{1}{2}p_{12}\right)^2}\Bigg) + (2\leftrightarrow 3) + (2\leftrightarrow 4)\,.
\end{aligned}
\tag{4.3.22}
$$

We could proceed to extract the box numerator factors for χPT, but it is a lot simpler to directly use the form (4.3.22) in the double-copy.

BI Self-Dual 1-loop 4-point Integrand and Amplitude. We obtain the self-dual loop-integrand for BI theory via the BCJ double-copy by replacing the box-color factors in (4.3.22) by the YM numerators (4.3.11). Since these are symmetric in the external labels, we have

$$
c_{ijkl}^{(\text{box})} \to \frac{1}{\lambda^4}n_{1234}^{(\text{box})}(l) = \frac{2g_{\text{YM}}^4}{\lambda^4}\frac{[12][34]}{\langle 12\rangle\langle 34\rangle}\left(\mu^2\right)^2 = \frac{2g_{\text{YM}}^4}{\lambda^4}\frac{[12]^2[34]^2}{s^2}(\mu^2)^2\,.
\tag{4.3.23}
$$

This results in significant simplifications, in particular all dependence on the 4d part of the loop-momentum in the numerator cancels. We are left with

$$
\begin{aligned}
\mathcal{I}_4^{\text{BI}}\left(1_{a_1}\, 2_{a_2}\, 3_{a_3}\, 4_{a_4}; l, \mu^2\right) &= \mathcal{I}_4^{\chi\text{PT}}\left(1_{a_1}\, 2_{a_2}\, 3_{a_3}\, 4_{a_4}; l, \mu^2\right)\Big|_{c^{(\text{box})}\to\lambda^{-4} n^{(\text{box})\,\text{YM}}} \\
&= \frac{1}{\Lambda^8}[12]^2[34]^2 \frac{\left(\mu^2\right)^2}{\left(l-\frac{1}{2}p_{12}\right)^2\left(l+\frac{1}{2}p_{12}\right)^2} + (2\leftrightarrow 3) + (2\leftrightarrow 4)\,,
\end{aligned}
\tag{4.3.24}
$$

where we used the identification (4.3.10). The loop integral can be evaluated using the dimension-shifting technique of [130]. The details of this are given in Appendix J. Using equation (J.0.11) with $p=2$, we find

$$
\int \frac{d^{4-2\epsilon} l}{(4\pi)^{4-2\epsilon}} \frac{(\mu^2)^2}{(l-\frac{1}{2}p_{12})^2(l+\frac{1}{2}p_{12})^2} = \frac{i}{(4\pi)^{2-\epsilon}}\epsilon(1-\epsilon)\frac{\Gamma(-2+\epsilon)\Gamma^2(3-\epsilon)}{\Gamma(6-2\epsilon)} s^{2-\epsilon} = \frac{-is^2}{960\pi^2}\,,
\tag{4.3.25}
$$

which finally gives

$$
\boxed{\mathcal{A}_4^{\text{1-loop BI}}\left(1_\gamma^+\, 2_\gamma^+\, 3_\gamma^+\, 4_\gamma^+\right) = \frac{-i}{960\pi^2}\Big([12]^2[34]^2 s^2 + [13]^2[24]^2 t^2 + [14]^2[23]^2 u^2\Big) + \mathcal{O}(\epsilon).}
\tag{4.3.26}
$$

In this and subsequent formulae we suppress the Λ-dependent prefactor, this can be easily restored by dimensional analysis. The result (4.3.26) agrees exactly with the result obtained by generalized unitarity in [12] and offers an interesting different path to exploring BI at loop-level.

4.4 1-loop SD and NSD Amplitudes from Unitarity

We briefly review results from [12] for the all-multiplicity self-dual and next-to-self-dual 1-loop amplitudes of pure Born-Infeld theory.

4.4.1 4-point and Counterterm

The self-dual amplitude (4.3.26) was derived in [12] and can also be computed directly from Feynman rules (see Appendix K). The next-to-self-dual 4-point amplitude vanishes [12]

$$
\boxed{\mathcal{A}_4^{\text{1-loop BI}}\left(1_\gamma^+\, 2_\gamma^+\, 3_\gamma^+\, 4_\gamma^-\right) = 0 + \mathcal{O}(\epsilon)\,.}
\tag{4.4.1}
$$

The vanishing of $\mathcal{A}_4^{\text{1-loop BI}}\left(1_\gamma^+\, 2_\gamma^+\, 3_\gamma^+\, 4_\gamma^-\right)$ is easy to understand: the amplitude has to be local, but little group scaling, Bose symmetry, and dimensional analysis show that there is no such possible local matrix element.

The non-vanishing of $\mathcal{A}_4^{\text{1-loop BI}}(1_\gamma^+\,2_\gamma^+\,3_\gamma^+\,4_\gamma^+)$, given in (4.3.26), indicates a potential violation of EM duality at loop-order. However, there is a local operator $\partial^4 F^4$ that generates this matrix element. We can construct this operator explicitly by using the spinorized form of the field strengths to make direct contact with the matrix elements in spinor helicity formalism. Define

$$F_+ = \frac{1}{2}(F^{\mu\nu} + i\tilde{F}^{\mu\nu})\sigma_{\mu\nu}, \qquad F_- = \frac{1}{2}(F^{\mu\nu} - i\tilde{F}^{\mu\nu})\overline{\sigma}_{\mu\nu}, \tag{4.4.2}$$

with $\sigma^{\mu\nu} = \frac{1}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)$, $\overline{\sigma}^{\mu\nu} = \frac{1}{4}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu)$ and $\tilde{F}^{\mu\nu} = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$. Then the Feynman rule for a positive/negative helicity external photon with momentum p is simply[4]

$$F_+ \;\leftrightarrow\; \sqrt{2}|p]\,[p| \qquad \text{and} \qquad F_- \;\leftrightarrow\; \sqrt{2}|p\rangle\langle p|\,. \tag{4.4.3}$$

The $\partial^4 F^4$ operator that cancels the loop-contribution (4.3.26) is

$$S_{\text{BI}} \to S_{\text{BI}} + \frac{1}{7680\pi^2\Lambda^8}\int \mathrm{d}^4x \left[(\partial_\mu F_{+\alpha\beta})(\partial^\mu F_+^{\alpha\beta})\right]^2 + \text{h.c.} \tag{4.4.4}$$

This then restores the duality symmetry at 4-point 1-loop order.

4.4.2 n-point

In a recent paper [12], we computed 1-loop amplitudes in the self-dual and next-to-self-dual sectors for any number of external states, using a combination of powerful modern methods. In the self-dual sector of pure BI, the 1-loop integrand is

$$\begin{aligned}
&\mathcal{I}_{2n}^{\text{BI}}(1_\gamma^+ 2_\gamma^+ \cdots 2n_\gamma^+; l, \mu^2) \qquad\qquad (4.4.5)\\
&= \left(\frac{1}{2}\right)^{n-1} [12]^2[34]^2 \ldots [2n-1, 2n]^2 \frac{(-\mu^2)^n}{\prod_{i=1}^n \left[\left(l - \sum_{j=1}^{2i} p_j\right)^2 + \mu^2\right]} + \mathcal{P}(2, 3, \ldots, 2n)\,,
\end{aligned}$$

[4] The reference spinors of the polarizations drop out because $F_{\mu\nu}$ is gauge invariant.

where the $\mathcal{P}(2,3,\ldots,2n)$ stands for all permutations of momentum labels $2,3,\ldots,2n$. Using the integral (J.0.3), it was shown in [12] that (4.4.5) integrates to the local expression

$$\boxed{\begin{aligned}&\mathcal{A}_{2n}^{\mathrm{BI}_4\ \text{1-loop}}\left(1_\gamma^+\,2_\gamma^+\ldots 2n_\gamma^+\right)\\ &\quad=\frac{i}{32\pi^2}\left(-\frac{1}{2}\right)^{n-1}\frac{1}{n(n+1)(n+2)(n+3)}\\ &\quad\times\Bigg[[12]^2[34]^2\ldots[2n-1,2n]^2\left(\sum_{i<j}^{n}\sum_{k<l}^{n}a_{ijkl}\left(\sum_{m=2i+1}^{2j}p_m\right)^2\left(\sum_{m=2k+1}^{2l}p_m\right)^2\right)\\ &\qquad+\mathcal{P}(2,3,\ldots,2n)\Bigg]+\mathcal{O}(\epsilon)\,,\end{aligned}} \tag{4.4.6}$$

with

$$a_{ijkl}=\begin{cases}1 & \text{if all } i,j,k,l \text{ are different}\\ 2 & \text{if exactly 2 of } i,j,k,l \text{ are identical}\\ 4 & \text{if } i=k \text{ and } j=l\end{cases}\,. \tag{4.4.7}$$

It is straightforward to check that this result matches the results of the explicit calculations for the case of $n=2$ presented above.

In the next-to-self-dual sector, the 1-loop amplitudes in pure BI theory are

$$\boxed{\begin{aligned}&\mathcal{A}_{2n}^{\mathrm{BI}_4\ \text{1-loop}}\left(1_\gamma^+\,2_\gamma^+\ldots(2n-1)_\gamma^+\,2n_\gamma^-\right)\\ &=\frac{i}{32\pi^2}\frac{(n-2)!}{(n+2)!}\left(-\frac{1}{2}\right)^{n-1}\frac{[12]^2\ldots[2n-3\;2n-2]^2[2n-1|p_{2n-2}+p_{2n-3}|2n\rangle^2}{s_{2n,2n-2,2n-3}}\\ &\times\Bigg[\sum_{i<j}^{n-2}\sum_{k<l}^{n-2}a_{ijkl}\left(\sum_{m=2i+1}^{2j}p_m\right)^2\left(\sum_{m=2k+1}^{2l}p_m\right)^2+4\sum_{i\leq j}^{n-2}\left(\sum_{m=1}^{2i}p_m\right)^2\left(\sum_{m=1}^{2j}p_m\right)^2\\ &+2\sum_{i=1}^{n-2}\sum_{k<l}^{n-2}a_{i(n-1)kl}\left(\sum_{m=1}^{2i}p_m\right)^2\left(\sum_{m=2k+1}^{2l}p_m\right)^2\Bigg]+\mathcal{P}(1,2,\ldots,2n-1)+\mathcal{O}(\epsilon)\\ &+\text{local terms.}\end{aligned}} \tag{4.4.8}$$

The local terms in this amplitude have been computed explicitly and can be found in equations (4.24)-(4.27) in [12].

As shown in [12], the 1-loop next-to-self-dual $+\cdots+-$ amplitude (3.4.39) has simple poles on which it factorizes into self-dual $+\cdots++$ amplitudes times a 4-point tree-level BI amplitude, e.g.

$$\operatorname*{Res}_{p_f^2=0} \mathcal{A}_{2n}^{\mathrm{BI}_4\ \text{1-loop}} \left(1_\gamma^+ \dots (2n-1)_\gamma^+ \, 2n_\gamma^-\right) \tag{4.4.9}$$
$$= \mathcal{A}_{2n-2}^{\mathrm{BI}_4\ \text{1-loop}} \left(1_\gamma^+ \dots (2n-3)_\gamma^+ \ (p_f)_\gamma^+\right) \times \mathcal{A}_4^{\mathrm{BI}_4} \left((-p_f)_\gamma^- \ (2n-2)_\gamma^+ \, (2n-1)_\gamma^+ \, (2n)_\gamma^-\right).$$

There are no other poles of any kind in amplitudes in the self-dual and next-to-self-dual sectors. Therefore, if local counterterms are chosen to set all 1-loop self-dual amplitudes to zero, then the next-to-self-dual 1-loop amplitudes are also local and can therefore be removed by local finite counterterms as well. This means that there is no violation of EM duality at the 1-loop level in the self-dual and next-to-self-dual sectors.

4.5 Supersymmetric (D)BI and MHV Amplitudes at 1-loop

We present supersymmetric extensions of BI theory and derive the $U(1)$ EM duality charges of the states in the supermultiplets. We then use the result from Section 4.4 for the self-dual 1-loop integrand of pure BI theory to construct a conjectured expression for the MHV 1-loop integrand of $\mathcal{N}=4$ DBI using the dimension-shifting relation of [130]. This integrates to a local polynomial expression for the MHV 1-loop amplitudes in $\mathcal{N}=4$ DBI that agrees at $n=4,6$ with known results.

4.5.1 Supersymmetric Born-Infeld

Consider Born-Infeld theory supersymmetrically coupled to N_f Weyl fermions and N_s complex scalars. We have

$$\begin{aligned}
&\mathcal{N}=1\ \text{BI:} && N_f=1 \quad \text{and} \quad N_s=0\,,\\
&\mathcal{N}=2\ \text{DBI:} && N_f=2 \quad \text{and} \quad N_s=1\,,\\
&\mathcal{N}=4\ \text{DBI:} && N_f=4 \quad \text{and} \quad N_s=3\,.
\end{aligned} \tag{4.5.1}$$

The scalars of the $\mathcal{N}=2$ and $\mathcal{N}=4$ supermultiplets are Dirac-Born-Infeld scalars, hence the switch of name from supersymmetric BI theory to the more commonly used supersymmetric DBI.

As we have discussed, 4d pure non-supersymmetric BI has electromagnetic duality symmetry that acts as a $U(1)$ symmetry on the on-shell photon states. In supersymmetric BI, this becomes a $U(1)_R$ symmetry. Suppose the supercharge changes the charge by r, then if the highest weight state in the multiplet has helicity h and charge q

$$\begin{array}{lcccc}
\text{state:} & |h\rangle & |h-\tfrac{1}{2}\rangle & |h-1\rangle & \dots \\
U(1)_R: & q & q-r & q-2r & \dots
\end{array} \tag{4.5.2}$$

CPT conjugate states must have opposite charges. In particular, if the multiplet is CPT self-conjugate, as is the case for the $\mathcal{N} = 4$ vector multiplet, then we must have $-q = q - 4r$, i.e. $r = q/2$. With $q = 1$, as in pure BI, this fixes the $U(1)_R$ charges of the multiplet to be

$$\begin{array}{lccccc} \text{state:} & |1\rangle & |\tfrac{1}{2}\rangle & |0\rangle & |-\tfrac{1}{2}\rangle & |-1\rangle \\ U(1)_R: & 1 & \tfrac{1}{2} & 0 & -\tfrac{1}{2} & -1 \end{array} \tag{4.5.3}$$

which means that the $U(1)_R$-charges coincide with the helicity labels. In addition, the $\mathcal{N} = 4$ theory admits an $SU(4)_R$ symmetry under which the vectors are singlets; so the non-abelian R-symmetry is not an electromagnetic duality symmetry.

When applied to $\mathcal{N} = 4$ SYM, one can exclude the existence of a $U(1)_R$: the reason is that the cubic gluon interactions give rise to non-vanishing 3-particle amplitudes with helicities $++-$ and $--+$. This requires the vector charge, and hence all the other $U(1)_R$ charges, to vanish.[5]

In $\mathcal{N} = 4$ DBI, the $U(1)_R$ is allowed; there are for example no cubic interactions to forbid it. The existence of the $U(1)_R$ was noted in a CHY formulation of the $\mathcal{N} = 4$ DBI amplitudes in [80].

In the supersymmetric theories (4.5.1), the Ward identities associated with the conservation of the $U(1)_R$ charge are, for the special case of amplitudes with only external photons, exactly the same as (4.1.1). Since self-dual and next-to-self dual amplitudes vanish in any supersymmetric theory, independent of the existence of a duality symmetry, the simplest class of potentially non-trivially duality-violating amplitudes are therefore the MHV sector starting at 6-point. Hence we now turn to study the MHV amplitudes.

4.5.2 All-Multiplicity 1-loop MHV Amplitudes in $\mathcal{N} = 4$ DBI

In this section we present a conjecture for the all-multiplicity 1-loop integrand of the MHV sector of $\mathcal{N} = 4$ DBI in $d = 4-2\epsilon$. As we argue, the expression we write down follows from combining two well-known conjectures, the dimension-shifting relation between self-dual and maximally supersymmetric Yang-Mills [130], and the 1-loop version of the BCJ double-copy [35] applied to Born-Infeld models (4.1.10). At $n = 4$ and $n = 6$, where alternative explicit results are available for comparison, we find exact agreement.

It was conjectured in [130] that the 1-loop MHV integrand of $\mathcal{N} = 4$ SYM is related to the 1-loop self-dual integrand of pure YM theory as[6]:

$$\mathcal{I}_n^{\mathcal{N}=4\text{ SYM}}\big(1^+2^+\cdots i^-\cdots j^-\cdots n^+; l, \mu^2\big) = \frac{\langle ij\rangle^4}{2(\mu^2)^2}\mathcal{I}_n^{\text{YM}}\big(1^+2^+\cdots n^+; l, \mu^2\big). \tag{4.5.4}$$

[5] A parallel argument can be used to prove that there can be no $U(1)$ R-symmetry of $\mathcal{N} = 8$ supergravity due to the graviton 3-particle self-interactions. See Section 4.8.2.

[6] Since finishing this paper, a proof of this conjecture has been presented by Britto, Jehu, and Orta [131].

The relation was proven for $n \leq 6$ and evidence was provided for its validity at any multiplicity [130]. We can write the 1-loop integrand for self-dual Yang-Mills in BCJ form

$$\mathcal{I}_n^{\text{YM}}\big(1^+2^+\cdots n^+; l, \mu^2\big) = \sum_i \frac{c_i\, n_i^{\text{YM}}\big(1^+2^+\cdots n^+; l, \mu^2\big)}{d_i}, \tag{4.5.5}$$

where the sum over i is taken over all trivalent 1-loop graphs with c_i and d_i the corresponding color factors and denominators respectively. If the 1-loop BCJ conjecture is correct, then we can always find a so-called *generalized gauge* in which the numerators satisfy kinematic Jacobi relations [35]

$$n_i^{\text{YM}} + n_j^{\text{YM}} + n_k^{\text{YM}} = 0 \qquad \Longleftrightarrow \qquad c_i + c_j + c_k = 0. \tag{4.5.6}$$

If we assume that such numerators exist then we can define

$$n_i^{\mathcal{N}=4\,\text{SYM}}\big(1^+2^+\cdots i^-\cdots j^-\cdots n^+; l, \mu^2\big) \;\equiv\; \frac{\langle ij\rangle^4}{2(\mu^2)^2}\, n_i^{\text{YM}}\big(1^+2^+\cdots n^+; l, \mu^2\big). \tag{4.5.7}$$

Then, further assuming the dimension-shifting relation (4.5.4), it follows that

$$\mathcal{I}_n^{\mathcal{N}=4\,\text{SYM}}\big(1^+\cdots i^-\cdots j^-\cdots n^+; l, \mu^2\big) = \sum_i \frac{c_i\, n_i^{\mathcal{N}=4\,\text{SYM}}\big(1^+\cdots i^-\cdots j^-\cdots n^+; l, \mu^2\big)}{d_i}. \tag{4.5.8}$$

The objects (4.5.7) are BCJ numerators for $\mathcal{N}=4$ super Yang-Mills in some generalized gauge. Furthermore, since they are constructed by multiplying by an overall factor, these numerators must also satisfy the kinematic Jacobi relation (4.5.6). If the loop-level BCJ conjecture is correct then we can generate an expression for the MHV 1-loop integrand of $\mathcal{N}=4$ DBI by replacing the color factors c_i in (4.5.8) with BCJ numerators of χPT (in any generalized gauge). This gives the following relation

$$\mathcal{I}_n^{\mathcal{N}=4\,\text{DBI}}\big(1^+2^+\cdots i^-\cdots j^-\cdots n^+; l, \mu^2\big) \;=\; \frac{\langle ij\rangle^4}{2(\mu^2)^2}\, \mathcal{I}_n^{\text{BI}}\big(1^+2^+\cdots n^+; l, \mu^2\big). \tag{4.5.9}$$

Using the explicit all-multiplicity expression for the self-dual integrand (4.4.5), we then use (4.5.9) to conjecture the following all-multiplicity expression for the 1-loop integrand in the MHV sector of $\mathcal{N}=4$ DBI

$$\boxed{\begin{aligned}&\mathcal{I}_{2n}^{\mathcal{N}=4\,\text{DBI}}\left(1_\gamma^+ 2_\gamma^+\cdots i_\gamma^-\cdots j_\gamma^-\cdots 2n_\gamma^+; l, \mu^2\right)\\ &= \left(-\frac{1}{2}\right)^n \langle ij\rangle^4\,[12]^2\cdots[2n-1,2n]^2\,\frac{(\mu^2)^{n-2}}{\prod_{i=1}^n\left[\left(l-\sum_{j=1}^{2i}p_j\right)^2+\mu^2\right]} + \mathcal{P}\left(2,3,\cdots,2n\right).\end{aligned}} \tag{4.5.10}$$

The 1-loop double-copy construction was tested successfully at 4-point for the self-dual amplitude in Section 4.3.2. When the result (4.3.24) for $\mathcal{I}_4^{\mathrm{BI}}\left(1_\gamma^+ 2_\gamma^+ 3_\gamma^+ 4_\gamma^+; l, \mu^2\right)$ is applied in (4.5.9), we obtain the 4-point MHV 1-loop integrand

$$\mathcal{I}_4^{\mathcal{N}=4\,\mathrm{BI}}\left(1_\gamma^+ 2_\gamma^+ 3_\gamma^- 4_\gamma^-; l, \mu^2\right) = \frac{1}{2}[12]^2\langle 34\rangle^2\left(s^2\mathcal{I}_2[p_{12}] + t^2\mathcal{I}_2[p_{13}] + u^2\mathcal{I}_2[p_{14}]\right), \tag{4.5.11}$$

where $\mathcal{I}_2$ is a scalar bubble integrand, whose integral I_2 in $4-2\epsilon$ dimensions is given in (J.0.11). Thus in the small-ϵ expansion we find

$$\boxed{\mathcal{A}_4^{\text{1-loop}\,\mathcal{N}\,=\,4\text{ DBI}}(1_\gamma^+\, 2_\gamma^+\, 3_\gamma^-\, 4_\gamma^-) = \frac{1}{2}[12]^2\langle 34\rangle^2\Big[s^2 I_2(s) + t^2 I_2(t) + u^2 I_2(u)\Big] + \mathcal{O}(\epsilon)\,.} \tag{4.5.12}$$

The amplitude is UV divergent, and it is in fact the only MHV amplitude of (D)BI that has non-vanishing 4d cuts. Unitarity requires that these cuts factor into physical tree-amplitudes. Even though the complete integrand (4.5.10) is scheme-dependent, the values of these 4d cuts are not, and therefore give a non-trivial check on the proposal (4.5.9). In the following subsection we verify explicitly that the cut-constructible part of the 4-point MHV amplitude, constructed from the known tree-amplitudes, agrees exactly with (4.5.12).

For $n > 2$, i.e. for 6-point and higher, the integrand vanishes as $\mu \to 0$, hence it has vanishing 4d cuts. Using the integral (J.0.9) derived in Appendix J, we integrate (4.5.10) for $n > 2$ to find the rational local expression

$$\boxed{\begin{aligned}&\mathcal{A}_{2n}^{\text{1-loop}\,\mathcal{N}\,=\,4\text{ DBI}}\left(1_\gamma^+ 2_\gamma^+ \cdots i_\gamma^- \cdots j_\gamma^- \cdots 2n_\gamma^+\right)\\ &= \frac{i}{16\pi^2}\frac{(-1)^{n+1}\langle ij\rangle^4}{2^n(n-1)(n-2)}\left([12]^2\cdots[2n-1,2n]^2 + \mathcal{P}\left(2,3,\cdots,2n\right)\right) + \mathcal{O}(\epsilon).\end{aligned}} \tag{4.5.13}$$

In the following we compare the 4-point MHV 1-loop result (4.5.12) with the prediction from unitarity and discuss the associated divergence and infinite local counterterm. We also compare our prediction for the 6-point MHV 1-loop amplitude (4.5.13) with explicit results obtained in [123] using the dimensional reduction of M5-brane tree-amplitudes. We find complete agreement in both cases.

4.5.3 1-loop MHV in BI with $\mathcal{N}$-Fold SUSY and Counterterms

4-point. Consider a Born-Infeld model with N_v vectors coupled supersymmetrically to N_f Weyl fermions and N_s complex scalars. The 4-point MHV amplitude in this model has non-vanishing 4d cuts and it is therefore fairly straightforward to calculate from unitarity. We include the details for the calculation in Appendix K.0.4. The result is

$$\boxed{\begin{aligned}\mathcal{A}_4^{\text{1-loop}}(1_\gamma^+\,2_\gamma^+\,3_\gamma^-\,4_\gamma^-) &= [12]^2\langle 34\rangle^2\left[\frac{N_v}{2}s^2 I_2(s) + \Big(\frac{N_v}{5}+\frac{N_f}{20}+\frac{N_s}{30}\Big)\big(t^2 I_2(t)+u^2 I_2(u)\big)\right]\\ &= \frac{1}{\epsilon}\frac{i}{16\pi^2}[12]^2\langle 34\rangle^2\left[\frac{N_v}{2}s^2 + \Big(\frac{N_v}{5}+\frac{N_f}{20}+\frac{N_s}{30}\Big)\big(t^2+u^2\big)\right] + \mathcal{O}(1)\,.\end{aligned}} \tag{4.5.14}$$

For $N_v = 1$ and $N_f = N_s = 0$ we obtain the pure Born-Infeld MHV amplitude. The $\mathcal{N} = 1, 2, 4$ results are likewise obtained by setting $N_v = 1$ and using (4.5.1). In particular, $N_v = 1$, $N_f = 4$, and $N_s = 3$, reproduces the $\mathcal{N} = 4$ DBI result (4.5.12), a non-trivial test of the conjectured relation (4.5.9). The 4-point MHV 1-loop amplitude of $\mathcal{N} = 4$ DBI was calculated previously by Shmakova [95] with the same result (4.5.12).

In order to absorb the $1/\epsilon$ divergence in the 4-point MHV amplitudes, it follows from simple power-counting that we need a local counterterm of the form $\partial^4 F^4$. Little group scaling, Bose symmetry, and dimensional analysis show that there are two independent local matrix elements, so there are two independent $\partial^4 F^4$ operators on-shell. The general counterterm amplitude takes the form

$$\mathcal{A}_4^{\text{ct}}\big(1_\gamma^+\,2_\gamma^+\,3_\gamma^-\,4_\gamma^-\big) = a\,[12]^2\langle 34\rangle^2 s^2 + b\,[12]^2\langle 34\rangle^2(t^2+u^2)\,, \tag{4.5.15}$$

where a and b are constants. With particular choices of a and b, we can cancel the UV divergence for all choices of N_f and N_s.

Imposing $\mathcal{N} = 4$ supersymmetry, the matrix element (4.5.15) must satisfy the supersymmetry Ward identity

$$\mathcal{A}_4^{\text{ct}}\big(1_\gamma^+\,2_\gamma^+\,3_\gamma^-\,4_\gamma^-\big) = \frac{[12]^4}{[13]^4}\,\mathcal{A}_4^{\text{ct}}\big(1_\gamma^+\,2_\gamma^-\,3_\gamma^+\,4_\gamma^-\big)\,, \tag{4.5.16}$$

which requires

$$as^2 + bt^2 + bu^2 = bs^2 + at^2 + bu^2 \quad\Longrightarrow\quad (a-b)(s^2-t^2) = 0\,. \tag{4.5.17}$$

So this is possible only if $a = b$. In other words, there is only one $\partial^4 F^4$ counterterm compatible with $\mathcal{N} = 4$ supersymmetry. Thus, in $\mathcal{N} = 4$ DBI, the UV divergence must be proportional to $s^2 + t^2 + u^2$, exactly as it is in (4.5.12). The counterterms associated with (4.5.15) are easy to construct using spinorized fields (4.4.2) and external line Feynman rules (4.4.3). We find

$$\begin{aligned}\mathrm{Tr}(\partial_\mu F_+ \partial^\mu F_+)\mathrm{Tr}(\partial_\nu F_- \partial^\nu F_-) &\longrightarrow 4[12]^2\langle 34\rangle^2\, s^2\,,\\ \mathrm{Tr}(\partial_\mu F_+ \partial_\nu F_+)\mathrm{Tr}(\partial^\mu F_- \partial^\nu F_-) &\longrightarrow 2[12]^2\langle 34\rangle^2\,(t^2+u^2)\,,\end{aligned} \tag{4.5.18}$$

where the trace refers to the spinor indices, i.e. $\mathrm{Tr}(\partial_\mu F_+ \partial^\mu F_+) = \partial_\mu (F_+)_a{}^b \partial^\mu (F_+)_b{}^a$. Linear combinations of these two operators cancel the $1/\epsilon$ divergence in the 1-loop amplitude (4.5.14).

For the $\mathcal{N} = 4$ supersymmetric case, the counterterm takes a particularly recognizable form in terms of the 8-rank t_8-tensor known from the open string amplitude. Specifically,

$$\begin{aligned}
&(t_8)_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}\partial_\alpha F^{\mu_1\nu_1}\partial_\beta F^{\mu_2\nu_2}\partial^\alpha F^{\mu_3\nu_3}\partial^\beta F^{\mu_4\nu_4} \\
&\quad = 2\,\mathrm{Tr}(\partial_\mu F_+\partial^\mu F_+)\mathrm{Tr}(\partial_\nu F_-\partial^\nu F_-) + 4\mathrm{Tr}(\partial_\mu F_+\partial_\nu F_+)\mathrm{Tr}(\partial^\mu F_-\partial^\nu F_-) \\
&\quad \longrightarrow 8[12]^2\langle 34\rangle^2\,(s^2+t^2+u^2)\,,
\end{aligned} \tag{4.5.19}$$

giving the $\mathcal{N} = 4$ supersymmetric matrix element (4.5.12).[7]

6-point. The 6-point 1-loop MHV amplitude of $\mathcal{N} = 4$ DBI was recently calculated from explicit CHY formulae for M5-brane tree-amplitudes using two methods [123]. First, by dimensionally reducing the forward limit of 8-point M5-brane tree-amplitudes, and second, using generalized unitarity by imposing consistency with the M5-brane tree-amplitudes on 6d cuts. Their result agrees with (4.5.13). It was also noted in [123] that the general form of the kinematic polynomial in (4.5.13) is the only possible one compatible with the requirements of power counting, little group scaling, and supersymmetry. Our result (4.5.13) is the exact result for the 1-loop MHV amplitude of $\mathcal{N} = 4$ DBI, including its normalization, using the conjectured relation (4.5.9).

The self-dual and next-to-self-dual sectors vanish in the presence of any amount of supersymmetry, hence the MHV amplitudes present the first potentially duality-violating sector in $\mathcal{N} = 4$ DBI. For multiplicities beyond 4-point, the MHV result (4.5.13) has the important feature that it is completely local and therefore it *can* be removed by the addition of a finite local counterterm. Thus the MHV duality-violating $\mathcal{N} = 4$ DBI amplitudes can be set to zero, providing yet another piece of evidence that electromagnetic duality may be preserved at 1-loop.

4.6 Rational Loop Amplitudes and Finite Counterterms

Given the explicit results (3.4.15) and (3.4.39) for the SD and NSD duality-violating 1-loop amplitudes in pure BI theory and the MHV 1-loop amplitudes (4.5.13) of $\mathcal{N} = 4$ DBI, there is a clear motivation to attempt a general *proof* that duality violation at 1-loop is always removable by adding an appropriate set of finite local counterterms.

As we discuss in more detail below, at 1-loop, all duality-violating amplitudes are purely rational functions of spinor-helicity brackets. The problem of determining whether-or-not such rational functions are removable by adding higher-derivative (duality-violating) local operators to the

[7]The $(t_8)(\partial F)^4$ operator presented in [95] has a different index contraction that produces matrix elements with the wrong helicity structure.

classical action can be rephrased as the problem of whether their kinematic singularity structure resembles that of a tree-level scattering amplitude; we refer to such amplitudes as *tree-like*. Explicitly, we define tree-like to mean that the rational 1-loop amplitudes $\mathcal{A}_n$ have two important properties:

- *Factorization*: If it is possible to find lower multiplicity on-shell amplitudes $\mathcal{A}_{n_L}$ and $\mathcal{A}_{n_R}$ with $n_L + n_R = n + 2$ that can be glued together into an expression of the form

$$\sum_X \mathcal{A}_{n_L}(..., P_X)\mathcal{A}_{n_R}(-P_{\overline{X}}, ...), \tag{4.6.1}$$

  where we sum over all physical states X, $\overline{X}$ denotes a state with CP conjugate quantum numbers, and the remaining n external states coincide with those of $\mathcal{A}_n$, then $\mathcal{A}_n$ must contain a (simple) pole at $P_X^2 = 0$ with this expression as its residue.

- *Locality*: The rational amplitude $\mathcal{A}_n$ contains no additional *spurious* singularities.

These properties are guaranteed to hold for any expression constructed using Feynman rules derived from a local, Lorentz-invariant Lagrangian of a unitary model. This includes both the genuine tree-level approximation to S-matrix elements as well as the contributions from higher-derivative counterterms equivalent in Λ counting to 1-loop order in the EFT. As a consequence, if the purely rational 1-loop amplitudes are not tree-like, then no choice of finite local counterterms can cancel them. It is not *a priori* obvious what the singularity structure of rational loop amplitudes should be.

An illustrative example of a non-tree-like rational amplitude is provided by the 1-loop self-dual Yang-Mills amplitude at $n = 4$ [132]

$$\mathcal{A}_n^{\mathrm{YM}}\left[1_g^+\, 2_g^+\, 3_g^+\, 4_g^+\right] = -\frac{i}{96\pi^2}\frac{[12][34]}{\langle 12\rangle\langle 34\rangle}. \tag{4.6.2}$$

Such an expression has complex multi-collinear singularity when $|1\rangle \propto |2\rangle \propto |3\rangle \propto |4\rangle$ that does not correspond to a physical factorization singularity, and hence this expression is *non-tree-like*. The physical interpretation of this result is that such an amplitude is *scheme-independent*, and cannot be removed by the addition of local counterterms to the Yang-Mills action.

To make the problem explicit we briefly review the discussion of [133]. Consider the representation of the $d = 4 - 2\epsilon$ dimensional loop integrand constructed using Feynman rules. This is a sum over terms of the form

$$\frac{N[l, q_i]}{(l+q_1)^2(l+q_2)^2...(l+q_k)^2}, \tag{4.6.3}$$

where q_i are region momenta. We consider only 1PI contributions since singularities arising from propagators which do not contain l in the chopped off parts are not relevant for the discussion.

Hence the Feynman numerator $N[l]$ is a polynomial in both l and q_i. We split the numerator into two pieces

$$N[l] = N^{(0)}[l] + N^{(1)}[l]\,, \tag{4.6.4}$$

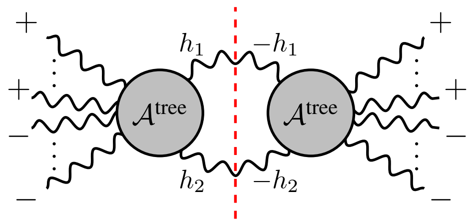
where $N^{(1)}[l] \to 0$ as $\epsilon \to 0$. The $N^{(1)}$ part may generate a contribution (which in [133] is called $\mathcal{R}_2$) from ϵ/ϵ cancellations against UV and IR divergences. Since this integrand is 1PI, the UV contributions from $N^{(1)}$ are always polynomial. The remaining contribution (called $\mathcal{R}_1$ in [133]) arises from the rational parts of the triangle and box master scalar integrals generated by tensor reduction of the $N^{(0)}$ part of the numerator. One perspective on this is given by old-fashioned Passarino-Veltman reduction [32]. Intermediate steps in the reduction of tensor integrals generates Gram determinant factors in the region momenta which in general contain spurious singularities. The general form of such rational contributions is quite complicated and not known explicitly in general. What is known is that they are not always tree-like, even in the case where the amplitudes are purely rational.

The pure BI results (3.4.15) and (3.4.39) and the MHV result (4.5.13) in $\mathcal{N} = 4$ DBI show that that these duality-violating 1-loop contributions are tree-like and can be set to zero by finite local counterterms. That expressions like (4.6.2) arise in the self-dual sector of Yang-Mills (and also Einstein gravity [134]) but not at 1-loop in Born-Infeld, as far as we know, could be taken as a hint that there is an essential difference between these models which is responsible for the absence of non-tree-like rational terms in the latter. A clue as to what this might be comes from the analysis in [135] of the factorization properties of QCD amplitudes. There, an argument was given (which assumed QCD-like power counting, though perhaps not in an essential way) that *IR finite* 1PI integrals do not generate spurious singularities. If such an argument can be extended to integrals with Born-Infeld-like power counting, then it would imply that all duality-violating 1-loop amplitudes are tree-like rational functions. Here the essential property that distinguishes Born-Infeld from Yang-Mills or Einstein Gravity is the absence of 3-particle interactions that generate Feynman integrals with soft or collinear IR divergences at 1-loop.

At present the statement:

no 3-point interactions + no 4d cuts $\Rightarrow$ tree-like rational amplitudes

remains a conjecture, but if proven it implies that all duality-violating 1-loop amplitudes in Born-Infeld can be completely cancelled by adding an appropriate set of finite local counterterms. To make the argument explicit, consider a 2-particle 4d cut of a 1-loop amplitude in Born-Infeld:



(4.6.5)

Here h_1 and h_2 are the helicities of the on-shell photons exchanged across the cut. Let the number of *external* positive and negative helicity photons on the LHS of the cut be $n_L^{\pm}$ and similarly on the RHS, $n_R^{\pm}$. Because each tree sub-amplitude satisfies the duality constraints (4.1.1), we have

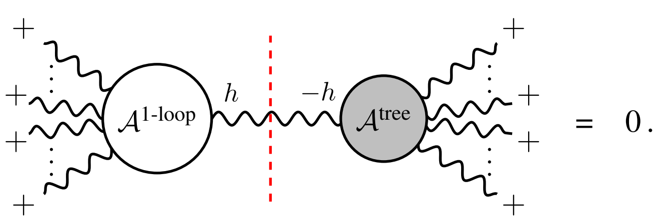
$$n_L^+ = n_L^- - h_1 - h_2\,, \quad \text{and} \quad n_R^+ = n_R^- + h_1 + h_2\,. \tag{4.6.6}$$

Hence for the overall amplitude

$$n^+ = n_L^+ + n_R^+ = n_L^- + n_R^- = n^-\,. \tag{4.6.7}$$

This means that the 4d cut can only be non-zero when the 1-loop amplitude obeys the constraint $n^+ = n^-$ of duality. Any triple and quadruple 4d cuts must necessarily obey the same constraint, since they are further restrictions of the 2-particle 4d cuts. Thus any duality-violating 1-loop amplitude of Born-Infeld has vanishing 4d cuts and therefore must be purely rational.

We now argue that any duality-violating 1-loop amplitude in BI theory can be set to zero by finite local counterterms *assuming* that this class of amplitudes have only standard factorizations into an on-shell tree amplitude and an on-shell 1-loop amplitude, i.e. that they are tree-like. Consider first the 1-loop self-dual amplitude. Any factorization channel has vanishing residue since the tree amplitude on the RHS is necessarily duality-violating and therefore vanishes:

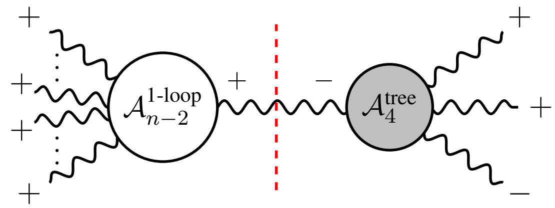


(4.6.8)

Because there are no factorization channels, the self-dual amplitude must be polynomial.

Given tree-level duality, the 1-loop amplitudes with only a single negative helicity photon $\mathcal{A}_n(-+\cdots+)$ factorize on simple poles into an $(n-2)$-point 1-loop self-dual amplitude $\mathcal{A}_{n-2}(+\cdots+)$

and a 4-point tree amplitude $\mathcal{A}_4(++--)$:



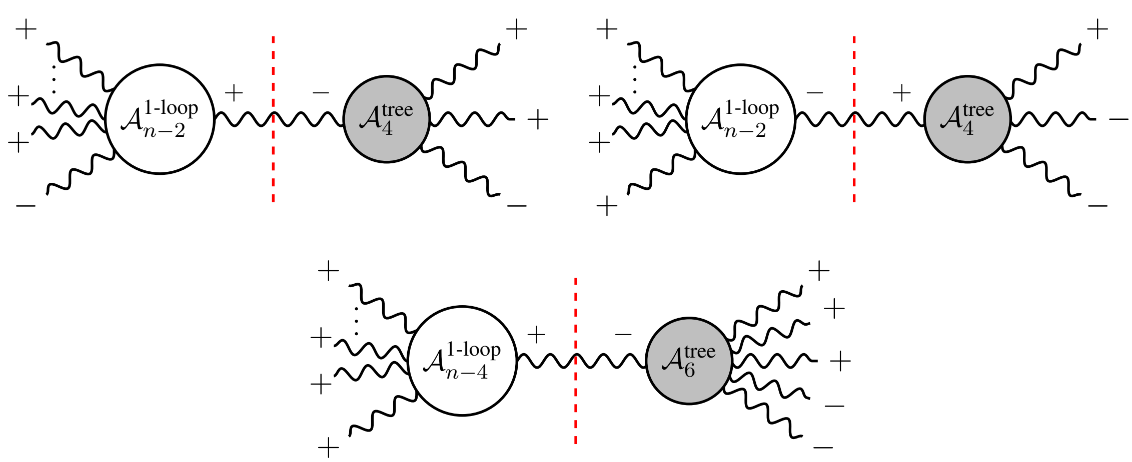
$$.\qquad (4.6.9)$$

The explicit expressions (3.4.15) and (3.4.39) for $\mathcal{A}_n^{\text{1-loop}}(++\cdots+)$ and $\mathcal{A}_n^{\text{1-loop}}(-+\cdots+)$ precisely have the above structure. In particular, the SD 1-loop amplitude (3.4.15) is polynomial and the NSD 1-loop amplitude (3.4.39) has precisely the factorization poles (4.6.9) plus polynomial terms.

This structure means that we can add local finite counterterms to the action so that the self-dual and next-to-self-dual 1-loop amplitudes are set to zero. Here and below, this means that the amplitudes vanish up to order $\mathcal{O}(\epsilon)$ in dimensional regularization. Henceforth, let us suppose this has been done, i.e.

$$\mathcal{A}_n^{\text{1-loop}}(++\cdots+) = \mathcal{O}(\epsilon) \quad \text{and} \quad \mathcal{A}_n^{\text{1-loop}}(-+\cdots+) = \mathcal{O}(\epsilon)\,. \qquad (4.6.10)$$

Consider now the MHV amplitudes $\mathcal{A}_n^{\text{1-loop}}(--+\cdots+)$. Electromagnetic duality of the BI tree amplitudes dictate that any factorization must involve either the self-dual or the next-to-self-dual 1-loop amplitudes; we write out the options explicitly



(4.6.11)

Since we have set the RHS 1-loop amplitudes to zero (4.6.10), there can be no contribution (at $\mathcal{O}(1)$) to the MHV 1-loop amplitude with $n > 4$. It must therefore be polynomial and we can set it to zero with the help of local finite counterterms, i.e. for $n > 4$

$$\mathcal{A}_n^{\text{1-loop}}(- - + \cdots +) = \mathcal{O}(\epsilon)\,. \tag{4.6.12}$$

The 4-point MHV amplitude was calculated explicitly in Section 4.5 and as it has non-vanishing 4d cuts, it is UV divergent.

It is clear that one can now proceed to check the factorization channels of the NMHV 1-loop amplitude and see that EM duality of the tree factor always requires the 1-loop sub-amplitude to be SD, NSD, or MHV. Since they vanish to order $\mathcal{O}(\epsilon)$, the NMHV 1-loop amplitude must be polynomial and we can proceed to set it to zero for $n > 6$,

$$\mathcal{A}_n^{\text{1-loop}}(- - - + \cdots +) = \mathcal{O}(\epsilon)\,. \tag{4.6.13}$$

For $n = 6$, this argument fails because the 1-loop NMHV amplitude has non-vanishing 4d cuts and hence it is not a rational function.

The argument extends in the obvious way to N^{k}MHV until the point where the duality-preserving amplitude with $n^+ = n^-$ is reached. The duality-conserving amplitudes have non-vanishing 4d cuts and the factorization argument no longer applies.

In the presence of any amount of supersymmetry, this argument continues to hold. In this case, (4.6.10) is modified to

$$\mathcal{A}_n^{\text{1-loop}}(+ + \cdots +) = 0 \quad \text{and} \quad \mathcal{A}_n^{\text{1-loop}}(- + \cdots +) = 0\,. \tag{4.6.14}$$

For the factorization of MHV amplitudes in (4.6.11), the RHS 1-loop amplitudes vanish again, this time to any order in ϵ by (4.6.14). Thus the supersymmetric MHV amplitude can be removed by adding a finite local counterterm. This prediction is explicitly verified by the MHV 1-loop amplitudes (4.5.13) with $n \geq 6$ in $\mathcal{N} = 4$ DBI: they are indeed tree-like and the duality-violation can be removed by finite local counterterms. The argument extends as before to N^{k}MHV until $k = \frac{n}{2} - 2$, which is the duality-preserving sector.

In conclusion, assuming that tree-like factorization of rational 1-loop amplitudes holds in BI theory, there exists a scheme (i.e. a set of finite local counterterms) in which any duality-violating 1-loop amplitudes vanish. If true, this means that EM duality can be preserved at 1-loop order. As discussed in the Introduction, from one perspective this is surprising, since EM duality is an on-shell symmetry of the equations of motion rather than a traditional off-shell symmetry of the (covariant) action.

Moreover, this analysis made no use of special properties of Born-Infeld beyond EM duality. It therefore applies to any 4d EM duality invariant model of nonlinear electrodynamics, such as the infinitely many models of the form (4.2.9) which additionally satisfy the Gaillard-Zumino condition (4.1.4).

4.7 Higher Derivative Corrections as a Double-Copy

The KLT relations [34] give closed-string tree amplitudes as sums of products of open-string tree amplitudes. In the limit of infinite string tension ($\alpha' \to 0$), these relations reduce to field theory KLT relations that express (super)gravity tree amplitudes as sums of products of tree amplitudes of two (not necessarily the same) gauge theories. In this section, we study the field theory KLT relations in the context of the double-copy (4.1.5) of Yang-Mills theory and χPT. In particular, we extend the double-copy relation to higher-derivative order with the purpose of examining the double-copy construction of the infinite and finite counterterms discussed in this chapter. We also compare our results with the string effective action.

4.7.1 KLT Double-Copy

The field theory KLT relation takes the form

$$\mathcal{A}_n^{A\otimes B} = \sum_{\alpha,\beta} \mathcal{A}_n^{A}[\alpha]\, S[\alpha|\beta] \widetilde{\mathcal{A}_n^{B}}\, [\beta]\,, \tag{4.7.1}$$

where A and B are theories with color-structure subject to constraints that we review below. The sum over α and β label sets of $(n-3)!$ independent color orderings for the partial amplitudes.[8] The KLT kernel $S[\alpha|\beta]$ is order $n-3$ in the Mandelstam variables in the field theory limit, but has an all-order in α' expression in string theory.

The amplitudes in the theories A and B must satisfy a number of non-trivial conditions for (4.7.1) to hold; not all theories with color-structure can be double-copied. First of all, the amplitudes in the right-hand side of (4.7.1) are color-ordered partial amplitudes. In the context of our discussion here, this means that the full tree amplitudes admit an expansion of the form

$$\mathcal{A}_n(12\ldots n) = \sum_{\sigma\in S_{n-1}} \mathrm{Tr}\left(t^{a_{\sigma_1}} t^{a_{\sigma_2}} \ldots t^{a_{\sigma_{n-1}}} t^{a_n}\right) \mathcal{A}_n[\sigma, n]\,, \tag{4.7.2}$$

where S_{n-1} is the symmetric group of order $n-1$. In addition, the partial amplitudes must satisfy the following constraints, which reduces the number of independent partial amplitudes to $(n-3)!$. These additional relations are

[8] We use square brackets for the arguments of a partial color-ordered amplitude and round brackets for the arguments of a full amplitude.

- Cyclicity,

$$\mathcal{A}_n[1\,2\ldots n] = \mathcal{A}_n[2\,3\ldots n\,1] = \mathcal{A}_n[3\,4\ldots n\,1\,2] = \ldots\,, \tag{4.7.3}$$

  as should be evident from the cyclicity of the trace of gauge group generators in (4.7.2).

- Kleiss-Kuijf (KK) relations [136],

$$\mathcal{A}_n[1\,\beta\,2\,\alpha] = (-1)^{|\beta|} \sum_{\sigma\in\alpha\Delta\beta^T} \mathcal{A}_n[1\,2\,\sigma]\,, \tag{4.7.4}$$

  where $|\beta|$ is the length of β and $\alpha\,\Delta\,\beta^T$ is the shuffle product of α and β in reverse order. The special case of α being the empty list is the reflection relations. When β has length 1, (4.7.4) simply gives the $U(1)$ decoupling identity.

- Fundamental Bern-Carrasco-Johansson (BCJ) identities [13],

$$\sum_{i=2}^{n-1} \left(\sum_{j=2}^{i} s_{jn} \right) \mathcal{A}_n[1\,2\ldots i, n, i+1\ldots n-1] = 0\,. \tag{4.7.5}$$

In the following, we restrict our study to 4-point amplitudes. For those the combined KK and BCJ relations give

$$\mathcal{A}_4[1234] = \frac{t}{u}\mathcal{A}_4[1243] = \frac{t}{s}\mathcal{A}_4[1324] = \frac{t}{u}\mathcal{A}_4[1342] = \frac{t}{s}\mathcal{A}_4[1423] = \mathcal{A}_4[1432]\,. \tag{4.7.6}$$

All other orderings are cyclic permutations of the ones given here. At 4-point, the explicit form of the field theory KLT relation (4.7.1) can be written

$$\mathcal{A}_4^{A\otimes B}(1234) \;=\; -\frac{1}{\Lambda^2} s\mathcal{A}_4^A[1234]\mathcal{A}_4^B[1243] \;=\; -\frac{1}{\Lambda^2}\frac{su}{t}\mathcal{A}_4^A[1234]\mathcal{A}_4^B[1234]\,, \tag{4.7.7}$$

using in the second step from the identities (4.7.6). Throughout this section we use the dimensionful scale Λ in place of the physical couplings $g_{\rm YM}$, λ and f_π, these can restored straightforwardly at the end of the calculation.

We now turn to the study of higher-derivative corrections to the 4-point amplitudes of BI theory from the double-copy.

4.7.2 Higher-Derivative Corrections to Born-Infeld

To extend the field theory KLT construction to include higher-derivative corrections, one must define what the double-copy means when higher-order terms are included. In a *top-down* approach, one uses the string theory prescription with α'-corrections to both the KLT kernel and the BCJ relations. In a bottom-up approach, one parametrizes all possible higher-derivative corrections to the KLT kernel and BCJ relations and subject them to consistency conditions. In either approach, We find that for the 4-point calculations presented here, the absence of the first sub-leading correction means that we can work with the uncorrected field theory relations (4.7.6) and (4.7.7) without any effect on the results presented here.[9]

With this setup, let us now consider the leading order higher-derivative corrections to the 4-point amplitudes of χPT and YM theory that satisfy the constraints of cyclicity, KK relations, and uncorrected BCJ relations (4.7.6). Additionally, we impose locality, unitarity, and the absence of higher-spin states in any factorization channels. The details are presented in Appendix M. We find:

- χPT (also obtained in [5]):

$$\mathcal{A}_4^{\chi\text{PT}}[1234] = \frac{1}{\Lambda^2} t \left(c_0 + \frac{c_4}{\Lambda^4} \left(s^2 + t^2 + u^2 \right) + \frac{c_6}{\Lambda^6} stu + \mathcal{O}(\Lambda^{-8}) \right) . \tag{4.7.8}$$

- YM

$$\mathcal{A}_4^{\text{YM}}[1^+2^+3^-4^-] = \frac{[12]^2\langle 34\rangle^2}{su} \left(\tilde{a}_0 + \frac{\tilde{a}_4}{\Lambda^4} tu + \frac{\tilde{a}_6}{\Lambda^6} stu + \mathcal{O}(\Lambda^{-8}) \right) , \tag{4.7.9}$$

$$\mathcal{A}_4^{\text{YM}}[1^+2^+3^+4^-] = \frac{1}{\Lambda^2} \frac{[12]^2[3|p_1|4\rangle^2}{su} \left(\tilde{b}_0 + \frac{\tilde{b}_6}{\Lambda^6} stu + \mathcal{O}(\Lambda^{-8}) \right) , \tag{4.7.10}$$

$$\begin{aligned} \mathcal{A}_4^{\text{YM}}[1^+2^+3^+4^+] = {} & \frac{\tilde{c}_2}{\Lambda^2} \frac{[12]^2[34]^2 s + [13]^2[24]^2 t + [14]^2[23]^2 u}{su} \\ & + \frac{\tilde{c}_6}{\Lambda^6} t \left([12]^2[34]^2 + [13]^2[24]^2 + [14]^2[23]^2 \right) + \mathcal{O}(\Lambda^{-8}) \, . \end{aligned} \tag{4.7.11}$$

The leading contributions in $\mathcal{A}_4^{\text{YM}}[1^+2^+3^+4^+]$ and $\mathcal{A}_4^{\text{YM}}[1^+2^+3^+4^-]$, as well as the sub-leading contribution of $\mathcal{A}_4^{\text{YM}}[1^+2^+3^-4^-]$ were also calculated in [137]. An important feature of these results is that contributions of order $1/\Lambda^4$ are absent both in the χPT amplitude (4.7.8) (where it is excluded by the BCJ constraints) and in the SD YM amplitude (4.7.11) (where the one BCJ permissible term at this order has a pole corresponding to the exchange of a massless spin-3 particle, hence we exclude it). Substituting the above results in the double-copy formula (4.7.7)

[9] At higher-orders in the derivative expansion, one needs corrected versions of the BCJ relations and KLT kernel.

gives the following results for the amplitudes of Born-Infeld,

$$\begin{aligned}\mathcal{A}_4^{\mathrm{BI}}(1^+2^+3^-4^-) &= -\frac{[12]^2\langle 34\rangle^2}{\Lambda^4}\bigg(\tilde{a}_0 + \frac{1}{\Lambda^4}\left(2\tilde{a}_0 c_4 s^2 + (\tilde{a}_4 - 2\tilde{a}_0 c_4)\, tu\right) \\ &\quad + \frac{1}{\Lambda^6}\left(\tilde{a}_6 + \tilde{a}_0 c_6\right) stu + \mathcal{O}(\Lambda^{-8})\bigg)\,,\end{aligned} \tag{4.7.12}$$

$$\mathcal{A}_4^{\mathrm{BI}}(1^+2^+3^+4^-) = -\frac{[12]^2[3|p_1|4\rangle^2}{\Lambda^6}\left(\tilde{b}_0 + \frac{\tilde{b}_0 c_4}{\Lambda^4}\left(s^2+t^2+u^2\right) + \frac{\tilde{b}_0 c_6 + \tilde{b}_6}{\Lambda^6} stu + \mathcal{O}(\Lambda^{-8})\right), \tag{4.7.13}$$

$$\begin{aligned}\mathcal{A}_4^{\mathrm{BI}}(1^+2^+3^+4^+) &= -\frac{\tilde{c}_2}{\Lambda^6}\left([12]^2[34]^2 s + [13]^2[24]^2 t + [14]^2[23]^2 u\right) \\ &\quad - \frac{\tilde{c}_6 + 3\tilde{c}_2 c_4}{\Lambda^{10}} stu\left([12]^2[34]^2 + [13]^2[24]^2 + [14]^2[23]^2\right) + \mathcal{O}(\Lambda^{-12})\,.\end{aligned} \tag{4.7.14}$$

At the leading order $\mathcal{O}(\Lambda^{-4})$, only the duality-conserving MHV amplitude (4.7.12) is non-zero and it matches the leading Born-Infeld amplitude if $\tilde{a}_0 = -1$. This illustrates the idea of symmetry enhancement in the double-copy; we discuss this further in Section 4.8.2.

The sub-leading contribution of the MHV amplitude (4.7.12) is at the same order, $1/\Lambda^8$, as the 1-loop result of (4.5.14). In fact, with the choice $c_4 = \frac{1}{32\pi^2\epsilon}\left(\frac{7N_v}{10} + \frac{N_f}{20} + \frac{N_s}{30}\right)$ and $\tilde{a}_4 = -\frac{1}{16\pi^2\epsilon}\left(\frac{3N_v}{10} - \frac{N_f}{20} - \frac{N_s}{30}\right)$ the two results match. This means that the infinite counterterm necessary for the cancellation of the 1-loop UV divergence of $\mathcal{A}_4(1^+2^+3^-4^-)$ can be obtained from a double-copy construction.

The amplitudes of (4.7.13) and (4.7.14) have leading-order contributions $\mathcal{O}(\Lambda^{-6})$, which is higher than the leading tree-level BI amplitude but lower than any possible 1-loop contribution at $\mathcal{O}(\Lambda^{-8})$. The SD 4-point amplitude (4.7.13) has no contribution at order $\mathcal{O}(\Lambda^{-8})$. This makes sense because there is no possible local finite operator at this order that can give rise to a NSD amplitude. This is also why the 1-loop NSD amplitude (4.4.1) vanishes.

It is very interesting that the $1/\Lambda^8$-term is missing from the SD amplitude (4.7.14). This stems from the lack of $1/\Lambda^4$ contributions in (4.7.8) and (4.7.11), as well as the lack of $1/\Lambda^4$ terms in the corrected BCJ and KLT relations. However, such a $1/\Lambda^8$-term is needed to restore EM duality by cancelling the non-zero result of the SD 1-loop 4-point amplitude (4.3.26). Thus, the finite quartic counterterm, required to restore electromagnetic duality at 1-loop level, *cannot* be obtained from a double-copy construction. We comment further on the potential implications of this result in Section 4.8.1.

Finally, let us briefly comment on 6-point. In the context of $\mathcal{N} = 4$ DBI, the 6-point 1-loop MHV amplitude of the n-point result (4.5.13) is polynomial and can be cancelled by a finite local counterterm that we have explicitly constructed using the KLT double-copy with higher-derivative corrections. This means that some counterterms needed to restore electromagnetic duality can be constructed via the double-copy with higher-derivatives while others cannot.

4.7.3 Comparison with the String Theory Effective Action

BI theory is the leading field-strength-dependent part of the open string effective action [90]. Higher-derivative corrections to this action have been obtained by considering the action at finite α', both in case of the bosonic open string [90] and the superstring [128]. We now compare these results, with our construction of higher-derivative corrections to the BI model via the KLT product in Section 4.7.2. This is done with the identification $\Lambda^{-2} = 2\pi\alpha'$.

Bosonic open string. The leading results at 4-point in the duality-violating sector, (4.7.13) and (4.7.14), with choice of the Wilson coefficients, e.g.

$$\tilde{b}_0 = \frac{1}{2} \qquad \text{and} \qquad \tilde{c}_2 = -\frac{1}{3} \tag{4.7.15}$$

agree with the bosonic open string action of [90].

For the MHV amplitude, the KLT construction gives no $1/\Lambda^6$ term, so the leading order correction is order $1/\Lambda^8$. This is consistent with the open string effective action [90].

Abelian Z-theory. Bosonic open string amplitudes have been constructed via the KLT double-copy of abelian Z-theory and Yang-Mills with certain higher-derivative corrections [138]. In order for this to be consistent with our construction in the previous section, the 4-point amplitude in χPT with higher derivative corrections (4.7.8) must reduce to the abelian Z-theory result [89]. This is indeed the case upon choosing

$$c_0 = -\frac{1}{2} \qquad c_4 = -\frac{1}{192} \qquad c_6 = \frac{3\zeta_3}{16\pi^3}\,. \tag{4.7.16}$$

Yang-Mills. Similarly, the 4-point Yang-Mills amplitudes (4.7.9), (4.7.10) and (4.7.11) are found to agree with the corrected Yang-Mills amplitudes used in [138] when

$$\tilde{a}_0 = -1 \qquad \tilde{a}_4 = -1 \qquad \tilde{b}_0 = 1 \qquad \tilde{b}_6 = 0 \qquad \tilde{c}_2 = -\frac{2}{3} \qquad \tilde{c}_6 = -1\,. \tag{4.7.17}$$

Thus the results in Section 4.7.2 reduce to the KLT construction of bosonic string amplitudes in [138] with a specific choice of free parameters.

Superstring. The SD and NSD sectors vanish in any supersymmetric context, so at 4-point we only have the MHV sector to compare with. Furthermore, supersymmetry constrains the YM amplitude (4.7.9), such that $\tilde{a}_4 = 0$. This term in the amplitude comes from the F^3 correction of YM, which is only allowed in the absence of supersymmetry. At the leading orders, the Born-Infeld MHV amplitude of (4.7.12) agrees with the superstring MHV amplitude [128] for the choice

$$\tilde{a}_0 = -\frac{1}{2} \qquad\qquad c_4 = \frac{1}{96}\,. \tag{4.7.18}$$

There is no contribution at order $\alpha'^3 \sim \Lambda^{-6}$.

In [89, 139], it was shown that the superstring amplitudes can be calculated as the KLT product of Yang-Mills theory and Z-theory. This too can be mapped onto the construction in Section 4.7.2 with a particular choice of Wilson coefficients.

4.8 Discussion

In this chapter we have discussed various aspects the physics of the hidden electromagnetic duality symmetry of D3-brane worldvolume effective field theories. In particular, our focus has been on analyzing the consequences of such a symmetry for the physically observable S-matrix elements at sub-leading order in the EFT expansion. These sub-leading contributions are of two kinds: first, loop-level contributions from massless degrees-of-freedom present in the IR and, second, higher-derivative tree-level, or α' corrections, from integrating out massive states in the UV completion. Our work represents a first investigation of electromagnetic duality symmetry in this context and there are many avenues for further exploration. Here we first outline a number of open questions and then comment on similar duality symmetries in supergravity, both with respect to the double-copy and ideas of oxidation of symmetries from 3d to 4d.

4.8.1 Open Questions

Loop-Level BCJ Double-Copy. In Section 4.3.2 we presented the first explicit example of a loop-level BCJ double-copy for a non-gravitational model. Using known color-kinematics duality satisfying numerators for self-dual Yang-Mills at 4-point, together with simple but non-color-kinematics duality manifesting numerators of χPT, we found that the result precisely matches the known self-dual Born-Infeld amplitude (3.4.15) at all orders in the ϵ-expansion. Since the loop-level double-copy [35] remains a conjecture, this successful matching can be taken as evidence that it extends to 1-loop in non-gravitational examples such as BI $=$ YM $\otimes$ χPT. It would be useful to have further examples, beyond 4-point and beyond 1-loop. Some of the relevant color-kinematics duality satisfying self-dual Yang-Mills numerators are known [140] but are quite complicated. A potentially simpler approach would be to construct color-kinematics duality satisfying numerators for χPT and form a double-copy with a simpler BCJ representation of the

Yang-Mills amplitudes.

1-Loop Dimension-Shifting Relation. Using the mysterious dimension-shifting relation between 1-loop integrands of self-dual Yang-Mills and the MHV sector of $\mathcal{N}=4$ super Yang-Mills together with the loop-level BCJ double-copy we have conjectured a representation of the integrand for the MHV sector of 1-loop $\mathcal{N}=4$ DBI (4.5.10) at all multiplicities. At $n=4$ the UV divergent result matches the physical amplitude we obtained from 4d cuts (and previously calculated in [95]) and for $n=6$ it agrees exactly with expressions recently derived from the dimensional reduction of M5-brane tree amplitudes in the forward limit [123]. At higher-multiplicity our expression remains a conjecture, and it is important to verify its validity. With only even-point amplitudes and its lack of IR divergences at 1-loop, it is even possible that this simpler example could provide insight into the mechanism behind the dimension-shifting relations.

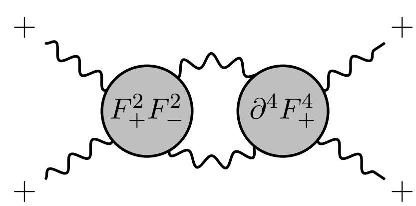
IR Behavior and Tree-Like 1-loop Amplitudes. In Section 4.6 we presented a conjecture, motivated by the analysis of 1-loop amplitudes in QCD [135], that the absence of IR divergent Feynman integrals (as a consequence of the absence of 3-particle interactions) implies that purely rational 1-loop amplitudes in Born-Infeld are "tree-like" (in the sense defined in Section 4.6). As we showed in Section 4.6, if this conjecture is true, then in a 4d theory with classical electromagnetic duality, the 1-loop rational duality-violating amplitudes can *always* be cancelled by adding finite local counterterms. Our explicit all-multiplicity results for the 1-loop SD and NSD amplitudes of pure BI theory and the MHV sector of $\mathcal{N}=4$ DBI are evidence of the conjecture. If this limited conjecture is proven, an understanding of the structure of duality-violating amplitudes at 2-loops and beyond remains lacking. The status of electromagnetic duality symmetries of interacting quantum field theories at all-orders of perturbation theory is generally unknown.

Color-Kinematics vs. Electromagnetic Duality? Finally, in Section 4.7.2 we have constructed the leading higher-derivative operators of Yang-Mills and χPT compatible with the tree-level KLT product. Contrary to the leading-order result, we find that generic double-copy constructible, higher-derivative operators do not conserve the duality charge. This is not surprising, duality invariance of the double-copy is not manifest at leading-order and is one of many examples of an unexpected *symmetry enhancement* through the double-copy. It would be interesting if there was some better understanding of which Yang-Mills higher-derivative corrections lead to enhanced symmetries and which do not, and if there was some formulation of the double-copy that made this feature manifest.

In some sense the result of the higher-derivative double-copy analysis is the worst of both worlds. The double-copy does not automatically generate duality satisfying amplitudes beyond leading-order, but neither does it generate all possible higher-derivative corrections to Born-Infeld. In particular, the matrix element corresponding to the finite local counterterm (4.4.4) needed to restore duality invariance at 4-point in the self-dual sector of pure Born-Infeld cannot be generated. This suggests a potential tension between electromagnetic duality and color-kinematics duality at loop-level.

The central problem is whether or not there exists a regularization scheme that is compatible with both notions of *duality*. At present, the only known explicit examples of the loop-level BCJ double-copy make use of dimensional regularization and it is unclear if this is a strict requirement. Since electromagnetic duality is only a symmetry in exactly 4d, we would expect that it is broken in a generic dimensional scheme. If the duality symmetry is non-anomalous at loop-level then there are two logical possibilities: first, that there exists a *special* dimensional regularization scheme which preserves the symmetry or second, that electromagnetic duality must be restored by including additional finite counterterm contributions, some of which may not be obtained via a double copy. Since there is presently no known duality preserving dimensional regularization scheme we will turn to an analysis of finite counterterms produced via the double-copy.

For example, if we were to calculate a 2-loop amplitude, the 4-point self-dual amplitude must receive contributions from diagrams with an effective 1-loop topology and a single insertion of the counterterm (4.4.4) of the form:



(4.8.1)

If the loop-level BCJ double-copy generates the complete loop amplitude it must include such contributions to the integrand. Here we find a problem. On any of the non-vanishing 4d cuts of this integrand, one half of the cut must be the local matrix element of the counterterm (4.4.4) that we know cannot be generated as a tree-level BCJ double-copy. It is not clear that the existence of color-kinematics duality satisfying loop-level numerators strictly implies the existence of color-kinematics duality satisfying tree-level numerators on each cut. So this observation can at most be taken as indicative of a potential obstruction, rather than a firm argument. If such *counterterm diagrams* cannot be generated by loop-level BCJ double-copy, and must be added to the amplitude by hand, then the advantages of the loop-level double-copy as a calculational tool are diminished. The calculation of the complete amplitude at L-loops requires a non-double-copy construction of an $L-1$-loop integrand with a counterterm insertion.

Conceptually this suggests two distinct definitions of Born-Infeld at the quantum level. One which is defined by the BCJ double-copy at all-loops but violates electromagnetic duality beginning at 1-loop, and another which is not a BCJ double-copy beyond tree-level but satisfies the selection rule (4.1.1) at all loop orders. At this stage such a dichotomy is just speculation with some supporting evidence. We do not even know if the double-copy construction is generally valid at higher-loops in Born-Infeld, or if we can continue to restore duality symmetry by adding further counterterms. As a further note, let us point out that the 6-point MHV amplitude $\mathcal{N} = 4$ DBI can be set to zero by a finite local counterterm that we can in fact produce in a KLT-construction with higher-derivative corrections. It is not known if supersymmetry plays a role in this context. These questions clearly deserve further detailed study.

4.8.2 Supergravity: Double-Copy and Symmetry Oxidation

Supergravity with extended supersymmetry is an important class of models in which duality symmetries play a significant role. We review briefly how non-abelian R-symmetry in pure supergravity theories can be understood as duality symmetry, how it emerges as a symmetry enhancement in the double-copy of gauge theories, the partial breaking of duality-symmetry by higher-derivative terms, and we comment on the proposed oxidation of 3d symmetries to 4d supergravity.

Duality Symmetry in 4d Supergravity. The 4d super Poincaré algebra with $\mathcal{N}$ supercharges admits a maximal R-symmetry extension locally of the form $SU(\mathcal{N})_R \times U(1)_R$, which may or may not be realized in a given interacting model. In 4d extended supergravity, the graviton supermultiplet contains massless vector bosons, graviphotons $|\gamma^{\pm}\rangle$. The positive helicity states $|\gamma^{+}\rangle$ transform in an $\binom{\mathcal{N}}{2}$ dimensional representation of $SU(\mathcal{N})_R$ while the negative helicity states $|\gamma^{-}\rangle$ transform in the complex conjugate representation. Even if the full $SU(\mathcal{N})_R$ symmetry is realized in observables, it may not be manifest in the action. This depends on whether the $SU(\mathcal{N})_R$ representation is real or complex. In the former case it may lift to an off-shell symmetry acting on the field strength tensors. In the latter case, only the restriction to a real representation of a subgroup can possibly be realized off-shell and the remaining symmetries are seen only as duality invariance of the equations of motion. Explicitly, for pure 4d supergravities, the graviphotons are in real representations for $\mathcal{N} = 2, 4$ and complex representations for $\mathcal{N} = 5, 6, 8$.

Due to its self-interactions, the graviton cannot carry $U(1)_R$ charge. Therefore if q is the $U(1)_R$ charge of the supercharges $\mathcal{Q}$, a state schematically of the form $\mathcal{Q}^k|h^{+2}\rangle$ has $U(1)_R$ charge kq. By CPT, the charges of the negative helicity multiplet must have the opposite signs of those for the positive helicity multiplet. If q is non-zero, the graviphoton must be charged under the $U(1)_R$, but such a symmetry is not possible off-shell at the level of the action. Hence, if present, the $U(1)_R$ must act in pure supergravity as a duality symmetry.

$\mathcal{N} = 8$ supergravity has a single CPT self-conjugate multiplet. The $U(1)_R$ charge of the state $|h^{-2}\rangle = \mathcal{Q}^8|h^2\rangle$ is $8q$, and since the graviton must be uncharged it must be that $q = 0$. Hence, $\mathcal{N} = 8$ supergravity cannot have a $U(1)_R$ duality symmetry and its maximal R-symmetry group is $SU(8)_R$. As the graviphotons transform non-trivially under $SU(8)_R$, only an $SO(8)_R$ subgroup can be realized off-shell and the rest of the group acts as a non-abelian duality symmetry.

Duality Symmetry in the Double-Copy. The S-matrix of $\mathcal{N}=8$ supergravity can be obtained as the double-copy

$$\big(\mathcal{N}=8 \text{ supergravity}\big) = \big(\mathcal{N}=4 \text{ super Yang-Mills}\big) \otimes \big(\mathcal{N}=4 \text{ super Yang-Mills}\big)\,. \tag{4.8.2}$$

In $\mathcal{N}=4$ super Yang-Mills, the vectors (gluons) are uncharged, so the $SU(4)_R$ is *not* an electromagnetic duality symmetry, but a regular global symmetry realized off-shell. In the double-copy (4.8.2), the LHS directly inherits the $SU(4)_R \times SU(4)_R$ R-symmetry under which the graviphotons of the $\mathcal{N}=8$ supermultiplet are charged. The KLT relations enhance $SU(4)_R \times SU(4)_R$ to the $SU(8)$ R-symmetry of the $\mathcal{N}=8$ supergravity tree amplitudes.

The 70 scalars of $\mathcal{N}=8$ supergravity are Goldstone modes of the spontaneous breaking of $E_{7(7)}$ to $SU(8)_R$. There are no states in the $\mathcal{N}=4$ SYM spectrum that have vanishing soft limits, and generically the individual terms in the KLT relations have $\mathcal{O}(1)$ soft limits. However, in the KLT sum, a cancellation takes place to ensure the vanishing soft limits of the Goldstone bosons. This is an example of a more general pattern: the double-copy of two states with soft behavior σ_1 and σ_2 respectively results in a state with soft behavior $\sigma \geq \sigma_1 + \sigma_2 + 1$. This illustrates another type of symmetry enhancement in the double-copy.

Generic higher-derivative corrections to $\mathcal{N}=8$ supergravity need not respect the $SU(8)_R$. In tree-level string theory on T^6, the presence of the dilaton in the α'-corrections breaks the $SU(8)_R$ to $SU(4)_R \times SU(4)_R$, for example through operators such as $\alpha'^3 e^{-6\phi} R^4$. The $SU(4)_R \times SU(4)_R$ *is* a global symmetry of the tree-level closed string amplitudes with exclusively massless external states; open string amplitudes with 4d massless $\mathcal{N}=4$ states have $SU(4)_R$ symmetry and the closed string tree amplitudes inherit $SU(4)_R \times SU(4)_R$ via KLT [72].[10] Note that this group with its 30 generators is larger than the 28-dimensional $SO(8)_R \subset SU(8)_R$ that can be realized off-shell.

The existence of the $SU(4)_R \times SU(4)_R$ global symmetry in the 4d closed string tree amplitudes has an interesting origin. The 4d α'-corrections explicitly break $E_{7(7)}$ to the $SO(6,6)$ of the T^6. The spontaneous breaking of $SO(6,6)$ to $SU(4)_R \times SU(4)_R$ produces 36 Goldstone bosons and these are exactly the scalars that in the double-copy are arise from the 6×6 SYM scalars in the 4d massless spectrum of the open string. For these 36 Goldstone scalars, the enhancement of the soft limits from $\mathcal{O}(1)$ to vanishing takes place in KLT, but it does not happen for the dilaton and axion (obtained from the double-copy of opposite helicity gluons) or the remaining 32 scalars (from the double-copy of opposite helicity gluinos). This has been verified by explicit calculations [72].

[10] The global symmetry $SU(4)_R \times SU(4)_R$ is absent in the higher-genus superstring amplitudes (as expected in a theory of quantum gravity).

The two types of double-copy symmetry enhancements discussed here are in direct parallel with those studied in this chapter for Born-Infeld theory. In the double-copy (4.1.5) of $\mathcal{N} = 4$ DBI from $\mathcal{N} = 4$ SYM times χPT, the $U(1)_R$ electromagnetic duality symmetry is emergent and so are the enhanced $\mathcal{O}(p^2)$ and $\mathcal{O}(|p\rangle)$ soft limits of the DBI scalars and Akulov-Volkov fermions respectively. That the double-copy provides these symmetry enhancements may appear like magic but given that the low-energy theories in these cases have these symmetries, it is also just a necessary condition for the double-copy to work at all. With regard to higher-derivative corrections, we have seen that the double-copy generically gives duality-violating operators, however, the particular operator needed to restore duality invariance at 4-point 1-loop order in BI theory cannot be produced by the double-copy.

There are other cases of emergent symmetries in double-copy constructions. In the double-copy

$$\big(\text{gravity} \oplus \text{dilaton} \oplus \text{axion}\big) = \text{YM} \otimes \text{YM} \tag{4.8.3}$$

the dilaton has an emergent $\mathbb{Z}_2$ *dilaton parity*, which has no analogue in Yang-Mills. From one perspective this can be seen as an inherited property of the $SU(8)_R$ symmetry in the truncation of $\mathcal{N} = 8$ supergravity to gravity plus the dilaton-axion.

A quite interesting case is that of $\mathcal{N} = 4$ supergravity:

$$(\mathcal{N} = 4 \text{ supergravity}) = (\mathcal{N} = 4 \text{ SYM}) \otimes \text{YM}\,. \tag{4.8.4}$$

The two real scalars of $\mathcal{N} = 4$ supergravity should be thought of as a dilaton-axion pair. They live in the $SU(1,1)/U(1)_R$ coset and as such they are the two Goldstone bosons of the breaking of $SU(1,1)$ to $U(1)_R$. The $U(1)_R$ emerges as a classical electromagnetic duality symmetry in the double-copy construction. At loop-level, the $U(1)_R$ was said to be anomalous [141], however, it was recently shown that the $U(1)_R$-violation can be removed by finite local counterterms [118], thus actually restoring the $U(1)_R$. This is very relevant for the study of the UV structure of $\mathcal{N} = 4$ supergravity. There is a clear parallel to the possible removal of the BI and super-DBI electromagnetic duality violations at 1-loop level studied in this chapter.

For completeness, let us note that some recent discussions of S-duality in the context of the double-copy can be found in [142, 143].

Dimensional Oxidation in Supergravity. In Section 4.2 we proved that the 3d $U(1)$ symmetry of the 3d M2-brane theory oxidizes to electromagnetic $U(1)$ symmetry of the 4d D3-brane. It is interesting to consider the analogue of dimensional oxidation — or absence thereof — in extended supergravity.

In this context the notion of dimensional oxidation has a longer history [144–146]. As noted above, $\mathcal{N}=8$ supergravity in 4d has a sector of 70 scalars forming a sigma model on the coset space $E_{7(7)}/SU(8)_R$, with the additional 58 bosonic degrees-of-freedom transforming in linear representations of $SU(8)_R$. When dimensionally reduced to $d=3$, all 128 bosonic degrees-of-freedom become sigma model scalars parametrizing the coset space $E_{8(8)}/SO(16)$ [144]. There have recently appeared constructions, based on light-cone superspace, claiming that this enhanced symmetry oxidizes to $d=4$ at the level of the action [147]. Here we focus on a particular $U(1)$ subgroup and its manifestation in the physical S-matrix. The linearly realized $SO(16)$ in $d=3$ is rank 8, while the analogous linearly realized $SU(8)$ in $d=4$ is rank 7. There must therefore be an additional, conserved, additive charge generated by dimensional reduction. Indeed, this has been demonstrated explicitly using the CHY construction of the tree-level S-matrix of maximal supergravity; only amplitudes in the helicity conserving sector are non-vanishing when the external momenta are restricted to a 3d subspace [148]. This would-be $U(1)$ symmetry would enhance the $SU(8)$ R-symmetry in $d=4$ to $U(8)$, analogously to the way the duality symmetry of the D3-brane enhances the $SU(4)$ to $U(4)$. But this symmetry is clearly broken (helicity non-conserving amplitudes in $d=4$ are generically non-vanishing), so why do we have oxidation in one case and not the other? In other words, why does the recursive argument given in Section 4.2 fail for $\mathcal{N}=8$ supergravity?

Certainly the recursive part of the argument remains valid, n-point tree-level graviton scattering amplitudes scale as z^{2-2n} under a *generic* holomorphic all-line shift. The failure is in the base case of the induction. The lowest multiplicity amplitudes in gravity are 3-point amplitudes of the form

$$\mathcal{A}_3^{\mathcal{N}=8\text{ SUGRA}}\left(1_h^{+2}\,2_h^{+2}\,3_h^{-2}\right), \tag{4.8.5}$$

which are non-zero despite being in the helicity non-conserving sector. Unlike the duality-violating 4-point amplitudes (4.2.11), such an amplitude vanishes when all of the momenta are restricted to a 3d subspace [31]. If the would-be $U(1)$ symmetry (together with the rest of the $E_{8(8)}$ symmetry) does indeed oxidize to $d=4$ at the level of the action, then its implications for the physical S-matrix must be more subtle than the strong form of oxidation demonstrated for the D3-brane in Section 4.2.

CHAPTER 5

Constraints on a Massive Double-Copy and Applications to Massive Gravity

5.1 A Massive BCJ Double Copy

The essence of the *double-copy* is the existence (or conjectured existence) of a map from the physical observables $\mathcal{O}$ of a pair of models A and B, each with a non-Abelian internal symmetry structure, to physical observables in some other model $A \otimes B$, without such a symmetry

$$\mathcal{O}_A \times \mathcal{O}_B \mapsto \mathcal{O}_{A\otimes B}. \tag{5.1.1}$$

The original and best-studied example of such a map is given by the construction of Kawai-Lewellen-Tye (KLT), relating tree-level open and closed string scattering amplitudes [34], and the associated field theory limit ($\alpha' \to 0$) relating Yang-Mills and Einstein gravity. For example at 4-point

$$\mathcal{M}_4^{\text{Grav}}(1,2,3,4) = -s_{14}\mathcal{A}_4^{\text{YM}}[1,2,3,4]\,\mathcal{A}_4^{\text{YM}}[1,3,2,4]\,. \tag{5.1.2}$$

The double-copy has subsequently been extended to non-Abelian gauge theories with matter fields in non-adjoint representations [149, 150], to non-linear sigma models and D-brane world-volume EFTs [85], generalized to loop-level [35] and even extended to classes of classical solutions [151]. See [152] and references therein for a comprehensive review of recent developments. More than a theoretical curiosity, there are often significant practical advantages to making use of such a map whenever it is available. Recent use of a generalized double-copy construction for Feynman integrands in $\mathcal{N} = 8$ supergravity allowed the first explicit calculation of 4-point, 5-loop scattering amplitudes [108], a feat that is practically impossible to replicate by other, presently available means.

It is therefore a timely and relevant theoretical problem to understand the potential scope for generalizing the double-copy, and demarcating the boundary between those models which admit a double-copy structure and those which do not. In this chapter we will be concerned with the problem of generalizing the field theory double-copy relation for tree-level scattering amplitudes to models with massive particles in the spectrum. Our central result is the demonstration that, when massive particles are present, *color-kinematics duality is not enough* to guarantee a physically well-defined double-copy. We present in detail an explicit example, massive Yang-Mills, for which color-kinematics duality satisfying numerators exist (up to at least $n = 5$, where n is the number of external partucles), but for which the BCJ double-copy prescription generates expressions with non-physical spurious singularities.

To understand the generalization we propose in this chapter, it is useful to first review the well-known construction of the double-copy for tree-level scattering amplitudes in pure Yang-Mills first described by Bern, Carrasco and Johansson (BCJ) [13]. We begin by organizing tree-level scattering amplitudes as a sum over trivalent graphs[1]

$$\mathcal{A}_4\left(1^{a_1},2^{a_2},3^{a_3},4^{a_4}\right) = \frac{c_{12}n_{12}}{s_{12}} + \frac{c_{13}n_{13}}{s_{13}} + \frac{c_{14}n_{14}}{s_{14}}. \tag{5.1.3}$$

This form of the amplitude reveals the remarkable, hidden property of *color-kinematics duality*, the numerators satisfy a sum rule

$$n_{12} + n_{13} + n_{14} = 0, \tag{5.1.4}$$

mirroring the Jacobi relation of the color factors

$$c_{12} + c_{13} + c_{14} = 0. \tag{5.1.5}$$

Perhaps even more remarkably, making the replacement $c_i \to n_i$ gives an expression

$$\mathcal{M}_4\left(1,2,3,4\right) = \frac{n_{12}^2}{s_{12}} + \frac{n_{13}^2}{s_{13}} + \frac{n_{14}^2}{s_{14}}, \tag{5.1.6}$$

which coincides with a scattering amplitude in a model of Einstein gravity coupled to a massless dilaton and Kalb-Ramond two-form. At higher multiplicity ($n \geq 5$) there will be multiple, independent color Jacobi relations, corresponding to various different choices of triples of trivalent graphs with related topology, the generalization of (5.1.4) being that the (signed) sum of the numerators of these triples must vanish. The fundamental result of BCJ was to prove this *BCJ*

[1] We will use the following convention $c_{12} = f^{a_1a_2b}f^{a_3a_4b}$, $c_{13} = f^{a_1a_3b}f^{a_4a_2b}$ and $c_{14} = f^{a_1a_4b}f^{a_2a_3b}$. We also use Mandelstam invariants with all outgoing momenta, i.e. $s_{ij} = (p_i + p_j)^2$.

double-copy and the property of color-kinematics duality persist at all multiplicity [13].

The BCJ numerators given by the trivalent organization of the amplitude are non-unique; as a consequence of the color Jacobi relation (5.1.5), the amplitude (5.1.3) is unchanged by a so-called *generalized gauge transformation*. For example for $n = 4$

$$n_{12} \to n_{12} + s_{12}\Delta, \quad n_{13} \to n_{13} + s_{13}\Delta, \quad n_{14} \to n_{14} + s_{14}\Delta, \tag{5.1.7}$$

where Δ is an arbitrary function of the Mandelstam invariants. The BCJ numerators given by constructing the amplitude using standard Feynman rules, in general, do not satisfy the kinematic Jacobi relations. The challenge in applying the double-copy is to find an appropriate set of generalized gauge transformations which produce numerators which do satisfy the kinematic Jacobi relation. However, there is no *a priori* guarantee that such a generalized gauge can be found. This can be illustrated in the simplest case at $n = 4$, suppose the Feynman-rule constructed numerators satisfy

$$n_{12} + n_{13} + n_{14} = \mathcal{E}, \tag{5.1.8}$$

for some $\mathcal{E}$. Then making a generalized gauge transformation

$$n_{12} + n_{13} + n_{14} \to n_{12} + n_{13} + n_{14} + \Delta\left(s_{12} + s_{13} + s_{14}\right) = \mathcal{E}, \tag{5.1.9}$$

where we have used the kinematic identity $s_{12} + s_{13} + s_{14} = 0$. We conclude that if $\mathcal{E} \neq 0$, then there exists an obstruction to finding a generalized gauge in which the numerators satisfy the kinematic Jacobi relation (5.1.4). As we will review in Section 5.2, at all multiplicities such obstructions are absent only if the color-ordered partial amplitudes of the model satisfy an infinite set of (generalized) gauge-invariant identities known as the *BCJ relations*. Color-kinematics duality is therefore a special property enjoyed by some models and not others.

This statement can be clearly illustrated in the context of an explicit example, first described in [153]. Consider a model of a $U(N)$ Yang-Mills theory coupled to a massless, adjoint, Majorana fermion in d-dimensions. For $n = 4$ scattering with four external fermions we find

$$\mathcal{E} \propto (\gamma_\mu)_{a_1a_2}(\gamma^\mu)_{a_3a_4} + (\gamma_\mu)_{a_2a_3}(\gamma^\mu)_{a_1a_4} + (\gamma_\mu)_{a_3a_1}(\gamma^\mu)_{a_2a_4}, \tag{5.1.10}$$

where γ_μ form some representation of the d-dimensional Clifford algebra. This expression is zero only in dimensions $d = 3, 4, 6$ and 10, and therefore only in those dimensions does the model described satisfy color-kinematics duality. Said another way, in $d \neq 3, 4, 6$ or 10 the scattering amplitudes are perfectly physical, but there is an obstruction to finding a generalized gauge in which the BCJ numerators satisfy the kinematic Jacobi relation, and consequently there is no well-defined notion of a double-copy.

While beautifully simple, it is not at all obvious that expressions like (5.1.6), and more importantly its generalizations to higher multiplicity, are actually physical scattering amplitudes. In particular, this construction fails to manifest *locality* in the form of the absence of spurious, non-propagator-like, singularities and the factorization of amplitudes on propagator-like, physical singularities. For $n \geq 5$ the generalized gauge functions needed to bring the local form of BCJ numerators generated by Feynman rules, to a color-kinematics duality satisfying representation, can in principle be arbitrarily complicated, non-local functions. There is an indirect argument that the result of the double-copy should be an expression with the locality properties of a scattering amplitude. Here we must make two additional assumptions about the color structure: the gauge group is $U(N)$ and all of the external states are in the adjoint representation[2]. This covers pure Yang-Mills and its supersymmetrizations, but excludes other known examples of the double-copy, such as QCD-like models with matter fields in the fundamental representation [149, 150]. Throughout this chapter we will always make these assumptions, leaving possible generalizations to future work. As shown explicitly in [13] it is possible to prove that, with these additional assumptions, the BCJ double-copy is equivalent to the KLT double-copy. By making a convenient choice of the basis of partial amplitudes in the KLT sum, this form of the double-copy manifests the absence of spurious singularities[3].

Clearly however, the BCJ form of the double-copy (5.1.3) would manifest locality if we could find a generalized gauge in which all of the numerators are simultaneously local functions. While it may be an empirical fact that among the diverse range of color-kinematics duality compatible models, such local numerators can often be found, we are not aware of a general argument that this should always be possible. The existence of local numerators is then possibly a stronger assumption than color-kinematics duality, but at least for those models which admit a KLT representation of the double-copy, it is also an unnecessary assumption. The proof of the equivalence of the BCJ and KLT double-copies requires only that the numerators satisfy the kinematic Jacobi, and makes no assumption about the locality structure thereof; if duality satisfying numerators can only be found in a non-local form then this just means that any spurious singularities must cancel in the sum over trivalent graph contributions. In the context of the familiar massless double-copy we conclude that, in addition to the usual S-matrix axioms of *locality*, *unitarity (factorization)*, *Lorentz invariance*, as well as the assumption that the model has the required color or flavor symmetry structure to admit a KLT form of the double-copy (5.1.3), the property of color-kinematics duality is a necessary and sufficient condition for the double-copy to be a physical scattering amplitude. One of the main results of this chapter is an explicit demonstration that when, in addition

[2]One can also apply this argument to $SU(N)$ gauge groups, but here we need the additional assumption that there are no multi-trace contributions to tree-level scattering amplitudes. For simplicity, for the remainder of the chapter we will assume that the gauge group is $U(N)$.

[3]Equation (5.1.2) illustrates the main idea. The only possible non-physical singularity that could appear in this expression is a possible doubling of the $s_{14} = 0$ pole, but this is clearly removed by a corresponding zero in the KLT kernel.

to the above assumptions, massive states are present in the spectrum, color-kinematics duality is no longer a sufficient condition to avoid spurious, non-physical, singularities in the double-copy.

The BCJ construction has a natural extension to models containing massive states, for which various special cases have been considered previously [150, 154–163]. To our knowledge, no completely general description of a massive BCJ double-copy, and the associated constraints, has been given. In particular, the case of double-copying amplitudes in theories with no massless particles has not been studied before. This chapter is a first step towards such a description, and an exploration of the various problems that may arise.

The direct analogue of the BCJ form of the amplitude for models with a uniform, non-zero mass spectrum is[4]

$$\mathcal{A}_4^{m\neq 0}\left(1^{a_1},2^{a_2},3^{a_3},4^{a_4}\right)=\frac{c_{12}n_{12}}{s_{12}+m^2}+\frac{c_{13}n_{13}}{s_{13}+m^2}+\frac{c_{14}n_{14}}{s_{14}+m^2}. \tag{5.1.11}$$

To construct the *massive double-copy* of such a model, we will follow closely the discussion above, and try to construct numerators which satisfy the kinematic Jacobi relation $n_{12}+n_{13}+n_{14}=0$. If we succeed, we make the replacement $c_i\to n_i$ and construct the would-be massive double-copy

$$\mathcal{M}_4^{m\neq 0}\left(1,2,3,4\right)=\frac{n_{12}^2}{s_{12}+m^2}+\frac{n_{13}^2}{s_{13}+m^2}+\frac{n_{14}^2}{s_{14}+m^2}. \tag{5.1.12}$$

The central problem in this chapter will be to understand the conditions under which expressions such as (5.1.12) and its natural generalization to higher multiplicity, define physical scattering amplitudes. At this point we make a simple observation: if we suppose that a BCJ representation of our massive model is constructed, perhaps using Feynman rules, with numerators satisfying

$$n_{12}+n_{13}+n_{14}=\mathcal{E}, \tag{5.1.13}$$

then by making the following generalized gauge transformation

$$\begin{aligned}
n_{12}\to\hat{n}_{12}&\equiv n_{12}+\frac{1}{m^2}(s_{12}+m^2)\mathcal{E}\\
n_{13}\to\hat{n}_{13}&\equiv n_{13}+\frac{1}{m^2}(s_{13}+m^2)\mathcal{E}\\
n_{14}\to\hat{n}_{14}&\equiv n_{14}+\frac{1}{m^2}(s_{14}+m^2)\mathcal{E},
\end{aligned} \tag{5.1.14}$$

the amplitude (5.1.11) is invariant but the transformed numerators satisfy

$$\hat{n}_{12}+\hat{n}_{13}+\hat{n}_{14}=0. \tag{5.1.15}$$

[4] Throughout this chapter we will use the mostly-plus metric convention $\eta_{\mu\nu}=\text{diag}\left(-1,+1,+1,+1\right)$.

This generalized gauge transformation is well-defined for all $m \neq 0$ and all $\mathcal{E}$, so we can *always* find a generalized gauge that realizes color-kinematics duality! Since this argument relied only on knowledge of the spectrum, it applies independently of the details of the interactions. Contrary to the $m = 0$ case where color-kinematics duality was a special property only found in a subset of models, usually with various special constraints on the spectrum of states and the associated interactions, for $m \neq 0$ it is no constraint at all. As we will see, this situation is indeed too good to be true. In Section 5.2 we will rewrite the would-be double-copy (5.1.12) in a KLT-like form with a kernel given by the inverse of a matrix of massive bi-adjoint scalar amplitudes [14, 164], we find that: (*i*) the double-copy generically introduces non-local, spurious singularities for $n \geq 5$, and (*ii*) for $n = 4$, gives a physical scattering amplitude, but fails to reduce to the standard double-copy in an appropriate $m \to 0$ decoupling limit.

These two major conclusions are not quite on equal footing. The result (*i*) is fatal for any would-be double-copy with $n \geq 5$, it means that the result of applying the proposed massive generalization of the BCJ double-copy is an expression that could not have been calculated as a tree amplitude of a local quantum field theory. In Section 5.4 we will extend our analysis to allow for a more complicated spectrum of states with possibly different masses, and provide evidence that if, in addition to the assumptions enumerated above, the masses satisfy a certain quadratic constraint then the problems with violations of locality are removed. The result (*ii*) is interesting, but does not mean that the massive double-copy for $n = 4$ is non-physical. That something dramatic happens as $m \to 0$ could have been anticipated from the fact that the generalized gauge transformation (5.1.14) is singular in this limit. In the double-copied expression these inverse powers of mass will appear as coupling constants multiplying certain higher-derivative interactions that diverge as $m \to 0$. There is nothing illegal about this, indeed as we review in Appendix N, interactions involving massive particles with spin ≥ 1 generically diverge in the massless limit as some inverse power of the mass. In such cases a non-singular massless limit may be defined as an appropriate double-scaling or *decoupling* limit in which the coupling constants of the model are chosen to vanish with an appropriate positive power of the mass. Result (*ii*) can then be more accurately stated as the observation that the decoupling limit does not *commute* with the massive double-copy. Interestingly, under the additional constraints on the spectrum postulated in Section 5.4 to ameliorate the non-locality in $n \geq 5$ particle scattering, we find that the decoupling limits and the massive double-copy do commute.

As a theoretical laboratory for making explicit calculations, we consider the physically motivated example of a model of massive Yang-Mills. As we explain in detail in Section 5.3.1, by considering the reduction to the familiar massless double-copy in the high-energy or Goldstone boson equivalence limit, there is a plausible expectation that massive Yang-Mills double copies to a model of de Rham-Gabadadze-Tolley or *dRGT massive gravity* [15] coupled to a massive dilaton and a massive two-form. The primary conclusion of the analysis of this example is that no miraculous cancellation of the spurious singularities takes place, and the proposed massive double-copy fails to generate physical scattering amplitudes for $n \geq 5$. We will conclude by revisiting the logic of the above argument in Section 5.4 and demonstrate that if, in addition to color-kinematics duality, the spectrum of masses satisfies certain constraints, then a local massive double-copy does indeed exist.

While this work was in its final stages, the preprint [165] by Momeni, Rumbutis and Tolley appeared with some overlapping results at 4-point level.

5.2 Massive KLT Formula

Under the assumptions outlined in the Introduction ($U(N)$ symmetry, external states in the adjoint representation, color-kinematics duality and the usual S-matrix axioms) it is possible to rewrite the BCJ double-copy as a KLT formula. In Section 5.2.1 we show how this well-known argument can be extended to the proposed massive double-copy. This provides us with a different representation of the would-be double-copy in which the analysis of the singularity structure is more transparent. In Section 5.2.2 we show that, term-by-term, the massive KLT sum contains spurious singularities and argue that a miraculous cancellation would need to take place for the final expression to contain only physical singularities.

5.2.1 Equivalence of Massive BCJ and Massive KLT

For any model with $U(N)$ symmetry, with asymptotic states in the adjoint representation, there exists a convenient decomposition of tree-amplitudes into *single trace* or *color-ordered partial* amplitudes of the form

$$\mathcal{A}_n\left(1^{a_1},...,n^{a_n}\right) = \sum_{\sigma \in S_{n-1}} \mathrm{Tr}\left[T^{a_1}T^{a_{\sigma(2)}}...T^{a_{\sigma(n)}}\right]\mathcal{A}_n\left[1\ \sigma(2)...\sigma(n)\right]. \tag{5.2.1}$$

Without any further assumptions, the resulting $(n-1)!$ partial amplitudes are generically independent.

The existence of a BCJ representation of the form (5.1.3) requires us to make the somewhat artificial assumption that the only color tensors which appear in vertex functions are contractions of the $U(N)$ structure constants f^{abc}, as is the case for example in Yang-Mills. Various generalizations of the BCJ double-copy relaxing this assumption have been considered in the literature [152], but in this chapter we will analyze only this simple Yang-Mills-like case.

Assuming that such a BCJ representation exists, then the number of linearly independent partial amplitudes can be shown to be reduced to *at least* $(n-2)!$. This reduction is accomplished by an additional set of linear constraints, known as the *Kleiss-Kuijf (KK) relations* [136]. If we further assume that the model satisfies color-kinematics duality, then by following the kinematic Jacobi analogue of the construction of the Dixon-Del Duca-Maltoni (DDM) basis [166], the number of linearly independent BCJ numerators is likewise seen to be $(n-2)!$.

If both representations exist, then since there are equal numbers of BCJ numerators and partial amplitudes, we should be able to translate between them by a linear transformation. For example, for $n=4$ for some arbitrary choice of numerators and partial amplitudes

$$\begin{pmatrix} \mathcal{A}_4[1234] \\ \mathcal{A}_4[1324] \end{pmatrix} = \begin{pmatrix} \frac{1}{s_{12}+m^2}+\frac{1}{s_{14}+m^2} & \frac{1}{s_{14}+m^2} \\ -\frac{1}{s_{14}+m^2} & -\frac{1}{s_{13}+m^2}-\frac{1}{s_{14}+m^2} \end{pmatrix} \begin{pmatrix} n_{12} \\ n_{13} \end{pmatrix}. \tag{5.2.2}$$

If $m \neq 0$, then the propagator matrix has full-rank and so we can *solve* for the kinematic Jacobi-satisfying numerators

$$\begin{pmatrix} n_{12} \\ n_{13} \end{pmatrix} = \begin{pmatrix} \frac{1}{s_{12}+m^2}+\frac{1}{s_{14}+m^2} & \frac{1}{s_{14}+m^2} \\ -\frac{1}{s_{14}+m^2} & -\frac{1}{s_{13}+m^2}-\frac{1}{s_{14}+m^2} \end{pmatrix}^{-1} \begin{pmatrix} \mathcal{A}_4[1234] \\ \mathcal{A}_4[1324] \end{pmatrix}. \tag{5.2.3}$$

When $m=0$ however, the propagator matrix has rank 1, and no such inversion is possible. In this case, the massless propagator matrix has a null-vector, and so we can make the replacement

$$\begin{pmatrix} n_{12} \\ n_{13} \end{pmatrix} \to \begin{pmatrix} \hat{n}_{12} \\ \hat{n}_{13} \end{pmatrix} = \begin{pmatrix} n_{12} \\ n_{13} \end{pmatrix} + \Delta \begin{pmatrix} s_{12} \\ s_{13} \end{pmatrix}, \tag{5.2.4}$$

for any function Δ. The existence of such null-vectors is indicative of an important difference between the massive and massless cases. For $m \neq 0$ the construction of numerators satisfying the kinematic Jacobi relations requires a *complete* fixing of the generalized gauge freedom. For $m=0$, this requires only a partial fixing. We can use this residual freedom to impose the gauge-fixing conditions $\hat{n}_{13}=0$, and solve for $\hat{n}_{12}$. From (5.2.2) with $m=0$, we have two different expressions for $\hat{n}_{12}$ which must be equal, leading to the so-called *fundamental BCJ identity*

$$s_{12}\mathcal{A}_4\,[1234] = s_{13}\mathcal{A}_4[1324]. \tag{5.2.5}$$

In general, the BCJ identities reduce the number of linearly independent partial amplitudes to $(n-3)!$ [14,87]. We can run this argument in both directions, reaching the well-known conclusion that the fundamental BCJ relations are *necessary* and *sufficient* conditions for the existence of color-kinematics duality satisfying BCJ numerators. In the massive case, due to the absence any residual generalized gauge freedom, there is no analogue of these identities. This is another way of saying that the constraint of color-kinematics duality is trivialized for models with a uniform massive spectrum.

This analysis generalizes naturally to n-point. For a model with uniform mass spectrum and $m \neq 0$ there is a linear relation of the form (5.2.2) relating the $(n-2)!$ DDM bases of partial amplitudes and the kinematic numerators with an $(n-2)! \times (n-2)!$ propagator matrix. We believe that this matrix is always of full-rank, but do not have a proof of this fact. An explicit expression for the 6×6 massive propagator matrix at $n = 5$ is given in Appendix O, from which the rank can be verified to be 6.

We will now proceed to derive a KLT form of the double-copy for the $m \neq 0$ case. We first rewrite the BCJ double-copy (5.1.11) in matrix form

$$\mathcal{A}_4^{A\otimes B}(1,2,3,4) = \begin{pmatrix} n_{12}^A & n_{13}^A \end{pmatrix} \begin{pmatrix} \frac{1}{s_{12}+m^2} + \frac{1}{s_{14}+m^2} & -\frac{1}{s_{14}+m^2} \\ -\frac{1}{s_{14}+m^2} & \frac{1}{s_{13}+m^2} + \frac{1}{s_{14}+m^2} \end{pmatrix} \begin{pmatrix} n_{12}^B \\ n_{13}^B \end{pmatrix}, \tag{5.2.6}$$

where we have already used the assumed kinematic Jacobi relations to express $n_{14} = -n_{12} - n_{13}$. Combining this with our solution for the numerators (5.2.3) gives

$$\begin{aligned} &\mathcal{A}_4^{A\otimes B}(1,2,3,4) \\ &\quad = \begin{pmatrix} \mathcal{A}_4^A[1234] & \mathcal{A}_4^A[1324] \end{pmatrix} \begin{pmatrix} \frac{1}{s_{12}+m^2} + \frac{1}{s_{14}+m^2} & -\frac{1}{s_{14}+m^2} \\ -\frac{1}{s_{14}+m^2} & \frac{1}{s_{13}+m^2} + \frac{1}{s_{14}+m^2} \end{pmatrix}^{-1} \begin{pmatrix} \mathcal{A}_4^B[1234] \\ \mathcal{A}_4^B[1324] \end{pmatrix}, \end{aligned} \tag{5.2.7}$$

which is of KLT form, with the matrix in the middle acting as a *massive KLT kernel*.

A similar calculation can be performed at 5-point, both to calculate the 6 independent BCJ numerators from a DDM basis of 6 partial amplitudes and to calculate the 5-point KLT kernel. The details of this calculation are presented in Appendix O.

While we are in principle finished, to illustrate the robustness of this proposed generalization of the double-copy, we will now derive the same formula through a different line of argument. Somewhat recently, the massless KLT kernel was understood to be the inverse of a $(n-3)! \times (n-3)!$ matrix of tree-level scattering amplitudes of the following $U(N) \times U(\tilde{N})$ invariant model of massless scalars transforming in the bi-adjoint representation [14, 164]

$$\mathcal{L} = -\frac{1}{2}\left(\partial_\mu \phi^{aa'}\right)^2 - g f^{abc} \tilde{f}^{a'b'c'} \phi^{aa'} \phi^{bb'} \phi^{cc'}. \tag{5.2.8}$$

These amplitudes admit a *double* color-ordering

$$\mathcal{A}_n^{\phi^3}\left(1^{a_1 a_1'}, ..., n^{a_n a_n'}\right) = \sum_{\alpha,\beta \in S_{n-1}} \mathrm{Tr}\left[T^{a_1} T^{a_{\alpha(2)}} ... T^{a_{\alpha(n)}}\right] \mathrm{Tr}\left[\tilde{T}^{a_1'} \tilde{T}^{a'_{\beta(2)}} ... \tilde{T}^{a'_{\beta(n)}}\right] \mathcal{A}_n^{\phi^3}[\alpha|\beta]. \tag{5.2.9}$$

The partial amplitudes $\mathcal{A}_n^{\phi^3}[\alpha|\beta]$ are indexed by two orderings and can be constructed efficiently via a simple diagrammatic procedure [164]. Regarding $\mathcal{A}_n^{\phi^3}[\alpha|\beta]$ as an $(n-1)! \times (n-1)!$

matrix of all possible orderings not related by a cyclic permutation, it can be shown to have rank $(n-3)!$ [14]. The null vectors correspond to separate row and column KK and BCJ relations. For example at 4-point

$$s_{12}\mathcal{A}_4^{\phi^3}[1234|1234] = s_{13}\mathcal{A}_4^{\phi^3}[1324|1234]. \tag{5.2.10}$$

The central result of [14] was to prove that a BCJ-independent $(n-3)! \times (n-3)!$ sub-matrix has full-rank, and moreover has an inverse which is precisely equal to the KLT kernel in the given BCJ basis. The massless KLT formula can then be succinctly formulated as

$$\mathcal{A}_n^{A\otimes B}(1,2,\cdots,n) = \sum_{\alpha,\beta} \mathcal{A}_n^A[\alpha] \, \left(\mathcal{A}^{\phi^3}\right)^{-1}[\alpha|\beta]\mathcal{A}_n^B[\beta], \tag{5.2.11}$$

where α and β range over, possibly distinct, BCJ bases of orderings of length $(n-3)!$. This suggests a second, *a priori* independent, massive generalization of the KLT formula. Let us now investigate what happens in a *massive bi-adjoint scalar theory*

$$\mathcal{L} = -\frac{1}{2}\left(\partial_\mu \phi^{aa'}\right)^2 - \frac{1}{2}m^2\phi^{aa'}\phi^{aa'} - gf^{abc}\tilde{f}^{a'b'c'}\phi^{aa'}\phi^{bb'}\phi^{cc'}. \tag{5.2.12}$$

Amplitudes in the massive theory are constructed using the same diagrammatic rules used for the massless theory [164], but with the massless propagators replaced with their massive counterparts. For example,

$$\mathcal{A}_4^{\phi^3}[1234|1234] = \frac{1}{s_{12}+m^2} + \frac{1}{s_{14}+m^2}, \tag{5.2.13}$$

$$\mathcal{A}_4^{\phi^3}[1234|1324] = -\frac{1}{s_{14}+m^2}. \tag{5.2.14}$$

The 5-point matrix of bi-adjoint scalar amplitudes can be found in Appendix O. The primary difference between the massless and massive bi-adjoint scalar amplitudes is in the number of independent color-orderings. In the massive theory, DDM orderings are independent and the $(n-2)! \times (n-2)!$ matrix of bi-adjoint scalar amplitudes has full-rank. Since this matrix is invertible, there is a natural conjecture for a massive KLT formula. At 4-point this takes the explicit form

$$
\begin{aligned}
&\mathcal{A}_4^{A\otimes B}(1,2,3,4) \\
&= \begin{pmatrix} \mathcal{A}_4^A[1234] & \mathcal{A}_4^A[1324] \end{pmatrix} \begin{pmatrix} \mathcal{A}_4^{\phi^3}[1234|1234] & \mathcal{A}_4^{\phi^3}[1234|1324] \\ \mathcal{A}_4^{\phi^3}[1234|1324] & \mathcal{A}_4^{\phi^3}[1324|1324] \end{pmatrix}^{-1} \begin{pmatrix} \mathcal{A}_4^B[1234] \\ \mathcal{A}_4^B[1324] \end{pmatrix} \\
&= \frac{1}{m^2}\mathcal{A}_4^A[1234]\left(m^2+s_{12}\right)\left(\mathcal{A}_4^B[1234]\left(2m^2+s_{12}\right)-\mathcal{A}_4^B[1324]\left(m^2+s_{13}\right)\right) \\
&\quad + \frac{1}{m^2}\mathcal{A}_4^A[1324]\left(m^2+s_{13}\right)\left(-\mathcal{A}_4^B[1234]\left(m^2+s_{12}\right)+\mathcal{A}_4^B[1324]\left(2m^2+s_{13}\right)\right). \qquad (5.2.15)
\end{aligned}
$$

Remarkably, this formula coincides exactly with the one we arrived at from the massive BCJ double-copy in (5.2.7).

Proceeding to 5-point, the explicit comparison of KLT and BCJ forms of the double-copy can be repeated using the results of Appendix O. We find that, again, the KLT kernel from the massive BCJ double-copy is precisely the inverse of massive bi-adjoint scalar amplitudes.

Generalizing this result to n-particle scattering, the KLT formulation of the massive double-copy takes the form

$$
\mathcal{A}_n^{A\otimes B}(1,2,\cdots,n) = \sum_{\alpha,\beta} \mathcal{A}_n^A[\alpha]\ \left(\mathcal{A}_n^{\phi^3}\right)^{-1}[\alpha|\beta]\ \mathcal{A}_n^B[\beta], \qquad (5.2.16)
$$

where α and β now range over all $(n-2)!$ DDM color orderings and $\mathcal{A}_n^{\phi^3}[\alpha|\beta]$ is a matrix of amplitudes of massive bi-adjoint scalar theory.

We will now close this subsection with a brief discussion about the relation between the massless KLT formula (5.2.11) and the $m\to 0$ limit of the new massive KLT formula (5.2.16). Before doing so there is a subtlety in this discussion we should address. For generic massive field theories, in particular those containing particles with spin ≥ 1, the naive massless limit, with $m\to 0$ and all couplings held fixed, may not exist. As reviewed in detail in Appendix N, for *any* model, a regular massless limit may be defined as an appropriate double-scaling or *decoupling limit*. Throughout this chapter, this is simply referred to as the *massless limit*.

Expanding the kernel of the $n=4$ formula around the $m=0$ limit gives

$$
\begin{aligned}
\mathcal{A}_4^{A\otimes B}(1,2,3,4) =& \frac{1}{m^2}\left(s_{12}\mathcal{A}_4^A[1234]-s_{13}\mathcal{A}_4^A[1324]\right)\left(s_{12}\mathcal{A}_4^B[1234]-s_{13}\mathcal{A}_4^B[1324]\right) \\
&+\left(3s_{12}\mathcal{A}_4^A[1234]\,\mathcal{A}_4^B[1234]+3s_{13}\mathcal{A}_4^A[1324]\,\mathcal{A}_4^B[1324]\right. \\
&\left.+s_{14}\left(\mathcal{A}_4^A[1234]\,\mathcal{A}_4^B[1324]+\mathcal{A}_4^A[1324]\,\mathcal{A}_4^B[1234]\right)\right)+\mathcal{O}(m^2). \qquad (5.2.17)
\end{aligned}
$$

The coefficient of the leading $\mathcal{O}(m^{-2})$ term is recognizable as a product of factors that would vanish if the models A and B were massless and satisfied the fundamental BCJ relations. If we take the massless limit of models A and B, which is finite as $m \to 0$ by assumption, then this term in the KLT formula is divergent. There are then two logical possibilities: (*i*) this leading term is non-zero, and so the double-scalings needed to regularize the massless limit before and after the double-copy do not agree, or (*ii*) this term is zero because at least one of the models A or B satisfy the fundamental BCJ identity in the massless limit. In the latter case, if both A and B satisfy the fundamental BCJ identity in the massless limit, then the massive KLT formula reduces to the familiar massless KLT relation,

$$\mathcal{A}_4^{A\otimes B}(1,2,3,4) = -\, s_{14}\mathcal{A}^A[1234]\mathcal{A}^B[1324] + \mathcal{O}(m^2)\,, \tag{5.2.18}$$

and we see that the double-copy and the massless limit commute. Since this required an additional assumption, we conclude that this property does not follow from color-kinematics duality alone.

5.2.2 Spurious Singularities

We have seen so far that our proposed massive KLT formula (5.2.16) does not require BCJ-type constraints in order to define a double-copy. In Appendix P we prove that, assuming models A and B have the usual locality and factorization properties, the formula (5.2.16) contains only simple poles at the locations of physical singularities and the resulting double-copy amplitudes factor correctly into the product of lower point amplitudes. These properties might suggest that (5.2.16) can double-copy any massive theory into a different local theory, but this is not the case. Locality requires not only that the amplitude should contain physical singularities, but also that there are no additional *spurious singularities*. Since these do not occur in the partial amplitudes of models A and B by assumption, they can only appear in the KLT kernel, which we will now analyze in detail.

In general the inverse of a matrix, M^{-1}, equals the matrix of cofactors times $1/\text{det}M$, where the cofactors are sums of products of elements of M. In the massive (massless) KLT kernel, $M = \mathcal{A}_n^{\phi^3}$, and the elements are physical scattering amplitudes of the massive (massless) bi-adjoint scalar theory, which have only physical singularities. Thus, any spurious singularities in the kernel must be a result of zeros of det $\mathcal{A}_n^{\phi^3}$.

Let us first understand how such potential spurious singularities are avoided in the massless KLT kernel. Here the BCJ relations restrict us to a subset of the DDM basis, and as a result, not all physical poles are present in $\mathcal{A}_n^{\phi^3}[\alpha|\beta]$. Thus some physical poles must appear as zeros of the determinant of $\mathcal{A}_n^{\phi^3}[\alpha|\beta]$, while others will appear in the matrix of cofactors. For example at

4-point we have

$$\mathcal{A}_4^{\phi^3}[1234|1234] = -\frac{s_{13}}{s_{12}s_{14}} \tag{5.2.19}$$

$$\Rightarrow \det \mathcal{A}_4^{\phi^3}[1234|1234] = -\frac{s_{13}}{s_{12}s_{14}}. \tag{5.2.20}$$

Thus there is one zero of the determinant $s_{13} = 0$ and it is a physical pole. Due to the color-ordering constraints, consistency with locality requires that $\mathcal{A}_4[1234]$ does not have a pole at $s_{13} = 0$. The *missing* pole in the double-copied amplitude is therefore provided by the zero of the determinant at $s_{13} = 0$.

A similar structure exists at 5-point. Consider BCJ orderings like that in [152], $[13524]$ and $[13542]$. This gives

$$\det \mathcal{A}_5^{\phi^3}[\alpha|\beta] = -\frac{s_{23}s_{15}s_{34}}{s_{12}s_{13}s_{14}s_{24}s_{45}s_{35}s_{25}}. \tag{5.2.21}$$

Again we find that zeros of the determinant $s_{23} = s_{15} = s_{34} = 0$, all correspond to physical poles. In addition, the color-ordering requires $\mathcal{A}_5[13524]$ and $\mathcal{A}_5[13542]$ to have no poles at these locations. Thus, also at 5-point, the zeros of the determinant contribute simple physical poles at locations otherwise excluded by color-ordering constraints.

Let us now investigate what happens to our proposed massive KLT formula at 4-point. We begin by choosing a DDM basis of orderings $([1234], [1324])$. This gives,

$$\det\ \mathcal{A}_4^{\phi^3}[\alpha|\beta] = \frac{m^2}{(s_{12}+m^2)(s_{13}+m^2)(s_{14}+m^2)}, \tag{5.2.22}$$

which has no zeros and thus no spurious pole can arise from the 4-point double-copy. The reciprocal of this determinant does contain a $1/m^2$ factor in the double-copied amplitude. This is exactly the factor we found at the end of the previous subsection in the expansion around the $m \to 0$ limit (5.2.17), and is responsible for the failure of the double-copy and massless decoupling limit to commute.

The absence of any additional *kinematic* zeroes in the determinant has the interesting consequence that any massive theory, satisfying the assumptions enumerated in the Introduction, can be inserted into the massive KLT formula to obtain a 4-point amplitude of a local theory.

At 5-point, we are less lucky. Consider a basis of DDM orderings $[13\sigma(245)]$ where σ runs over all 6 permutations of $(2, 4, 5)$, also used in [152]. Here we find

$$\det \mathcal{A}_5^{\phi^3}[\alpha|\beta] = \frac{m^8}{\prod_i \mathcal{D}_i}\mathcal{P}(s_{ij}, m^2), \tag{5.2.23}$$

where

$$\prod_i \mathcal{D}_i = \left(m^2+s_{12}\right)^2\left(m^2+s_{13}\right)^2\left(m^2+s_{14}\right)^2\left(m^2+s_{23}\right)^2\left(m^2+s_{24}\right)^2$$
$$\left(m^2+s_{15}\right)^2\left(m^2+s_{45}\right)^2\left(m^2+s_{35}\right)^2\left(m^2+s_{25}\right)^2\left(m^2+s_{34}\right)^2, \tag{5.2.24}$$

and

$$\begin{aligned}
\mathcal{P}(s_{ij},m^2) =& 320m^8+36m^6(9s_{12}+4(s_{13}+s_{14}+s_{23}+s_{24}))\\
&+m^4\left(117s_{12}^2+108s_{12}(s_{13}+s_{14}+s_{23}+s_{24})+4\left(s_{13}(13s_{14}+4s_{23}+17s_{24})\right.\right.\\
&\qquad\left.\left.+4s_{13}^2+4s_{14}^2+17s_{14}s_{23}+4s_{14}s_{24}+4s_{23}^2+13s_{23}s_{24}+4s_{24}^2\right)\right)\\
&+2m^2\left(9s_{12}^3+13s_{12}^2(s_{13}+s_{14}+s_{23}+s_{24})+s_{12}\left(s_{13}(10s_{14}+6s_{23}+17s_{24})\right.\right.\\
&\qquad\left.+4s_{13}^2+4s_{14}^2+s_{14}(17s_{23}+6s_{24})+2(2s_{23}+s_{24})(s_{23}+2s_{24})\right)\\
&\qquad+2\left(s_{13}^2(s_{14}+2s_{24})+s_{13}\left(s_{14}^2+s_{14}(s_{23}+s_{24})+s_{24}(s_{23}+2s_{24})\right)\right.\\
&\qquad\left.\left.+s_{23}\left(s_{24}(s_{14}+s_{23})+2s_{14}(s_{14}+s_{23})+s_{24}^2\right)\right)\right)\\
&+2s_{24}\left(s_{23}\left(s_{12}^2+s_{12}(s_{13}+s_{14})-s_{13}s_{14}\right)+s_{12}(s_{12}+s_{13})(s_{12}+s_{13}+s_{14})\right)\\
&+\left(s_{12}(s_{12}+s_{13}+s_{14})+s_{23}(s_{12}+s_{14})\right)^2+s_{24}^2(s_{12}+s_{13})^2.
\end{aligned} \tag{5.2.25}$$

Here, $\mathcal{D}_i$ contains all the physical poles and $\mathcal{P}$ is a quartic polynomial in Mandelstams. Allowing one of the five independent Mandelstam variables to vary, holding the other four fixed, we find that there are four zeros of the determinant that *do not* correspond to physical poles. As a result, unless the amplitudes $\mathcal{A}_5[13\sigma(245)]$ conspire to cancel these spurious poles when we sum over the whole DDM basis, the proposed massive KLT formula will not give us amplitudes of a local theory. We expect that the presence of spurious poles will persist at higher-point.

It is interesting to note that quartic polynomial $\mathcal{P}$ vanishes when the external momenta p_i are restricted to three dimensions, pointing to possible relations between amplitudes in the DDM basis. Thus one cannot immediately conclude that spurious singularities arise when double-copying three-dimensional massive theories via the construction in this section.

This analysis of the equivalent KLT form of the proposed massive double-copy reveals a dangerous tension with locality. As we have argued, color-kinematics duality satisfying BCJ numerators exist (at least up to $n = 5$) for generic models with uniform non-zero mass spectra. But such a double-copy will contain spurious singularities unless magical cancellations take place to remove them. Such cancellations will necessarily require additional relations among the DDM basis of partial amplitudes. Since there is no analogue of the usual BCJ relations, themselves a consequence of color-kinematics duality in massless models, these relations must be genuinely new constraints.

5.3 Massive Gravity and $(\text{Massive Yang-Mills})^2$

To definitively establish that color-kinematics duality is not a sufficient condition for a double-copy to be physical, it is enough to construct a single explicit counterexample. In this section we analyze in detail the *massive Yang-Mills* EFT and demonstrate that a BCJ representation of the scattering amplitudes with color-kinematics duality satisfying numerators exists, at least up to 5-point. We see that 3- and 4-point scattering amplitudes generated by the double-copy can be interpreted as coming from a theory of dRGT massive gravity and show that at 5-point the would-be double-copied amplitude contains spurious singularities.

5.3.1 Physical Motivation

To understand the model we consider and the independent physical arguments that suggest a massive double-copy should be sensible, it is useful to begin with a slightly more general class of models. We consider models with a *global* $U(N)$ symmetry with a spectrum of spin-1 states of mass m transforming in the adjoint representation. To ensure the existence of a standard BCJ representation (5.1.11), we will restrict to interactions in which the color indices are contracted using only the (totally anti-symmetric) structure constants f^{abc}. The most general such model with parity-conserving interaction terms of mass dimension up to four is given by the Lagrangian[5]

$$\mathcal{L} = -\frac{1}{4}\left(\partial_{[\mu}A^a_{\nu]}\right)^2 - \frac{1}{2}m^2 A^a_\mu A^{a\mu} - g f^{abc} A^a_\mu A^b_\nu \partial^\mu A^{c\nu} - \frac{1}{4}g' f^{abe} f^{cde} A^a_\mu A^{\mu c} A^b_\nu A^{\nu d}. \tag{5.3.1}$$

Models of this kind with massive spinning states are generically only valid as low-energy effective descriptions. The associated scattering amplitudes violate perturbative unitarity bounds at a parametrically low energy scale unless special tunings of couplings are made or additional states such as Higgs bosons are introduced to *soften* the UV behaviour. An efficient way to observe this is to study high-energy fixed angle, 2-to-2 scattering amplitudes. Here we use explicit center-of-mass frame kinematics with polarization vectors,

$$\begin{aligned} \epsilon^{(\pm)}_\mu(p^i) &= (0, \mp\cos\theta^i, -i, \pm\sin\theta^i) \\ \epsilon^{(0)}_\mu(p^i) &= \frac{1}{m}(p, E\sin\theta^i, 0, E\cos\theta^i), \end{aligned} \tag{5.3.2}$$

and momenta

$$p^i_\mu = (E, p\sin\theta^i, 0, p\cos\theta^i), \tag{5.3.3}$$

with $i = 1,2,3,4$ labeling the external particles scattering at angles $\theta^1 = 0$, $\theta^2 = \pi$, $\theta^3 = \theta$, $\theta^4 = \theta - \pi$. The worst behaved choice for the polarizations is given by purely longitudinal

[5] In this chapter we will use the Lie algebra conventions $[T^a, T^b] = i f^{abc} T^c$ and $\text{Tr}[T^a T^b] = \delta^{ab}$.

scattering[6]

$$\mathcal{A}(0000) = \frac{1}{4m^4}\left(g^2 - g'\right)\left[c_{12}\left(2s^2 + 2st - t^2\right) + c_{13}\left(s^2 - 2st - 2t^2\right)\right] + \frac{1}{4m^2}\left[c_{12}(4g'(2s+3t) - g^2(8s+13t)) + c_{13}s(4g' - 3g^2)\right] + \mathcal{O}\left(s^0\right), \quad (5.3.4)$$

where we have parametrized the expression in terms of the $m \to 0$ limit of the Mandelstam invariants

$$s \equiv 4E^2, \quad t \equiv 2E^2(\cos(\theta) - 1). \quad (5.3.5)$$

We see that for generic values of g' the scattering amplitudes grow like E^4 at high-energies, but for a specific tuning, $g' = g^2$, this is improved to E^2. If this tuning is made, the generic Lagrangian (5.3.1) simplifies to

$$\mathcal{L} = -\frac{1}{4}\left(F^a_{\mu\nu}\right)^2 - \frac{1}{2}m^2 A^a_\mu A^{a\mu}, \quad (5.3.6)$$

where

$$F^a_{\mu\nu} \equiv \partial_{[\mu}A^a_{\nu]} + g f^{abc} A^b_\mu A^c_\nu, \quad (5.3.7)$$

and defines the model we will study in this section under the name massive Yang-Mills.

The improved high-energy behaviour of this tuning has a nice physical explanation. The massive Yang-Mills model has a simple (perturbative) UV completion as a particular limit of a Higgsed gauge theory. We begin with a model of scalar fields $\phi^{aa'}$ transforming in the bi-adjoint representation of $U(N)_L \times U(N)_R$ with a Higgs potential

$$\mathcal{L} = -\frac{1}{2}\left(\partial_\mu \phi^{aa'}\right)^2 + \lambda v^2 \phi^{aa'}\phi^{aa'} - \frac{\lambda}{2}\left(\phi^{aa'}\phi^{aa'}\right)^2. \quad (5.3.8)$$

When $\lambda > 0$ and $v^2 > 0$, the $U(N)_L \times U(N)_R$ symmetry is spontaneously broken to a $U(N)$ subgroup. Without loss of generality the vacuum expectation value can be taken to have the form

$$\langle \phi^{aa'} \rangle = \frac{v}{N}\delta^{aa'}, \quad (5.3.9)$$

for which the unbroken subgroup $U(N)_V$ is generated by the "vector-like" combinations[7]

$$(T^i_V)^{aa'bb'} = (T^i_L)^{ab}\delta^{a'b'} + \delta^{ab}(T^i_R)^{a'b'}. \quad (5.3.10)$$

If we gauge the orthogonal, broken "axial-like" subgroup $U(N)_A$ generated by

$$(T^i_A)^{aa'bb'} = (T^i_L)^{ab}\delta^{a'b'} - \delta^{ab}(T^i_R)^{a'b'}, \quad (5.3.11)$$

[6] Here we are using a shorthand notation $\mathcal{A}(s_1 s_2 s_3 s_4) \equiv \mathcal{A}_4\left(1^{a_1}_{s_1}, 2^{a_2}_{s_2} \to 3^{a_3}_{s_3}, 4^{a_4}_{s_4}\right)$, where a_i are adjoint indices and $s_i = +, -, 0$ is the polarization.

[7] Here the adjoint generators are defined as $(T^i_L)^{ab} = f^{iab}$ and $(T^i_R)^{a'b'} = f^{ia'b'}$.

then in unitary gauge the associated $U(N)_A$ gauge bosons acquire masses $m_A \sim gv$, while preserving the unbroken global $U(N)_V$ symmetry under which they transform in the adjoint representation. The remaining $N^2(N^2-1)$ Higgs scalars have masses $m_H \sim \lambda^{1/2}v$, and in the limit $\lambda \to \infty$ with v held fixed, decouple, with the low-energy dynamics of the massive vector bosons described by the massive Yang-Mills EFT.

The Goldstone boson equivalence theorem [167] tells us that the high-energy scattering of longitudinal vector modes of a spontaneously broken gauge theory must match the high-energy limit of a coset sigma model describing the same symmetry breaking pattern. In this case the coset is $(U(N)_L \times U(N)_R)/U(N)_V$, which is coincidentally the coset defining *Chiral Perturbation Theory* (χPT) [168], with the well-known Lagrangian

$$\mathcal{L} = \frac{f_\pi^2}{2}\text{Tr}\left[\partial_\mu U^\dagger \partial^\mu U\right], \quad U(x) \equiv \exp\left(\frac{i}{f_\pi}T^a\pi^a(x)\right). \tag{5.3.12}$$

The 2-to-2 scattering amplitude in this model is given by the simple expression

$$\mathcal{A}_4(1,2,3,4) = \frac{1}{4f_\pi^2}\left(-c_{12}t + c_{13}s\right), \tag{5.3.13}$$

which precisely matches (5.3.4) in the limit $g' = g^2$, if the pion decay constant is identified as $f_\pi \sim m/g$.

Massive Yang-Mills is not only a special EFT because it has softer than expected high-energy growth. As the above discussion indicates, in the high-energy limit the scattering amplitudes coincide with those of χPT, which is among the special class of massless models exhibiting color-kinematics duality [85, 87], as can be verified explicitly using (5.3.13). As a consequence, in the high-energy limit the massive Yang-Mills amplitudes can be double-copied to give the scattering amplitudes of the special Galileon [6],

$$\left(\lim_{E\gg m}\mathcal{A}_n^{\text{mYM}}\right) \otimes \left(\lim_{E\gg m}\mathcal{A}_n^{\text{mYM}}\right) = \mathcal{A}_n^{\text{sGal}}. \tag{5.3.14}$$

Galileons were originally discovered in the context of the DGP model of modified gravity [169], but were later found to arise naturally in the decoupling limit of ghost-free massive gravity [61]. On the basis of this observation, it seems natural to speculate that there exists some model of a massive spin-2 or massive gravity, which matches the special Galileon amplitudes at high-energies and can be constructed as a double-copy

$$\mathcal{M}_n^{\text{mGrav}} \equiv \mathcal{A}_n^{\text{mYM}} \otimes_m \mathcal{A}_n^{\text{mYM}}. \tag{5.3.15}$$

An immediate problem with this is that we do not know what the symbol $\otimes_m$, denoting a massive double-copy, is supposed to mean. One property it should have, *if* this story is self-consistent, is that it *commutes* with the high-energy limit, meaning

$$\lim_{E\gg m}\left(\mathcal{A}_n^{\mathrm{mYM}}\otimes_m\mathcal{A}_n^{\mathrm{mYM}}\right)\stackrel{!}{=}\left(\lim_{E\gg m}\mathcal{A}_n^{\mathrm{mYM}}\right)\otimes\left(\lim_{E\gg m}\mathcal{A}_n^{\mathrm{mYM}}\right), \tag{5.3.16}$$

where $\otimes$ on the right-hand-side is the familiar massless double-copy. In the Introduction (5.1.11), we described a natural generalization of the BCJ double-copy based on color-kinematics duality, to models with massive states, and in Section 5.2 constructed an equivalent KLT-like formula. In this section we will demonstrate explicitly that such a double-copy does *not* have the property (5.3.16) and moreover, for $n>4$ does not produce a physical scattering amplitude that can be matched to a local Lagrangian.

5.3.2 3-point Amplitudes and Asymptotic States

Before considering the dynamical content of the double-copy, we first need to understand the mapping of states in the asymptotic Hilbert space. Massive Yang-Mills is a model of a massive vector boson, with 3 on-shell degrees of freedom in $d=4$. The Hilbert space of asymptotic one-particle states is spanned by the space of plane-wave solutions to the linearized equations of motion. In the present context it is convenient to represent the basis of linearly independent plane-wave solutions using the massive spinor formalism of [23]. In this approach, the 3 independent spin states are collected together into a rank-2, totally symmetric $SU(2)$ little group tensor. Explicitly,

$$A_\mu^{a\,IJ}(x)=c^a\epsilon_\mu^{IJ}(p)e^{ip\cdot x},\quad \text{where}\quad \epsilon_\mu^{IJ}(p)=-\frac{1}{2\sqrt{2}}\tilde{\lambda}_{\dot{\alpha}}^{(I}\overline{\sigma}_\mu^{\dot{\alpha}\alpha}\lambda_\alpha^{J)}. \tag{5.3.17}$$

The double-copy of such a plane-wave solution is given simply by replacing the color factor c^a with a second copy of the polarization vector,

$$A_\mu^{a\,IJ}(x)\otimes A_\nu^{b\,KL}(x)=\mathfrak{h}_{\mu\nu}^{IJKL}(x)\equiv\epsilon_\mu^{IJ}(p)\epsilon_\nu^{KL}(p)e^{ip\cdot x}. \tag{5.3.18}$$

Where (5.3.17) transforms in an irreducible representation of $SU(2)$, the double-copy (5.3.18) transforms in a reducible representation. Such a plane-wave double-copy is equivalent to a tensor product of one-particle Hilbert spaces, for which standard decomposition of representations of $SU(2)$ gives the physical spectrum of the double-copy

$$\mathbf{3}\otimes\mathbf{3}=\mathbf{5}\oplus\mathbf{3}\oplus\mathbf{1}. \tag{5.3.19}$$

Hence we expect the double-copy of massive Yang-Mills to describe a model of a massive *graviton* $h_{\mu\nu}$ (spin-2) coupled to a massive *Kalb-Ramond two-form* $B_{\mu\nu}$ (spin-1) and a massive *dilaton*

ϕ (spin-0). It is most convenient to first calculate the scattering amplitudes for the reducible $\mathfrak{h}$-states, and project out the physical states as needed. To extract the physical spectrum of the double-copy we use the following *projection operators*[8]

$$(P_h)^{K_1K_2K_3K_4}_{I_1I_2J_1J_2} = \frac{1}{24}\delta^{(K_1K_2K_3K_4)}_{I_1I_2J_1J_2},\ (P_B)^{K_1K_2}_{I_1I_2J_1J_2} = \frac{1}{\sqrt{2}}\epsilon_{I_1J_1}\delta^{(K_1K_2)}_{I_2J_2},$$
$$(P_\phi)_{I_1I_2J_1J_2} = \frac{1}{\sqrt{3}}\epsilon_{I_1J_1}\epsilon_{I_2J_2}. \tag{5.3.20}$$

The physical polarization tensor of the two-form is antisymmetric $\epsilon^{(B)}_{\mu\nu} = -\epsilon^{(B)}_{\nu\mu}$, and consequently gives a non-vanishing contribution to amplitudes in the double-copy only if there are an even number of such states. Equivalently, the two-form has a $\mathbb{Z}_2$ symmetry, which allows us to form a consistent truncation containing only the graviton and dilaton modes.

Since the polarization tensors in the truncated model are symmetric we can represent the amplitudes using a convenient shorthand. We suppress the little-group indices by making the replacement $\epsilon^{I_iI_i}_\mu(p_i) \to z^i_\mu$; the amplitude is then a rational function of the following elementary building blocks:

$$p_{ij} \equiv p^i_\mu p^{j\mu}, \quad z_{ij} \equiv z^i_\mu z^{j\mu}, \quad zp_{ij} \equiv z^i_\mu p^{j\mu}. \tag{5.3.21}$$

Extracting the physical graviton and dilaton states amounts to the replacement rules,

$$z^i_\mu z^i_\nu \to \epsilon_{\mu\nu}(p_i) \qquad \text{(Massive Graviton)}$$
$$z^i_\mu z^i_\nu \to \frac{1}{\sqrt{3}}\left(\eta_{\mu\nu} + \frac{p_{i\mu}p_{i\nu}}{m^2}\right) \qquad \text{(Massive Dilaton).} \tag{5.3.22}$$

We begin with the double-copy of 3-point scattering amplitudes. This is of course unconstrained by color-kinematics duality, but will be important for reconstructing the massive gravity Lagrangian from the 4-point amplitudes. A local BCJ representation of the massive Yang-Mills amplitudes can be efficiently constructed using the Feynman rules given in Appendix Q. The cubic Yang-Mills amplitude is given by

$$\mathcal{A}_3 = 2g\big(z_{23}zp_{12} + z_{13}zp_{23} + z_{12}zp_{31}\big). \tag{5.3.23}$$

The gravitational amplitude is given by squaring the Yang-Mills amplitude and replacing the coupling constants as $g^2 \to \frac{1}{2M_p}$, giving

$$\mathcal{M}_3 = \frac{2}{M_p}\big(z_{23}zp_{12} + z_{13}zp_{23} + z_{12}zp_{31}\big)^2. \tag{5.3.24}$$

[8]The normalization constants can be fixed by requiring that the completeness relation for polarizations gives the same sum over states before and after projecting onto physical states.

Using (5.3.22) we can extract from this the cubic amplitudes for physical states. The on-shell cubic amplitude for 3 gravitons is formally identical to the massless case, given by:

$$\mathcal{M}(1_h, 2_h, 3_h) = \frac{2}{M_p}\bigg(\epsilon_{1\mu\nu}{\epsilon_2}^{\mu\nu}\epsilon_{3\alpha\beta}{p_1}^{\alpha}{p_1}^{\beta} + 2\,{p_2}^{\mu}\epsilon_{1\mu\nu}{\epsilon_2}^{\nu\alpha}\epsilon_{3\alpha\beta}{p_1}^{\beta} \\ + \text{cyclic permutations of } (1,2,3)\bigg). \tag{5.3.25}$$

The amplitude for 2 gravitons and 1 dilaton is given by

$$\mathcal{M}_3\left(1_h, 2_h, 3_\phi\right) = -\frac{\sqrt{3}}{2M_p}m^2\epsilon_{1\mu\nu}{\epsilon_2}^{\mu\nu}. \tag{5.3.26}$$

We see that this expression vanishes as $m \to 0$, recovering the expected massless amplitude. It is interesting to note that the $\mathbb{Z}_2$ *dilaton parity* of the massless double-copy only emerges in the massless limit. Therefore when $m \neq 0$ we cannot make a further consistent truncation to the gravity sector. The on-shell cubic amplitudes for 1 graviton and 2 dilatons is given by

$$\mathcal{M}_3\left(1_h, 2_\phi, 3_\phi\right) = \frac{3}{2M_p}\epsilon_{1\mu\nu}{p_2}^{\mu}{p_2}^{\nu}. \tag{5.3.27}$$

This vertex appears in both the massive and massless cases. The on-shell cubic amplitude for 3 dilatons is given by

$$\mathcal{M}_3\left(1_\phi, 2_\phi, 3_\phi\right) = -\frac{11\sqrt{3}}{8M_p}m^2. \tag{5.3.28}$$

This cubic dilaton vertex is also unique to the massive case and does not appear in the massless case.

5.3.3 4-point Amplitudes and High Energy Behavior

A BCJ representation of the 4-point amplitude is straightforwardly generated from the Feynman rules in Appendix Q. This gives the following massive kinematic numerators

$$\begin{aligned} n_{12} =& [(\epsilon_1\cdot\epsilon_2)p_1^\mu + 2(\epsilon_1\cdot p_2)\epsilon_2^\mu - (1\leftrightarrow 2)]\left(g_{\mu\nu} + \frac{(-{p_1}_\mu - {p_2}_\mu)({p_3}_\nu + {p_4}_\nu)}{m^2}\right) \\ & \times [(\epsilon_3\cdot\epsilon_4){p_3}^\nu + 2(\epsilon_3\cdot p_4){\epsilon_4}^\nu - (3\leftrightarrow 4)] \\ & + (s+m^2)[(\epsilon_1\cdot\epsilon_3)(\epsilon_2\cdot\epsilon_4) - (\epsilon_1\cdot\epsilon_4)(\epsilon_2\cdot\epsilon_3)], \end{aligned} \tag{5.3.29}$$

with the first two lines coming from the exchange diagrams and the third line coming from the contact diagram. The other numerators are found by taking

$$n_{13} = n_{12}|_{1\to3\to2\to1}, \quad n_{14} = n_{12}|_{1\to2\to3\to1}. \tag{5.3.30}$$

The $1/m^2$ term in the massive vector propagator vanishes, and so these numerators are formally identical to the Feynman rule-generated expressions for massless Yang-Mills. As a consequence of this formal equivalence, together with the fact that at 4-point in massless Yang-Mills, all generalized gauges satisfy the kinematic Jacobi identity, we find that this Feynman rule generated expression for the mass deformed numerators (5.3.29) and (5.3.30), just happens to be in the unique generalized gauge to satisfy the massive kinematic Jacobi relation

$$n_{12} + n_{13} + n_{14} = 0. \tag{5.3.31}$$

The 4-point massive gravity amplitude is then given by

$$\mathcal{M}_4 = \frac{1}{4M_p^2}\left(\frac{n_{12}^2}{s_{12}+m^2} + \frac{n_{13}^2}{s_{13}+m^2} + \frac{n_{14}^2}{s_{14}+m^2}\right). \tag{5.3.32}$$

The explicit expressions for the physical scattering amplitudes are rather complicated and are given explicitly in Appendix R[9].

We expect the double-copy procedure for massive Yang-Mills to give a ghost-free theory of massive gravity[10]. Generic ghost free massive gravity without coupling to a dilaton, also known as dRGT, propagates 5 degrees of freedom, has two free parameters in $D = 4$, and is given by the action

$$S = \frac{M_P^{D-2}}{2}\int d^D x\Big[(\sqrt{-g}R) - \sqrt{-g}\frac{1}{4}m^2 W(g,\mathcal{K})\Big], \tag{5.3.33}$$

where

$$W(g,\mathcal{K}) = \sum_{n=2}^{n=D} \alpha_n \mathcal{L}_n^{TD}(\mathcal{K}), \tag{5.3.34}$$

brackets mean trace with respect to the full metric, $\alpha_2 = -4$, and the rest of the coefficients are arbitrary [171, 172]. The tensor $\mathcal{K}^\mu_\nu(g,H)$ is given by

$$\mathcal{K}^\mu_\nu = \delta^\mu_\nu - \sqrt{\delta^\mu_\nu - H^\mu_\nu} = \sum_{n=1}^{\infty} d_n (H^n)^\mu_\nu, \quad d_n = -\frac{(2n)!}{(1-2n)(n!)^2 4^n}, \tag{5.3.35}$$

where indices are raised by the full metric $g^\mu_\nu = \gamma^\mu_\nu + h^\mu_\nu$, the background metric is γ^μ_ν, and $H^\mu_\nu = g^\mu_\nu - \tilde{\gamma}^\mu_\nu$ is the Stückelberg replacement for h^μ_ν. The quantity $\mathcal{L}_n^{TD}(\Pi)$ can be written as

[9] These results are in agreement with those that appeared recently in [165].
[10] See [170] for a review of massive gravity.

total derivatives when $\Pi = \partial_\mu \partial_\nu \phi$. These total derivative combinations are unique up to an overall constant and can be found using the recursion relation

$$\mathcal{L}_n^{TD}(\Pi) = -\sum_{m=1}^{n} (-1)^m \frac{n!}{(n-m)!} \Pi_{\mu\nu}^m \mathcal{L}_{n-m}^{TD}(\Pi), \tag{5.3.36}$$

with $\mathcal{L}_0^{TD} = 1$.

Massive gravity with the most generic potential without the dRGT tuning has an extra scalar degree of freedom that is ghostly and 4-point scattering amplitudes that grow with center-of-mass energy like E^{10}. However, the dRGT tuning, which leaves only 2 free parameters, removes the ghostly degree of freedom and improves the high energy behavior to scale with energy as E^6 [173, 174]. Another common parameterization of dRGT massive gravity is given in [175]. The leading high energy behavior in this parameterization, for the tree-level 4-point amplitude for dRGT massive gravity, is given by:

$$\mathcal{M}(1^+1^+1^+1^+) = -\frac{3}{32}(1-4c_3)s^3 \tag{5.3.37}$$

$$\mathcal{M}(1^+1^+1^-1^-) = \mathcal{M}(1^-1^-1^+1^+) = \frac{9}{32}(1-4c_3)^2 st(s+t) \tag{5.3.38}$$

$$\mathcal{M}(2^+000) = \frac{1}{\sqrt{6}}(c_3+8d_5)st(s+t) \tag{5.3.39}$$

$$\mathcal{M}(1^+1^+00) = \frac{1}{32}s\Big(2\big(1-8c_3+48c_3^2+64d_5\big)t(s+t) - 3(1-4c_3)^2 s^2\Big) \tag{5.3.40}$$

$$\mathcal{M}(1^+1^-00) = \frac{1}{96}s\big((1+12c_3)^2+384d_5\big)st(s+t) \tag{5.3.41}$$

$$\mathcal{M}(0000) = \frac{1}{6}\big(1+4c_3(9c_3-1)+64d_5\big)st(s+t), \tag{5.3.42}$$

where the polarization tensors, $\epsilon_{\mu\nu}^{(a)}$, have been split into two tensor modes $(a = 2^+, 2^-)$, two vector modes $(a = 1^+, 1^-)$, and one scalar mode $(a = 0)$, the relation between the free parameters of dRGT are given by:

$$\alpha_3 = -2c_3 \ \text{ and } \ \alpha_4 = -4d_5, \tag{5.3.43}$$

and the polarization tensors are chosen to be:

$$\begin{aligned} \epsilon_{\mu\nu}^{(2\pm)} &= \epsilon_\mu^{(\pm)}\epsilon_\nu^{(\pm)}, \\ \epsilon_{\mu\nu}^{(1\pm)} &= \frac{1}{\sqrt{2}}\left(\epsilon_\mu^{(\pm)}\epsilon_\nu^{(0)} + \epsilon_\mu^{(0)}\epsilon_\nu^{(\pm)}\right) \\ \epsilon_{\mu\nu}^{(0)} &= \frac{1}{\sqrt{6}}\left(\epsilon_\mu^{(+)}\epsilon_\nu^{(-)} + \epsilon_\mu^{(-)}\epsilon_\nu^{(+)} + 2\epsilon_\mu^{(0)}\epsilon_\nu^{(0)}\right). \end{aligned} \tag{5.3.44}$$

Indeed the 3-point amplitude (5.3.24) corresponds to dRGT massive gravity with

$$\alpha_3 = -\frac{1}{2} \text{ or } c_3 = \frac{1}{4}. \tag{5.3.45}$$

This value is also the one picked out in the eikonal approximation analysis needed to avoid superluminal propagation as shown in [176] and is the "partially massless" α_3 [171, 177].

With the new cubic vertices that appear in the massive case, there are new scattering channels that appear in the quartic amplitudes that would not appear in the massless case. In agreement with the general discussion in Section P, we find that all quartic amplitudes factorize properly on the poles into products of the corresponding 3-point amplitudes. For example, in the 4-graviton scattering amplitude, we find contributions from diagrams corresponding to the s, t, u channels mediated by both a massive graviton and a dilaton, due to the non-vanishing cubic coupling with 2 gravitons and 1 dilaton. The 4-graviton amplitude matches that of massive gravity with the coefficients

$$\alpha_4 = \frac{7}{48} \text{ or } d_5 = -\frac{7}{192}, \tag{5.3.46}$$

plus the additional channels mediated by the dilaton.

At first glance, it may appear that a field redefinition could mix the cubic $hh\phi$ vertex and massive gravity quartic interactions, leading to the choice of α_4 to not be uniquely specified. Since amplitudes are unaffected by field redefinition, we consider the difference between the double-copied amplitude and the dRGT massive gravity amplitude with $\alpha_3 = -\frac{1}{2}$ and α_4 left unspecified. We find terms proportional to $\sim (48\alpha_4 - 7)Tr[\epsilon_1 \cdot \epsilon_2 \cdot \epsilon_3 \cdot \epsilon_4]$. This structure cannot be altered by introducing scalar channel diagrams and thus, requiring that it vanish picks out the remaining parameter to be $\alpha_4 = \frac{7}{48}$.

The leading high energy behavior of the amplitudes for graviton-graviton scattering in the massive double-copy goes as:

$$\mathcal{M}(2^+000) = -\frac{1}{24\sqrt{6}} st(s+t) \tag{5.3.47}$$

$$\mathcal{M}(1^+1^+00) = -\frac{1}{48} st(s+t) \tag{5.3.48}$$

$$\mathcal{M}(1^+1^-00) = \frac{1}{48} st(s+t) \tag{5.3.49}$$

$$\mathcal{M}(0000) = \frac{7}{144} st(s+t). \tag{5.3.50}$$

For the value of c_3 picked out by the double-copy, the high energy behavior of the 4-point amplitudes for massive gravity, (5.3.37) through (5.3.42), is improved for amplitudes where all the polarizations of the external particles are vector modes, scaling as E^4 rather than E^6. The dilaton affects the coefficient of $\mathcal{M}(0000)$, the amplitude where all the external particles are scalar modes. Without the dilaton, this amplitude would behave as $\mathcal{M}(0000) = -\frac{1}{72}st(s+t)$. All other amplitudes behave as they would without the dilaton and are consistent with the above c_3 and d_5 values in expressions (5.3.37) through (5.3.42).

One immediate and important result from (5.3.47) is that the conjectured property (5.3.16) does not hold for the BCJ double-copy. In the Goldstone boson equivalence limit for massive Yang-Mills, only the spin-1 longitudinal mode contributes at E^2. If (5.3.16) held, we would expect only the scattering of a single scalar mode to contribute at E^6 in the double-copy. From (5.3.47), we see explicitly that this is not the case.

In the 4-point amplitude where all the external particles are dilatons, there will be s, t, u channels mediated by a massive graviton, as well as s, t, u channels mediated by a dilaton, and a 4-dilaton contact term. The massless case only has the channels mediated by the massless graviton.

The 4-point amplitude with 2 gravitons and 2 dilatons exists in the massless and massive case. In the massless case, this 4-point amplitude has graviton exchange channels via the $h\phi\phi$ and hhh vertices and dilaton exchange channels via two $h\phi\phi$ vertices, plus a contact term $hh\phi\phi$. In the massive case, there will be additional graviton exchange channels via two $hh\phi$ vertices, as well as dilaton exchange channels via the vertices $hh\phi$ and $\phi\phi\phi$.

The 4-point amplitudes with 3 gravitons and 1 dilaton or 1 graviton and 3 dilatons are unique to the massive case and involve all possible exchange diagrams with dilaton propagators, as well as graviton propagators, and with additional $hhh\phi$ and $h\phi\phi\phi$ contact terms.

The high energy behavior of all the amplitudes scales with energy like $\sim E^6$ or less and the amplitudes that scale like E^6 take the special galileon form $st(s+t)$ [6]. As another example, the leading high energy behavior of $h\phi\phi\phi$ amplitudes is shown below:

$$\mathcal{M}(2^+\phi\phi\phi) = -\frac{st(s+t)}{96\sqrt{3}} \tag{5.3.51}$$

$$\mathcal{M}(0\phi\phi\phi) = -\frac{11st(s+t)}{288\sqrt{2}}. \tag{5.3.52}$$

All the 4-point graviton and dilaton amplitudes resulting from the double-copy are given in Appendix R.

5.3.4 5-point Amplitudes and Non-Physical Singularities

As discussed in Section P, 5-point massive gravity amplitudes constructed via the massive KLT formula are guaranteed to factorize correctly into 4- and 3-point amplitudes (listed in Appendix R and Section 5.3.2 respectively). Nonetheless as we saw at 4-point, checking factorization at 5-point is a good cross-check of our more general results, in particular those of Appendix P.

We begin by choosing a DDM basis of orderings $[13\,\sigma(2,4,5)]$ where σ runs over all possible permutations. Using the Feynman rules of massive Yang-Mills, we then calculate partial amplitudes and use the inverse of bi-adjoint scalar matrix (O.0.3) to construct 5-point all-graviton amplitudes. The inverse of (O.0.3) is unwieldy so we do the following numerical tests of factorization.

One can choose an independent basis of building blocks from the set of all $(\epsilon_i \cdot \epsilon_j)$, $(\epsilon_i \cdot p_j)$ and $(p_i \cdot p_j)$. We then assign numeric values to all these kinematic structures except one, without loss of generality let's call this $(p_1 \cdot p_2)$. One can then evaluate the 5-point amplitude on this set of kinematic data and check that the residue on physical pole $(p_1 \cdot p_2) = -\frac{m^2}{2}$ is exactly what one would expect

$$\operatorname*{Res}_{s_{12}=-m^2} \mathcal{M}_5(12345) \stackrel{!}{=} \sum_X \mathcal{M}_3\left(12(-P_{12})_X\right) \times \mathcal{M}_4\left((P_{12})_{\bar{X}}345\right), \tag{5.3.53}$$

where X can either be a dilaton or a graviton. As expected from the general discussion in Appendix P we find that the would-be 5-point amplitude factors as expected on physical poles.

While the correct factorization of the 5-point amplitude is promising, we saw in Section 5.2.2 that the KLT kernel suffers from non-physical poles arising from the determinant of the matrix of bi-adjoint scalar amplitudes. These singularities (5.2.24) can only be removed if special cancellations occur between amplitudes in the theory we are double-copying and the KLT kernel.

In the context of this explicit example, we can proceed with our numerical analysis to check for example, whether all poles in $(p_1 \cdot p_2)$ are physical. This can be done by evaluating the KLT formula on an incomplete set of kinematic data that leaves $(p_1 \cdot p_2)$ unspecified. One can then check if all singularities in $(p_1 \cdot p_2)$ are accounted for by locality. We find that this is not the case and that the resulting 5-point amplitude $\mathcal{M}_5$ *does have spurious poles*. The singularity structure takes exactly the form (5.2.24) which can be recast as

$$\mathcal{P}(s_{ij}, m^2) = \alpha_1 s_{12}^4 + \alpha_2 s_{12}^3 + \alpha_3 s_{12}^2 + \alpha_4 s_{12} + \alpha_5\,, \tag{5.3.54}$$

where α_i are functions of the mass and other Mandelstam variables. Since this polynomial does not easily factor into rational roots, it is useful to choose special kinematic configurations where it factors more readily. In these cases, the exact locations of the spurious poles can be found and the amplitude evaluated on such a non-physical pole gives a nonzero residue.

Thus, no miraculous cancellations occur in massive Yang-Mills to get rid of spurious singularities. In particular this means that in its current form, *massive Yang-Mills does not sensibly double-copy to massive gravity*.

Furthermore, if we attempt to save the double-copy, by for example, adding a 5-point contact contribution to cancel these non-physical poles, we find no improvement. Consider for example adding a new operator at 5-point, such that

$$\tilde{\mathcal{A}}_5[13542] = \mathcal{A}_5[13245] + \frac{\alpha g^3}{m^2}(p_1 \cdot \epsilon_3)(\epsilon_1 \cdot \epsilon_2)(\epsilon_3 \cdot \epsilon_5), \tag{5.3.55}$$

with contributions to the other orderings determined by relabeling. Here α is a free coefficient. The powers of m^2 have been introduced to correct the mass dimension, this would correspond to adding a term $\sim \partial A^5$ to the massive Yang-Mills Lagrangian.

We find that there is *no way* to tune α to remove any of the spurious singularities. Since it is unclear whether this statement still holds for arbitrary combinations of the other 28 possible ∂A^5 structures, we cannot strictly rule out the possibility of a massive Yang-Mills 5-point operator removing non-physical poles from the KLT product. Nonetheless, our calculation is indicative that this may not be possible.

5.4 Locality and the Spectral Condition

We have seen that the proposed massive KLT construction (5.2.16) is in serious tension with locality. In general, the inverse of the matrix of KK independent massive bi-adjoint scalar amplitudes contains spurious, non-physical singularities (5.2.24). For the full KLT sum to be free of these non-physical singularities, additional non-trivial constraints must be imposed. These conditions are not met in the case of massive Yang-Mills, because as we saw in Section 5.3.4, the resulting 5-point massive gravity amplitude is not local. Thus, despite the existence of color-kinematics duality satisfying numerators for all KK satisfying models, the resulting would-be double copies only correspond to physical amplitudes if additional constraints are imposed.

To better understand these additional constraints, let us first look at the massless case. Here the additional constraints are the fundamental BCJ relations and color-kinematics duality satisfying numerators can only be found in theories whose amplitudes are BCJ-compatible. In the language of bi-adjoint scalar theory, the double-copy formulation gives rise to physical amplitudes only if the $(n-2)! \times (n-2)!$ matrix of bi-adjoint scalar amplitudes $\mathcal{A}^{\phi^3}[\alpha|\beta]$ has rank $(n-3)!$, which we will refer to as *minimal rank*[11]. In addition, only theories whose amplitudes satisfy the fundamental BCJ relations, which arise as null vectors of the singular matrix of bi-adjoint scalar amplitudes, can be double-copied.

In the massive case, a matrix of bi-adjoint scalar amplitudes that has minimal rank can be constructed if a specific condition on the masses, given by the equation $\det \mathcal{A}^{\phi^3}[\alpha|\beta] = 0$ is met. We will call this the *spectral condition*. The null vectors of this matrix will then give rise to massive

[11] In this work we only consider states that transform in the adjoint representation. The value $(n-3)!$ for the minimal rank may be modified if particles in other representations are present.

BCJ relations. On the basis of this observation, we propose the following:

Conjecture: *The KLT prescription for double-copying models with massive states generates physical amplitudes without spurious singularities, and reduces smoothly to the massless double-copy in an appropriate* $m \to 0$ *decoupling limit, if the associated bi-adjoint scalar matrix has minimal rank.*

In this section we will illustrate the consequences of imposing these conditions on models at $n = 4$ and $n = 5$. We will see how this alternative construction has both a commuting decoupling limit and the absence of spurious singularities, providing evidence in support of our conjecture above.

5.4.1 4-point Spectral Condition

We will begin with a model that has a more general spectrum of massive or massless states. We denote the external states m_i and the intermediate masses being exchanged on a factorization channel as m_{ij}. The only assumption we will make is the existence of a BCJ representation of the form

$$\mathcal{A}_4\left(1^{a_1}, 2^{a_2}, 3^{a_3}, 4^{a_4}\right) = \frac{c_{12}n_{12}}{s_{12} + m_{12}^2} + \frac{c_{13}n_{13}}{s_{13} + m_{13}^2} + \frac{c_{14}n_{14}}{s_{14} + m_{14}^2}. \tag{5.4.1}$$

Implicitly built into this expression is the assumption that only states with mass m_{12}^2 are exchanged in the s_{12}-channel and so forth. This is not completely general and an interesting open problem is to construct an appropriate generalization of the BCJ form for models with multiple mass states exchanged in a single channel. We now choose a DDM basis $([1234], [1324])$, in which the matrix of bi-adjoint scalar amplitudes is

$$\mathcal{A}_4^{\phi^3}[\alpha|\beta] = \begin{pmatrix} \frac{1}{s_{12}+m_{12}^2} + \frac{1}{s_{14}+m_{14}^2} & -\frac{1}{s_{14}+m_{14}^2} \\ -\frac{1}{s_{14}+m_{14}^2} & \frac{1}{s_{13}+m_{13}^2} + \frac{1}{s_{14}+m_{14}^2} \end{pmatrix}. \tag{5.4.2}$$

Taking the determinant, gives

$$\det \mathcal{A}_4^{\phi^3} = \frac{m_{12}^2 + m_{13}^2 + m_{14}^2 - m_1^2 - m_2^2 - m_3^2 - m_4^2}{(s_{12} + m_{12}^2)(s_{13} + m_{13}^2)(s_{14} + m_{14}^2)}. \tag{5.4.3}$$

Clearly $\mathcal{A}^{\phi^3}[\alpha|\beta]$ is full-rank and non-singular, i.e. $\det \mathcal{A}^{\phi^3}[\alpha|\beta]$ does not vanish, for generic mass spectra. In keeping with our conjecture, we want to reduce the rank of $\mathcal{A}^{\phi^3}[\alpha|\beta]$ to (4-3)!=1, which is the minimal rank at 4-point order. This is achieved by imposing the following condition on the mass spectrum of the theory,

$$m_{12}^2 + m_{13}^2 + m_{14}^2 = m_1^2 + m_2^2 + m_3^2 + m_4^2. \tag{5.4.4}$$

This is the *4-point spectral condition*. It is interesting to note that the spectrum of massive Yang-Mills does not satisfy this condition. We will see later that this is what led the double-copy and decoupling limit to fail to commute when studying massive (Yang-Mills)^2.

On imposing the spectral condition, $\mathcal{A}^{\phi^3}[\alpha|\beta]$ becomes singular and is no longer invertible. As a result, we must eliminate one row and one column to produce an invertible matrix of bi-adjoint scalar amplitudes. This is consistent only if all such choices give the same result. For example, we could remove the second row and second column, the resulting KLT formula is then

$$\mathcal{M}_4\,(1,2,3,4) = -\frac{(s_{12}+m_{12}^2)(s_{14}+m_{14}^2)}{s_{13}+(m_1^2+m_2^2+m_3^2+m_4^2-m_{12}^2-m_{14}^2)}\,\mathcal{A}_4[1,2,3,4]^2. \tag{5.4.5}$$

If however, we choose to eliminate the second row and the first column, we find

$$\mathcal{M}_4\,(1,2,3,4) = -(s_{14}+m_{14}^2)\mathcal{A}_4[1,2,3,4]\mathcal{A}_4[1,3,2,4]. \tag{5.4.6}$$

Equating these formulae constructs a *massive* version of the fundamental BCJ relation

$$(s_{12}+m_{12}^2)\mathcal{A}_4[1,2,3,4] = (s_{13}+m_{13}^2)\mathcal{A}_4[1,3,2,4], \tag{5.4.7}$$

where we have used the spectral condition to rewrite the relation in a more compact form.

As we prove in Appendix S, an equivalent way to derive the massive BCJ relation is by studying the null vector of $\mathcal{A}^{\phi^3}[\alpha|\beta]$ which is

$$\vec{n} = \begin{pmatrix} -s_{12}-m_{12}^2 \\ s_{13}+m_{13}^2 \end{pmatrix}. \tag{5.4.8}$$

Setting the dot product of this vector with the DDM basis to zero then gives the BCJ relation,

$$\vec{n}\cdot(\mathcal{A}_5[1234] \quad \mathcal{A}_5[1324]) = (s_{12}+m_{12}^2)\mathcal{A}_4[1,2,3,4]-(s_{13}+m_{13}^2)\mathcal{A}_4[1,3,2,4] = 0. \tag{5.4.9}$$

We are now in a position to study the singularity structure of the KLT formula (5.4.5). The first aspect of the formula that we note is the absence of spurious poles, i.e. all poles are at physical locations. To ensure locality, we can study the amplitude in the neighbourhood of its three physical poles. For example,

$$\begin{aligned}\underset{s_{12}=-m_{12}^2}{\text{Res}}\mathcal{M}_4\,(1,2,3,4) &= -\frac{(s_{14}+m_{14}^2)}{s_{13}+m_{13}^2}\mathcal{A}_4[1,2,-P_{12}]^2\mathcal{A}_3[P_{12},3,4]^2 \\ &= \mathcal{A}_4[1,2,-P_{12}]^2\mathcal{A}_3[P_{12},3,4]^2 \\ &= \mathcal{M}_3\,(1,2,-P_{12})\,\mathcal{M}_3\,(P_{12},3,4)\,,\end{aligned} \tag{5.4.10}$$

where we have used $s_{13}+m_{13}^2=-s_{14}-m_{14}^2$ on the s_{12} pole. Thus the amplitude factorizes correctly on the s_{12} pole. Factorization on the s_{13} and s_{14} pole follow in a similar manner.

It is easy to see that these forms of the massive BCJ relations and KLT formula smoothly reduce to the massless ones when all external and intermediate masses, m_i and m_{ij} are taken to zero. As a result, this version of the massive double-copy *does commute with the decoupling limit*. Thus for any pair of massive BCJ-compatible theories $A^{(m)}$ and $B^{(m)}$ that satisfy the spectral condition, one can construct a local theory,

$$C^{(m)}=A^{(m)}\otimes_m B^{(m)}\,, \tag{5.4.11}$$

where $\otimes_m$ is our conjectured massive KLT formalism. This will reduce in the decoupling limit to

$$\lim_{m\to0}C^{(m)}=\lim_{m\to0}\left(A^{(m)}\otimes_m B^{(m)}\right)=\left(\lim_{m\to0}A^{(m)}\right)\otimes\left(\lim_{m\to0}B^{(m)}\right)\,, \tag{5.4.12}$$

where $\otimes$ denotes the massless KLT double-copy.

As we saw in Section 5.2.1, the massive KLT and massive BCJ double copies are equivalent. Let us now understand our conjecture from the perspective of the BCJ double-copy. We begin by considering the effect of a generalized gauge transformation on the BCJ representation. The amplitude is invariant under the following replacements

$$\begin{aligned}
n_{12}&\to n_{12}+(s_{12}+m_{12}^2)\Delta\\
n_{13}&\to n_{13}+(s_{13}+m_{13}^2)\Delta\\
n_{14}&\to n_{14}+(s_{14}+m_{14}^2)\Delta,
\end{aligned} \tag{5.4.13}$$

for any function Δ. Putting these together we find the kinematic Jacobi sum of numerators transforms as

$$n_{12}+n_{13}+n_{14}\to n_{12}+n_{13}+n_{14}+\left(m_{12}^2+m_{13}^2+m_{14}^2-m_1^2-m_2^2-m_3^2-m_4^2\right)\Delta. \tag{5.4.14}$$

If the spectral condition is not satisfied then we can always find a generalized gauge in which the numerators satisfy color-kinematics duality by using,

$$\Delta=-\frac{n_{12}+n_{13}+n_{14}}{(m_{12}^2+m_{13}^2+m_{14}^2-m_1^2-m_2^2-m_3^2-m_4^2)}. \tag{5.4.15}$$

If the spectral condition is satisfied, however, then there is no choice of Δ that can construct numerators that satisfy the kinematic Jacobi relations from ones that do not. Hence the existence of kinematic Jacobi-satisfying numerators is a non-trivial constraint on the space of BCJ-like models, equivalent to imposing the massive fundamental BCJ relations.

At 4-point, we saw that there is a well-chosen BCJ basis in which the KLT kernel is polynomial, and therefore together with the discussion in Section 5.2.2, the resulting formula defines an amplitude with only physical singularitites. The BCJ version of this statement is that if the spectral condition is satisfied, and there exist color-kinematics duality satisfying numerators, then the BCJ double-copy is free of spurious singularities.

It is clear that a model with a uniform mass spectrum like massive Yang-Mills could only satisfy the 4-point spectral condition if all of the states have zero mass. For more complicated models, with states of multiple masses, the constraints are very restrictive. We will now illustrate these constraints with a few examples.

Example 1: Compton Scattering

Consider a model such as Yang-Mills minimally coupled to a complex adjoint scalar with mass $m \neq 0$. There are three factorization channels contributing to the Compton amplitude $g + \phi \to g + \phi$:

(5.4.16)

The first diagram contributes twice, corresponding to exchanging the labels on the gluons. Here the spectral condition is satisfied since for the external states

$$m_1^2 + m_2^2 + m_3^2 + m_4^2 = 2m^2, \tag{5.4.17}$$

while for the internal states

$$m_{12}^2 + m_{13}^2 + m_{14}^2 = 2m^2. \tag{5.4.18}$$

We must keep in mind that the spectral condition is only a conjectured *necessary* condition for the existence of a local double-copy, not a sufficient one. For a theory to produce a local double-copy, it must also satisfy the BCJ relations. The fact that a sensible double-copy of Compton scattering amplitudes can be defined only if the theory satisfies the massive BCJ relations (5.4.7) was first observed in [155].

Explicitly the color-ordered amplitudes [154]

$$\begin{aligned}\mathcal{A}_4\left[1_\phi, 2_g^+, 3_g^-, 4_{\overline{\phi}}\right] &= -\frac{\langle 3|p_1|2]^2}{s_{23}(s_{12}+m^2)}\\ \mathcal{A}_4\left[1_\phi, 3_g^-, 2_g^+, 4_{\overline{\phi}}\right] &= -\frac{\langle 3|p_1|2]^2}{s_{23}(s_{13}+m^2)},\end{aligned} \tag{5.4.19}$$

satisfy the massive BCJ relation (5.4.7). According to our conjecture the double-copy and the massless limit should commute in such a case. Indeed, taking the massive double-copy and then the massless limit

$$\mathcal{M}_4^{m\neq 0}\left(1_\phi, 2_h^+, 3_h^-, 4_{\overline{\phi}}\right) = \frac{\langle 3|p_1|2]^4}{(s_{12}+m^2)(s_{13}+m^2)} \xrightarrow{m=0} \frac{\langle 3|p_1|2]^4}{s_{12}s_{13}}, \tag{5.4.20}$$

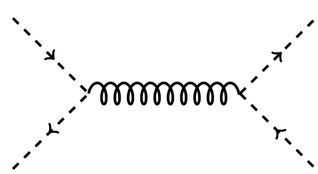
compared to taking the massless limit and then the double-copy

$$s_{14}\mathcal{A}_4^{m=0}\left[1_\phi, 2_g^+, 3_g^-, 4_{\overline{\phi}}\right]\mathcal{A}_4^{m=0}\left[1_\phi, 3_g^-, 2_g^+, 4_\phi\right] = \frac{\langle 3|p_1|2]^4}{s_{12}s_{13}}, \tag{5.4.21}$$

gives the same result.

Example 2: Bhabha Scattering

In the same model as the previous example we can consider Bhabha scattering $\phi+\overline{\phi}\to\phi+\overline{\phi}$ which has two contributing factorization channels related by relabelling:



(5.4.22)

Here the spectral condition is not satisfied since for the external states

$$m_1^2+m_2^2+m_3^2+m_4^2 = 4m^2, \tag{5.4.23}$$

while for the internal states

$$m_{12}^2+m_{13}^2+m_{14}^2 = 0. \tag{5.4.24}$$

Since the spectral condition is not satisfied, there are no associated fundamental BCJ conditions. Similar to the 4-point massive Yang-Mills calculation, we can find color-kinematics duality satisfying numerators and take a massive double-copy, but such an amplitude should not have a smooth $m\to 0$ limit. It is instructive to see this explicitly. We begin with the tree-amplitude calculated using ordinary Feynman rules for a minimally coupled scalar

$$\mathcal{A}_4\left(1_{\phi}^{a_1}, 2_{\overline{\phi}}^{a_2}, 3_{\phi}^{a_3}, 4_{\overline{\phi}}^{a_4}\right) = c_{12}\frac{s_{13}-s_{14}}{s_{12}} + c_{14}\frac{s_{12}-s_{13}}{s_{14}}. \tag{5.4.25}$$

The corresponding BCJ numerators,

$$\begin{aligned} n_{12} &= s_{13} - s_{14} \\ n_{13} &= 0 \\ n_{14} &= s_{12} - s_{13}, \end{aligned} \tag{5.4.26}$$

do not satisfy the kinematic Jacobi relation. We can construct numerators which do, however, by making a generalized gauge transformation

$$\begin{aligned} \hat{n}_{12} &= s_{13} - s_{14} + \frac{1}{4m^2}s_{12}(s_{12}-s_{14}) \\ \hat{n}_{13} &= \frac{1}{4m^2}s_{13}(s_{12}-s_{14}) \\ \hat{n}_{14} &= s_{12} - s_{13} + \frac{1}{4m^2}s_{14}(s_{12}-s_{14}). \end{aligned} \tag{5.4.27}$$

Forming the massive BCJ double-copy, we find

$$\begin{aligned} \mathcal{M}_4\left(1_{\phi}, 2_{\overline{\phi}}, 3_{\phi}, 4_{\overline{\phi}}\right) =& \frac{(s_{13}-s_{14})^2}{s_{12}} + \frac{(s_{12}-s_{13})^2}{s_{14}} \\ &+ 4m^2 + 4s_{12} + 2s_{13} + \frac{1}{4m^2}\left(4s_{12}^2 + 4s_{12}s_{13} + s_{13}^2\right). \end{aligned} \tag{5.4.28}$$

While this is a perfectly physical scattering amplitude, the massive double-copy has generated a contact contribution corresponding to a local operator of the form $\frac{1}{m^2 M_p^2}(\partial\phi)^4$, which diverges as $m \to 0$.

Example 3: Kaluza-Klein Theory

An important class of examples arises from the dimensional reduction of the massless KLT relations in higher dimensions, some of which have already been discussed in [156–159, 163]. This has the effect of generating a Kaluza-Klein tower of states and vertices that conserve Kaluza-Klein number. This conservation law manifests as a conservation of mass at each vertex. For concreteness, consider a $d = 5$ scalar model compactified on $\mathbb{R}^4 \times S^1$, and take for example the scattering process $1 + 2 \to 3 + 4$, where all of the external states are right-moving ($p_i^4 = +m_i$)

states. At the vertices the masses satisfy the sum rules

$$\begin{aligned}
m_1 + m_2 &= m_{12} \\
m_1 - m_3 &= m_{13} \\
m_1 - m_4 &= m_{14} \\
m_1 + m_2 &= m_3 + m_4 .
\end{aligned} \tag{5.4.29}$$

In this case as well, the spectral condition holds with no further constraints,

$$\begin{aligned}
\Rightarrow m_{12}^2 + m_{13}^2 + m_{14}^2 &= 3m_1^2 + m_2^2 + m_3^2 + m_4^2 + 2m_1 m_2 - 2m_1 m_4 - 2m_1 m_3 \\
&= 3m_1^2 + m_2^2 + m_3^2 + m_4^2 - 2m_1^2 \\
&= m_1^2 + m_2^2 + m_3^2 + m_4^2 .
\end{aligned} \tag{5.4.30}$$

Thus any theory that arises as a dimensional reduction of a massless BCJ-compatible theory will automatically satisfy the spectral condition and thus it will give a local double-copy. Such a model gives a *complete* example, for which every scattering amplitude satisfies the spectral constraints, and moreover, if the higher-dimensional model satisfies the massless BCJ relations then so too will the lower-dimensional Kaluza-Klein model. We leave as future work the problem of determining if there are additional complete examples which are not obtained by dimensional reduction.

5.4.2 5-point Spectral Conditions

Locality places the strongest constraints on the massive double-copy. As was exemplified in Section 5.3.4, demanding the existence of color-kinematics duality satisfying 5-point numerators is not a strong enough condition to ensure locality of double-copied 5-point amplitudes. A natural question is what conditions need to be satisfied at 5-point in order for the resulting double-copied amplitude to be local.

We set the calculation up in a manner similar to the 4-point case. We assume the existence of a BCJ representation and allow for general external and intermediate masses, m_i and m_{ij} respectively. Here the masses m_{ij} are exchanged on the ij 2-particle channel. We can then write down a bi-adjoint scalar matrix (O.0.3) where each propagator $s_{ij} + m^2$ is now replaced by $s_{ij} + m_{ij}^2$.

We know that 5-point amplitudes need to factorize on 2-particle channels to give 4-point amplitudes. At 4-point, we saw that locality is only ensured by requiring that the matrix of bi-adjoint scalar amplitudes is singular. This is achieved via the so-called spectral condition (5.4.4). On demanding that this condition is satisfied on every possible 4-point amplitude that could result on a factorization channel, we come up with the following set of conditions,

$$m_{ij}^2 + m_{ik}^2 + m_{jk}^2 = m_i^2 + m_j^2 + m_k^2 + m_{pq}^2 \tag{5.4.31}$$

for each triplet i, j, k and where p, q are the leftover elements in $\{1, 2, 3, 4, 5\}$. There are ${}^5C_3 = 10$ such relations, but they are not all independent. We can reduce them to 5 independent conditions,

$$\begin{aligned}
m_{15}^2 &= 2m_1^2 - m_{12}^2 - m_{13}^2 - m_{14}^2 + m_2^2 + m_3^2 + m_4^2 + m_5^2 \\
m_{25}^2 &= m_1^2 - m_{12}^2 + 2m_2^2 - m_{23}^2 - m_{24}^2 + m_3^2 + m_4^2 + m_5^2 \\
m_{34}^2 &= 2m_1^2 - m_{12}^2 - m_{13}^2 - m_{14}^2 + 2m_2^2 - m_{23}^2 - m_{24}^2 + 2m_3^2 + 2m_4^2 + m_5^2 \\
m_{35}^2 &= -m_1^2 + m_{12}^2 + m_{14}^2 - m_2^2 + m_{24}^2 - m_4^2 \\
m_{45}^2 &= -m_1^2 + m_{12}^2 + m_{13}^2 - m_2^2 + m_{23}^2 - m_3^2.
\end{aligned} \tag{5.4.32}$$

We will refer to these as the *5-point spectral conditions*. These conditions indeed make the bi-adjoint scalar matrix singular. Further, they reduce the rank of the $(n-2)! \times (n-2)! = 6 \times 6$ matrix from full-rank to minimal rank, $(n-3)! = 2$.

As we show in Appendix S, the null vectors of the bi-adjoint scalar matrix give us the 5-point massive BCJ relations,

$$\mathcal{A}_5[13452] = \left(-\frac{m_{12}^2 + s_{12}}{m_{34}^2 + s_{34}} + \frac{m_{35}^2 + s_{35}}{m_{34}^2 + s_{34}}\right)\mathcal{A}_5[13542] + \left(\frac{m_{14}^2 + s_{14}}{m_{34}^2 + s_{34}}\right)\mathcal{A}_5[13524]\,, \tag{5.4.33}$$

$$\begin{aligned}
\mathcal{A}_5[13425] = &\left(\frac{(m_{12}^2 + s_{12})(m_{45}^2 + s_{45})}{(m_{15}^2 + s_{15})(m_{34}^2 + s_{34})}\right)\mathcal{A}_5[13542] \\
&+ \left(\frac{m_{14}^2 + s_{14}}{m_{15}^2 + s_{15}} - \frac{(m_{12}^2 + s_{12})(m_{14}^2 + s_{14})}{(m_{15}^2 + s_{15})(m_{34}^2 + s_{34})}\right)\mathcal{A}_5[13524]\,,
\end{aligned} \tag{5.4.34}$$

$$\begin{aligned}
\mathcal{A}_5[13245] = &\left(\frac{m_{12}^2 + s_{12}}{m_{15}^2 + s_{15}} - \frac{(m_{12}^2 + s_{12})(m_{14}^2 + s_{14})}{(m_{15}^2 + s_{15})(m_{23}^2 + s_{23})}\right)\mathcal{A}_5[13542] \\
&+ \left(\frac{(m_{14}^2 + s_{14})(m_{25}^2 + s_{25})}{(m_{15}^2 + s_{15})(m_{23}^2 + s_{23})}\right)\mathcal{A}_5[13524]\,,
\end{aligned} \tag{5.4.35}$$

$$\mathcal{A}_5[13254] = \left(-\frac{m_{12}^2 + s_{12}}{m_{23}^2 + s_{23}}\right)\mathcal{A}_5[13542] + \left(-\frac{m_{12}^2 + s_{12}}{m_{23}^2 + s_{23}} - \frac{m_{24}^2 + s_{24}}{m_{23}^2 + s_{23}}\right)\mathcal{A}_5[13524]\,, \tag{5.4.36}$$

with the understanding that m_{15}, m_{25}, m_{34}, m_{35} and m_{45} are given by the spectral conditions (5.4.32).

Choosing any 2×2 submatrix $\mathcal{A}^{\phi^3}[\alpha|\beta]$ of the bi-adjoint scalar matrix is now invertible and can be used to define a local double-copy. For example,

$$\mathcal{A}_5^{A\otimes B}(12345) = \sum_{\alpha,\beta=[13542],[13524]} \mathcal{A}_5^{A}[\alpha]\, \mathcal{A}^{\phi^3}[\alpha|\beta]^{-1}\, \mathcal{A}_5^{A}[\beta], \tag{5.4.37}$$

where

$$\mathcal{A}^{\phi^3}[\alpha|\beta] = \begin{pmatrix} \frac{1}{D_1}+\frac{1}{D_{12}}+\frac{1}{D_2}+\frac{1}{D_6}+\frac{1}{D_9} & -\frac{1}{D_{12}}-\frac{1}{D_9} \\ \\ -\frac{1}{D_{12}}+\frac{1}{D_9} & \frac{1}{D_{12}}+\frac{1}{D_3}+\frac{1}{D_4}+\frac{1}{D_5}+\frac{1}{D_9} \end{pmatrix}, \tag{5.4.38}$$

and D_i are as defined in Appendix O.

To explicitly see that the resulting amplitude is local, we perform the following tests. First, we look at the denominator of the resulting KLT formula,

$$(s_{15}+m_{15}^2)(m_{23}^2+s_{23})(s_{34}+m_{34}^2), \tag{5.4.39}$$

again with the understanding that m_{ij} satisfy the spectral conditions (5.4.32) and note that the KLT formula only has poles in physical locations.

Second, we must check that $\mathcal{A}_5^{A\otimes B}(12345)$ factorizes correctly on all poles. Let us look at an example. Consider the pole $s_{23}\to -m_{23}^2$,

$$\begin{aligned}\operatorname*{Res}_{s_{23}=-m_{23}^2} \mathcal{A}_5^{A\otimes B}(12345) =& \frac{(m_{14}^2+s_{14})\,(m_{12}^2+m_{13}^2+s_{12}+s_{13})}{(s_{15}+m_{15}^2)} \Big[\mathcal{A}_5[13542]\left(m_{12}^2+s_{12}\right) \\ &+ \mathcal{A}_5[13524]\left(m_{12}^2+m_{24}^2+s_{12}+s_{24}\right)\Big]^2.\end{aligned} \tag{5.4.40}$$

The massive BCJ relation (5.4.36) tells us that the expression in the square brackets is $\mathcal{A}_5[13254]$ which factorizes into $\mathcal{A}_3[32(-P_{23})] \times \mathcal{A}_4[P_{23}541]$ on the pole to give

$$\begin{aligned}\operatorname*{Res}_{s_{23}=-m_{23}^2} \mathcal{A}_5^{A\otimes B}(12345) =& \frac{(m_{14}^2+s_{14})\,(m_{12}^2+m_{13}^2+s_{12}+s_{13})}{(s_{15}+m_{15}^2)} \left(\mathcal{A}_3[32(-P_{23})]\, \mathcal{A}_4[P_{23}541]\right)^2 \\ =& \mathcal{A}_3^{A\otimes B}(32(-P_{23})) \times \mathcal{A}_4^{A\otimes B}(P_{23}541)\,,\end{aligned} \tag{5.4.41}$$

where we have used the 4-point KLT formula in the last step. Thus the amplitude factorizes correctly on the $s_{23}=-m_{23}^2$ pole.

One can proceed in a similar manner (either with or without the help of massive BCJ relations) to determine that the 5-point KLT formula (5.4.37) factorizes correctly on all poles. Thus, *given a theory that satisfies the 5-point spectral conditions, the KLT formula constructs local amplitudes*, giving us a sensible definition of the 5-point double-copy.

5.4.3 Non-minimal Rank

There is a new possibility that arises at higher-point which is not present at 4-point. This is the ability to reduce the rank of a bi-adjoint scalar matrix from full-rank $(n-2)!$ not to minimal rank $(n-3)!$, but somewhere in between $(n-2)!$ and $(n-3)!$. Since this too makes the $(n-2)! \times (n-2)!$ matrix singular, one might imagine this to be an alternate approach to the massive double-copy that does not require all four BCJ relations to hold. Indeed such a procedure *does not* give rise to local amplitudes. Let us understand how this works at 5-point.

By imposing all-but-one of the spectral conditions (5.4.32), the rank of the 5-point bi-adjoint scalar matrix reduces from 6 to 4, rather than minimal rank 2. For example, let us choose not to impose the spectral condition on m_{34}^2. Since the resulting expressions are difficult to manipulate analytically, we proceed in a particular kinematic configuration where all-but-one (let us say s_{12}) independent Mandelstam variables are fixed.

We can now check the behaviour of the double-copied amplitude as we approach the pole $s_{12} = -m_{12}^2$. We want the double-copied amplitude to factorize as,

$$\operatorname*{Res}_{s_{12}=-m_{12}^2} \mathcal{A}_5^{A\otimes B}(12345) = \mathcal{A}_3^{A\otimes B}(12(-P_{12})) \times \mathcal{A}_4^{A\otimes B}(P_{12}345). \tag{5.4.42}$$

We find that this condition is not met unless,

$$m_{34}^2 = 2m_1^2 - m_{12}^2 - m_{13}^2 - m_{14}^2 + 2m_2^2 - m_{23}^2 - m_{24}^2 + 2m_3^2 + 2m_4^2 + m_5^2, \tag{5.4.43}$$

which is exactly the spectral condition that we left out. Thus, by not imposing all of the BCJ relations, we do not construct local amplitudes.

This supports our conjecture: only by imposing all BCJ relations, i.e. reducing the bi-adjoint scalar matrix to minimal rank, can we construct local amplitudes via the KLT formula.

5.5 Discussion

The proposition of a KLT construction for the double-copy of massive particles opens up many areas of exploration and application. In Section 5.2.2, we noted that our argument for the existence of spurious singularities in the proposed double-copy formula (5.2.11) does not apply to three-dimensional theories. This suggests that the prescription (5.2.11) might be healthy in 3d, despite suffering from spurious singularities in 4d and higher, with a possible example being the conjectured double copy construction of topologically massive gravity from topologically massive gauge theory [178].

In Section 5.3, we see that a putative double-copy of massive Yang-Mills is not well-behaved due to the presence of spurious singularities in the would-be double-copied 5-point amplitude. Further work is needed to address the problems brought to light here. For example, one could investigate the possibility of addition of 5-point operators or new degrees of freedom to the massive Yang-Mills EFT to construct a local double-copy.

Another interesting question is what happens when the bi-adjoint Higgs model presented in Section 5.3 is double copied with itself. It has been shown that the high energy behaviour of a theory of Λ_3 massive gravity cannot be improved by introducing vector or scalar interactions [179]. Therefore, we expect the double-copy of the bi-adjoint Higgs model to fail. A better understanding of the precise nature of this failure would be interesting.

An important assumption that lead to the derivation of the mass spectral conditions presented in Section 5.4 was that a unique mass is exchanged in each factorization channel. We know that a massless KLT formula can be constructed that allows for the exchange of particles of multiple masses on each channel [164]. It would be interesting to see how this construction generalizes to the case of massive external particles and more general spectra.

In addition, we would like to better understand the landscape of theories that produce a local double-copy. We saw examples of dimensionally reduced BCJ-compatible theories in which the Kaluza-Klein tower of massive states and interactions between them manifestly satisfy the spectral condition and hence result in local double-copied amplitudes. We would like to understand whether there are double-copy-compatible theories that do not result from a dimensional reduction.

Finally, in Section 5.4, we saw that spurious singularities are removed *if* the spectral conditions and massive BCJ relations are satisfied. However, we know that massive bi-adjoint scalar theory trivially provides an explicit counter-example to making the converse statement, since it will produce a local, massive double-copy even if the spectral conditions are not satisfied. It is therefore an interesting open problem to determine if there exist further, non-trivial, examples of massive models which double-copy to physical scattering amplitudes but do not satisfy the spectral condition. One pathway to such a construction would be to try and find a model which admits a local, off-shell representation of the kinematic algebra, similar to [180, 181]. Since the numerators of such a model are local by construction, it is clear from the BCJ form of the double-copy that no spurious poles can be generated. Even more interestingly, given such a set of local, kinematic Jacobi satisfying numerators, we can always form a heterotic double-copy with the numerators of a generic, spectral condition violating, massive model. Since the result does not depend on the generalized gauge used for the numerators of the latter, they can always be taken to be the local representation given by Feynman rules, and so even in this case, we see that no spurious poles can be generated. We see then that constructing even a single example of a model with a local, off-shell representation of the kinematic algebra, is sufficient to generate an infinite number of examples of healthy, massive double-copies. We leave this and similar investigations to future work.

APPENDIX A

Derivation of Subtracted Recursion Relations

In this appendix, we derive the manifestly local form (2.4.9) of the subtracted recursion relations. For a given factorization channel, consider from the recursion relations (2.4.8) the expression

$$\frac{\hat{\mathcal{A}}_L^{(I)}(z_I^{\pm})\hat{\mathcal{A}}_R^{(I)}(z_I^{\pm})}{F(z_I^{\pm})P_I^2(1-z_I^{\pm}/z_I^{\mp})} = \sum_{z_I=z_I^{\pm}} \mathrm{Res}_{z=z_I} \frac{\hat{\mathcal{A}}_L^{(I)}(z)\hat{\mathcal{A}}_R^{(I)}(z)}{z\,F(z)\,\hat{P}_I^2} = \oint_{\mathcal{C}} dz\,\frac{\hat{\mathcal{A}}_L^{(I)}(z)\hat{\mathcal{A}}_R^{(I)}(z)}{z\,F(z)\,\hat{P}_I^2}\,, \qquad \text{(A.0.1)}$$

where the contour surrounds only the two poles $z_I^{\pm}$. The second equality is non-trivial and deserves clarification. In the second expression, the subamplitudes $\hat{\mathcal{A}}_L^{(I)}(z)$ and $\hat{\mathcal{A}}_R^{(I)}(z)$ are only defined precisely on the residue values $z = z_I^{\pm}$ for which the internal momentum $\hat{P}_I$ is on-shell; in general one cannot just think of $\hat{\mathcal{A}}_{L,R}^{(I)}(z)$ as functions of z. However, in the product $\hat{\mathcal{A}}_L^{(I)}(z)\hat{\mathcal{A}}_R^{(I)}(z)$, one can eliminate the internal momentum $\hat{P}_I$ in favor of the n shifted external momenta by using momentum conservation. Then the resulting expression can be analytically continued in z away from the residue value. This is implicitly what has been done in performing the second step in (A.0.1).

Let us assess the large-z behavior of the integrand in (A.0.1). The L and R subamplitudes have couplings g_L and g_R such that $g_L g_R = g_n$, with g_n the coupling of $\mathcal{A}_n$. Their mass-dimensions are related as $[g_L] + [g_R] = [g_n]$. Hence, using $n_L + n_R = n + 2$ and (2.4.16), we find that the numerator behaves at large z as

$$\hat{\mathcal{A}}_L^{(I)}(z)\hat{\mathcal{A}}_R^{(I)}(z) \to z^{D_L} z^{D_R} = z^{6-n-[g_n]-\sum_{i=1}^n s_i - 2s_P} = z^{D+2-2s_P}\,, \qquad \text{(A.0.2)}$$

where s_P denotes the spin of the particle exchanged on the internal line and D is the large z behavior of the $\mathcal{A}_n$ which we know satisfies $D - \sum_{i=1}^n \sigma_i < 0$, by the assumption that the amplitude $\mathcal{A}_n$ is recursively constructible by the criterion (2.4.10). We therefore conclude that the integrand in (A.0.1) behaves as $z^{D-1-\sum_{i=1}^n \sigma_i - 2s_P}$, i.e. it goes to zero as $1/z^2$ or faster. Hence, there is no simple pole at $z \to \infty$.

If we deform the contour, we get the sum over all poles $z \neq z_I^\pm$ in $\hat{\mathcal{A}}_L^{(I)}(z)\hat{\mathcal{A}}_R^{(I)}(z)/(z\,F(z)\,\hat{P}_I^2)$. Let us *assume that* $\mathcal{A}_L^{(I)}$ *and* $\mathcal{A}_R^{(I)}$ *are both local*: they have no poles and hence we pick up exactly the simple poles at $z = 0$ and $z = 1/a_i$ for $i = 1, 2, \ldots, n$. We then conclude that the soft recursion relations take the form

$$\mathcal{A}_n = \sum_I \sum_{z'=0,\frac{1}{a_1},\ldots,\frac{1}{a_n}} \sum_{|\psi^{(I)}\rangle} \text{Res}_{z=z'} \frac{\hat{\mathcal{A}}_L^{(I)}(z)\hat{\mathcal{A}}_R^{(I)}(z)}{z\,F(z)\,\hat{P}_I^2}\,, \tag{A.0.3}$$

where $F(z) = \prod_{i=1}^n (1 - a_i z)^{\sigma_i}$. This form of the recursion relation is manifestly rational in the momenta.

Note that only the $z = 0$ residues give pole terms in $\mathcal{A}_n$. Therefore the sum of the $1/a_i$ residues over all channels must be a local polynomial in the momenta. For example, it is valid for the reconstruction of the 6-point scalar amplitude of NLSM, but not for the reconstruction of an 8-point amplitude.

APPENDIX B

Explicit expressions for amplitudes

In this appendix, we present expressions for the 4- and 6-point amplitudes of the theories discussed in the main text. The 6-point amplitudes were reconstructed with the 4-point ones as input, by means of the subtracted recursion relations and the the supersymmetry Ward identities also discussed in the main text.

B.0.1 Supersymmetric $\mathbb{CP}^1$ NLSM

Below, we list the amplitudes for the $\mathbb{CP}^1$ $\mathcal{N}=1$ supersymmetric NLSM. This model is discussed in Section 2.7 as an illustration of our methods.

The 4-point amplitudes are:

$$\mathcal{A}_4(1_Z 2_{\bar{Z}} 3_Z 4_{\bar{Z}}) = \frac{1}{\Lambda^2} s_{13}\,, \tag{B.0.1}$$

$$\mathcal{A}_4(1_Z 2_{\bar{Z}} 3^+_\psi 4^-_\psi) = -\frac{1}{\Lambda^2}\,[23]\,\langle 24\rangle = \frac{1}{2\Lambda^2}\langle 4|p_1 - p_2|3]\,, \tag{B.0.2}$$

$$\mathcal{A}_4(1^+_\psi 2^-_\psi 3^+_\psi 4^-_\psi) = -\frac{1}{\Lambda^2}\,[13]\,\langle 24\rangle\,. \tag{B.0.3}$$

They serve as the input for computing the 6-point amplitudes recursively:

$$\begin{aligned}&\mathcal{A}_6(1_Z 2_{\bar{Z}} 3_Z 4_{\bar{Z}} 5_Z 6_{\bar{Z}})\\ &\quad= \frac{1}{\Lambda^4}\left[\left(\frac{s_{13}s_{46}}{p_{123}^2} + (1\leftrightarrow 5) + (3\leftrightarrow 5)\right) + (2\leftrightarrow 4) + (2\leftrightarrow 6) + 3p_{135}^2\right],\end{aligned} \tag{B.0.4}$$

$$\mathcal{A}_6(1_Z 2_{\bar{Z}} 3_Z 4_{\bar{Z}} 5^+_\psi 6^-_\psi)$$

$$= \frac{1}{\Lambda^4}\Bigg[\left(\frac{s_{13}\,[54]\,\langle 46\rangle}{p_{123}^2} + (2\leftrightarrow 4)\right) - \left(\frac{s_{24}\,[51]\,\langle 16\rangle}{p_{156}^2} + (1\leftrightarrow 3)\right)$$
$$- \left(\left(\frac{[54]\,\langle 4|p_{126}|2]\langle 26\rangle}{p_{126}^2} + (1\leftrightarrow 3)\right) + (2\leftrightarrow 4)\right) + \langle 6|p_{135}|5]\Bigg]\,, \tag{B.0.5}$$

$$\mathcal{A}_6(1_Z 2_{\bar{Z}} 3_\psi^+ 4_\psi^- 5_\psi^+ 6_\psi^-)$$
$$= \frac{1}{\Lambda^4}\Bigg[-\left(\frac{[31]\,\langle 1|p_{123}|5]\langle 46\rangle}{p_{123}^2} - (3\leftrightarrow 5)\right) + \left(\frac{[35]\,\langle 4|p_{126}|2]\langle 26\rangle}{p_{126}^2} - (4\leftrightarrow 6)\right)$$
$$- \left(\left(\frac{[51]\,\langle 16\rangle\,[32]\,\langle 24\rangle}{p_{156}^2} - (3\leftrightarrow 5)\right) - (4\leftrightarrow 6)\right)\Bigg]\,, \tag{B.0.6}$$

$$\mathcal{A}_6(1_\psi^+ 2_\psi^- 3_\psi^+ 4_\psi^- 5_\psi^+ 6_\psi^-)$$
$$= \frac{1}{\Lambda^4}\Bigg[\left(\frac{[13]\,\langle 2|p_{123}|5]\langle 46\rangle}{p_{123}^2} - (1\leftrightarrow 5) - (3\leftrightarrow 5)\right) - (2\leftrightarrow 4) - (2\leftrightarrow 6)\Bigg]\,. \tag{B.0.7}$$

Note that only the pure scalar amplitudes and the 2-fermion amplitudes have local terms. The 6-point amplitudes satisfy the NMHV supersymmetry Ward identities in (2.6.21)-(2.6.23).

B.0.2 Supersymmetric Dirac-Born-Infeld Theory

The amplitudes of $\mathcal{N} = 1$ supersymmetric Dirac-Born-Infeld theory are all recursively constructible. The 4-point amplitudes are

$$\mathcal{A}_4(1_Z 2_{\bar{Z}} 3_Z 4_{\bar{Z}}) = \frac{1}{\Lambda^4} s_{13}^2\,, \tag{B.0.8}$$
$$\mathcal{A}_4(1_Z 2_{\bar{Z}} 3_\psi^+ 4_\psi^-) = \frac{1}{\Lambda^4} s_{13}\,[32]\,\langle 24\rangle = \frac{1}{2\Lambda^4} s_{13}\langle 4|p_1 - p_2|3]\,, \tag{B.0.9}$$
$$\mathcal{A}_4(1_\psi^+ 2_\psi^- 3_\psi^+ 4_\psi^-) = -\frac{1}{\Lambda^4} s_{13}\,[13]\,\langle 24\rangle\,. \tag{B.0.10}$$

and the results of soft subtracted recursion for the 6-point amplitudes are

$$\mathcal{A}_6(1_Z 2_{\bar{Z}} 3_Z 4_{\bar{Z}} 5_Z 6_{\bar{Z}})$$
$$= \frac{1}{\Lambda^8}\Bigg[\left(\frac{s_{13}^2 s_{46}^2}{p_{123}^2} + (1\leftrightarrow 5) + (3\leftrightarrow 5)\right) + (2\leftrightarrow 4) + (2\leftrightarrow 6) - p_{135}^6\Bigg]\,, \tag{B.0.11}$$

$$\mathcal{A}_6(1_Z 2_{\bar{Z}} 3_Z 4_{\bar{Z}} 5_\psi^+ 6_\psi^-)$$
$$= \frac{1}{\Lambda^8}\Bigg[\left(\left(\frac{s_{26}s_{35}\,[54]\,\langle 4|p_{126}|1]\langle 16\rangle}{p_{126}^2} + (1\leftrightarrow 3)\right) + (2\leftrightarrow 4)\right) + \left(\frac{s_{13}^2 s_{46}\,[54]\,\langle 46\rangle}{p_{123}^2} + (2\leftrightarrow 4)\right)$$
$$- \left(\frac{s_{15}s_{24}^2\,[51]\,\langle 16\rangle}{p_{156}^2} + (1\leftrightarrow 3)\right) + \left(s_{13}s_{24} - (s_{13}+s_{24})\,p_{135}^2\right)\langle 6|p_{24}|5]\Bigg]\,, \tag{B.0.12}$$

$$\mathcal{A}_6(1_Z 2_{\bar{Z}} 3_\psi^+ 4_\psi^- 5_\psi^+ 6_\psi^-)$$
$$= \frac{1}{\Lambda^8}\Bigg[(s_{24}+s_{26})\,p_{135}^2\,[35]\,\langle 46\rangle - \left(\left(\frac{s_{15}s_{24}\,[51]\,\langle 16\rangle\,[32]\,\langle 24\rangle}{p_{156}^2} - (3\leftrightarrow 5)\right) - (4\leftrightarrow 6)\right)$$

$$-\left(\frac{s_{13}s_{46}\,[32]\,\langle 2|p_{123}|5]\langle 46\rangle}{p_{123}^2} - (3\leftrightarrow 5)\right) + \left(\frac{s_{26}s_{35}\,[35]\,\langle 4|p_{126}|2]\langle 26\rangle}{p_{126}^2} - (4\leftrightarrow 6)\right)\Bigg], \tag{B.0.13}$$

$$\mathcal{A}_6(1_\psi^+2_\psi^-3_\psi^+4_\psi^-5_\psi^+6_\psi^-)$$
$$= \frac{1}{\Lambda^8}\left[\left(\frac{s_{13}s_{46}\,[13]\,\langle 2|p_{123}|5]\langle 46\rangle}{p_{123}^2} - (1\leftrightarrow 5) - (3\leftrightarrow 5)\right) - (2\leftrightarrow 4) - (2\leftrightarrow 6)\right]. \tag{B.0.14}$$

The 6-point amplitudes satisfy the NMHV supersymmetry Ward identities in (2.6.21)-(2.6.23). As in the case of the NLSM, only the pure scalar amplitudes and the 2-fermion amplitudes have local terms.

B.0.3 Supersymmetric Born-Infeld Theory

In this subsection, we list the amplitudes of Born-Infeld theory. This theory is the leading order contribution to the effective field theory of a Goldstone $\mathcal{N} = 1$ vector multiplet. The 4-point amplitudes are

$$\mathcal{A}_4(1_\psi^+2_\psi^-3_\psi^+4_\psi^-) = -\frac{1}{\Lambda^4}\,[13]\,\langle 24\rangle s_{13}\,, \tag{B.0.15}$$
$$\mathcal{A}_4(1_\psi^+2_\psi^-3_\gamma^+4_\gamma^-) = \frac{1}{\Lambda^4}\,[13]\,[23]\,\langle 24\rangle^2 = -\frac{1}{2\Lambda^4}[13]\langle 4|p_1-p_2|3]\langle 24\rangle\,, \tag{B.0.16}$$
$$\mathcal{A}_4(1_\gamma^+2_\gamma^-3_\gamma^+4_\gamma^-) = \frac{1}{\Lambda^4}\,[13]^2\,\langle 24\rangle^2\,. \tag{B.0.17}$$

Except for the all-vector amplitudes, all amplitudes are constructible with soft subtracted recursion. The all-vector amplitudes are the amplitudes of Born-Infeld theory, and they are fixed in terms of the other amplitudes using the supersymmetry Ward identities. In particular, at 6-points, we use (2.6.23) and the remaining five identities in (2.6.21)-(2.6.23) are used as checks. The results are

$$\mathcal{A}_6(1_\psi^+2_\psi^-3_\psi^+4_\psi^-5_\psi^+6_\psi^-)$$
$$= \frac{1}{\Lambda^8}\left[\left(\frac{s_{13}s_{46}\,[13]\,\langle 2|p_{123}|5]\langle 46\rangle}{p_{123}^2} - (1\leftrightarrow 5) - (3\leftrightarrow 5)\right) - (2\leftrightarrow 4) - (2\leftrightarrow 6)\right], \tag{B.0.18}$$

$$\mathcal{A}_6(1_\gamma^+2_\gamma^-3_\psi^+4_\psi^-5_\psi^+6_\psi^-)$$
$$= \frac{1}{\Lambda^8}\Bigg[\left(\frac{s_{46}\,[13]^2\,\langle 2|p_{123}|5]\langle 23\rangle\langle 46\rangle}{p_{123}^2} - (3\leftrightarrow 5)\right) + \left(\frac{s_{35}\,[14]\,[35]\,\langle 6|p_{124}|1]\langle 24\rangle^2}{p_{124}^2} - (4\leftrightarrow 6)\right)$$
$$-\left(\left(\frac{[13]\,[14]\,\langle 4|p_{134}|5]^2\langle 52\rangle\langle 26\rangle}{p_{134}^2} - [13]\,\langle 2|p_{35}|1]\langle 6|p_{46}|5]\langle 24\rangle - (3\leftrightarrow 5)\right) - (4\leftrightarrow 6)\right)\Bigg], \tag{B.0.19}$$

$$\mathcal{A}_6(1_\gamma^+2_\gamma^-3_\gamma^+4_\gamma^-5_\psi^+6_\psi^-)$$

$$= \frac{1}{\Lambda^8}\Bigg[\Bigg(\frac{[13]^2\,\langle 2|p_{123}|5]^2\langle 54\rangle\langle 46\rangle}{p_{123}^2} + (2\leftrightarrow 4)\Bigg) + \Bigg(\frac{[35]\,[36]\,\langle 6|p_{124}|1]^2\langle 24\rangle^2}{p_{124}^2} + (1\leftrightarrow 3)\Bigg)$$
$$+ \Bigg(\Bigg(\frac{[15]^2\,[36]\,\langle 2|p_{125}|3]\langle 25\rangle\langle 46\rangle^2}{p_{125}^2} + (1\leftrightarrow 3)\Bigg) + (2\leftrightarrow 4)\Bigg) + [13]^2\,\langle 6|p_{24}|5]\langle 24\rangle^2\Bigg], \tag{B.0.20}$$

$$\mathcal{A}_6(1^+_\gamma 2^-_\gamma 3^+_\gamma 4^-_\gamma 5^+_\gamma 6^-_\gamma)$$
$$= \frac{1}{\Lambda^8}\Bigg[\Bigg(\frac{[13]^2\langle 2|p_{123}|5]^2\langle 46\rangle^2}{p_{123}^2} + (1\leftrightarrow 5) + (3\leftrightarrow 5)\Bigg) + (2\leftrightarrow 4) + (2\leftrightarrow 6)\Bigg]. \tag{B.0.21}$$

In this case, only $\mathcal{A}_6(1^+_\gamma 2^-_\gamma 3^+_\gamma 4^-_\gamma 5^+_\psi 6^-_\psi)$ has local terms.

B.0.4 Supersymmetric Quartic Galileon Theory

Below, we list the amplitudes of an $\mathcal{N} = 1$ supersymmetric quartic Galileon. This model was discussed in detail in [18] and reviewed in Section 2.9. The 4-point amplitudes are

$$\mathcal{A}_4(1_Z 2_{\bar{Z}} 3_Z 4_{\bar{Z}}) = \frac{1}{\Lambda^6} s_{12} s_{13} s_{23}\,, \tag{B.0.22}$$
$$\mathcal{A}_4(1_Z 2_{\bar{Z}} 3^+_\psi 4^-_\psi) = \frac{1}{\Lambda^6} s_{12} s_{23}\,[32]\,\langle 24\rangle = \frac{1}{2\Lambda^6} s_{12} s_{23} \langle 4|p_1 - p_2|3]\,, \tag{B.0.23}$$
$$\mathcal{A}_4(1^+_\psi 2^-_\psi 3^+_\psi 4^-_\psi) = -\frac{1}{\Lambda^6}\,[13]\,\langle 24\rangle s_{12} s_{23}\,. \tag{B.0.24}$$

At 6-point, only the amplitudes with at most two fermions are constructible with soft subtracted recursion relations. The remaining ones are fixed by the supersymmetry Ward identities (2.6.21)-

(2.6.23), and we find

$$
\begin{aligned}
&\mathcal{A}_6(1_Z 2_{\bar{Z}} 3_Z 4_{\bar{Z}} 5_Z 6_{\bar{Z}}) \\
&\quad = \frac{1}{\Lambda^{12}} \left[\left(\frac{s_{12}s_{13}s_{23}s_{45}s_{46}s_{56}}{p_{123}^2} + (1 \leftrightarrow 5) + (3 \leftrightarrow 5) \right) + (2 \leftrightarrow 4) + (2 \leftrightarrow 6) \right], \qquad \text{(B.0.25)} \\
&\mathcal{A}_6(1_Z 2_{\bar{Z}} 3_Z 4_{\bar{Z}} 5_\psi^+ 6_\psi^-) \\
&\quad = \frac{1}{\Lambda^{12}} \left[\left(\frac{s_{12}s_{13}s_{23}s_{45}s_{56}\,[54]\,\langle 46\rangle}{p_{123}^2} + (2 \leftrightarrow 4) \right) - \left(\frac{s_{16}s_{23}s_{24}s_{34}s_{56}\,[51]\,\langle 16\rangle}{p_{156}^2} + (1 \leftrightarrow 3) \right) \right. \\
&\qquad\qquad \left. + \left(\left(\frac{s_{12}s_{16}s_{34}s_{45}\,[53]\,\langle 3|p_{126}|2]\langle 26\rangle}{p_{126}^2} + (1 \leftrightarrow 3) \right) + (2 \leftrightarrow 4) \right) \right], \qquad \text{(B.0.26)} \\
&\mathcal{A}_6(1_Z 2_{\bar{Z}} 3_\psi^+ 4_\psi^- 5_\psi^+ 6_\psi^-) \\
&\quad = \frac{1}{\Lambda^{12}} \left[\left(\frac{[31]\,\langle 1|p_{46}|5]\langle 46\rangle}{p_{123}^2} - (3 \leftrightarrow 5) \right) + \left(\frac{[35]\,\langle 4|p_{16}|2]\langle 26\rangle}{p_{126}^2} - (4 \leftrightarrow 6) \right) \right. \\
&\qquad\qquad \left. - \left(\left(\frac{[32]\,\langle 24\rangle\,[51]\,\langle 16\rangle}{p_{156}^2} - (3 \leftrightarrow 5) \right) - (4 \leftrightarrow 6) \right) \right], \qquad \text{(B.0.27)} \\
&\mathcal{A}_6(1_\psi^+ 2_\psi^- 3_\psi^+ 4_\psi^- 5_\psi^+ 6_\psi^-) \\
&\quad = \frac{1}{\Lambda^{12}} \left[\left(\frac{[13]\,\langle 2|p_{13}|5]\langle 46\rangle}{p_{123}^2} - (1 \leftrightarrow 5) - (3 \leftrightarrow 5) \right) - (2 \leftrightarrow 4) - (2 \leftrightarrow 6) \right]. \qquad \text{(B.0.28)}
\end{aligned}
$$

None of the amplitudes have local terms.

B.0.5 Chiral Perturbation Theory

Below, we list the color-ordered amplitudes of the $\frac{U(N)\times U(N)}{U(N)}$ sigma model, with higher derivative corrections, referred to as chiral perturbation theory in the main text. Different color orderings are related to the ones listed by momentum relabelling. At 4-point we have

$$
\mathcal{A}_4[1,2,3,4] = \frac{g_2}{\Lambda^2} t + \frac{g_6}{\Lambda^6} t \left(s^2 + t^2 + u^2\right) + \frac{g_8}{\Lambda^8} s t^2 u + \mathcal{O}(\Lambda^{-10}) \qquad \text{(B.0.29)}
$$

and at 6-point

$$
\begin{aligned}
&\mathcal{A}_6[1,2,3,4,5,6] \\
&= \frac{g_2^2}{\Lambda^4} \left[\frac{s_{13}s_{46}}{p_{123}^2} + \frac{s_{24}s_{15}}{p_{234}^2} + \frac{s_{35}s_{26}}{p_{345}^2} - s_{24} - s_{26} - s_{46} \right] \\
&+ \frac{g_2 g_6}{\Lambda^8} \left[\frac{s_{13}s_{46}}{p_{123}^2} \left(s_{12}^2 + s_{13}^2 + s_{23}^2 + s_{45}^2 + s_{46}^2 + s_{56}^2 \right) \right. \\
&\quad + \frac{s_{24}s_{15}}{p_{234}^2} \left(s_{23}^2 + s_{24}^2 + s_{34}^2 + s_{56}^2 + s_{15}^2 + s_{16}^2 \right) + \frac{s_{35}s_{26}}{p_{345}^2} \left(s_{34}^2 + s_{35}^2 + s_{45}^2 + s_{16}^2 + s_{26}^2 + s_{12}^2 \right) \\
&\quad - 2\big(s_{26}^3 + s_{23}s_{26}^2 + s_{25}s_{26}^2 + s_{34}s_{26}^2 + s_{45}s_{26}^2 + s_{23}^2 s_{26} + s_{25}^2 s_{26} + s_{34}^2 s_{26} + s_{35}^2 s_{26} + s_{45}^2 s_{26}
\end{aligned}
$$

$$
\begin{aligned}
&\quad + s_{23}s_{34}s_{26} + s_{23}s_{35}s_{26} + s_{25}s_{35}s_{26} + s_{34}s_{36}s_{26} + s_{23}s_{45}s_{26} + s_{34}s_{45}s_{26} + s_{36}s_{45}s_{26} \\
&\quad + s_{46}^3 + s_{24}s_{25}^2 + s_{24}s_{35}^2 + s_{24}s_{45}^2 + s_{23}s_{46}^2 + s_{25}s_{46}^2 + s_{34}s_{46}^2 + s_{35}s_{46}^2 + s_{36}s_{46}^2 \\
&\quad + s_{45}s_{46}^2 + s_{24}s_{35}s_{36} + s_{25}^2s_{46} + s_{34}^2s_{46} + s_{35}^2s_{46} + s_{36}^2s_{46} + s_{45}^2s_{46} + s_{23}s_{25}s_{46} \\
&\quad + s_{25}s_{34}s_{46} + s_{23}s_{45}s_{46} + s_{34}s_{45}s_{46} + s_{35}s_{45}s_{46} + s_{36}s_{45}s_{46}\Big) \\
&- 4\Big(s_{24}^3 + s_{25}s_{24}^2 + s_{35}s_{24}^2 + s_{45}s_{24}^2 + s_{23}^2s_{24} + s_{34}^2s_{24} + s_{36}^2s_{24} + s_{23}s_{25}s_{24} + s_{25}s_{34}s_{24} \\
&\quad + s_{23}s_{35}s_{24} + s_{25}s_{35}s_{24} + s_{34}s_{35}s_{24} + s_{26}s_{36}s_{24} + s_{23}s_{45}s_{24} + s_{25}s_{45}s_{24} + s_{34}s_{45}s_{24} \\
&\quad + s_{35}s_{45}s_{24} + s_{36}s_{45}s_{24} + s_{23}s_{25}s_{26} + s_{25}s_{26}s_{34} + s_{25}s_{26}s_{45} + s_{23}^2s_{46} + s_{25}s_{26}s_{46} \\
&\quad + s_{23}s_{34}s_{46} + s_{23}s_{35}s_{46} + s_{34}s_{35}s_{46} + s_{23}s_{36}s_{46} + s_{25}s_{36}s_{46} + s_{26}s_{36}s_{46} + s_{34}s_{36}s_{46} \\
&\quad + s_{35}s_{36}s_{46} + s_{25}s_{45}s_{46} + s_{26}s_{45}s_{46}\Big) \\
&- 6\Big(s_{23}s_{24}^2 + s_{34}s_{24}^2 + s_{36}s_{24}^2 + s_{26}^2s_{24} + s_{46}^2s_{24} + s_{23}s_{26}s_{24} + s_{25}s_{26}s_{24} + s_{23}s_{34}s_{24} \\
&\quad + s_{26}s_{34}s_{24} + s_{23}s_{36}s_{24} + s_{25}s_{36}s_{24} + s_{26}s_{45}s_{24} + s_{25}s_{46}s_{24} + s_{35}s_{46}s_{24} + s_{45}s_{46}s_{24} \\
&\quad + s_{26}s_{46}^2 + s_{25}s_{34}s_{36} + s_{25}s_{36}s_{45} + s_{26}^2s_{46} + s_{23}s_{26}s_{46} + s_{26}s_{34}s_{46}\Big) \\
&- 8s_{24}\Big(s_{24}s_{26} + s_{34}s_{36} + s_{23}s_{46} + s_{24}s_{46} + s_{34}s_{46} + s_{36}s_{46}\Big) - 12s_{24}s_{26}s_{46}\Bigg] + \mathcal{O}(\Lambda^{-10})\,.
\end{aligned}
\tag{B.0.30}
$$

These amplitudes are discussed in further detail in Section 2.9.4.

APPENDIX C

Recursion Relations and Ward Identities

We show that if the seed amplitudes of a recursive theory satisfy a set of Ward identities, then all recursively constructible n-point amplitudes also satisfy them. For Abelian groups, this follows from two features:

(a) additive charges have Ward identities that simply state that the sum of charges of the states in an amplitude must vanish.

(b) CPT conjugate states sitting on either end of a factorization channel have equal and opposite charges.

Hence recursion will result in amplitudes that respect the Abelian symmetry so long as the seed amplitudes do.

Now consider Ward identities generated by elements of a semi-simple Lie algebra. In the root space decomposition of the algebra, we can choose a triplet of generators: raising operators $\mathcal{T}_+$, lowering operators $\mathcal{T}_-$, and "diagonal" $\mathcal{T}_0$ generators, for each positive root that satisfy the algebra

$$[\mathcal{T}_+, \mathcal{T}_-] = \mathcal{T}_0 \,, \quad [\mathcal{T}_+, \mathcal{T}_0] = -2\mathcal{T}_+ \,, \quad [\mathcal{T}_-, \mathcal{T}_0] = 2\mathcal{T}_- \,. \tag{C.0.1}$$

In order for representations of this algebra to be physical, CPT must be an algebra automorphism. The CPT charge conjugation generator $\mathcal{C}$ must also flip the sign of the additive $\mathcal{T}_0$-charge. So we determine the action of $\mathcal{C}$ to be

$$\begin{aligned} \mathcal{C} \cdot \mathcal{T}_0 \cdot X &= -\mathcal{T}_0 \cdot \mathcal{C} \cdot X = -\mathcal{T}_0 \cdot \tilde{X} \,, \\ \mathcal{C} \cdot \mathcal{T}_+ \cdot X &= -\mathcal{T}_- \cdot \mathcal{C} \cdot X = -\mathcal{T}_- \cdot \tilde{X} \,, \\ \mathcal{C} \cdot \mathcal{T}_- \cdot X &= -\mathcal{T}_+ \cdot \mathcal{C} \cdot X = -\mathcal{T}_+ \cdot \tilde{X} \,, \end{aligned} \tag{C.0.2}$$

where X is a physical state and we have defined the conjugate state $\tilde{X}$ to be the charge conjugate of X, i.e. $\tilde{X} = \mathcal{C} \cdot X$.

If the S-matix is recursively constructible (at some order in the derivative expansion) then each n-point amplitude is given as a sum over factorization singularities with residues given in terms of a product of amplitudes with fewer external states

$$\mathcal{A}_n(1, \cdots, n) = \sum_I \sum_X \operatorname*{Res}_{z=z_I^\pm} \frac{\hat{\mathcal{A}}_L^{(I)}(z)\hat{\mathcal{A}}_R^{(I)}(z)}{z\hat{P}_I(z)^2 F(z)}\,, \tag{C.0.3}$$

where I labels all possible factorization channels and X the exchanged internal states. Since $\mathcal{T}_0$ is diagonal, the Ward identity generated by $\mathcal{T}_0$ works just like in the Abelian case – charges can be assigned to the physical states and recursion preserves this charge in any n-point amplitude. More complicated are the non-diagonal generators $\mathcal{T}_\pm$. For simplicity, we present the argument explicitly for $SU(2)_R$ Ward identities as they apply to the $\mathcal{N} = 2$ NLSM described in Section 2.7.2. For $SU(2)_R$, the action of $\mathcal{T}_+$ on the fermion helicity states is given in (2.7.25). The scalar and vectors are singlets under $SU(2)_R$.

The statement of the $SU(2)_R$ Ward identity is that $\mathcal{T}_+ \cdot \mathcal{A}_n(1, ..., n) = 0$. The inductive assumption is that this holds true for the lower-point amplitudes in the recursive expression for $\mathcal{A}_n(1, ..., n)$. We already know from Section 2.7.2 that $SU(2)_R$ is a symmetry of the 3- and 4-point amplitudes, so that provides the basis of induction.

The action of $\mathcal{T}_+$ on the recursive expression for an n-point amplitude is

$$\mathcal{T}_+ \cdot \mathcal{A}_n(1, ..., n) \equiv \sum_{i=1}^n (-1)^{P_i} \mathcal{A}_n(1, ..., \mathcal{T}_+ \cdot i, ..., n) \tag{C.0.4}$$

$$= \sum_I \sum_X \operatorname*{Res}_{z=z_I^\pm} \Bigg[\sum_{i\in I} (-1)^{P_i} \frac{\hat{\mathcal{A}}_L^{(I)}(\ldots, \mathcal{T}_+ \cdot i, \ldots, X)\hat{\mathcal{A}}_R^{(I)}(\ldots)}{z\hat{P}_I(z)^2 F(z)}$$

$$+ \sum_{i\notin I} (-1)^{P_i} \frac{\hat{\mathcal{A}}_L^{(I)}(\ldots)\hat{\mathcal{A}}_R^{(I)}(\tilde{X}, \ldots, \mathcal{T}_+ \cdot i, \ldots)}{z\hat{P}_I(z)^2 F(z)} \Bigg]\,, \tag{C.0.5}$$

where $P_i = 0$ or 1 corresponds to the additional signs in the prefactors for the action of $\mathcal{T}_+$ as given in Table 2.7.25. We now prove that this expression vanishes channel by channel. Without loss of generality, we will show that the contribution from the $(1 \ldots k)^\pm$ channel vanishes independently, where $+$ means the contribution from the $z^\pm$ residue. The argument follows for all other factorization channels by replacing $(1 \ldots k)^\pm$ by $I^\pm$. For the $(1 \ldots k)$-channel, the relevant part of (C.0.4) that we want to show vanishes is

$$\sum_X \Bigg[\bigg(\sum_{i=1}^{k}(-1)^{P_i}\hat{\mathcal{A}}_L(1,\dots,\mathcal{T}_+\cdot i,\dots,k,X)\bigg)\hat{\mathcal{A}}_R(\tilde{X},k+1,\dots,n) \\ +\hat{\mathcal{A}}_L(1,\dots,k,X)\bigg(\sum_{i=k+1}^{n}(-1)^{P_i}\hat{\mathcal{A}}_R(\tilde{X},k+1,\dots,\mathcal{T}_+\cdot i,\dots,n)\bigg)\Bigg]. \tag{C.0.6}$$

By the inductive assumption, the lower-point amplitudes respect the $\mathcal{T}_+$ Ward identities

$$\sum_{i=1}^{k}(-1)^{P_i}\hat{\mathcal{A}}_L(1,\dots,\mathcal{T}_+\cdot i,\dots,k,X) \;=\; (-1)^{P_X+1}\hat{\mathcal{A}}_L(1,\dots,k,\mathcal{T}_+\cdot X)\,, \tag{C.0.7}$$

and similarly for $\hat{\mathcal{A}}_R$. Using this relation and splitting the sum over particles X allows us to rewrite (C.0.6) as

$$\begin{aligned}&-\sum_X(-1)^{P_X}\Big[\hat{\mathcal{A}}_L(1,\dots,k,\mathcal{T}_+\cdot X)\hat{\mathcal{A}}_R(\tilde{X},k+1,\dots,n)\Big]\\ &-\sum_{X'}(-1)^{P_{\tilde{X}'}}\Big[\hat{\mathcal{A}}_L(1,\dots,k,X')\hat{\mathcal{A}}_R(\mathcal{T}_+\cdot\tilde{X}',k+1,\dots,n)\Big]\,.\end{aligned} \tag{C.0.8}$$

In the second line we have made a change of dummy summation variable that we now exploit further.

It is non-trivial, but turns out to be true for $SU(2)_R$ as we have explicitly checked, that if we define $X'=\mathcal{T}_+\cdot X$ and sum over X instead of X', the second line of (C.0.8) gives exactly the same result. We can then write (C.0.8) as

$$\begin{aligned}-\sum_X\Big[&(-1)^{P_X}\hat{\mathcal{A}}_L(1,\dots,k,\mathcal{T}_+\cdot X)\hat{\mathcal{A}}_R(\tilde{X},k+1,\dots,n)\\ &+(-1)^{P_{\tilde{X}'}}\hat{\mathcal{A}}_L(1,\dots,k,\mathcal{T}_+\cdot X)\hat{\mathcal{A}}_R(\mathcal{T}_+\cdot C\cdot\mathcal{T}_+\cdot X,k+1,\dots,n)\Big]\,.\end{aligned} \tag{C.0.9}$$

Since $\mathcal{T}_+\cdot C\cdot\mathcal{T}_+\cdot X=\mathcal{T}_+\cdot\mathcal{T}_-\cdot\tilde{X}$, this becomes

$$\begin{aligned}-\sum_X\Big[&(-1)^{P_X}\hat{\mathcal{A}}_L(1,\dots,k,\mathcal{T}_+\cdot X)\hat{\mathcal{A}}_R(\tilde{X},k+1,\dots,n)\\ &+(-1)^{P_{\mathcal{T}_-\cdot\tilde{X}}+Q_{\tilde{X}}+1}\hat{\mathcal{A}}_L(1,\dots,k,\mathcal{T}_+\cdot X)\hat{\mathcal{A}}_R(\mathcal{T}_+\cdot\mathcal{T}_-\cdot\tilde{X},k+1,\dots,n)\Big]\,.\end{aligned} \tag{C.0.10}$$

where Q_X refers to the prefactors for the action of $\mathcal{T}_-$ as given in Table 2.7.25. This vanishes when $\mathcal{T}_+\cdot\mathcal{T}_-\cdot\tilde{X}=\tilde{X}$ and $P_{\mathcal{T}_-\cdot\tilde{X}}+Q_{\tilde{X}}=0$ for any state X such that $\mathcal{T}_+\cdot X\neq 0$. For $SU(2)_R$, we can check explicitly that these conditions are satisfied. The only states for which $\mathcal{T}_+\cdot X\neq 0$ are $X=\psi^{2+}$ and ψ_1^-. Their conjugates are $\tilde{X}=\psi_2^-$ and ψ^{2+}, respectively, and by (2.7.25) we have

$$\mathcal{T}_+ \cdot \mathcal{T}_- \cdot \psi^{1+} = \mathcal{T}_+ \cdot \psi^{2+} = \psi^{1+} \qquad \mathcal{T}_+ \cdot \mathcal{T}_- \cdot \psi_2^- = \mathcal{T}_+ \cdot \psi_1^- = \psi_2^-\,, \tag{C.0.11}$$

$$P_{\mathcal{T}_- \cdot \psi^{1+}} + Q_{\psi^{1+}} = 0 + 0 = 0 \qquad P_{\mathcal{T}_- \cdot \psi_2^-} + Q_{\psi_2^-} = 1 + 1 = 0 \ (\text{mod } 2)\,. \tag{C.0.12}$$

If follows that from the inductive step that all amplitudes satisfy the $SU(2)_R$ Ward identities when the seed amplitudes do.

APPENDIX D

Amplitude Relations in Supersymmetric NLSM

Below are explicit formulae, derived from $\mathcal{N} = 2$ supersymmetry Ward identities, for all amplitudes in this model with total spin ≤ 2 expressed as linear combinations of amplitudes with strictly greater total spin. Collectively these formulae allow us to construct every tree-level amplitude in the $\mathcal{N} = 2$ $\mathbb{CP}^1$ sigma model using unsubtracted recursion. The needed relations are:

$$\mathcal{A}_{2n}\left(1_Z, 2_{\bar{Z}}, 3_Z, 4_{\bar{Z}} ..., (2n)_{\bar{Z}}\right) = \sum_{k=1}^{n-1} \frac{\langle 1, 2k+1\rangle}{\langle 12\rangle} \mathcal{A}_{2n}\left(1_Z, 2^-_{\psi_1}, 3_Z, 4_{\bar{Z}}, ..., (2k+1)^+_{\psi^1}, ..., (2n)_{\bar{Z}}\right)$$

$$\mathcal{A}_{2n}\left(1^+_{\psi^1}, 2^-_{\psi_1}, 3_Z, 4_{\bar{Z}}, ..., (2n)_{\bar{Z}}\right) = \sum_{k=1}^{n-1} \frac{[2, 2k+2]}{[21]} \mathcal{A}_{2n}\left(1^+_{\gamma}, 2^-_{\psi_1}, 3_Z, 4_{\bar{Z}}, ..., (2k+2)^-_{\psi_2}, ..., (2n)_{\bar{Z}}\right)$$

$$\begin{aligned}&\mathcal{A}_{2n+1}\left(1^+_{\gamma}, 2^+_{\gamma}, 3_Z, 4_{\bar{Z}}, ..., (2n+1)_Z\right) \\ &\qquad = \sum_{k=1}^{n-2} \frac{\langle 3, 2k+3\rangle}{\langle 34\rangle} \mathcal{A}_{2n+1}\left(1^+_{\gamma}, 2^+_{\gamma}, 3_Z, 4^-_{\psi_2}, 5_Z, ..., (2k+3)^+_{\psi^2}, ..., (2n+1)_Z\right)\end{aligned}$$

$$\mathcal{A}_{2n}\left(1^+_{\psi^1}, 2^-_{\gamma}, 3^+_{\psi^2}, 4_{\bar{Z}}, 5_Z, ..., (2n)_{\bar{Z}}\right) = \sum_{k=1}^{n-1} \frac{[3, 2k+2]}{[31]} \mathcal{A}_{2n}\left(1^+_{\gamma}, 2^-_{\gamma}, 3^+_{\psi^2}, 4_{\bar{Z}}, ..., (2k+2)^-_{\psi_2}, ..., (2n)_{\bar{Z}}\right)$$

$$\mathcal{A}_{2n}\left(1^+_{\gamma}, 2^-_{\gamma}, 3_Z, 4_{\bar{Z}}, 5_Z, ..., (2n)_{\bar{Z}}\right) = \sum_{k=1}^{n-1} \frac{[1, 2k+2]}{[13]} \mathcal{A}_{2n}\left(1^+_{\gamma}, 2^-_{\gamma}, 3^+_{\psi^2}, 4_{\bar{Z}}, ..., (2k+2)^-_{\psi_2}, ..., (2n)_{\bar{Z}}\right)$$

$$\begin{aligned}\mathcal{A}_{2n+1}\left(1^+_{\gamma}, 2^+_{\psi^1}, 3^+_{\psi^2}, 4_Z, 5_{\bar{Z}}, ..., (2n+1)_{\bar{Z}}\right) &= -\frac{\langle 42\rangle}{\langle 45\rangle} \mathcal{A}_{2n+1}\left(1^+_{\gamma}, 2^+_{\gamma}, 3^+_{\psi^2}, 4_Z, 5^-_{\psi_2}, 6_Z, ..., (2n+1)_{\bar{Z}}\right) \\ &+ \sum_{k=1}^{n-2} \frac{\langle 4, 2k+4\rangle}{\langle 45\rangle} \mathcal{A}_{2n+1}\left(1^+_{\gamma}, 2^+_{\psi^1}, 3^+_{\psi^3}, 4_Z, 5^-_{\psi_2}, 6_Z, ..., (2k+4)^+_{\psi^2}, ..., (2n+1)_{\bar{Z}}\right)\end{aligned}$$

$$\mathcal{A}_{2n}\left(1^+_{\psi^1}, 2^-_{\psi_1}, 3^+_{\psi^2}, 4^-_{\psi_2}, 5_Z, 6_{\bar{Z}}, ..., (2n)_{\bar{Z}}\right) = \frac{[32]}{[31]} \mathcal{A}_{2n}\left(1^+_{\gamma}, 2^-_{\gamma}, 3^+_{\psi^2}, 4^-_{\psi_2}, 5_Z, 6_{\bar{Z}}, ..., (2n)_{\bar{Z}}\right)$$

$$
\begin{aligned}
&\qquad\qquad + \sum_{k=1}^{n-2} \frac{[3,2k+4]}{[31]} \mathcal{A}_{2n}\left(1_{\gamma}^{+}, 2_{\psi_1}^{-}, 3_{\psi^2}^{+}, 4_{\psi_2}^{-}, 5_Z, ..., (2k+4)_{\psi_2}^{-}, ..., (2n)_{\bar{Z}}\right) \\
&\mathcal{A}_{2n}\left(1_{\psi^1}^{+}, 2_{\psi_1}^{-}, 3_{\psi^1}^{+}, 4_{\psi_1}^{-}, 5_Z, 6_{\bar{Z}}, ..., (2n)_{\bar{Z}}\right) = \frac{[42]}{[41]} \mathcal{A}_{2n}\left(1_{\gamma}^{+}, 2_{\gamma}^{-}, 3_{\psi^1}^{+}, 4_{\psi_1}^{-}, 5_Z, 6_{\bar{Z}}, ..., (2n)_{\bar{Z}}\right) \\
&\qquad\qquad + \sum_{k=1}^{n-2} \frac{[4,2k+4]}{[41]} \mathcal{A}_{2n}\left(1_{\gamma}^{+}, 2_{\psi_1}^{-}, 3_{\psi^1}^{+}, 4_{\psi_1}^{-}, 5_Z, ..., (2k+4)_{\psi_2}^{-}, ..., (2n)_{\bar{Z}}\right) \\
&\mathcal{A}_{2n+1}\left(1_{\psi^1}^{+}, 2_{\psi^1}^{+}, 3_{\psi^2}^{+}, 4_{\psi^2}^{+}, 5_{\bar{Z}}, 6_Z, ..., (2n+1)_{\bar{Z}}\right) \\
&\qquad = -\frac{\langle 21\rangle}{\langle 25\rangle} \mathcal{A}_{2n+1}\left(1_{\gamma}^{+}, 2_{\psi^1}^{+}, 3_{\psi^2}^{+}, 4_{\psi^2}^{+}, 5_{\psi_2}^{-}, 6_Z, 7_{\bar{Z}}, ..., (2n+1)_{\bar{Z}}\right) \\
&\qquad + \sum_{k=1}^{n-2} \frac{\langle 2,2k+4\rangle}{\langle 25\rangle} \mathcal{A}_{2n+1}\left(1_{\psi^1}^{+}, 2_{\psi^1}^{+}, 3_{\psi^2}^{+}, 4_{\psi^2}^{+}, 5_{\psi_2}^{-}, 6_Z, ..., (2k+4)_{\psi^2}^{+}, ..., (2n+1)_{\bar{Z}}\right).
\end{aligned}
\tag{D.0.1}
$$

APPENDIX E

Alternative Approach to Contact Terms: Massive KLT Relations

As discussed in Section 3.2.2, the tree-level amplitudes of Born-Infeld in d-dimensions are given by the KLT product

$$\text{BI}_d = \text{YM}_d \otimes_{\text{KLT}} \chi\text{PT}_d, \tag{E.0.1}$$

where χPT_d denotes the $\frac{SU(N)\times SU(N)}{SU(N)}$ non-linear sigma model in d-dimensions. Beginning with $d = 6$ we can (formally) calculate tree amplitudes in BI_6 from the tree amplitudes for YM_6 and χPT_6 using the dimension independent form of the KLT product. Since we do not require the completely general 6d Born-Infeld amplitudes, only the configuration in Figure 3.1, we can dimensionally reduce the 6d KLT relations into a form of *massive* KLT relations by separating the 4d and extra-dimensional components of the momenta. This amounts to taking the dimension independent form the KLT relations and making the replacements

$$s_{1i} \to s_{1i} + \mu^2, \quad s_{nj} \to s_{nj} + \mu^2, \tag{E.0.2}$$

where $i \neq n$ and $j \neq 1$ (Note that we are defining our Mandelstam invariants as $s_{ij} \equiv (p_i + p_j)^2$). Using this prescription the needed KLT relations

$$\text{mDBI}_4 = \text{YM+mAdj}_4 \otimes_{\text{KLT}} \text{m}\chi\text{PT}_4, \tag{E.0.3}$$

up to $n = 8$ take the explicit form [96]

$$\mathcal{A}_4^{\text{mDBI}_4}\left(1_\phi, 2_\gamma, 3_\gamma, 4_{\overline{\phi}}\right) = (s_{12}+\mu^2)\mathcal{A}_4^{\text{YM+mAdj}_4}[1_\phi, 2_g, 3_g, 4_{\overline{\phi}}]\mathcal{A}_4^{\text{m}\chi\text{PT}_4}\left[\mathbf{1}, 2, \mathbf{4}, 3\right], \tag{E.0.4}$$

$$\begin{aligned}
&\mathcal{A}_6^{\text{mDBI}_4}\left(1_\phi, 2_\gamma, 3_\gamma, 4_\gamma, 5_\gamma, 6_{\overline{\phi}}\right) \\
&\quad = (s_{12}+\mu^2)s_{45}\mathcal{A}_6^{\text{YM+mAdj}_4}[1_\phi, 2_g, 3_g, 4_g, 5_g, 6_{\overline{\phi}}] \\
&\qquad \times \left(s_{35}\mathcal{A}_6^{\text{m}\chi\text{PT}_4}\left[\mathbf{1}, 5, 3, 4, \mathbf{6}, 2\right] + (s_{34}+s_{35})\mathcal{A}_6^{\text{m}\chi\text{PT}_4}\left[\mathbf{1}, 5, 4, 3, \mathbf{6}, 2\right]\right) \\
&\qquad + \mathcal{P}\left(2, 3, 4\right),
\end{aligned} \tag{E.0.5}$$

$$\begin{aligned}
&\mathcal{A}_8^{\text{mDBI}_4}\left(1_\phi, 2_\gamma, 3_\gamma, 4_\gamma, 5_\gamma, 6_\gamma, 7_\gamma, 8_{\overline{\phi}}\right) \\
&\quad = (s_{12}+\mu^2)s_{67}\mathcal{A}_8^{\text{YM+mAdj}_4}[1_\phi, 2_g, 3_g, 4_g, 5_g, 6_g, 7_g, 8_{\overline{\phi}}] \\
&\qquad \times \Big[(s_{13}+\mu^2)s_{14}\Big(s_{57}\mathcal{A}_8^{\text{m}\chi\text{PT}_4}[\mathbf{1}, 7, 5, 6, \mathbf{8}, 2, 3, 4] \\
&\qquad\qquad\qquad + (s_{57}+s_{56})\mathcal{A}_8^{\text{m}\chi\text{PT}_4}[\mathbf{1}, 7, 6, 5, \mathbf{8}, 2, 3, 4]\Big) \\
&\qquad\quad + (s_{13}+\mu^2)(s_{14}+s_{34}+\mu^2)\Big(s_{57}\mathcal{A}_8^{\text{m}\chi\text{PT}_4}[\mathbf{1}, 7, 5, 6, \mathbf{8}, 2, 4, 3] \\
&\qquad\qquad\qquad + (s_{57}+s_{56})\mathcal{A}_8^{\text{m}\chi\text{PT}_4}[\mathbf{1}, 7, 6, 5, \mathbf{8}, 2, 4, 3]\Big) \\
&\qquad\quad + (s_{14}+\mu^2)(s_{13}+s_{23}+\mu^2)\Big(s_{57}\mathcal{A}_8^{\text{m}\chi\text{PT}_4}[\mathbf{1}, 7, 5, 6, \mathbf{8}, 3, 2, 4] \\
&\qquad\qquad\qquad + (s_{57}+s_{56})\mathcal{A}_8^{\text{m}\chi\text{PT}_4}[\mathbf{1}, 7, 6, 5, \mathbf{8}, 3, 2, 4]\Big) \\
&\qquad\quad + (s_{13}+s_{23}+\mu^2)(s_{14}+s_{24}+\mu^2)\Big(s_{57}\mathcal{A}_8^{\text{m}\chi\text{PT}_4}[\mathbf{1}, 7, 5, 6, \mathbf{8}, 3, 4, 2] \\
&\qquad\qquad\qquad + (s_{57}+s_{56})\mathcal{A}_8^{\text{m}\chi\text{PT}_4}[\mathbf{1}, 7, 6, 5, \mathbf{8}, 3, 4, 2]\Big) \\
&\qquad\quad + (s_{13}+\mu^2)(s_{14}+s_{24}+s_{34}+\mu^2)\Big(s_{57}\mathcal{A}_8^{\text{m}\chi\text{PT}_4}[\mathbf{1}, 7, 5, 6, \mathbf{8}, 4, 2, 3] \\
&\qquad\qquad\qquad + (s_{57}+s_{56})\mathcal{A}_8^{\text{m}\chi\text{PT}_4}[\mathbf{1}, 7, 6, 5, \mathbf{8}, 4, 2, 3]\Big) \\
&\qquad\quad + (s_{13}+s_{23}+\mu^2)(s_{14}+s_{34}+s_{24}+\mu^2)\Big(s_{57}\mathcal{A}_8^{\text{m}\chi\text{PT}_4}[\mathbf{1}, 7, 5, 6, \mathbf{8}, 4, 3, 2] \\
&\qquad\qquad\qquad + (s_{57}+s_{56})\mathcal{A}_8^{\text{m}\chi\text{PT}_4}[\mathbf{1}, 7, 6, 5, \mathbf{8}, 4, 3, 2]\Big)\Big] \\
&\qquad + \mathcal{P}\left(2, 3, 4, 5, 6\right).
\end{aligned} \tag{E.0.6}$$

In the mχPT amplitudes bolded momenta denote massive particles.

Note that these expressions differ by an overall sign from the expressions given in [31] due to our conventions for the Mandelstam invariants. Below we will describe the calculation of both $\text{YM} + \text{mAdj}_4$ and $\text{m}\chi\text{PT}_4$ amplitudes and then give the result of the double copy.

E.0.1 YM+mAdj$_4$ from Massive BCFW

The needed tree-level amplitudes of YM+mAdj can be calculated using BCFW recursion from 3-point input. Since this model should have only marginal couplings between the gluons and massive adjoint scalar, the tree-level amplitudes are completely fixed by gauge invariance. This approach was first used in [182], below we give a brief review.

The seed amplitudes for the recursion are

$$\mathcal{A}_3^{\text{YM+mAdj}_4}[1_\phi, 2_g^+, 3_{\overline{\phi}}] = -\frac{[2|p_1|q\rangle}{\langle 2q\rangle}, \quad \mathcal{A}_3^{\text{YM+mAdj}_4}[1_\phi, 2_g^-, 3_{\overline{\phi}}] = \frac{[\tilde{q}|p_1|2\rangle}{[\tilde{q}\; 2]}, \tag{E.0.7}$$

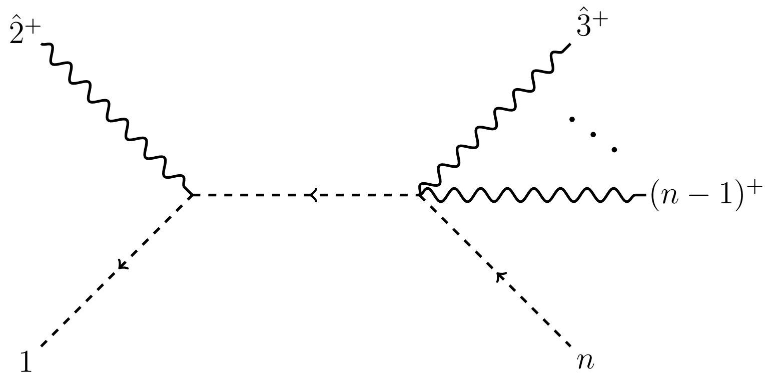
where $|q\rangle$ and $|\tilde{q}]$ are arbitrary. We want to calculate NSD amplitudes

$$\mathcal{A}_n^{\text{YM+mAdj}_4}[1_\phi, 2_g^+, 3_g^+ \dots, (n-1)_g^+, n_{\overline{\phi}}], \tag{E.0.8}$$

using a BCFW shift

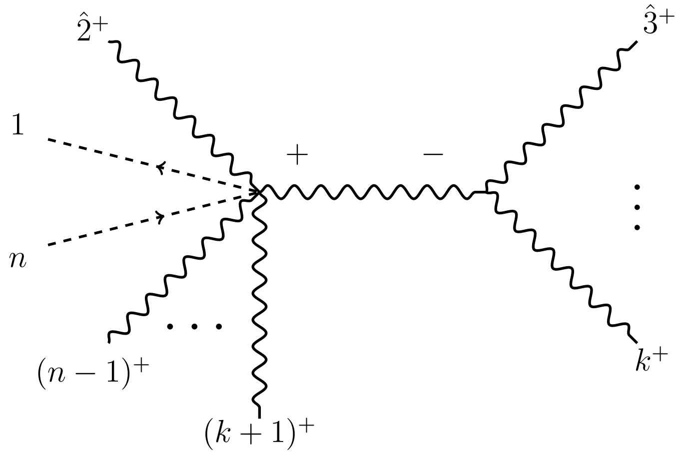
$$|\hat{2}\rangle = |2\rangle - z|3\rangle, \quad |\hat{3}] = |3] + z|2]. \tag{E.0.9}$$

With the given color-ordering (and the fact that the shifted lines must sit on opposite sides of the factorization diagram) there are two types of factorization channel which could contribute:



(E.0.10)

and



(E.0.11)

Interestingly, the second diagram never contributes. The argument for this is has two parts, first we consider diagrams with $k > 4$. In this case the right-hand amplitude is of the form $\mathcal{A}_{k-1}^{\text{YM+mAdj}_4}[-,+,\ldots,+]$ which vanishes at tree-level. For the case $k = 4$ the right-hand amplitude is simply the pure Yang-Mills 3-point amplitude[1]

$$\mathcal{A}_3^{\text{YM+mAdj}_4}\left[(-\hat{p}_{34})_g^-, \hat{3}_g^+, 4_g^+\right] = \frac{[\hat{3}4]^3}{[4,-\hat{p}_{34}][-\hat{p}_{34},\hat{3}]}. \tag{E.0.12}$$

On the factorization channel we have $[\hat{3}4] = 0$ and therefore this amplitude vanishes. So we see that only a single factorization channel contributes at each recursive step. Explicitly the BCFW recursion relation takes the form

$$\begin{aligned}&\mathcal{A}_n^{\text{YM+mAdj}_4}\left[1_\phi, 2_g^+, 3_g^+, \ldots, (n-1)_g^+, n_{\overline{\phi}}\right]\\ &\quad= \frac{\mathcal{A}_3^{\text{YM+mAdj}_4}[1_\phi, \hat{2}_g^+, (-\hat{p}_{12})_{\overline{\phi}}]\mathcal{A}_{n-1}^{\text{YM+mAdj}_4}[(\hat{p}_{12})_\phi, \hat{3}_g^+, 4_g^+, \ldots, (n-2)_g^+, (n-1)_{\overline{\phi}}]}{s_{12}+\mu^2}.\end{aligned} \tag{E.0.13}$$

We will now use this to calculate the amplitudes up to $n = 8$. Here (and subsequently) we will use the convenient shorthand notation

$$p_{1,k} \equiv p_{12\ldots k}, \qquad D_n \equiv \langle 23\rangle\langle 34\rangle\ldots\langle n-2,n-1\rangle(s_{12}+\mu^2)(s_{123}+\mu^2)\ldots(s_{12\ldots n-2}+\mu^2). \tag{E.0.14}$$

At 4-point we need both the NSD and MHV amplitudes

$$\mathcal{A}_4^{\text{YM+mAdj}_4}[1_\phi, 2_g^+, 3_g^+, 4_{\overline{\phi}}] = -\frac{\mu^2[23]}{\langle 23\rangle(s_{12}+\mu^2)}, \tag{E.0.15}$$

[1]Here and subsequently, we use the convention $|-p] = i|p]$ and $|-p\rangle = i|p\rangle$. This is because the prescription for dimensional reduction to $3d$ we use in Section 3.3.2 requires that we treat the angle and square spinors "democratically". A consequence of this convention choice is that the Parke-Taylor amplitudes acquire an additional factor of -1 for an even number of external states.

and

$$\mathcal{A}_4^{\text{YM+mAdj}_4}[1_\phi, 2_g^+, 3_g^-, 4_{\bar{\phi}}] = -\frac{\langle 3|p_1|2]^2}{s_{23}(s_{12}+\mu^2)}. \tag{E.0.16}$$

At 6-point we will only need amplitudes in the NSD sector

$$\mathcal{A}_6^{\text{YM+mAdj}_4}\left[1_\phi, 2_g^+, 3_g^+, 4_g^+, 5_g^+, 6_{\bar{\phi}}\right] = -\frac{\mu^2[2|p_1\cdot p_{23}\cdot p_{45}\cdot p_6|5]}{D_6}. \tag{E.0.17}$$

Similarly at 8-point

$$\begin{aligned}
&\mathcal{A}_8^{\text{YM+mAdj}_4}\left[1_\phi, 2_g^+, 3_g^+, 4_g^+, 5_g^+, 6_g^+, 7_g^+, 8_{\bar{\phi}}\right] \\
&\quad = \frac{1}{D_8}\left[-(\mu^2)^3[2|p_1\cdot p_{23}\cdot p_{67}\cdot p_8|7] + (\mu^2)^2[2|p_1\cdot p_{23}\cdot p_{4,8}\cdot p_{5,8}\cdot p_{67}\cdot p_8|7]\right. \\
&\qquad\quad + (\mu^2)^2[2|p_1\cdot p_{23}\cdot p_{5,8}\cdot p_{6,8}\cdot p_{67}\cdot p_8|7] \\
&\qquad\quad \left. -\mu^2[2|p_1\cdot p_{23}\cdot p_{4,8}\cdot p_{5,8}\cdot p_{5,8}\cdot p_{6,8}\cdot p_{67}\cdot p_8|7]\right].
\end{aligned} \tag{E.0.18}$$

All multiplicity results for these amplitudes have been calculated in [183], but we will not need explicit expressions beyond 8-points.

E.0.2 mχPT$_4$ from Soft Limits and Dimensional Reduction

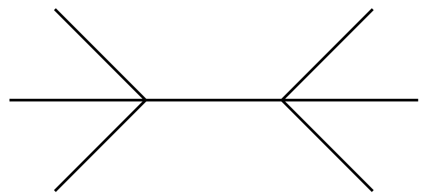
The needed tree level amplitudes for χPT$_d$ can be calculated using the *soft bootstrap* approach [1,5,6,29,38,54,74,84,184–187]. While it is certainly possible to setup formal recursion relations analogous to the BCFW recursion used above (this is the so-called *subtracted recursion* [38, 53]), in practice since this is such a simple model there is a more efficient approach. We note that locality is manifest in the χPT amplitudes, and so we can treat the contact terms of lower-point amplitudes as "vertex rules", gluing them together in a diagrammatic expansion. This will automatically generate expressions with the correct factorization properties (which can be verified straightforwardly *post hoc* by computing residues), the remaining ambiguity is contained in the contact terms. These ambiguous contributions can then be determined by imposing the Adler zero, that is, single soft limit which vanish at $\mathcal{O}(p)$ [27].

We start with the flavor-ordered 4-point amplitude

$$\mathcal{A}_4^{\chi\text{PT}_d}[1,2,3,4] = s_{13}. \tag{E.0.19}$$

With the dimensionful coupling suppressed, the χPT$_d$ tree-amplitudes take a dimension independent form. Similar to the definition of mDBI$_4$ we define the model mχPT$_4$ as the tree amplitudes of χPT$_6$ with momenta in the configuration given in Figure 3.1. Operationally these amplitudes are calculated using the replacement rules (E.0.2), on the $\chi\mathrm{PT}_d$ amplitudes, similar to the way we derived the massive KLT relations above.

Now we turn to the explicit calculation of the 6-point $\chi\mathrm{PT}_d$ amplitude. In this case the factoring part of the amplitude corresponds to diagrams with a unique topology



(E.0.20)

There are three inequivalent cyclic permutations of the external labels $[1,2,3,4,5,6]$, so the factoring part of the six point amplitude has the form

$$\mathcal{A}_6^{\chi\mathrm{PT}_d}[1,2,3,4,5,6]\Big|_{\text{factoring}} = \frac{s_{13}s_{46}}{s_{123}} + \frac{s_{24}s_{51}}{s_{234}} + \frac{s_{35}s_{62}}{s_{345}}. \tag{E.0.21}$$

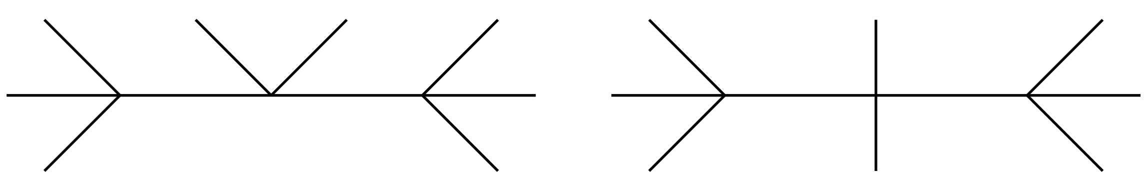
This differs from the full answer by a possible contact term. Such a contact contribution is fixed by demanding that the amplitude vanishes in the soft limit of each particle. It is straightforward to verify that the following expression satisfies all of the aforementioned properties

$$\mathcal{A}_6^{\chi\mathrm{PT}_d}[1,2,3,4,5,6] = \frac{s_{13}s_{46}}{s_{123}} + \frac{s_{24}s_{51}}{s_{234}} + \frac{s_{35}s_{62}}{s_{345}} - s_{135}. \tag{E.0.22}$$

We can then convert this into an $\mathrm{m}\chi\mathrm{PT}_4$ amplitude with particles 1 and 5 massive for later use in the KLT product

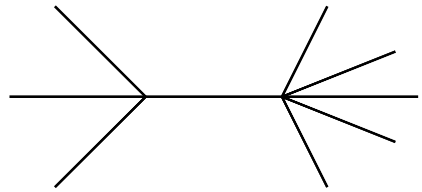
$$\mathcal{A}_6^{\mathrm{m}\chi\mathrm{PT}_4}[\mathbf{1},2,3,4,\mathbf{5},6] = \frac{(s_{13}+\mu^2)s_{46}}{s_{123}+\mu^2} + \frac{s_{24}s_{51}}{s_{234}} + \frac{(s_{35}+\mu^2)s_{62}}{s_{345}+\mu^2} - s_{135}. \tag{E.0.23}$$

For $n = 8$ there are three distinct factorization topologies we need to consider, two constructed from 4-point vertices



(E.0.24)

and one from a 4-point and a 6-point vertex



(E.0.25)

It is straightforward to write down the factoring part of this amplitude

$$\mathcal{A}_8^{\chi\mathrm{PT}_d}[1,2,3,4,5,6,7,8]\Big|_{\text{factoring}} = \frac{s_{13}s_{1235}s_{68}}{s_{123}s_{678}} + \frac{1}{2}\left(\frac{s_{13}s_{48}s_{57}}{s_{123}s_{567}}\right) - \frac{s_{13}s_{468}}{s_{123}} + \mathcal{C}\left(1,2,3,4,5,6,7,8\right). \tag{E.0.26}$$

where $\mathcal{C}$ denotes the sum over all *cyclic* permutations. The contact terms we need to add can be found straightforwardly by taking soft limits, the result is

$$\mathcal{A}_8^{\chi\mathrm{PT}_d}[1,2,3,4,5,6,7,8] = \left[\frac{s_{13}s_{1235}s_{68}}{s_{123}s_{678}} + \frac{1}{2}\left(\frac{s_{13}s_{48}s_{57}}{s_{123}s_{567}}\right) - \frac{s_{13}s_{468}}{s_{123}} + \mathcal{C}\left(1,2,3,4,5,6,7,8\right)\right] + s_{2468}. \tag{E.0.27}$$

Constructing the mχPT$_4$ amplitude with particle 1 and 5 massive gives

$$\begin{aligned}
&\mathcal{A}_8^{\mathrm{m}\chi\mathrm{PT}_4}[\mathbf{1},2,3,4,\mathbf{5},6,7,8] \\
&= \frac{(s_{13}+\mu^2)s_{1235}s_{68}}{(s_{123}+\mu^2)s_{678}} + \frac{(s_{13}+\mu^2)s_{48}(s_{57}+\mu^2)}{(s_{123}+\mu^2)(s_{567}+\mu^2)} - \frac{(s_{13}+\mu^2)s_{468}}{s_{123}+\mu^2} + \frac{s_{24}s_{2346}(s_{71}+\mu^2)}{s_{234}(s_{781}+\mu^2)} \\
&\quad + \frac{s_{24}s_{51}s_{68}}{s_{234}s_{678}} - \frac{s_{24}s_{571}}{s_{234}} + \frac{(s_{35}+\mu^2)(s_{3457}+\mu^2)s_{82}}{(s_{345}+\mu^2)(s_{812}+\mu^2)} + \frac{(s_{35}+\mu^2)s_{62}(s_{71}+\mu^2)}{(s_{345}+\mu^2)(s_{781}+\mu^2)} \\
&\quad - \frac{(s_{35}+\mu^2)s_{682}}{s_{345}+\mu^2} + \frac{s_{46}(s_{4568}+\mu^2)(s_{13}+\mu^2)}{(s_{456}+\mu^2)(s_{123}+\mu^2)} + \frac{s_{46}s_{73}s_{82}}{(s_{456}+\mu^2)(s_{812}+\mu^2)} - \frac{s_{46}(s_{713}+\mu^2)}{s_{456}+\mu^2} \\
&\quad + \frac{(s_{57}+\mu^2)s_{5671}s_{24}}{(s_{567}+\mu^2)s_{234}} - \frac{(s_{57}+\mu^2)s_{824}}{s_{567}+\mu^2} + \frac{s_{68}(s_{6781}+\mu^2)(s_{35}+\mu^2)}{s_{678}(s_{345}+\mu^2)} - \frac{s_{68}s_{135}}{s_{678}} \\
&\quad + \frac{(s_{71}+\mu^2)(s_{7812}+\mu^2)s_{46}}{(s_{781}+\mu^2)(s_{456}+\mu^2)} - \frac{(s_{71}+\mu^2)s_{246}}{s_{781}+\mu^2} + \frac{s_{82}(s_{8123}+\mu^2)(s_{57}+\mu^2)}{(s_{812}+\mu^2)(s_{567}+\mu^2)} \\
&\quad - \frac{s_{82}(s_{357}+\mu^2)}{s_{812}+\mu^2} + s_{2468}.
\end{aligned} \tag{E.0.28}$$

Simple closed form expressions for all $\chi\mathrm{PT}_d$ amplitudes are not known, but this procedure is simple enough that it can be implemented efficiently to calculate amplitudes up to the desired multiplicity. As in the previous section we will only need explicit expressions up to $n = 8$.

E.0.3 Result of Double Copy

We can begin with the calculation of the 4-point amplitudes of mDBI_4, which are simple enough to be evaluated by hand without difficulty

$$\begin{aligned}\mathcal{A}_4^{\mathrm{mDBI}_4}\left(1_\phi, 2_\gamma^+, 3_\gamma^+, 4_{\overline{\phi}}\right) &= (s_{12}+\mu^2)\mathcal{A}_4^{\text{YM+mAdj}}[1_\phi, 2_g^+, 3_g^+, 4_{\overline{\phi}}]\mathcal{A}_4^{\text{m}\chi\text{PT}}[\mathbf{1}, 2, \mathbf{4}, 3] \\ &= (s_{12}+\mu^2)\left[-\frac{\mu^2[23]}{\langle 23\rangle(s_{12}+\mu^2)}\right][s_{23}] \\ &= -\mu^2[23]^2,\end{aligned} \tag{E.0.29}$$

and

$$\begin{aligned}\mathcal{A}_4^{\mathrm{mDBI}_4}\left(1_\phi, 2_\gamma^+, 3_\gamma^-, 4_{\overline{\phi}}\right) &= (s_{12}+\mu^2)\mathcal{A}_4^{\text{YM+mAdj}}[1_\phi, 2_g^+, 3_g^-, 4_{\overline{\phi}}]\mathcal{A}_4^{\text{m}\chi\text{PT}}[\mathbf{1}, 2, \mathbf{4}, 3] \\ &= (s_{12}+\mu^2)\left[-\frac{\langle 3|p_1|2]^2}{s_{23}(s_{12}+\mu^2)}\right][s_{23}] \\ &= -\langle 3|p_1|2]^2.\end{aligned} \tag{E.0.30}$$

We will also need the 4-point pure Born-Infeld amplitude. This can also be calculated with the (massless) KLT product using the 4-point Parke-Taylor gluon amplitude

$$\begin{aligned}\mathcal{A}_4^{\mathrm{mDBI}_4}\left(1_\gamma^+, 2_\gamma^+, 3_\gamma^-, 4_\gamma^-\right) &= s_{12}\left[-\frac{[12]^3}{[23][34][41]}\right][s_{23}] \\ &= [12]^2\langle 34\rangle^2.\end{aligned} \tag{E.0.31}$$

Notice that due to our convention choice (see comments in footnote 1), the Parke-Taylor amplitude above has an additional factor of -1.

Simplifying the massive KLT relations algebraically beyond 4-point is a daunting task. Fortunately it is straightforward to construct a general Ansatz for the higher-multiplicity amplitudes. Beginning with the NSD 6-point amplitude we know the answer should have the form

$$\begin{aligned}&\mathcal{A}_6^{\mathrm{mDBI}_4}\left(1_\phi, 2_\gamma^+, 3_\gamma^+, 4_\gamma^+, 5_\gamma^+, 6_{\overline{\phi}}\right) \\ &\quad = \frac{1}{4}\left[\frac{(\mu^2)^2[23]^2[45]^2}{s_{123}+\mu^2} + \mathcal{P}\left(2,3,4,5\right)\right] + c_6\mu^2\left([23]^2[45]^2 + [24]^2[35]^2 + [25]^2[34]^2\right).\end{aligned} \tag{E.0.32}$$

This expression has the correct factorization singularities consistent with the known 4-point amplitudes, and a polynomial ambiguity parametrized by a single coefficient c_6, as discussed above. To determine the coefficient c_6 we numerically evaluate the KLT sum (E.0.5) on several sets of randomly generated kinematic variables and compare with a numerical evaluation of the Ansatz.

For more than one choice of kinematics this overconstrains the problem and allows us to both verify the validity of the Ansatz and determine the value of the coefficient. Doing so we find that the Ansatz is valid and $c_6 = 0$; the amplitude is simply

$$\mathcal{A}_6^{\text{mDBI}_4}\left(1_\phi, 2_\gamma^+, 3_\gamma^+, 4_\gamma^+, 5_\gamma^+, 6_{\overline{\phi}}\right) = \frac{1}{4}\left[\frac{(\mu^2)^2[23]^2[45]^2}{s_{123}+\mu^2}\right] + \mathcal{P}\left(2,3,4,5\right). \tag{E.0.33}$$

Next we calculate the MHV 6-point amplitude. As discussed in Section 3.3.1, in this case there are *no* contact terms consistent with little group scaling and Bose symmetry. There is then no ambiguity in the answer, the result of gluing together the 4-point amplitudes on factorization channels is the unique correct result. We find

$$\begin{aligned}\mathcal{A}_6^{\text{mDBI}_4}&\left(1_\phi, 2_\gamma^+, 3_\gamma^+, 4_\gamma^+, 5_\gamma^-, 6_{\overline{\phi}}\right)\\ &= \frac{\mu^2}{2}\left[\frac{[23]^2\langle 5|p_6|4]^2}{s_{123}+\mu^2} + \frac{[34]^2\langle 5|p_1|2]^2}{s_{125}+\mu^2} + \frac{[34]^2\langle 5|p_{34}|2]^2}{s_{126}}\right] + \mathcal{P}\left(2,3,4\right).\end{aligned} \tag{E.0.34}$$

At 8-point the method is the same, we begin with the calculation of the NSD amplitude. Using the result $c_6 = 0$, we should use an Ansatz of the form

$$\begin{aligned}\mathcal{A}_8^{\text{mDBI}_4}&\left(1_\phi, 2_\gamma^+, 3_\gamma^+, 4_\gamma^+, 5_\gamma^+, 6_\gamma^+, 7_\gamma^+, 8_{\overline{\phi}}\right)\\ &= -\frac{1}{8}\left[\frac{(\mu^2)^3[23]^2[45]^2[67]^2}{(s_{123}+\mu^2)(s_{678}+\mu^2)}\right] + c_8\mu^2[23]^2[45]^2[67]^2 + \mathcal{P}\left(2,3,4,5,6,7\right).\end{aligned} \tag{E.0.35}$$

Explicit numerical evaluation of the massive KLT relations reveals the surprising result that $c_8 = 0$ also! Finally, as above the MHV 8-point amplitude is completely fixed by factorization

$$\begin{aligned}\mathcal{A}_8^{\text{mDBI}_4}&\left(1_\phi, 2_\gamma^+, 3_\gamma^+, 4_\gamma^+, 5_\gamma^+, 6_\gamma^+, 7_\gamma^-, 8_{\overline{\phi}}\right)\\ &= -\frac{(\mu^2)^2}{4}\left[\frac{[23]^2[45]^2\langle 7|p_8|6]^2}{(s_{123}+\mu^2)(s_{678}+\mu^2)} + \frac{[23]^2[45]^2\langle 7|p_{123}|6]^2}{(s_{123}+\mu^2)(s_{458}+\mu^2)} + \frac{[23]^2[45]^2\langle 7|p_1|6]^2}{(s_{167}+\mu^2)(s_{458}+\mu^2)}\right.\\ &\qquad\left. + \frac{[34]^2[56]^2\langle 7|p_{34}|2]^2}{s_{347}(s_{568}+\mu^2)} + \frac{[23]^2[56]^2\langle 7|p_{56}|4]^2}{s_{567}(s_{123}+\mu^2)}\right] + \mathcal{P}\left(2,3,4,5,6\right).\end{aligned} \tag{E.0.36}$$

APPENDIX F

Structure of Contact Terms

In Section 3.3.1 we argued, by a combination of dimensional analysis, little group scaling and requiring vanishing as $\mu^2 \to 0$, that contact terms could appear in the mDBI$_4$ amplitudes in the NSD sector in the form of some contraction of the form

$$\mathcal{A}_n^{\text{mDBI}_4}\left(1_\phi, 2_\gamma^+, \ldots, (n-1)_\gamma^+, n_{\overline{\phi}}\right)\Big|_{\text{contact}} \sim \mu^2 |2]^2 |3]^2 \ldots |n-1]^2, \tag{F.0.1}$$

where n is even. In this appendix we will give a short proof that there is a unique such contact term for each n. We begin by noting that any candidate term has the form of a sum over terms where each term is a sum over cyclic contractions of the spinors. For example for $n = 12$ typical terms might have the form

$$([23][34][45][56][67][72])\,([89][9,10][10,11][11,8])\,, \tag{F.0.2}$$

or

$$([23][34][42])\,([56][67][75])\,([89][9,10][10,11][11,8])\,. \tag{F.0.3}$$

Neither term by itself is a candidate contact term since it does not have the appropriate Bose symmetry. We should take expression (F.0.2) and symmetrize over each pair of spinors, beginning with 3 and 4 gives

$$([23][34][45] + [24][43][35])\,[56][67][72]\,([89][9,10][10,11][11,8])\,, \tag{F.0.4}$$

applying the Schouten identity then gives

$$= -[34]^2\,([25][56][67][72])\,([89][9,10][10,11][11,8])\,. \tag{F.0.5}$$

This has reduced a cyclic contraction of length 6 to a product of cyclic contractions of *strictly shorter* length. By Bose symmetrizing over all pairs of spinors we can reduce any possible contact term to a sum over product of cyclic contractions of length 2. Terms such as (F.0.3) with odd cyclic contractions vanish after Bose symmetrization. The final expression then has the unique form

$$\mathcal{A}_n^{\mathrm{mDBI}_4}\left(1_\phi, 2_\gamma^+, \ldots, (n-1)_\gamma^+, n_{\overline{\phi}}\right)\Big|_{\text{contact}} = c_n \mu^2 \left([23]^2[45]^2 \ldots [n-2, n-1]^2 + \ldots\right), \quad \text{(F.0.6)}$$

where $+\ldots$ denotes the sum over all ways of partitioning the set $\{2, \ldots, n\}$ into subsets of length 2. This completes the proof that there is a unique possible contact term at each multiplicity.

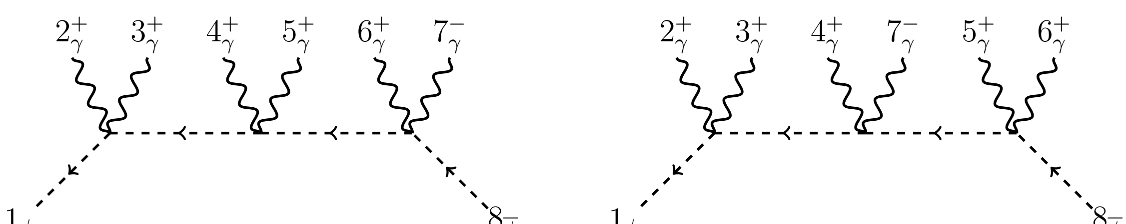
APPENDIX G

T-Duality Constraints on 8-point Amplitudes

Following our discussion in Section 3.3.2, we now investigate how T-duality constrains the 8-point amplitudes in mDBI$_4$. Begin with the dimensional reduction followed by the soft limit of particle 7 for the NSD 8-point mDBI$_4$ Ansatz

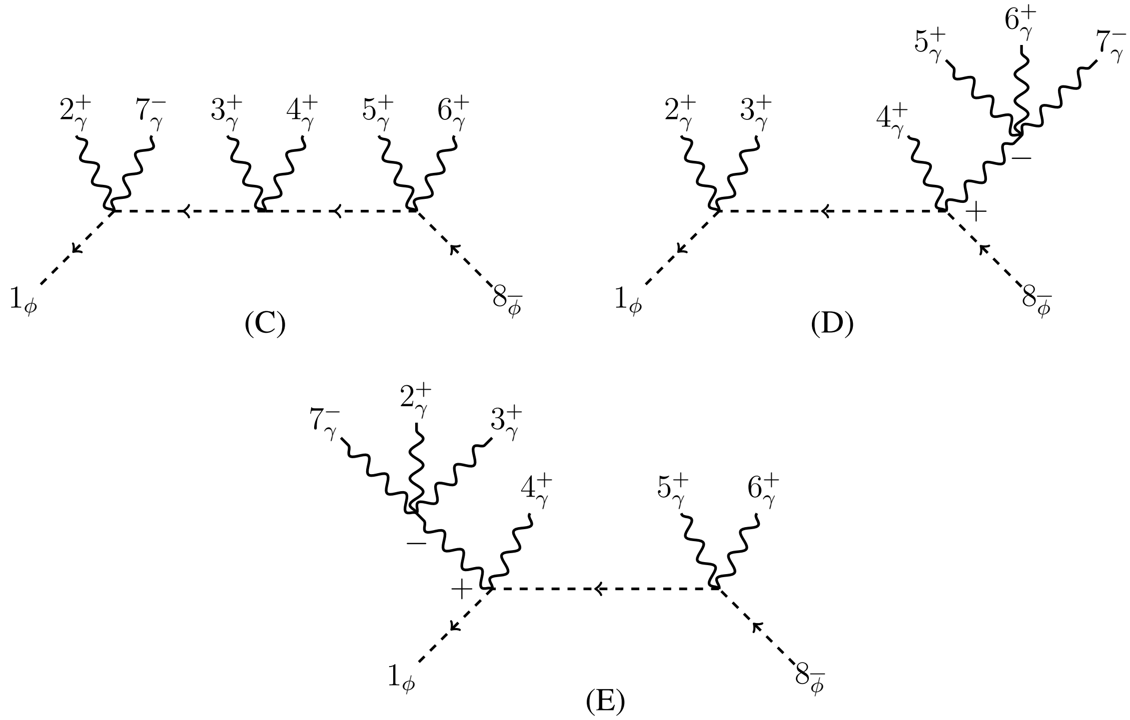
$$\begin{aligned}&\mathcal{A}_8^{\mathrm{mDBI}_4}\left(1_\phi, 2_\gamma^+, 3_\gamma^+, 4_\gamma^+, 5_\gamma^+, 6_\gamma^+, 7_\gamma^+, 8_{\overline{\phi}}\right)\\ &\xrightarrow{3d+\mathrm{soft}} -\frac{1}{8}\left[\frac{2(\mu^2)^3 s_{23}s_{45}(p_6\cdot p_7)}{(s_{123}+\mu^2)(s_{68}+\mu^2)}\right] + c_8\mu^2 s_{23}s_{45}s_{67} + \mathcal{P}\left(2,3,4,5,6,7\right).\end{aligned} \tag{G.0.1}$$

The MHV amplitude has a more complicated structure, there are more factorization graphs which are not related by permutations of external lines. Explicitly



(A) (B)

(G.0.2)



(G.0.3)

In this topological decomposition the amplitude has the form

$$\begin{aligned}\mathcal{A}_8^{\mathrm{mDBI}_4}&\left(1_\phi, 2_\gamma^+, 3_\gamma^+, 4_\gamma^+, 5_\gamma^+, 6_\gamma^+, 7_\gamma^-, 8_{\overline{\phi}}\right)\\ &= \mathcal{A}_{8(\mathrm{A})}^{\mathrm{mDBI}_4} + \mathcal{A}_{8(\mathrm{B})}^{\mathrm{mDBI}_4} + \mathcal{A}_{8(\mathrm{C})}^{\mathrm{mDBI}_4} + \mathcal{A}_{8(\mathrm{D})}^{\mathrm{mDBI}_4} + \mathcal{A}_{8(\mathrm{E})}^{\mathrm{mDBI}_4},\end{aligned} \tag{G.0.4}$$

where

$$\mathcal{A}_{8(\mathrm{A})}^{\mathrm{mDBI}_4} \xrightarrow{3d+\mathrm{soft}} \frac{-(\mu^2)^2 s_{23} s_{45}\left(2(p_7\cdot p_8)(s_{68}+\mu^2)+2\mu^2(p_7\cdot p_6)\right)}{(s_{123}+\mu^2)(s_{68}+\mu^2)} + \ldots \tag{G.0.5}$$

$$\mathcal{A}_{8(\mathrm{B})}^{\mathrm{mDBI}_4} \xrightarrow{3d+\mathrm{soft}} \frac{-(\mu^2)^2 s_{23} s_{56}\left(4(p_7\cdot p_{123})(p_4\cdot p_{123})-2s_{123}(p_4\cdot p_7)\right)}{(s_{123}+\mu^2)(s_{568}+\mu^2)} + \ldots \tag{G.0.6}$$

$$\mathcal{A}_{8(\mathrm{C})}^{\mathrm{mDBI}_4} \xrightarrow{3d+\mathrm{soft}} \frac{-(\mu^2)^2 s_{34} s_{56}\left(2(p_7\cdot p_1)(s_{12}+\mu^2)+2\mu^2(p_2\cdot p_7)\right)}{(s_{12}+\mu^2)(s_{568}+\mu^2)} + \ldots \tag{G.0.7}$$

$$\mathcal{A}_{8(\mathrm{D})}^{\mathrm{mDBI}_4} \xrightarrow{3d+\mathrm{soft}} -\frac{(\mu^2)^2 s_{23}\left(4(p_7\cdot p_{56})(p_4\cdot p_{56})-2s_{56}(p_4\cdot p_7)\right)}{s_{123}+\mu^2} + \ldots \tag{G.0.8}$$

$$\mathcal{A}_{8(\mathrm{E})}^{\mathrm{mDBI}_4} \xrightarrow{3d+\mathrm{soft}} -\frac{(\mu^2)^2 s_{56}\left(4(p_7\cdot p_{23})(p_4\cdot p_{23})-2s_{23}(p_4\cdot p_7)\right)}{s_{568}+\mu^2} + \ldots \tag{G.0.9}$$

Here $+\ldots$ corresponds to summing over all topologically inequivalent relabelings of the positive helicity photons. Note that we do not include a contact contribution, as discussed in Appendix F.

From the singularity structure it is clear that diagrams A, B and C must cancel against the contribution of the NSD amplitude. For diagrams A and C it is easy to pick out the relevant pieces proportional to $(\mu^2)^3$. For diagram B this is a little less obvious and requires a little algebra first.

The key idea is to recognize that there is something special about p_4 since it is the positive helicity particle in the middle of the diagram. We will see that something nice happens if we use momentum conservation and on-shellness to remove p_4 from the expression. That is we use

$$p_4 = -p_{123} - p_{568}, \tag{G.0.10}$$

and the on-shell constraint

$$p_4^2 = 0 \Rightarrow p_{123} \cdot p_{568} = -\frac{1}{2}\left(s_{123} + s_{568}\right). \tag{G.0.11}$$

Using this on the numerator of B gives

$$\begin{aligned} &4(p_7 \cdot p_{123})(p_4 \cdot p_{123}) - 2s_{123}(p_4 \cdot p_7) \\ &\quad = -2(p_7 \cdot p_{123})\left(s_{123} - s_{568}\right) + 2s_{123}(p_{123} \cdot p_7 + p_{568} \cdot p_7) \\ &\quad = 2(p_7 \cdot p_{123})(s_{568} + \mu^2) + 2(p_7 \cdot p_{568})(s_{123} + \mu^2) + 2\mu^2(p_4 \cdot p_7). \end{aligned} \tag{G.0.12}$$

We can therefore more usefully rewrite B in the form

$$\begin{aligned} &\mathcal{A}_{8(\mathrm{B})}^{\mathrm{mDBI}}\left(1_\phi, 2_\gamma^+, 3_\gamma^+, 4_\gamma^+, 5_\gamma^+, 6_\gamma^+, 7_\gamma^-, 8_{\overline{\phi}}\right) \\ &\quad \xrightarrow{3d+\text{soft}} \frac{-2(\mu^2)^3 s_{23}s_{56}(p_4 \cdot p_7)}{(s_{123}+\mu^2)(s_{568}+\mu^2)} - \frac{2(\mu^2)^2 s_{23}s_{56}(p_7 \cdot p_{123})}{s_{123}+\mu^2} - \frac{2(\mu^2)^2 s_{23}s_{56}(p_7 \cdot p_{568})}{s_{568}+\mu^2} + \dots \end{aligned} \tag{G.0.13}$$

We now see explicitly that the non-local contributions from the MHV amplitude cancel completely. What remains is a sum of terms with only a single propagator. This is important since we want the remaining terms to cancel against each other, this couldn't happen unless some of the singularities disappeared upon dimensional reduction and soft limits since the topologically distinct graphs, by definition, have distinct singularity structure.

To finish the calculation we pick a singularity and verify that the sum of all contributions vanishes. Due to charge conjugation symmetry all such calculations are identical so we only need to verify a single case explicitly. We will choose the singularity associated with $s_{123} = -\mu^2$, this receives contributions from diagrams A, B and D. Summing the relevant terms

$$
\begin{aligned}
& -\frac{2(\mu^2)^2 s_{23} s_{45} (p_8 \cdot p_7)}{s_{123}+\mu^2} - \frac{2(\mu^2)^2 s_{23} s_{56} (p_7 \cdot p_{123})}{s_{123}+\mu^2} \\
& \qquad - \frac{(\mu^2)^2 s_{23} \left(4(p_7 \cdot p_{56})(p_4 \cdot p_{56}) - 2 s_{56} (p_4 \cdot p_7)\right)}{s_{123}+\mu^2} + \mathcal{C}\,(4,5,6) \\
& = -\frac{2(\mu^2)^2 s_{23} s_{456} (p_8 \cdot p_7)}{s_{123}+\mu^2} - \frac{2(\mu^2)^2 s_{23} s_{456} (p_7 \cdot p_{123})}{s_{123}+\mu^2} - \frac{2(\mu^2)^2 s_{23} s_{456} (p_7 \cdot p_{456})}{s_{123}+\mu^2} \\
& = 0.
\end{aligned} \tag{G.0.14}
$$

As in the 6-point case we find that all of the factoring terms in the NSD and MHV mDBI$_4$ amplitudes cancel against each other and vanish in the T-dual soft configuration. Since the possible contact term is $\mathcal{O}\,(p_7)$, we must choose $c_8 = 0$ for compatibility with T-duality.

APPENDIX H

Evaluating Rational Integrals

A *rational integral* in this context is defined as an integral in $d = 4 - 2\epsilon$ dimensions, for which the integrand vanishes in $d = 4$. A powerful and general method for evaluating these integrals was given in [188] where the following *dimension shifting formula* was derived

$$\int \frac{\mathrm{d}^{4-2\epsilon}l}{(2\pi)^{4-2\epsilon}}(l^2_{-2\epsilon})^p f(l) = (4\pi)^p \frac{\Gamma(-\epsilon + p)}{\Gamma(-\epsilon)} \int \frac{\mathrm{d}^{4+2p-2\epsilon}l}{(2\pi)^{4+2p-2\epsilon}} f(l), \tag{H.0.1}$$

where $f(l)$ is some rational function of the d-dimensional loop momentum. This formula allows us to exchange integrals with explicit factors of $l^2_{-2\epsilon}$ for integrals without such factors evaluated in higher dimensions. The integral on the left-hand-side of (H.0.1) is formally defined as a tensor integral

$$\int \frac{\mathrm{d}^{4-2\epsilon}l}{(2\pi)^{4-2\epsilon}}(l^2_{-2\epsilon})^p f(l) \equiv \left(\prod_{i=1}^{p} g^{[-2\epsilon]}_{\mu_i\nu_i}\right) \int \frac{\mathrm{d}^{4-2\epsilon}l}{(2\pi)^{4-2\epsilon}} \left(\prod_{j=1}^{p} l^{\mu_j} l^{\nu_j}\right) f(l), \tag{H.0.2}$$

where $g^{[-2\epsilon]}_{\mu\nu}$ is the metric tensor projected onto the non-physical -2ϵ-dimensional momentum subspace. The utility of the formula (H.0.1) is that it gives an efficient way to bypass calculating tensor reduction for integrands of arbitrarily high-rank; in Chapter 2 all integrals can be exchanged using this method to either scalar or rank-2 tensor integrals. Even with this simplification, obtaining explicit results to all orders in ϵ is a very difficult problem, for which only a small fraction of the necessary integrals are known. At $\mathcal{O}(\epsilon^0)$ however, the formula (H.0.1) simplifies significantly and the right-hand-side depends only on the *divergent* part of the $d = 4 + 2p - 2\epsilon$-

dimensional integral

$$\int \frac{\mathrm{d}^{4-2\epsilon}l}{(2\pi)^{4-2\epsilon}}(l_{-2\epsilon}^2)^p f(l) = -(p-1)!(4\pi)^p \left[\int \frac{\mathrm{d}^{4+2p-2\epsilon}l}{(2\pi)^{4+2p-2\epsilon}} f\left(l\right)\right]_{1/\epsilon} + \mathcal{O}\left(\epsilon\right). \tag{H.0.3}$$

This is the key formula for obtaining explicit expressions for one-loop rational integrals. As we will see below the simplification arises from the fact that after Feynman parametrization the divergent part of the integral can be extracted as the trivial integration of a polynomial in Feynman parameters.

H.0.1 Rational Scalar n-gon Integral

In this section we present the explicit calculation of the rational scalar n-gon integral

$$I_n^{d=4-2\epsilon}\left[(\mu^2)^n;\{p_i\}\right] \equiv \int \frac{\mathrm{d}^{4-2\epsilon}l}{(2\pi)^{4-2\epsilon}} \frac{\left(l_{-2\epsilon}^2\right)^n}{\prod_{i=1}^n \left(l-\sum_{j=1}^i p_j\right)^2}, \tag{H.0.4}$$

where the external momenta p_i are massive. Using the dimension shifting formula (H.0.1) this is related to the massless scalar n-gon integral in $d = 4+2n-2\epsilon$ dimensions

$$= (4\pi)^n \frac{\Gamma(n-\epsilon)}{\Gamma(-\epsilon)} \int \frac{\mathrm{d}^{4+2n-2\epsilon}l}{(2\pi)^{4+2n-2\epsilon}} \frac{1}{\prod_{i=1}^n \left(l-\sum_{j=1}^i p_j\right)^2}. \tag{H.0.5}$$

The next step is to use Feynman parametrization and write the integral as

$$\begin{aligned} &= (4\pi)^n \frac{\Gamma(n-\epsilon)}{\Gamma(-\epsilon)}(n-1)! \\ &\quad \times \int_0^1 \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_n \left[\delta\left(\sum_{i=1}^n x_i - 1\right) \int \frac{\mathrm{d}^{4+2n-2\epsilon}l}{(2\pi)^{4+2n-2\epsilon}} \frac{1}{\left[\sum_{i=1}^n x_i \left(l-\sum_{j=1}^i p_j\right)^2\right]^n}\right]. \end{aligned} \tag{H.0.6}$$

After shifting the loop momentum by $l \to l + \sum_{i=1}^{n-1} \sum_{j=1}^{i} x_i p_j$ the denominator of the above integrand can be written as $\left[l^2 + \Delta\right]^n$ with

$$\begin{aligned}\Delta &= \sum_{i=1}^{n} x_i \left(1 - x_i\right) \left(\sum_{j=1}^{i} p_j\right)^2 - 2 \sum_{i<j}^{n} x_i x_j \left(\sum_{k=1}^{i} p_k\right) \cdot \left(\sum_{k=1}^{j} p_k\right) \\ &= - \sum_{i=1}^{n} x_i \left(1 - x_i\right) \left(\sum_{j=1}^{i} p_j\right) \cdot \left(\sum_{j=i+1}^{n} p_j\right) + 2 \sum_{i<j}^{n} x_i x_j \left(\sum_{k=1}^{i} p_k\right) \cdot \left(\sum_{k=j+1}^{n} p_k\right) \\ &= - \sum_{i<j}^{n} p_i \cdot p_j \left(\sum_{k=i}^{j-1} x_k\right) \left(1 - \sum_{k=i}^{j-1} x_k\right) . \end{aligned} \tag{H.0.7}$$

In the second line above, we used momentum conservation to write everything in terms of scalar products of two different momenta and in the third line, we rearranged the sums, writing explicitly the coefficient of each $p_i \cdot p_j$. To further simplify this, we substitute $1 = \sum_{i=1}^{n} x_i$ and we collect the coefficients of each product $x_i x_j$,

$$\Delta = - \sum_{i<j}^{n} x_i x_j \left(\sum_{k=i+1}^{j} p_k\right) \cdot \left(\sum_{k=1}^{i} p_k + \sum_{k=j+1}^{n} p_k\right) = \sum_{i<j}^{n} x_i x_j \left(\sum_{k=i+1}^{j} p_k\right)^2 , \tag{H.0.8}$$

where in the second step we used momentum conservation to write everything in terms of Mandelstam variables of adjacent momenta. Going back to (H.0.6) and using the standard integral

$$\int \frac{\mathrm{d}^{4+2n-2\epsilon} l}{(2\pi)^{4+2n-2\epsilon}} \frac{1}{\left[l^2 + \Delta\right]^n} = \frac{i}{(4\pi)^{n+2-\epsilon}} \frac{\Gamma\left(-2+\epsilon\right)}{(n-1)!} \Delta^{2-\epsilon}, \tag{H.0.9}$$

in full generality the rational integral (H.0.4) is given by the Feynman parameter integral

$$\begin{aligned} I_n^{d=4-2\epsilon}\left[(\mu^2)^n; \{p_i\}\right] &= \frac{i}{(4\pi)^{2-\epsilon}} \frac{\Gamma(n-\epsilon)\Gamma(-2+\epsilon)}{\Gamma(-\epsilon)} \\ &\times \int_0^1 \mathrm{d}x_1 \ldots \mathrm{d}x_n \, \delta\left(\sum_{i=1}^{n} x_i - 1\right) \left[\sum_{i<j}^{n} x_i x_j \left(\sum_{k=i+1}^{j} p_k\right)^2\right]^{2-\epsilon} . \end{aligned} \tag{H.0.10}$$

Only in certain special cases ($n = 2$ and $n = 3$) is this integral known to all orders in ϵ [189]. The leading $\mathcal{O}\left(\epsilon^0\right)$ contribution however, can be calculated explicitly for all n. It is given by

$$= -\frac{i}{32\pi^2}(n-1)! \int_0^1 \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_n \delta\left(\sum_{i=1}^{n} x_i - 1\right) \left[\sum_{i<j}^{n} x_i x_j \left(\sum_{k=i+1}^{j} p_k\right)^2\right]^2 + \mathcal{O}\left(\epsilon\right) . \tag{H.0.11}$$

We now have to perform the integration over the n Feynman parameters. For this we use the general formula

$$\int_0^1 \mathrm{d}x_1 \mathrm{d}x_2 \dots \mathrm{d}x_n \delta\left(\sum_{i=1}^n x_i - 1\right) x_1^{r_1} x_2^{r_2} \dots x_n^{r_n} = \frac{\Gamma(1+r_1)\Gamma(1+r_2)\dots\Gamma(1+r_n)}{\Gamma(n+r_1+r_2+\dots+r_n)} \,. \tag{H.0.12}$$

Special instances of this formula that are relevant for the calculations of this and the next subsection are the following

$$\int_0^1 \mathrm{d}x_1 \mathrm{d}x_2 \dots \mathrm{d}x_n \delta\left(\sum_{i=1}^n x_i - 1\right) x_1 x_2 x_3 x_4 = \frac{1}{(n+3)!} \,, \tag{H.0.13}$$

$$\int_0^1 \mathrm{d}x_1 \mathrm{d}x_2 \dots \mathrm{d}x_n \delta\left(\sum_{i=1}^n x_i - 1\right) x_1 x_2 x_3^2 = \frac{2}{(n+3)!} \,, \tag{H.0.14}$$

$$\int_0^1 \mathrm{d}x_1 \mathrm{d}x_2 \dots \mathrm{d}x_n \delta\left(\sum_{i=1}^n x_i - 1\right) x_1^2 x_2^2 = \frac{4}{(n+3)!} \,, \tag{H.0.15}$$

$$\int_0^1 \mathrm{d}x_1 \mathrm{d}x_2 \dots \mathrm{d}x_n \delta\left(\sum_{i=1}^n x_i - 1\right) x_1^3 x_2 = \frac{6}{(n+3)!} \,, \tag{H.0.16}$$

With these, we find that the integrated result takes the form

$$\begin{aligned} & I_n^{d=4-2\epsilon}\left[(\mu^2)^n; \{p_i\}\right] \\ & = -\frac{i}{32\pi^2} \frac{1}{n(n+1)(n+2)(n+3)} \sum_{i<j}^n \sum_{k<l}^n a_{ijkl} \left(\sum_{m=i+1}^j p_m\right)^2 \left(\sum_{m=k+1}^l p_m\right)^2 + \mathcal{O}(\epsilon) \,, \end{aligned} \tag{H.0.17}$$

where

$$a_{ijkl} = \begin{cases} 1 & \text{if all } i,j,k,l \text{ are different} \\ 2 & \text{if exactly 2 of } i,j,k,l \text{ are identical} \\ 4 & \text{if } i=k \text{ and } j=l \end{cases} \,. \tag{H.0.18}$$

H.0.2 Rational Rank-2 Tensor n-gon Integral

Similar to the case of the rational scalar n-gon integral, we present the explicit calculation of the rational rank-2 tensor n-gon integral

$$I_n^{d=4-2\epsilon}\left[\left(\mu^2\right)^{n-1}(u\cdot l)^2, \{p_i\}\right] \equiv \int \frac{\mathrm{d}^{4-2\epsilon} l}{(2\pi)^{4-2\epsilon}} \frac{\left(l_{-2\epsilon}^2\right)^{n-1}(u\cdot l)^2}{\prod_{i=1}^n \left(l - \sum_{j=1}^i p_j\right)^2}, \tag{H.0.19}$$

where u^μ is a 4-dimensional null vector. The dimension shifting formula (H.0.1) gives

$$= (4\pi)^{n-1}\,\frac{\Gamma(n-1-\epsilon)}{\Gamma(-\epsilon)}\int\frac{\mathrm{d}^{2+2n-2\epsilon}l}{(2\pi)^{2+2n-2\epsilon}}\frac{(u\cdot l)^2}{\prod_{i=1}^{n}\left(l-\sum_{j=1}^{i}p_j\right)^2}. \tag{H.0.20}$$

We can use the same Feynman parametrization trick as before to write the integral as

$$\begin{aligned}= (4\pi)^{n-1}\,\frac{\Gamma(n-1-\epsilon)}{\Gamma(-\epsilon)}\,(n-1)!\int_0^1 \mathrm{d}x_1\mathrm{d}x_2\ldots\mathrm{d}x_n\delta\left(\sum_{i=1}^{n}x_i-1\right)&\\ \times\int\frac{\mathrm{d}^{2+2n-2\epsilon}l}{(2\pi)^{2+2n-2\epsilon}}\frac{(u\cdot l)^2}{\left[\sum_{i=1}^{n}x_i\left(l-\sum_{j=1}^{i}p_j\right)^2\right]^n}&.\end{aligned} \tag{H.0.21}$$

After shifting the loop momentum by $l\to l+\sum_{i=1}^{n-1}\sum_{j=1}^{i}x_ip_j$, we get

$$\begin{aligned}= (4\pi)^{n-1}\,\frac{\Gamma(n-1-\epsilon)}{\Gamma(-\epsilon)}\,(n-1)!\int_0^1 \mathrm{d}x_1\mathrm{d}x_2\ldots\mathrm{d}x_n\delta\left(\sum_{i=1}^{n}x_i-1\right)&\\ \times\int\frac{\mathrm{d}^{2+2n-2\epsilon}l}{(2\pi)^{2+2n-2\epsilon}}\frac{(u\cdot l)^2+\left(\sum_{i=1}^{n-1}\sum_{j=1}^{i}x_i(u\cdot p_j)\right)^2}{[l^2+\Delta]^n}&,\end{aligned} \tag{H.0.22}$$

where $\Delta=\sum_{i<j}^{n}x_ix_j\left(\sum_{k=i+1}^{j}p_k\right)^2$ as before and all cross-terms have been dropped since they are odd in l. The first term integrates to an expression proportional to u^2 which is zero by assumption. The remaining terms have the form of the standard integral (J.0.1), so we can give a general expression for (H.0.19) as a integral over Feynman parameters

$$\begin{aligned}I_n^{d=4-2\epsilon}\left[\left(\mu^2\right)^{n-1}(u\cdot l)^2,\{p_i\}\right]&=\frac{i}{(4\pi)^{2-\epsilon}}\frac{\Gamma(n-1-\epsilon)\Gamma(-1+\epsilon)}{\Gamma(-\epsilon)}\\ \times\int_0^1\mathrm{d}x_1\mathrm{d}x_2\ldots\mathrm{d}x_n\delta\left(\sum_{i=1}^{n}x_i-1\right)&\left(\sum_{i=1}^{n-1}\sum_{j=1}^{i}x_iu\cdot p_j\right)^2\left[\sum_{i<j}^{n}x_ix_j\left(\sum_{k=i+1}^{j}p_k\right)^2\right]^{1-\epsilon}.\end{aligned} \tag{H.0.23}$$

As in the scalar case we can give explicit expressions for all n at $\mathcal{O}\left(\epsilon^0\right)$, using the Feynman-parameter integrals (H.0.13) - (H.0.16). With these, we find that the integrated result takes the form

$$I_n^{d=4-2\epsilon}\left[\left(\mu^2\right)^{n-1}(u\cdot l)^2,\{p_i\}\right] = \frac{i}{16\pi^2}\frac{1}{(n-1)n(n+1)(n+2)(n+3)}$$
$$\times\sum_{i<j}^{n}\left(\sum_{m=i+1}^{j}p_m\right)^2\left[\sum_{k<l}^{n}2a_{ijkl}\left(\sum_{m=1}^{k}u\cdot p_m\right)\left(\sum_{m=1}^{l}u\cdot p_m\right)+\sum_{k=1}^{n}b_{ijk}\left(\sum_{m=1}^{k}u\cdot p_m\right)^2\right], \tag{H.0.24}$$

where a_{ijkl} is as defined above and

$$b_{ijk} = \begin{cases} 2 & \text{if } i \neq k \text{ and } j \neq k \\ 6 & \text{if } i = k \text{ or } j = k \end{cases}. \tag{H.0.25}$$

APPENDIX I

Momentum Shift for Recursion Relations

In this appendix we provide an explicit $n \geq 6$ solution of the complex momentum shift presented in Section 4.2.3. In the spinor-helicity formalism, the shift takes the form (4.2.17) with n parameters a_i subject to the momentum conservation constraint

$$\sum_{i:\, h_i \geq 0} a_i |i][i| N + \sum_{j:\, h_j < 0} a_j N |j\rangle\langle j| = 0. \tag{I.0.1}$$

At first glance this is 4 independent constraints, but since it actually involves a projection to a 3d subspace, momentum conservation in the N^μ direction is automatically satisfied. We therefore have to solve for 3 of the a_i in terms of the remaining $n-3$. There are many ways to do this and each gives different, perfectly valid, momentum shifts. In this appendix, we present an explicit example of such a momentum shift.

For every duality-violating amplitude with $n \geq 6$ there must be at least 4 particles of the same helicity.[1] Without loss of generality we assume that the non-negative states have momentum labels $\{12 \cdots n_+\}$, with $n_+ \geq 4$. We then project the above matrix equation onto spinor bases spanned by $[1|$ and $[2|$ on the left and $\bar{N}|3]$ and $\bar{N}|4]$ on the right, where $\bar{N} = N^\mu \bar{\sigma}_\mu$. For this, we need the following identity

$$[a|N\bar{N}|b] = -[ab], \tag{I.0.2}$$

which follows from the assumption that N^μ is a unit vector. The resulting system of equations is

[1]For amplitudes with $n = 4, 5$ and fewer than 4 particles of the same helicity the steps to solve for $a_1,\ a_2,\ a_3$ are the same, but produce slightly different rational expressions.

then

$$\begin{pmatrix} 0 & [23][21] & 0 & [43][41] \\ 0 & [24][21] & [34][31] & 0 \\ [13][12] & 0 & 0 & [43][42] \\ [14][12] & 0 & [34][32] & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} -\sum_{i=5}^{n_+} a_i[i3][i1] - \sum_{j=n_++1}^{n} a_j[3|N|j\rangle[1|N|j\rangle \\ -\sum_{i=5}^{n_+} a_i[i4][i1] - \sum_{j=n_++1}^{n} a_j[4|N|j\rangle[1|N|j\rangle \\ -\sum_{i=5}^{n_+} a_i[i3][i2] - \sum_{j=n_++1}^{n} a_j[3|N|j\rangle[2|N|j\rangle \\ -\sum_{i=5}^{n_+} a_i[i4][i2] - \sum_{j=n_++1}^{n} a_j[4|N|j\rangle[2|N|j\rangle \end{pmatrix}. \tag{I.0.3}$$

This equation cannot be solved for $a_1, ..., a_4$ since the matrix on the left-hand-side has vanishing determinant. This is a consequence of the fact that only three of the momentum conservation equations are linearly independent. So we can simply drop the last equation and solve the simpler non-degenerate system

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{1}{[12][13]} \\ \frac{1}{[21][23]} & 0 & 0 \\ \frac{[24]}{[23][13][34]} & \frac{1}{[31][34]} & 0 \end{pmatrix} \begin{pmatrix} -\sum_{i=4}^{n_+} a_i[i3][i1] - \sum_{j=n_++1}^{n} a_j[3|N|j\rangle[1|N|j\rangle \\ -\sum_{i=4}^{n_+} a_i[i4][i1] - \sum_{j=n_++1}^{n} a_j[4|N|j\rangle[1|N|j\rangle \\ -\sum_{i=4}^{n_+} a_i[i3][i2] - \sum_{j=n_++1}^{n} a_j[3|N|j\rangle[2|N|j\rangle \end{pmatrix}. \tag{I.0.4}$$

Since we want an all-line shift we choose $a_4, ..., a_n$ to be any non-zero values. Importantly, the resulting shift (4.2.17) is a *rational* function of the spinor brackets and therefore it is relatively simple to implement, both analytically and numerically.

APPENDIX J

Loop Integrals in Dimensional Regularization

In this appendix, we present some useful 1-loop integrals in dimensional regularization.

J.0.1 Scalar n-gon in $4+2p-2\epsilon$ Dimensions

A well-known result is

$$\int \frac{\mathrm{d}^{4+2n-2\epsilon} l}{(2\pi)^{4+2n-2\epsilon}} \frac{1}{[l^2+\Delta]^n} = \frac{i}{(4\pi)^{n+2-\epsilon}} \frac{\Gamma(-2+\epsilon)}{(n-1)!} \Delta^{2-\epsilon}. \tag{J.0.1}$$

Using this result with $\epsilon \to \epsilon' = \epsilon + n - p$ gives

$$\boxed{\int \frac{\mathrm{d}^{4+2p-2\epsilon} l}{(2\pi)^{4+2p-2\epsilon}} \frac{1}{[l^2+\Delta]^n} = \frac{i}{(4\pi)^{p+2-\epsilon}} \frac{\Gamma(-2+\epsilon+n-p)}{(n-1)!} \Delta^{p+2-n-\epsilon}.} \tag{J.0.2}$$

This result can also be obtained from Peskin & Schroeder (A.44), but the form (J.0.2) is directly useful for us.

J.0.2 n-gon with p Powers of $l_{-2\epsilon}^2$ in $4-2\epsilon$ Dimensions

For an n-gon diagram with massive momenta P_i entering at each vertex, we now derive the following result:

$$\boxed{\begin{aligned} I_n^{(p)} &\equiv \int \frac{d^{4-2\epsilon}l}{(2\pi)^{4-2\epsilon}} \frac{(l_{-2\epsilon}^2)^p}{\prod_{i=1}^n \left(l-\sum_{j=1}^{2i} P_j\right)^2} \\ &= \frac{i}{(4\pi)^{2-\epsilon}} \frac{\Gamma(p-\epsilon)\Gamma(n-2-p+\epsilon)}{\Gamma(-\epsilon)} \int_0^1 dx_1 \cdots dx_n \delta\left(\sum_{i=1}^n x_i - 1\right) \Delta^{p+2-n-\epsilon}, \end{aligned}} \tag{J.0.3}$$

where

$$\Delta = \sum_{i<j}^n x_i x_j \left(\sum_{k=i+1}^j P_k\right)^2 . \tag{J.0.4}$$

The dimension-shifting formula [188] gives

$$I_n^{(p)} = (4\pi)^p \frac{\Gamma(p-\epsilon)}{\Gamma(-\epsilon)} \int \frac{d^{4+2p-2\epsilon}l}{(2\pi)^{4+2p-2\epsilon}} \frac{1}{\prod_{i=1}^n \left(l-\sum_{j=1}^{2i} P_j\right)^2} . \tag{J.0.5}$$

Introducing Feynman parameters this can be rewritten as

$$I_n^{(p)} = (4\pi)^p \frac{\Gamma(p-\epsilon)}{\Gamma(-\epsilon)} (n-1)! \int_0^1 dx_1 \cdots dx_n \delta\left(\sum_{i=1}^n x_i - 1\right) \int \frac{d^{4+2p-2\epsilon}l}{(2\pi)^{4+2p-2\epsilon}} \frac{1}{(l^2+\Delta)^n} \tag{J.0.6}$$

where the compact from of Δ given in (J.0.4) was derived in Appendix C of [12]. We evaluate the l-integral using (J.0.2) to get

$$I_n^{(p)} = \frac{i}{(4\pi)^{2-\epsilon}} \frac{\Gamma(p-\epsilon)\Gamma(n-2-p+\epsilon)}{\Gamma(-\epsilon)} \int_0^1 dx_1 \cdots dx_n \delta\left(\sum_{i=1}^n x_i - 1\right) \Delta^{p+2-n-\epsilon} . \tag{J.0.7}$$

This completes the derivation of (J.0.3).

When $p = n-2$, the integral (J.0.3) is particularly simple and the Feynman parameter integral can be carried out trivially:

$$\begin{aligned} I_n^{(n-2)} &= \frac{i}{(4\pi)^{2-\epsilon}} \frac{\Gamma(n-2-\epsilon)\Gamma(\epsilon)}{\Gamma(-\epsilon)} \int_0^1 dx_1 \cdots dx_n \delta\left(\sum_{i=1}^n x_i - 1\right) \Delta^{-\epsilon} \\ &\xrightarrow{\epsilon\to 0} -\frac{i}{(4\pi)^2} \frac{\Gamma(n-2)}{(n-1)!} + \mathcal{O}(\epsilon) \\ &= -\frac{i}{(4\pi)^2} \frac{1}{(n-1)(n-2)} + \mathcal{O}(\epsilon), \end{aligned} \tag{J.0.8}$$

i.e.

$$\boxed{I_n^{(n-2)} = \int \frac{d^{4-2\epsilon}l}{(2\pi)^{4-2\epsilon}} \frac{(l^2_{-2\epsilon})^{n-2}}{\prod_{i=1}^{n}\left(l - \sum_{j=1}^{2i} P_j\right)^2} = -\frac{i}{(4\pi)^2}\frac{1}{(n-1)(n-2)} + \mathcal{O}(\epsilon)\,.} \tag{J.0.9}$$

Another integral we repeatedly use in the maintext is the bubble integral

$$\begin{aligned} I_2^{(p)}(P^2) &= \int \frac{d^{4-2\epsilon}l}{(2\pi)^{4-2\epsilon}} \frac{(l^2_{-2\epsilon})^p}{(l-\frac{1}{2}P)^2(l+\frac{1}{2}P)^2} \\ &= \frac{i}{(4\pi)^{2-\epsilon}} \frac{\Gamma(p-\epsilon)\Gamma(-p+\epsilon)}{\Gamma(-\epsilon)} \int dx \left(P^2 x(1-x)\right)^{p-\epsilon}\,. \end{aligned} \tag{J.0.10}$$

In this simple case the Feynman integral can be evaluated exactly, giving

$$\boxed{I_2^{(p)}(P^2) = \frac{i}{(4\pi)^{2-\epsilon}} \frac{\Gamma(p-\epsilon)\Gamma(-p+\epsilon)\Gamma^2(1+p-\epsilon)}{\Gamma(-\epsilon)\Gamma(2+2p-2\epsilon)} (P^2)^{p-\epsilon}.} \tag{J.0.11}$$

APPENDIX K

Born-Infeld Amplitudes

In this appendix, we calculate the 1-loop amplitudes with 4 external photons of pure Born-Infeld, using a combination of supersymmetric decomposition, traditional Feynman-diagrammatic methods, and generalized unitarity.

K.0.1 Tree-Level Amplitudes in BI and DBI

We are primarily interested in amplitudes in pure BI theory, however, we exploit computational tricks that use the supersymmetric completion of BI theory to supersymmetric DBI. The bosonic part of the $\mathcal{N}=4$ DBI action truncated to one complex scalar (i.e. $\mathcal{N}=2$ DBI) takes the form

$$S_{\mathrm{D3}}[F_{\mu\nu}, Z, \bar{Z}] = -\Lambda^4 \int \mathrm{d}^4x \sqrt{-\det\left(\eta_{\mu\nu} + \Lambda^{-2}F_{\mu\nu} + \Lambda^{-4}\partial_{(\mu}Z\partial_{\nu)}\bar{Z}\right)}. \tag{K.0.1}$$

The 4-scalar scattering processes are related by supersymmetry Ward identities

$$\mathcal{A}_4\big(1_Z\, 2_Z\, 3_{\bar{Z}}\, 4_{\bar{Z}}\big) = \frac{\langle 12\rangle^2}{\langle 23\rangle^2}\mathcal{A}_4\big(1^+_\gamma\, 2_Z\, 3^-_\gamma\, 4_{\bar{Z}}\big) = \frac{\langle 12\rangle^2}{\langle 34\rangle^2}\mathcal{A}_4\left(1^+_\gamma\, 2^+_\gamma\, 3^-_\gamma\, 4^-_\gamma\right), \tag{K.0.2}$$

to the 4-photon amplitude

$$\mathcal{A}_4\big(1^+_\gamma\, 2^+_\gamma\, 3^-_\gamma\, 4^-_\gamma\big) = [12]^2\langle 34\rangle^2\,. \tag{K.0.3}$$

All other 4-particle amplitudes vanish at leading order (tree-level), including those that violate the conservation of the duality charge (4.1.1):

$$\mathcal{A}_4\big(1^+_\gamma\, 2^+_\gamma\, 3^+_\gamma\, 4^+_\gamma\big) = \mathcal{A}_4\big(1^-_\gamma\, 2^+_\gamma\, 3^+_\gamma\, 4^+_\gamma\big) = \mathcal{A}_4\big(1^+_\gamma\, 2_Z\, 3^+_\gamma\, 4_{\bar{Z}}\big) = 0\,. \tag{K.0.4}$$

More generally, an n-particle process vanishes unless it involves an equal number of positive and negative helicity photons (4.1.1).

K.0.2 Feynman Rules

The vertex function of supersymmetric DBI with 2 external photons and 2 scalars is given by

$$\begin{aligned}
&V^{\mu\nu}(p_1, p_2,p_3, p_4) \\
&\quad = -2i\big[- p_3^{\nu}p_4^{\mu}(p_1\cdot p_2) - p_3^{\mu}p_4^{\nu}(p_1\cdot p_2) + p_2^{\mu}p_4^{\nu}(p_1\cdot p_3) + p_2^{\mu}p_3^{\nu}(p_1\cdot p_4) + p_1^{\nu}p_4^{\mu}(p_2\cdot p_3) \\
&\qquad\qquad + p_1^{\nu}p_3^{\mu}(p_2\cdot p_4) - (p_1\cdot p_4)(p_2\cdot p_3)g^{\mu\nu} - (p_1\cdot p_3)(p_2\cdot p_4)g^{\mu\nu}\big] \\
&\qquad -2i\big[- p_1^{\nu}p_2^{\mu}(p_3\cdot p_4) + (p_1\cdot p_2)(p_3\cdot p_4)g^{\mu\nu}\big]\,.
\end{aligned} \tag{K.0.5}$$

Notice that the 2 expressions in square brackets are individually gauge invariant and correspond to the 2 independent (up to integration by parts and application of the equations of motion) Lagrangian operators with the desired mass dimension and external states. The polarization vectors in spinor-helicity variables are

$$\epsilon^{\mu}_{i+} = -\frac{\langle q_i|\bar{\sigma}^{\mu}|i]}{\sqrt{2}\langle q_i i\rangle} \quad \text{and} \quad \epsilon^{\mu}_{i-} = -\frac{\langle i|\bar{\sigma}^{\mu}|q_i]}{\sqrt{2}[i q_i]}, \tag{K.0.6}$$

where q_i are arbitrary reference spinors.

K.0.3 Duality-Violating 1-loop 4-Point Amplitudes

For the computation of 1-loop 4-point amplitudes in the following, we are going to use a combination of on-shell techniques and Feynman rules. The 4-vector Feynman rule (K.0.5) is rather involved, but we can use a trick to calculate the duality-violating 1-loop amplitudes using the (vector)^2-(scalar)^2 vertex instead. The reason is that in any supersymmetrization of BI theory, the $++++$ and $+++-$ amplitudes (K.0.4) must vanish at any loop-order. Thus, in $\mathcal{N}=1$ BI theory, the contributions from the vector in the loop must cancel against the contribution from the fermion, i.e.

$$\mathcal{N}=1: \quad \mathcal{A}_4^{\mathrm{V}}(+++\pm) + \mathcal{A}_4^{\mathrm{F}}(+++\pm) = 0\,. \tag{K.0.7}$$

Similarly, in $\mathcal{N}=2$ BI theory we have

$$\mathcal{N}=2: \quad \mathcal{A}_4^{\mathrm{V}}(+++\pm) + 2\mathcal{A}_4^{\mathrm{F}}(+++\pm) + \mathcal{A}_4^{\mathrm{S}}(+++\pm) = 0. \tag{K.0.8}$$

Here the superscripts indicate contributions from a vector (V), Weyl fermion (F), or complex scalar (S) running in the loop. It follows that the pure BI result must equal the result from only scalars running in the loop:

$$\mathcal{A}_4^{\mathrm{V}}(+++\pm) = -\mathcal{A}_4^{\mathrm{F}}(+++\pm) = \mathcal{A}_4^{\mathrm{S}}(+++\pm)\,. \tag{K.0.9}$$

We exploit this in the following two subsections.

Due to the form of the interaction, $\partial Z \partial \bar{Z} F F$, the general (vector)^2-(scalar)^2 vertex rule is proportional to two powers of the scalar-momenta, which we denote l_1 and l_2 in anticipation of the loop-calculations below. The general Feynman rule is given in Appendix Q and is gauge-invariant.

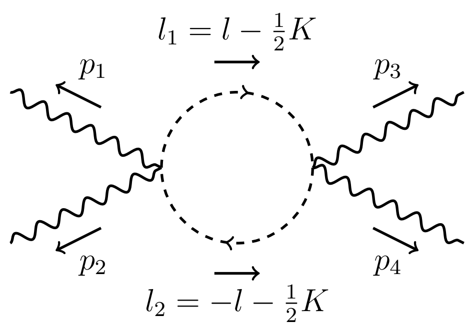
For later convenience, we write the vertex rules with on-shell photons and off-shell scalars compactly, using the spinor-helicity formalism, as

$$V(1_\gamma, 2_\gamma, l_{1Z}, l_{2\bar{Z}}) = V_{\mu\nu}(1_\gamma, 2_\gamma)\, l_1^\mu l_2^\nu\,, \tag{K.0.10}$$

where

$$\begin{aligned} V_{\mu\nu}(1_\gamma^+, 2_\gamma^+) &= -i[12]^2\eta_{\mu\nu} - \frac{1}{2} i \frac{[12]}{\langle 12\rangle}\,\mathrm{Tr}(p_1\bar{\sigma}_\mu p_2 \bar{\sigma}_\nu + p_1\bar{\sigma}_\nu p_2\bar{\sigma}_\mu) + 2i\frac{[12]}{\langle 12\rangle}(p_{1\mu}p_{2\nu} + p_{1\nu}p_{2\mu}) \\ V_{\mu\nu}(1_\gamma^+, 2_\gamma^-) &= i\langle 2|\bar{\sigma}_\mu|1]\langle 2|\bar{\sigma}_\nu|1]. \end{aligned} \tag{K.0.11}$$

Since there are no 3-point interactions in BI theory, only bubble diagrams contribute to the 1-loop 4-point amplitudes. We use the following convenient parametrization of the loop momenta



(K.0.12)

where $K^\mu = p_1^\mu + p_2^\mu$ and similarly for other permutations of the external lines. The diagram in (K.0.12) is given by the integral

$$\begin{aligned} I_4^S &= \int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{V\big(1_\gamma, 2_\gamma, l_1, l_2\big) V\big(3_\gamma, 4_\gamma, -l_1, -l_2\big)}{\big(l + \frac{1}{2}K\big)^2 \big(l - \frac{1}{2}K\big)^2} \\ &= V_{\mu\nu}(1_\gamma, 2_\gamma)\, V_{\rho\sigma}(3_\gamma, 4_\gamma) \int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{l_1^\mu l_2^\nu l_1^\rho l_2^\sigma}{\big(l + \frac{1}{2}K\big)^2 \big(l - \frac{1}{2}K\big)^2}\,, \end{aligned} \tag{K.0.13}$$

where the superscript S is used to denote that this integral corresponds to the s-channel contribution. Since $l_1 = l - \frac{1}{2}K$ and $l_2 = -l - \frac{1}{2}K$, it is very useful to observe that

- integrals with odd powers of loop momentum l vanish, and
- the photon-scalar vertices (K.0.11) have the property that

$$K^\mu V_{\mu\nu}(1_\gamma, 2_\gamma) = K^\nu V_{\mu\nu}(1_\gamma, 2_\gamma) = 0 \quad \text{for} \quad K^\mu = p_1^\mu + p_2^\mu\,, \tag{K.0.14}$$

  and similarly for the 34-vertex since $K^\mu = p_1^\mu + p_2^\mu = -p_3^\mu - p_4^\mu$.

This means that in the integral numerator we can replace $l_1^\mu l_2^\nu l_1^\rho l_2^\sigma$ by $l^\mu l^\nu l^\rho l^\sigma$.

In Appendix L we use the Passarino-Veltman integral-reduction method to compute the two tensor integrals that we need in Chapter 3, namely

$$\int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{l^\mu l^\nu}{\big(l + \frac{1}{2}K\big)^2 \big(l - \frac{1}{2}K\big)^2} = -\frac{1}{4(d-1)} \big[K^2 \eta^{\mu\nu} - K^\mu K^\nu\big]\; I_2(K^2)\,, \tag{K.0.15}$$

and

$$\begin{aligned}\int \frac{\mathrm{d}^d l}{(2\pi)^d} & \frac{l^\mu l^\nu l^\rho l^\sigma}{\big(l + \frac{1}{2}K\big)^2 \big(l - \frac{1}{2}K\big)^2} \\ &= \frac{1}{16(d^2-1)} \big[(K^2)^2 \eta^{(\mu\nu}\eta^{\rho\sigma)} - K^2 K^{(\mu} K^\nu \eta^{\rho\sigma)} + 3K^\mu K^\nu K^\rho K^\sigma\big]\; I_2(K^2)\,.\end{aligned} \tag{K.0.16}$$

Here I_2 is the scalar-bubble integral defined as

$$I_2(K^2) \equiv \int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{1}{\big(l + \frac{1}{2}K\big)^2 \big(l - \frac{1}{2}K\big)^2} = \frac{i}{16\pi^2}\left[\frac{1}{\epsilon} - \log(K^2)\right] + \mathcal{O}(1)\,. \tag{K.0.17}$$

The property (K.0.14) therefore implies the very simple result of the scalar-loop integral:

$$\begin{aligned} I_4^S &= \frac{(K^2)^2}{16(d^2-1)} V_{\mu\nu}(1_\gamma, 2_\gamma)\, V_{\rho\sigma}(3_\gamma, 4_\gamma)\, \eta^{(\mu\nu}\eta^{\rho\sigma)}\, I_2(K^2) \\ &= \frac{s^2}{240}\Big(V_\mu{}^\mu(1_\gamma, 2_\gamma)\, V_\nu{}^\nu(3_\gamma, 4_\gamma) + 2V_{\mu\nu}(1_\gamma, 2_\gamma)\, V^{\mu\nu}(3_\gamma, 4_\gamma)\Big)\, I_2(s)\,, \end{aligned} \tag{K.0.18}$$

where we have also used the symmetry of $V_{\mu\nu}$ under the exchange $\mu \leftrightarrow \nu$. Now it is a simple task to compute the desired duality-violating 1-loop amplitudes.

• For the next-to-self-dual case, we have $V_\nu{}^\nu(3_\gamma^+, 4_\gamma^-) = 0$, by the identity $\langle i|\bar\sigma^\mu|j]\langle k|\bar\sigma_\mu|l] =$

$2\langle ik\rangle\,[jl]$. It is also easy to show that the second term in (K.0.18) vanishes since

$$
\begin{aligned}
&V_{\mu\nu}(1_\gamma^+,2_\gamma^+)\,V^{\mu\nu}(3_\gamma^+,4_\gamma^-)\\
&\quad=-\,[12]^2\,\langle 4|\bar\sigma^\mu|3]\langle 4|\bar\sigma_\mu|3]-\frac{[12]}{\langle 12\rangle}\langle 1|\bar\sigma^\mu|2]\langle 2|\bar\sigma^\nu|1]\langle 4|\bar\sigma_\mu|3]\langle 4|\bar\sigma_\nu|3]+4\frac{[12]}{\langle 12\rangle}\langle 4|p_1|3]\langle 4|p_2|3]\\
&\quad=0\,.
\end{aligned}
\tag{K.0.19}
$$

Hence the entire s-channel diagram vanishes: $I_4^S(+++-)=0$. Since the t- and u-channel diagrams are simply permutations of the $+$-lines, we conclude that

$$
\boxed{\mathcal{A}_4^{\text{1-loop BI}}\left(1_\gamma^+2_\gamma^+3_\gamma^+4_\gamma^-\right)=0\,.}
\tag{K.0.20}
$$

• For the self-dual case, let us define

$$
Q_{\mu\nu\rho\sigma}\equiv V_{\mu\nu}(1_\gamma^+,2_\gamma^+)\,V_{\rho\sigma}(3_\gamma^+,4_\gamma^+)\,.
\tag{K.0.21}
$$

The external momenta and polarizations live in 4d, while the metric $\eta_{\mu\nu}$ arises from the loop-reduction and is therefore in $d=4-2\epsilon$ dimensions, so $\eta_{\mu\nu}\eta^{\mu\nu}=4-2\epsilon$. Using (K.0.11), we find

$$
Q_{\mu\ \nu}^{\ \mu\ \nu}=[12]^2\,[34]^2\left[4-\eta_{\mu\nu}\eta^{\mu\nu}\right]^2=4[12]^2[34]^2\,\epsilon^2\,,
\tag{K.0.22}
$$

and

$$
Q_{\mu\nu}{}^{\mu\nu}=Q_{\mu\nu}{}^{\nu\mu}=-2\epsilon\,[12]^2[34]^2\,.
\tag{K.0.23}
$$

Using the results (K.0.22) and (K.0.23), we arrive at

$$
Q_{\mu\ \nu}^{\ \mu\ \nu}+Q_{\mu\nu}{}^{\mu\nu}+Q_{\mu\nu}{}^{\nu\mu}=4[12]^2[34]^2\Big[-\epsilon+\epsilon^2\Big]\,.
\tag{K.0.24}
$$

Note that there are no $\mathcal{O}(1)$ terms in (K.0.24); this means that in the amplitude (K.0.18) the $1/\epsilon$ divergent terms and finite logarithms from (K.0.17) vanish for the diagram, as expected. We are left with finite rational terms generated from ϵ/ϵ anomalies. The s-diagram (K.0.18) simply evaluates to

$$
I_4^S=\frac{s^2}{240}\times 4[12]^2[34]^2\Big[-\epsilon+\epsilon^2\Big]\times I_2(s)
\tag{K.0.25}
$$

$$
=-\frac{i}{960\pi^2}[12]^2[34]^2s^2+O(\epsilon)\,.
\tag{K.0.26}
$$

Summing over the three diagrams gives the full amplitude

$$\mathcal{A}_4^{\text{1-loop BI}}(1_\gamma^+2_\gamma^+3_\gamma^+4_\gamma^+) = -\frac{i}{960\pi^2}\Big([12]^2[34]^2s^2 + [13]^2[24]^2t^2 + [14]^2[23]^2u^2\Big). \tag{K.0.27}$$

Note that we automatically have a more general result too: using (K.0.9) we can conclude that in a generalized BI theory with N_v vectors, N_f Weyl fermions, and N_s complex scalars, we have

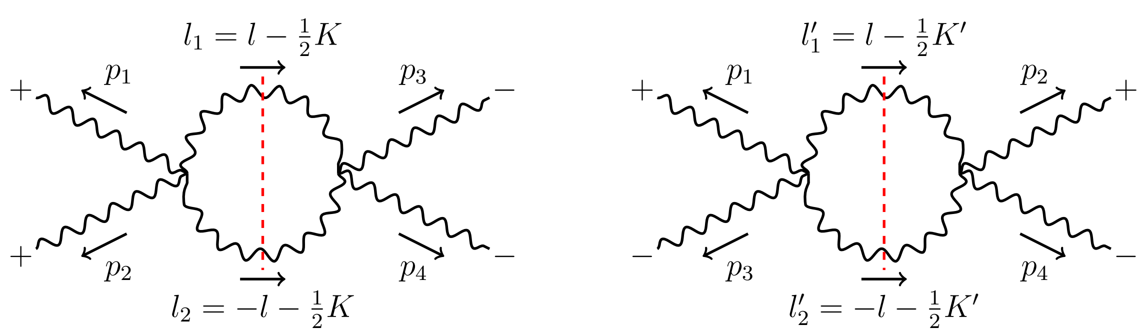
$$\begin{aligned}\mathcal{A}_4^{\text{1-loop gen BI}}(1_\gamma^+2_\gamma^+3_\gamma^+4_\gamma^+) &= \frac{-i}{960\pi^2}(N_v - N_f + N_s)\\ &\quad\times\Big([12]^2[34]^2s^2 + [13]^2[24]^2t^2 + [14]^2[23]^2u^2\Big). \\ \mathcal{A}_4^{\text{1-loop gen BI}}(1_\gamma^+2_\gamma^+3_\gamma^+4_\gamma^-) &= 0\,.\end{aligned} \tag{K.0.28}$$

Note that the self-dual amplitude vanishes when $(N_v - N_f + N_s)$ vanishes, as expected, for $\mathcal{N} = 1, 2, 4$ supersymmetry.

If we carry out the same calculation with external helicities $++--$, we would compute the contribution of a complex scalar in the loop of a 1-loop MHV amplitude. But that does not help us compute the 1-loop MHV amplitude in pure BI theory, since the supersymmetry trick we used to calculate the duality-violating amplitudes does not apply. Instead, we use the fact that the MHV amplitude is cut-constructible and compute it using unitarity.

K.0.4 1-loop MHV Amplitude without Supersymmetry

We compute the divergent part of the BI 1-loop MHV amplitude using unitarity. As a check, we also compute the MHV amplitude in a theory with N_v vectors, N_f fermions, and N_s complex scalars. The loop-integrand is ambiguous by additive terms that integrate to zero. We construct a representative of the loop-integrand that factorizes correctly on all cuts. We then use integral-reduction to relate the result to the scalar-bubble integral. For the 1-loop amplitude $\mathcal{A}_4^{\text{1-loop BI}}(1_\gamma^+, 2_\gamma^+, 3_\gamma^-, 4_\gamma^-)$ in pure BI theory, there are two distinct cuts to consider:



(K.0.29)

In the first diagram $K = p_1 + p_2$ and in the second $K' = p_1 + p_3$. In addition, one needs the $3 \leftrightarrow 4$ permutation of the second diagram. The cut-constructible part of the amplitude is then the sum of the s-, t-, and u-channel cuts C_s, C_t, and C_u. Note that each diagram comes with a symmetry factor of $1/2$.

The cut puts the internal lines l_1 and l_2 on-shell and each diagram factorizes into a product of tree-amplitudes for each possible on-shell state that can appear in the loop. For the s-channel diagram we have

$$\begin{aligned} C_s &= \mathcal{A}_4^{\text{tree}}\big(1_\gamma^+\, 2_\gamma^+\, l_{1\gamma}^-\, l_{2\gamma}^-\big)\mathcal{A}_4^{\text{tree}}\big(-l_{1\gamma}^+\; -l_{2\gamma}^+\, 3_\gamma^-\, 4_\gamma^-\big) \\ &= [12]^2\langle l_1 l_2\rangle^2 [l_1 l_2]^2 \langle 34\rangle^2 \\ &= [12]^2\langle 34\rangle^2 s^2, \end{aligned} \tag{K.0.30}$$

where we used that[1] $\langle l_1 l_2\rangle^2[l_1 l_2]^2 = \big((l_1+l_2)^2\big)^2 = (K^2)^2 = s^2$. Thus the contribution from the s-channel diagram is (including the symmetry factor of $1/2$)

$$I_s = \frac{1}{2}\int \frac{\mathrm{d}^d l}{(2\pi)^d}\frac{C_s}{\big(l+\frac{1}{2}K\big)^2\big(l-\frac{1}{2}K\big)^2} = \frac{1}{2}[12]^2\langle 34\rangle^2 s^2\, I_2(s), \tag{K.0.31}$$

where $I_2(s)$ is the scalar bubble integral (K.0.17). The t-channel cut is a little more involved. Here we have

$$\begin{aligned} C_t &= 2\mathcal{A}_4^{\text{tree}}\big(1_\gamma^+\, 3_\gamma^-\, {l'}_{1\gamma}^{+}\, {l'}_{2\gamma}^{-}\big)\mathcal{A}_4^{\text{tree}}\big(-{l'}_{1\gamma}^{-}\; -{l'}_{2\gamma}^{+}\, 2_\gamma^+\, 4_\gamma^-\big) \\ &= 2[1l'_1]^2\langle 3l'_2\rangle^2[2l'_2]^2\langle l'_1 4\rangle^2 \\ &= 2\langle 4|\bar\sigma_\mu|1]\langle 4|\bar\sigma_\nu|1]\langle 3|\bar\sigma_\rho|2]\langle 3|\bar\sigma_\sigma|2]\, {l'_1}^\mu {l'_1}^\nu {l'_2}^\rho {l'_2}^\sigma . \end{aligned} \tag{K.0.32}$$

The factor of 2 takes into account that one also has to include the opposite helicity assignments for the internal lines. Odd powers of the loop-momentum l vanish in the integral

$$I_t = \frac{1}{2}\int \frac{\mathrm{d}^d l}{(2\pi)^d}\frac{C_t}{\big(l+\frac{1}{2}K'\big)^2\big(l-\frac{1}{2}K'\big)^2} . \tag{K.0.33}$$

Again, we have included a symmetry factor of $1/2$. Using $l'_1 = l + \frac{1}{2}K'$ and $l'_2 = -l + \frac{1}{2}K'$, the integral can be simplified to the scalar bubble integral (K.0.17) with the help of the tensor integrals (K.0.15) and (K.0.16). After some manipulations of the resulting expressions, one arrives at the simple result

$$I_t = \frac{1}{5}[12]^2\langle 34\rangle^2 t^2\, I_2(t) . \tag{K.0.34}$$

The result of the u-channel diagram is obtained by taking $3 \leftrightarrow 4$ of I_t. We can now write the

[1] Note the ambiguity in this rewriting: we could also have written $\langle l_1 l_2\rangle^2[l_1 l_2]^2 = \big((l_1-l_2)^2\big)^2 = 16(l^2)^2$. The resulting two forms of the integrand differ by terms that integrate to zero.

divergent part of the 1-loop MHV 4-point amplitude in pure BI theory as

$$\mathcal{A}_4^{\text{1-loop BI}}(1_\gamma^+\,2_\gamma^+\,3_\gamma^-\,4_\gamma^-) = [12]^2\langle 34\rangle^2\Big[\frac{1}{2}s^2 I_2(s) + \frac{1}{5}t^2 I_2(t) + \frac{1}{5}u^2 I_2(u)\Big]\,. \tag{K.0.35}$$

Thus the divergent part is simply

$$\mathcal{A}_4^{\text{1-loop BI}}(1_\gamma^+\,2_\gamma^+\,3_\gamma^-\,4_\gamma^-) = \frac{1}{\epsilon}\frac{i}{16\pi^2}[12]^2\langle 34\rangle^2\Big[\frac{1}{2}s^2 + \frac{1}{5}t^2 + \frac{1}{5}u^2\Big] + \mathcal{O}(1)\,, \tag{K.0.36}$$

where the $\mathcal{O}(1)$-terms are regulator dependent. We have also calculated this amplitude using unitarity and the results agree.

K.0.5 1-loop MHV Amplitude with Supersymmetry

We now perform a check on our calculation by computing the MHV amplitude in a theory with fermions and scalars coupled to the vector supersymmetrically, i.e. such that the supersymmetry Ward identities hold:

$$\mathcal{A}_4\big(1_\gamma^+\,2_Z\,3_\gamma^-\,4_{\overline{Z}}\big) = \frac{\langle 23\rangle}{\langle 34\rangle}\mathcal{A}_4\big(1_\gamma^+\,2_\psi^+\,3_\gamma^-\,4_\psi^-\big) = \frac{\langle 23\rangle^2}{\langle 34\rangle^2}\mathcal{A}_4\big(1_\gamma^+\,2_\gamma^+\,3_\gamma^-\,4_\gamma^-\big)\,. \tag{K.0.37}$$

Amplitudes with two same-helicity vectors and two scalars or two fermions vanish; this means that there are no s-channel cuts when a fermion or scalar runs in the loop. The calculation of the t and u-channel cuts proceed as for the vector. With N_v vectors, N_f Weyl fermions, and N_s complex scalars, the result for the divergent part of the MHV 1-loop amplitude is then

$$\begin{aligned}\mathcal{A}_4^{\text{1-loop gen}}&(1_\gamma^+\,2_\gamma^+\,3_\gamma^-\,4_\gamma^-)\\ &= [12]^2\langle 34\rangle^2\left[\frac{N_v}{2}s^2 I_2(s) + \Big(\frac{N_v}{5}+\frac{N_f}{20}+\frac{N_s}{30}\Big)\big(t^2 I_2(t) + u^2 I_2(u)\big)\right]\\ &= \frac{1}{\epsilon}\frac{i}{16\pi^2}[12]^2\langle 34\rangle^2\left[\frac{N_v}{2}s^2 + \Big(\frac{N_v}{5}+\frac{N_f}{20}+\frac{N_s}{30}\Big)\big(t^2+u^2\big)\right] + \mathcal{O}(1)\,.\end{aligned} \tag{K.0.38}$$

As an independent check, we also calculated this amplitude using unitarity. It is also reproduced by the expression for the bubble coefficient given in [190] that connects the behavior of the tree amplitudes under BCFW shifts to the UV divergences at 1-loop.

With $\mathcal{N}=4$ supersymmetry, i.e. in $\mathcal{N}=4$ DBI, we have $N_v=1$, $N_f=4$, and $N_s=3$, which gives

$$\begin{aligned}\mathcal{A}_4^{\text{1-loop }\mathcal{N}=4}(1_\gamma^+\,2_\gamma^+\,3_\gamma^-\,4_\gamma^-) &= [12]^2\langle 34\rangle^2\frac{1}{2}\Big[s^2 I_2(s) + t^2 I_2(t) + u^2 I_2(u)\Big]\\ &= \frac{1}{\epsilon}\frac{i}{32\pi^2}[12]^2\langle 34\rangle^2\Big[s^2+t^2+u^2\Big] + \mathcal{O}(1)\,.\end{aligned} \tag{K.0.39}$$

This result for the 1-loop amplitude in $\mathcal{N} = 4$ DBI reproduces that found in [95].

APPENDIX L

Passarino-Veltman Integral Reduction

We use the Passarino-Veltman integral reduction method to evaluate the 1-loop integrals in section K. We outline the method here and use it to reduce the needed tensor integrals to scalar-bubble integrals.

The loop-integrands encountered in section K are all of the form

$$\mathcal{I} = \frac{N[l; p_i; \epsilon_i]}{\left(l - \frac{1}{2}K\right)^2 \left(l + \frac{1}{2}K\right)^2}\,, \tag{L.0.1}$$

where l is the loop-momentum, ϵ_i are polarization vectors, p_i are on-shell external momenta, and K is a sum of external momenta.

The numerator $N[l; p_i; \epsilon_i]$ is ambiguous by terms that integrate to zero: there are two types

Property 1. Any term with odd powers of the loop-momentum vanishes.

Property 2. Any term proportional to $\left(l - \frac{1}{2}K\right)^2$ or $\left(l + \frac{1}{2}K\right)^2$ vanishes. The reason is that an integral with such a factor can be put in the form of scaleless integrals

$$\int \frac{\mathrm{d}^d l}{(2\pi)^4} \frac{1}{l^2}, \quad \int \frac{\mathrm{d}^d l}{(2\pi)^4} \frac{l^\mu l^\nu}{l^2}, \quad \int \frac{\mathrm{d}^d l}{(2\pi)^4} (1), \quad \ldots \tag{L.0.2}$$

that are zero in dimensional regularization.

We now use these two properties to derive the integrals (K.0.15) and (K.0.16) we need for the computations in section K:

- Integral 1

$$\int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{(K\cdot l) l^{\mu_1}\cdots l^{\mu_n}}{\left(l+\frac{1}{2}K\right)^2\left(l-\frac{1}{2}K\right)^2} = 0\,. \tag{L.0.3}$$

Proof: if n is even, the integral vanishes by Property 1 above. If n is odd, write $(K\cdot l) = \left(l+\frac{1}{2}K\right)^2 - l^2 - \frac{1}{4}K^2$. Then the first term makes the integral vanish by Property 2, while the last two terms integrate to zero by Property 1.

- Integral 2

$$\int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{l^2}{\left(l+\frac{1}{2}K\right)^2\left(l-\frac{1}{2}K\right)^2} = -\frac{1}{4}K^2\, I_2(K^2)\,. \tag{L.0.4}$$

Proof: write $l^2 = \left(l+\frac{1}{2}K\right)^2 - (K\cdot l) - \frac{1}{4}K^2$. The first two term integrate to zero. So we are left with $-\frac{1}{4}K^2$ times the scalar bubble-integral $I_2(K^2)$.

- Integral 3 (the integral in (K.0.15))

$$\int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{l^\mu l^\nu}{\left(l+\frac{1}{2}K\right)^2\left(l-\frac{1}{2}K\right)^2} = -\frac{1}{4(d-1)}\left[K^2\eta^{\mu\nu} - K^\mu K^\nu\right]\, I_2(K^2)\,. \tag{L.0.5}$$

Proof: there are only two possible tensor structures for the integral, so we can write an ansatz as $A_1\eta^{\mu\nu}K^2 + A_2K^\mu K^\nu$, where A_1 and A_2 are constant numbers. Dot in K^μ and use Integral 1 (L.0.3) to conclude that $A_1 + A_2 = 0$. Then contract with $\eta_{\mu\nu}$ and use Integral 2 (L.0.4) to show that $A_1 d + A_2 = -\frac{1}{4}I_2(K^2)$. Solving for the unknown constants gives $A_1 = -A_2 = -\frac{1}{4(d-1)}I_2(K^2)$ and the result (L.0.5) follows.

- Integral 4

$$\int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{(l^2)^2}{\left(l+\frac{1}{2}K\right)^2\left(l-\frac{1}{2}K\right)^2} = \frac{1}{16}(K^2)^2\, I_2(K^2)\,. \tag{L.0.6}$$

Proof: rewrite $(l^2)^2 = \left(\left(l+\frac{1}{2}K\right)^2 - (K\cdot l) - \frac{1}{4}K^2\right)l^2$. The first two term integrates to zero by Properties 1 and 2, so this leaves $-\frac{1}{4}K^2$ times Integral 2. Then (L.0.6) simply follows from (L.0.4).

- Integral 5 (the integral in (K.0.16))

$$\begin{aligned}&\int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{l^\mu l^\nu l^\rho l^\sigma}{\left(l+\frac{1}{2}K\right)^2\left(l-\frac{1}{2}K\right)^2} \\ &\quad = \frac{1}{16(d^2-1)}\left[(K^2)^2\eta^{(\mu\nu}\eta^{\rho\sigma)} - K^2K^{(\mu}K^\nu\eta^{\rho\sigma)} + 3K^\mu K^\nu K^\rho K^\sigma\right]\, I_2(K^2)\,,\end{aligned} \tag{L.0.7}$$

where we have normalized the symmetrizations such that

$$\eta^{(\mu\nu}\eta^{\rho\sigma)} = \eta^{\mu\nu}\eta^{\rho\sigma} + \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}, \tag{L.0.8}$$

and $K^{(\mu}K^{\nu}\eta^{\rho\sigma)} = K^{\mu}K^{\nu}\eta^{\rho\sigma} + 5$ nontrivial perms.

Proof: the tensor structure allows only three possible terms, so we write an ansatz

$$B_1\,(K^2)^2\eta^{(\mu\nu}\eta^{\rho\sigma)} + B_2\,K^2K^{(\mu}K^{\nu}\eta^{\rho\sigma)} + B_3\,K^{\mu}K^{\nu}K^{\rho}K^{\sigma}, \tag{L.0.9}$$

for the value of (L.0.6). We know from Integral 1 that the integral must vanish whenever K is dotted in, so from dotting in $K_\mu K_\nu K_\rho K_\sigma$ and $K^{\mu}K^{\nu}\eta^{\rho\sigma}$ we learn that

$$3B_1 + 6B_2 + B_3 = 0 \quad \text{and} \quad (d+2)B_1 + (d+5)B_2 + B_3 = 0\,. \tag{L.0.10}$$

Finally we contract with $\eta^{\mu\nu}\eta^{\rho\sigma}$ and use Integral 4 to conclude that

$$d(d+2)B_1 + 2(d+2)B_2 + B_3 = \frac{1}{16}\,I_2(K^2)\,. \tag{L.0.11}$$

Solving for the coefficients B_1, B_2, and B_3 gives (L.0.7).

APPENDIX M

Higher-Derivative Corrections of Yang-Mills

In this appendix, we calculate possible higher-derivative corrections to the 4-point amplitudes of YM that are used in Section 4.7.2 to calculate corrections to 4-point BI amplitudes using the KLT double-copy.

We construct BCJ-compatible ansatze for three helicity configurations that have the following properties:

- **Relabeling:** Different color-orderings with the same helicity structure have to be related by momentum relabelling.
- **Locality:** The amplitude has at most simple poles at $s, t, u = 0$. Compatibility with BCJ relations (4.7.6) and locality dictate that $\mathcal{A}_4^{\mathrm{YM}}[1234]$ cannot have a pole at $t = 0$.
- **Unitarity:** 4-point amplitudes factorize on simple poles into products of 3-point amplitudes. Since 3-point amplitudes of massless particles are fixed by little group scaling, residues on the s- and u-channel poles are constrained by the spins of particles in our spectrum. Since no fermions can be exchanged and gravitons cannot carry a color-charge, any particles exchanged must have spin 0 and spin 1.

M.0.1 The $++++$ Amplitude

We begin with the 4-point self-dual YM amplitude. While it is well known that at leading order the self-dual YM amplitude is vanishing, there can exist non-zero higher-derivative terms, which we evaluate in this section.

Since all external states have the same helicity, all color orderings of the self-dual amplitude are related by momentum relabeling. With this in mind we write the following BCJ and locality-compatible ansatz,

$$\begin{aligned}\mathcal{A}_4^{\text{YM}}[1^+2^+3^+4^+] =& \frac{1}{su}\Bigg(\tilde{c}_0\left([12]^2[34]^2+[13]^2[24]^2+[14]^2[23]^2\right)\\&+\frac{\tilde{c}_2}{\Lambda^2}\left([12]^2[34]^2 s+[13]^2[24]^2 t+[14]^2[23]^2 u\right)\\&+\frac{\tilde{c}_4}{\Lambda^4}\left(s^2+t^2+u^2\right)\left([12]^2[34]^2+[13]^2[24]^2+[14]^2[23]^2\right)\\&+\frac{\tilde{c}_6}{\Lambda^6}stu\left([12]^2[34]^2+[13]^2[24]^2+[14]^2[23]^2\right)+\mathcal{O}(\Lambda^{-8})\Bigg)\,. \end{aligned} \tag{M.0.1}$$

One can show that $\tilde{c}_0$ produces a non-zero residue that does not correspond to a particle exchange and $\tilde{c}_4$ corresponds to a higher spin exchange, so they must both be zero by unitarity. Thus we finally conclude that the 4-point YM amplitude takes the form

$$\begin{aligned}\mathcal{A}_4^{\text{YM}}[1^+2^+3^+4^+] = \frac{\tilde{c}_2}{\Lambda^2}&\frac{[12]^2[34]^2 s+[13]^2[24]^2 t+[14]^2[23]^2 u}{su}\\&+\frac{\tilde{c}_6}{\Lambda^6}t\left([12]^2[34]^2+[13]^2[24]^2+[14]^2[23]^2\right)+\mathcal{O}(\Lambda^{-8})\,.\end{aligned} \tag{M.0.2}$$

The $\mathcal{O}(\Lambda^{-2})$ contribution of this amplitude was also calculated in [137] and our result is in perfect agreement. To our knowledge, the $\mathcal{O}(\Lambda^{-6})$ contribution is new.

M.0.2 The $+++-$ Amplitude

We now move on to the next-to-self-dual amplitude. We know this to be vanishing at leading order and here we calculate sub-leading higher-derivative corrections.

Since not all external particles have the same helicity, different color orderings can be related using cyclicity, in addition to simple relabeling. For example,

$$\mathcal{A}_4^{\text{YM}}[1^+2^+4^-3^+] = \mathcal{A}_4^{\text{YM}}[3^+1^+2^+4^-] = \mathcal{A}_4^{\text{YM}}[1^+2^+3^+4^-]\big|_{1\to3\to2\to1}\,. \tag{M.0.3}$$

Compatibility with the BCJ relations and locality fixes our ansatz to be,

$$\mathcal{A}_4^{\text{YM}}[1^+2^+3^+4^-] = \frac{1}{\Lambda^2}\frac{[12]^2[3|p_1|4\rangle^2}{su}\left(\tilde{b}_0+\frac{\tilde{b}_4}{\Lambda^4}\left(s^2+t^2+u^2\right)+\frac{\tilde{b}_6}{\Lambda^6}stu+\mathcal{O}(\Lambda^{-8})\right)\,. \tag{M.0.4}$$

The $\tilde{b}_4$ corresponds to the exchange of a spin-3 particle and is thus excluded by unitarity. This leaves us with

$$\mathcal{A}_4^{\text{YM}}[1^+2^+3^+4^-] = \frac{1}{\Lambda^2}\frac{[12]^2[3|p_1|4\rangle^2}{su}\left(\tilde{b}_0 + \frac{\tilde{b}_6}{\Lambda^6}stu + \mathcal{O}(\Lambda^{-8})\right). \tag{M.0.5}$$

It is interesting to note that the first term factorizes into a 3-point YM amplitude and a 3-point amplitude consistent with an F^3-type interaction. As in the self-dual case, we check the leading contribution to our result against the results of [137] and we find perfect agreement.

M.0.3 The $++--$ Amplitude

Finally, let us examine the MHV 4-point amplitude. Using momentum relabeling and cyclicity, we can rewrite the BCJ relations as

$$\begin{aligned}
&\frac{u}{t}\mathcal{A}_4^{\text{YM}}[1^+2^+3^-4^-] = \mathcal{A}_4^{\text{YM}}[2^+1^+3^-4^-] = \mathcal{A}_4^{\text{YM}}[1^+2^+3^-4^-]\Big|_{1\leftrightarrow 2} \text{ and}\\
&\frac{u}{t}\mathcal{A}_4^{\text{YM}}[1^+2^+3^-4^-] = \mathcal{A}_4^{\text{YM}}[1^+2^+4^-3^-] = \mathcal{A}_4^{\text{YM}}[1^+2^+3^-4^-]\Big|_{3\leftrightarrow 4}.
\end{aligned} \tag{M.0.6}$$

With the above constraints, a local ansatz for the amplitude is

$$\begin{aligned}
&\mathcal{A}_4^{\text{YM}}[1^+2^+3^-4^-]\\
&\quad = \frac{[12]^2\langle 34\rangle^2}{su}\left(\tilde{a}_0 + \frac{\tilde{a}_{21}}{\Lambda^2}s + \left(\frac{\tilde{a}_{41}}{\Lambda^4}s^2 - \frac{\tilde{a}_{42}}{\Lambda^4}tu\right) + \left(\frac{\tilde{a}_{61}}{\Lambda^6}s^3 - \frac{\tilde{a}_{62}}{\Lambda^6}stu\right) + \mathcal{O}(\Lambda^{-8})\right).
\end{aligned} \tag{M.0.7}$$

$\tilde{a}_{21}$, $\tilde{a}_{41}$ and $\tilde{a}_{61}$ correspond to exchanges of spin-2, spin-3 and spin-4 particles respectively and are hence all set to zero by unitarity. If we continue at higher orders in the derivative expansion, we see that this pattern continues. We either find local contributions to the amplitude or pole-terms with residues that correspond to the exchange of higher-spin particles.

To summarize, the YM 4-point amplitude with higher-derivative corrections has the form

$$\mathcal{A}_4^{\text{YM}}[1^+2^+3^-4^-] = \frac{[12]^2\langle 34\rangle^2}{su}\left(\tilde{a}_0 + \frac{\tilde{a}_4}{\Lambda^4}tu + \frac{\tilde{a}_6}{\Lambda^6}stu + \mathcal{O}(\Lambda^{-8})\right). \tag{M.0.8}$$

In the above, the leading term matches the well-known Parke-Taylor formula for Yang-Mills if we choose $\tilde{a}_0 = -g_{\text{YM}}^2$. Also note that we have redefined $\tilde{a}_{42} = -\tilde{a}_4$ and $\tilde{a}_{62} = -\tilde{a}_6$ to simplify the notation.

APPENDIX N

Massless Limits of Massive Theories

In this appendix, we will review taking massless limits of massive theories. If the massless limit in the Lagrangian of a higher spin (≥ 1) particle is taken by just setting $m \to 0$, degrees of freedom can be lost causing the limit to be discontinuous. Here we show how to take the massless limit in a continuous way that preserves the number of degrees of freedom. More extensive reviews can be found in [170, 191].

Example 1: Massive Photon

To start with, we examine the Lagrangian of a massive photon with a quartic interaction term,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu + g(A_\mu A^\mu)^2. \tag{N.0.1}$$

The mass term and interaction term break the $U(1)$ gauge symmetry $\delta A_\mu = \partial_\mu \Lambda$ that a massless photon would have. If we were to try to take the massless limit by setting $g, m \to 0$, the limit would not be continuous, as in 4 dimensions a massive photon has 3 degrees of freedom regardless of how small the mass, while a massless photon has 2 degrees of freedom.

In order to properly take the massless limit, the limit must be taken in a way that preserves the number of degrees of freedom. One way to explicitly see how the discontinuity arises is by using the Stückelberg trick. This involves introducing new fields in a way that makes the theory gauge invariant, but is still dynamically equivalent to the original theory. To do this, we make a replacement of the field patterned after the $U(1)$ gauge symmetry enjoyed by a massless photon:

$$A_\mu \to A_\mu + \frac{1}{m}\partial_\mu \phi. \tag{N.0.2}$$

This gives an action,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2\left(A_\mu + \frac{1}{m}\partial_\mu\phi\right)\left(A^\mu + \frac{1}{m}\partial^\mu\phi\right) + g\left(\left(A_\mu + \frac{1}{m}\partial_\mu\phi\right)\left(A^\mu + \frac{1}{m}\partial^\mu\phi\right)\right)^2, \tag{N.0.3}$$

which is gauge invariant under the transformations:

$$\delta A_\mu = \partial_\mu\lambda, \quad \delta\phi = -m\lambda. \tag{N.0.4}$$

This action, although it contains more fields, is completely equivalent to (N.0.1) since (N.0.3) is gauge invariant, and we can always choose unitary gauge, $\phi = 0$ to recover the original action. Expanding the action, we find:

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2A_\mu A^\mu - mA_\mu\partial^\mu\phi A^\mu - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + g(A_\mu A^\mu)^2 + 4\,\frac{g}{m}A_\mu A^\mu A_\nu\partial^\nu\phi \\ &+ \frac{g}{m^2}\left(4\,(A_\mu\partial^\mu\phi)^2 + 2A_\mu A^\mu\partial_\nu\phi\partial^\nu\phi\right) + 4\,\frac{g}{m^3}A_\mu\partial^\mu\phi\partial_\nu\phi\partial^\nu\phi + \frac{g}{m^4}\left(\partial_\mu\phi\partial^\mu\phi\right)^2.\end{aligned} \tag{N.0.5}$$

Now, when taking the massless limit, the lowest energy scale suppressing the interaction terms is $\Lambda = (m^4/g)^{1/4}$. If we now take the massless decoupling limit,

$$g, m \to 0, \quad \Lambda = \left(\frac{m^4}{g}\right)^{1/4}, \; A_\mu, \; \phi \,\text{fixed}, \tag{N.0.6}$$

we find all the interaction terms vanish except for the scalar self-interaction terms, giving the Lagrangian,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{\Lambda^4}\left(\partial_\mu\phi\partial^\mu\phi\right)^2, \tag{N.0.7}$$

the number of degrees of freedom is preserved, two in the form of a massless vector and one in the form of a scalar with a quartic self-interaction term, and the action is gauge invariant

$$\delta A_\mu = \partial_\mu\lambda, \quad \delta\phi = 0. \tag{N.0.8}$$

In taking the massless limit this way, it is obvious that the limit is smooth and that the massless limit of a massive vector is not just a massless vector, but is a massless vector plus a scalar.

Example 2: Linearized Massive Graviton

The linearized Lagrangian for ghost-free massive gravity is given by the action:

$$\mathcal{L}_m = -\frac{1}{2}\partial_\alpha h_{\mu\nu}\partial^\alpha h^{\mu\nu} + \partial_\alpha h_{\mu\nu}\partial^\nu h^{\mu\alpha} - \partial_\mu h \partial_\nu h^{\mu\nu} + \frac{1}{2}\partial_\mu h \partial^\mu h - \frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} - h^2). \tag{N.0.9}$$

One can easily see that the linearized Lagrangian for general relativity can be recovered by setting $m = 0$. However, when the massless limit is taken this way, just as in the massive vector case, degrees of freedom are lost as a massive graviton has 5, while a massless graviton only has 2 degrees of freedom. The gauge invariance that kills the extra degrees of freedom only appears when the mass is exactly zero. We can use the Stückelberg trick to take the massless limit in a way that preserves the number of degrees of freedom. To do this we make a replacement of the field patterned after the linearized diffeomorphism gauge symmetry:

$$h_{\mu\nu} \to h_{\mu\nu} + \frac{1}{m}\partial_\mu A_\nu + \frac{1}{m}\partial_\nu A_\mu + \frac{1}{m^2}\partial_\mu\partial_\nu\phi, \tag{N.0.10}$$

giving the Lagrangian:

$$\mathcal{L} = \mathcal{L}_m - \frac{1}{2}F_{\mu\nu}F^{\mu\nu} - 2m(h_{\mu\nu}\partial^\mu A^\nu - h\partial_\mu A^\mu) - 2(h_{\mu\nu}\partial^\mu\partial^\nu\phi - h\partial^2\phi). \tag{N.0.11}$$

This action is invariant under the gauge transformations:

$$\begin{aligned} \delta h_{\mu\nu} &= \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \quad \delta A_\mu = -m\xi_\mu, \\ \delta A_\mu &= \partial_\mu \lambda, \quad \delta\phi = -m\lambda. \end{aligned} \tag{N.0.12}$$

This Lagrangian is dynamically equivalent to (N.0.9) since it is gauge invariant and we can always choose the gauge $A_\mu = 0,\ \phi = 0$ to recover (N.0.9). Now if we take the massless limit, the Lagrangian is

$$\mathcal{L} = \mathcal{L}_{m=0} - \frac{1}{2}F_{\mu\nu}F^{\mu\nu} - 2(h_{\mu\nu}\partial^\mu\partial^\nu\phi - h\partial^2\phi), \tag{N.0.13}$$

and the gauge transformations are

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

$$\delta A_\mu = \partial_\mu \lambda \tag{N.0.14}$$

$$\delta\phi = 0. \tag{N.0.15}$$

We find the degrees of freedom break up into 2 tensor modes, 2 vector modes, and 1 scalar mode. So the massless limit of massive gravity is not massless gravity, but massless gravity plus extra degrees of freedom. The tensor modes and scalar mode are coupled. They can be decoupled using a field redefinition, $h_{\mu\nu} \to h_{\mu\nu} + \pi\eta_{\mu\nu}$. However, if the graviton is coupled to a stress-energy tensor this leads to non-minimal coupling to the stress energy tensor, the so-called vDVZ discontinuity [192].

Example 3: Massive Yang-Mills

The Stückelberg trick can be extended to non-Abelian theories as well. To demonstrate, we examine massive Yang-Mills,

$$\mathcal{L} = -\frac{1}{4}\left(F^a_{\mu\nu}\right)^2 - \frac{1}{2}m^2 A^a_\mu A^{a\mu}, \tag{N.0.16}$$

where

$$F^a_{\mu\nu} \equiv \partial_{[\mu}A^a_{\nu]} + g f^{abc} A^b_\mu A^c_\nu. \tag{N.0.17}$$

Without the mass term, the action would be gauge invariant under: $A_\mu \to RA_\mu R^\dagger + R\partial_\mu R^\dagger$, where $R = e^{-i\alpha_a T^a}$, T^a are the generators of the gauge group, and $\alpha_a(x)$ are gauge parameters, but the mass term breaks this gauge symmetry. Just as in the previous examples, we can make a field replacement patterned after the gauge symmetry to create a Lagrangian that is gauge invariant:

$$A_\mu \to UA_\mu U^\dagger + U\partial_\mu U^\dagger, \tag{N.0.18}$$

where $U = e^{-i\frac{g}{m}\pi_a T^a}$ and $\pi_a(x)$ are scalar fields. This can give interactions that go like:

$$\sim g\partial A^3 \quad \sim g^2 A^4, \quad \sim g^2\left(\frac{g}{m}\right)^{n-2} A^2\pi^n \quad \sim g\left(\frac{g}{m}\right)^{n-2} \partial A\pi^n \quad \sim \left(\frac{g}{m}\right)^{n-2} \partial^2\pi^n. \tag{N.0.19}$$

For $g < 1$, the lowest energy scale suppressing the interaction terms is given by $\sim \frac{m}{g}$. We can take the decoupling limit by sending:

$$g, m \to 0, \quad \frac{g}{m},\ A_\mu,\ \pi \text{ fixed}. \tag{N.0.20}$$

In this limit, the only terms that survive are the pure scalar interactions, given by the Lagrangian

$$\mathcal{L} = \frac{m^2}{g^2} Tr\left(\partial_\mu U^\dagger \partial^\mu U\right), \tag{N.0.21}$$

and the free vector fields.

Example 4: Non-linear Massive Gravity

As a final example, we will examine the massless decoupling limit of full non-linear massive gravity. Using the ghost-free potential for massive gravity, given by the action in (5.3.33), and assuming all fields are canonically normalized, one finds generic terms with n_h powers of $h_{\mu\nu}$, n_A powers of A_μ, and n_ϕ powers of ϕ is given by

$$\sim \Lambda_\lambda^{2-n_h-2n_A-3n_\phi} h^{n_h} (\partial A)^{n_A} (\partial^2\phi)^{n_\phi}, \tag{N.0.22}$$

where the term is suppressed by the scale

$$\Lambda_\lambda = (M_P m^{\lambda-1})^{1/\lambda}, \quad \lambda = \frac{3n_\phi + 2n_A + n_h - 4}{n_\phi + n_A + n_h - 2}. \tag{N.0.23}$$

Looking at interaction terms suppressed by the smallest scale, we find

$$\sim \frac{(\partial^2\phi)^3}{M_P m^4}, \tag{N.0.24}$$

corresponding to the decoupling limit:

$$m \to 0, \quad M_P \to \infty, \quad \Lambda_5,\ h_{\mu\nu},\ A_\mu,\ \phi \,\text{fixed}, \tag{N.0.25}$$

which corresponds to 4 point scattering amplitudes growing with energy like $\sim E^{10}$. However, miraculous cancellations occur and these all vanish with the ghost free potential [15, 193]. Similarly, interaction coming in at the next smallest scale, given by:

$$\sim \frac{(\partial^2\phi)^4}{(M_P m^3)^2}, \quad \frac{\partial A(\partial^2\phi)^2}{(M_P m^3)}, \tag{N.0.26}$$

which would correspond to 4 point scattering amplitudes growing with energy like E^8 also vanish.

The non-vanishing terms with the smallest suppression scale that survive the massless limit are given by:

$$\sim \frac{h(\partial^2\phi)^n}{(M_P m^2)^{n-1}}, \quad \frac{(\partial A)^2(\partial^2\phi)^n}{(M_P m^2)^n}, \tag{N.0.27}$$

and the gauge symmetry reduces to their linear form (N.0.14). This is found by taking the decoupling limit

$$m \to 0, \quad M_P \to \infty, \quad \Lambda_3,\ h_{\mu\nu},\ A_\mu,\ \phi \,\text{fixed}. \tag{N.0.28}$$

The remaining interactions give 4 point scattering amplitudes growing like E^6 [173]. In the full non-linear theory, the tensor and scalar modes cannot be fully decoupled from one another, and we get a scalar-tensor theory along with a scalar-vector theory.

APPENDIX O

Matrix of 5-point Bi-adjoint Scalar Amplitudes

The numerators are given by $\begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \\ n_6 \end{pmatrix} = M_n^{-1} \begin{pmatrix} \mathcal{A}_5[13542] \\ \mathcal{A}_5[13524] \\ \mathcal{A}_5[13254] \\ \mathcal{A}_5[13245] \\ \mathcal{A}_5[13425] \\ \mathcal{A}_5[13452] \end{pmatrix}$ where M_n is given by:

$$M_n =$$

$$\begin{pmatrix} \frac{1}{D_1} & \frac{1}{D_2} + \frac{1}{D_9} & -\frac{1}{D_9} & 0 & -\frac{1}{D_{12}} & \frac{1}{D_{12}} + \frac{1}{D_6} \\ 0 & -\frac{1}{D_9} & \frac{1}{D_3} + \frac{1}{D_9} & \frac{1}{D_4} & \frac{1}{D_{12}} + \frac{1}{D_5} & -\frac{1}{D_{12}} \\ \frac{1}{D_8} & -\frac{1}{D_2} - \frac{1}{D_8} & -\frac{1}{D_3} & -\frac{1}{D_{11}} - \frac{1}{D_4} & \frac{1}{D_{11}} & 0 \\ -\frac{1}{D_{14}} - \frac{1}{D_8} & \frac{1}{D_{14}} + \frac{1}{D_{15}} + \frac{1}{D_2} + \frac{1}{D_8} + \frac{1}{D_9} & -\frac{1}{D_{15}} - \frac{1}{D_9} & -\frac{1}{D_{14}} & \frac{1}{D_{14}} + \frac{1}{D_{15}} & -\frac{1}{D_{15}} \\ -\frac{1}{D_{13}} & -\frac{1}{D_{15}} - \frac{1}{D_9} & \frac{1}{D_{10}} + \frac{1}{D_{13}} + \frac{1}{D_{15}} + \frac{1}{D_3} + \frac{1}{D_9} & -\frac{1}{D_{10}} - \frac{1}{D_{13}} & -\frac{1}{D_{15}} & \frac{1}{D_{13}} + \frac{1}{D_{15}} \\ -\frac{1}{D_1} - \frac{1}{D_7} & -\frac{1}{D_2} & -\frac{1}{D_{10}} - \frac{1}{D_3} & \frac{1}{D_{10}} & 0 & -\frac{1}{D_7} \end{pmatrix}, \quad \text{(O.0.1)}$$

where $1/D_i$ corresponds to the propagators used in [152] for the 5-point trivalent graphs shown in Figure O.1. The 5-point amplitude can be calculated from the bi-adjoint scalar matrix as

$$\mathcal{A}_5^{A\otimes B}(1,2,3,4,5) = \begin{pmatrix} \mathcal{A}_5^A[13542] & \mathcal{A}_5^A[13524] & \mathcal{A}_5^A[13254] & \mathcal{A}_5^A[13245] & \mathcal{A}_5^A[13425] & \mathcal{A}_5^A[13452] \end{pmatrix} \left(\mathcal{A}_5^{\phi^3}\right)^{-1}[\alpha|\beta] \begin{pmatrix} \mathcal{A}_5^B[13542] \\ \mathcal{A}_5^B[13524] \\ \mathcal{A}_5^B[13254] \\ \mathcal{A}_5^B[13245] \\ \mathcal{A}_5^B[13425] \\ \mathcal{A}_5^B[13452] \end{pmatrix}, \tag{O.0.2}$$

where the bi-adjoint scalar matrix is given by:

$$\mathcal{A}_5^{\phi^3}[\alpha|\beta] =$$

$$\begin{pmatrix}
\frac{1}{D_1}+\frac{1}{D_{12}}+\frac{1}{D_2}+\frac{1}{D_6}+\frac{1}{D_9} & -\frac{1}{D_{12}}-\frac{1}{D_9} & -\frac{1}{D_2} & \frac{1}{D_2}+\frac{1}{D_9} & -\frac{1}{D_9} & -\frac{1}{D_1}-\frac{1}{D_2} \\
-\frac{1}{D_{12}}-\frac{1}{D_9} & \frac{1}{D_{12}}+\frac{1}{D_3}+\frac{1}{D_4}+\frac{1}{D_5}+\frac{1}{D_9} & -\frac{1}{D_3}-\frac{1}{D_4} & -\frac{1}{D_9} & \frac{1}{D_3}+\frac{1}{D_9} & -\frac{1}{D_3} \\
-\frac{1}{D_2} & -\frac{1}{D_3}-\frac{1}{D_4} & \frac{1}{D_{11}}+\frac{1}{D_2}+\frac{1}{D_3}+\frac{1}{D_4}+\frac{1}{D_8} & -\frac{1}{D_2}-\frac{1}{D_8} & -\frac{1}{D_3} & \frac{1}{D_2}+\frac{1}{D_3} \\
\frac{1}{D_2}+\frac{1}{D_9} & -\frac{1}{D_9} & -\frac{1}{D_2}-\frac{1}{D_8} & \frac{1}{D_{14}}+\frac{1}{D_{15}}+\frac{1}{D_2}+\frac{1}{D_8}+\frac{1}{D_9} & -\frac{1}{D_{15}}-\frac{1}{D_9} & -\frac{1}{D_2} \\
-\frac{1}{D_9} & \frac{1}{D_3}+\frac{1}{D_9} & -\frac{1}{D_3} & -\frac{1}{D_{15}}-\frac{1}{D_9} & \frac{1}{D_{10}}+\frac{1}{D_{13}}+\frac{1}{D_{15}}+\frac{1}{D_3}+\frac{1}{D_9} & -\frac{1}{D_{10}}-\frac{1}{D_3} \\
-\frac{1}{D_1}-\frac{1}{D_2} & -\frac{1}{D_3} & \frac{1}{D_2}+\frac{1}{D_3} & -\frac{1}{D_2} & -\frac{1}{D_{10}}-\frac{1}{D_3} & \frac{1}{D_1}+\frac{1}{D_{10}}+\frac{1}{D_2}+\frac{1}{D_3}+\frac{1}{D_7}
\end{pmatrix} \tag{O.0.3}$$

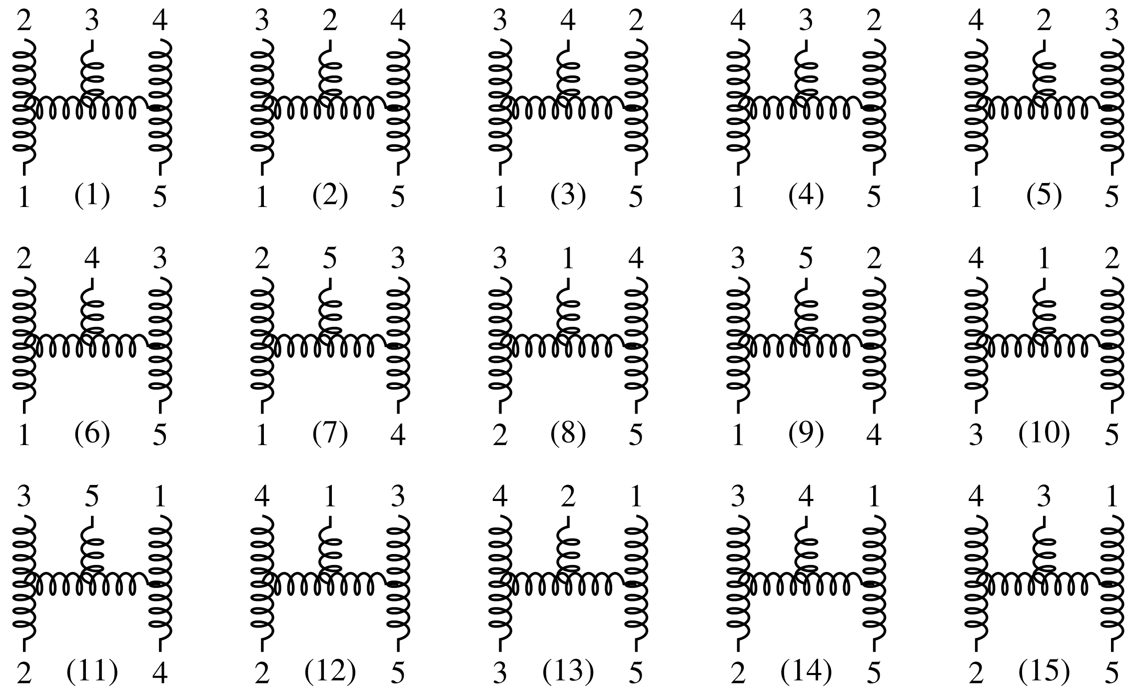


Figure O.1: Color-dressed tree-level 5-point amplitude organized using graphs with only cubic vertices.

APPENDIX P

Factorization on Physical Poles

An essential property of amplitudes in local theories is the presence of simple poles when intermediate momenta go on-shell and factorization of the amplitude into products of lower-point amplitudes in the associated residue. In this appendix, we will discuss how these properties are ensured in amplitudes generated by our proposed massive KLT formula (5.2.16). We will begin by analyzing factorization of (5.2.16) on two-particle channels, and then extend the result to multi-particle factorization.

We begin by assuming that theories A and B are local and that their amplitudes factorize correctly on two-particle channels,

$$\begin{aligned}\mathcal{A}_n^A[12,\sigma] &= \frac{\mathcal{A}_3^A[12,-P_{12}]\mathcal{A}_{n-1}^A[P_{12},\sigma]}{s_{12}+m^2} + \mathcal{O}\left((s_{12}+m^2)^0\right)\\ \mathcal{A}_n^B[12,\sigma] &= \frac{\mathcal{A}_3^B[12,-P_{12}]\mathcal{A}_{n-1}^B[P_{12},\sigma]}{s_{12}+m^2} + \mathcal{O}\left((s_{12}+m^2)^0\right),\end{aligned} \tag{P.0.1}$$

where there is an implicit sum over states on the right-hand-side.

Next without loss of generality, let us choose to study factorization on the s_{12} pole. We can further assume that we have chosen a DDM basis in which the first m elements have the form $[12\sigma(3,\cdots,n)]$ where σ is a permutation, and no other elements have 1 and 2 adjacent[1]. Thus only orderings in the first m rows and columns admit poles in s_{12} and we can resolve our matrix

[1]For example, the basis $[1\sigma(2,\cdots,n-1)n]$ where σ runs over all $(n-2)!$ permutations is a DDM basis with $(n-3)!$ elements of the form $[12\sigma(3,\cdots,n-1)n]$. One can then choose an ordering of basis elements such that the assumed property is fulfilled.

of bi-adjoint scalar amplitudes into blocks,

$$\mathcal{A}_n^{\phi^3}[\alpha|\beta] = \begin{pmatrix} P & Q^\top \\ Q & R \end{pmatrix}, \tag{P.0.2}$$

where P, Q and R are $m \times m$, $(n-m) \times m$ and $(n-m) \times (n-m)$ matrices respectively. Since s_{12} poles are not admitted by the last $(n-m)$ orderings, the P matrix contains all the elements with an s_{12} pole and Q and R do not contain any elements with an s_{12} pole. Locality and unitarity of bi-adjoint scalar theory then demands that elements of Q and R will have zero residue on the s_{12} pole, and a given element of P will have the form

$$\begin{aligned}\mathcal{A}_n^{\phi^3}[12, \sigma(3, \cdots, n)|12, \sigma'(3, \cdots, n)] =& \frac{\mathcal{A}_{n-1}^{\phi^3}[P_{12}, \sigma(3, \cdots, n)|P_{12}, \sigma'(3, \cdots, n)]}{s_{12} + m^2} \\ &+ \mathcal{O}\left((s_{12} + m^2)^0\right),\end{aligned} \tag{P.0.3}$$

near the pole. We will now assume that the orderings $[P_{12}, \sigma(3, \cdots, n)]$ form a DDM basis for $n-1$ particles $\{P_{12}, 3, 4, ..., n\}$[2]. Thus the blocks are characterized by their behavior as they approach the s_{12} pole,

$$P = \mathcal{O}\left((s_{12} + m^2)^{-1}\right), \quad Q = \mathcal{O}\left((s_{12} + m^2)^0\right), \quad R = \mathcal{O}\left((s_{12} + m^2)^0\right). \tag{P.0.4}$$

Various useful corollaries can be drawn. For example,

$$P^{-1} = \mathcal{O}\left((s_{12} + m^2)^1\right), \quad R^{-1} = \mathcal{O}\left((s_{12} + m^2)^0\right). \tag{P.0.5}$$

In fact, (P.0.3) allows us to be more specific, for P^{-1},

$$P^{-1}[12, \sigma|12, \sigma'] = (s_{12} + m^2)\left(\mathcal{A}_{n-1}^{\phi^3}\right)^{-1}[P_{12}, \sigma|P_{12}, \sigma'] + \mathcal{O}\left((s_{12} + m^2)^2\right), \tag{P.0.6}$$

where we will use the shorthand $\sigma = \sigma(3, \cdots, n)$ and $\sigma' = \sigma'(3, \cdots, n)$ for the rest of the section. Finally, using the geometric series formula for matrices, we get

$$(1 - P^{-1}Q^\top R^{-1}Q)^{-1} = 1 + \mathcal{O}\left((s_{12} + m^2)^1\right). \tag{P.0.7}$$

These properties, along with the blockwise inversion formula

$$\left(\mathcal{A}_n^{\phi^3}\right)^{-1}[\alpha|\beta] = \begin{pmatrix} P^{-1}(1 - P^{-1}Q^\top R^{-1}Q)^{-1} & -P^{-1}(1 - P^{-1}Q^\top R^{-1}Q)^{-1}Q^\top R^{-1} \\ -R^{-1}QP^{-1}(1 - P^{-1}Q^\top R^{-1}Q)^{-1} & R^{-1} + R^{-1}QP^{-1}(1 - P^{-1}Q^\top R^{-1}Q)^{-1}Q^\top R^{-1} \end{pmatrix}, \tag{P.0.8}$$

[2]Returning to the example DDM basis $[1\sigma(2, \cdots, n-1)n]$, we see that this condition is satisfied, i.e. $[P_{12}\sigma(3, \cdots, n-1)n]$ forms a DDM basis.

gives

$$\begin{pmatrix} P & Q^\top \\ Q & R \end{pmatrix}^{-1} = \begin{pmatrix} (s_{12}+m^2)\left(\mathcal{A}^{\phi^3}_{n-1}\right)^{-1}[P_{12},\sigma|P_{12},\sigma'] & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \mathcal{O}\left((s_{12}+m^2)^2\right) & \mathcal{O}\left((s_{12}+m^2)^1\right) \\ \mathcal{O}\left((s_{12}+m^2)^1\right) & \mathcal{O}\left((s_{12}+m^2)^0\right) \end{pmatrix}. \tag{P.0.9}$$

It is straightforward to see that only the elements in the top left block will multiply amplitudes $\mathcal{A}^A_3[12\sigma]$ and $\mathcal{A}^B_3[12\sigma']$ and hence only these could develop a pole at s_{12}. Thus the suppressed terms on the right-hand-side will not contribute on the factorization channel.

So in a neighborhood of the s_{12} pole,

$$\begin{aligned}
\mathcal{A}^{A\otimes B}_n &= \sum_{\alpha,\beta} \mathcal{A}^A_n[\alpha]\left(\mathcal{A}^{\phi^3}_n\right)^{-1}[\alpha|\beta]\mathcal{A}^B_n[\beta] \\
&= \sum_{\sigma,\sigma'} \frac{\mathcal{A}^A_3[12,-P_{12}]\mathcal{A}^A_{n-1}[P_{12},\sigma]\left(\mathcal{A}^{\phi^3}_{n-1}\right)^{-1}[P_{12},\sigma|P_{12},\sigma']\mathcal{A}^B_3[12,-P_{12}]\mathcal{A}^B_{n-1}[P_{12},\sigma']}{s_{12}+m^2} \\
&\quad + \mathcal{O}\left((s_{12}+m^2)^0\right) \\
&= \mathcal{A}^A_3[12,-P_{12}]\mathcal{A}^B_3[12,-P_{12}] \sum_{\sigma,\sigma'} \frac{\mathcal{A}^A_{n-1}[P_{12},\sigma]\left(\mathcal{A}^{\phi^3}_{n-1}\right)^{-1}[P_{12},\sigma|P_{12},\sigma']\mathcal{A}^B_{n-1}[P_{12},\sigma']}{s_{12}+m^2} \\
&\quad + \mathcal{O}\left((s_{12}+m^2)^0\right) \\
&= \frac{\mathcal{A}^{A\otimes B}_3(1,2,-P_{12})\mathcal{A}^{A\otimes B}_{n-1}(P_{12},3,...,n)}{s_{12}+m^2} + \mathcal{O}\left((s_{12}+m^2)^0\right),
\end{aligned} \tag{P.0.10}$$

where we have used the fact that for $n=3$, the formula (5.2.16) takes the simple form,

$$\mathcal{A}^{A\otimes B}_3(1,2,3) = \mathcal{A}^A_3[123]\mathcal{A}^B_3[123]. \tag{P.0.11}$$

Thus, on a two-particle channel, an n-point amplitude generated by the massive KLT formula factorizes into lower-point amplitudes also generated by (5.2.16), i.e. these amplitudes factorize into the correct lower-point amplitudes. Since we chose to study the s_{12} pole without loss of generality, this argument demonstrates factorization on all two-particle singularities.

This argument generalizes straightforwardly to multi-particle factorization. Without loss of generality we will consider factorization on the singularity

$$P^2 = m^2, \quad \text{where } P^\mu \equiv p_1^\mu + p_2^\mu + ... + p_{k-1}^\mu + p_k^\mu. \tag{P.0.12}$$

A double-ordered bi-adjoint scalar amplitude will contain such a singularity only if both its orderings have $\{1,2,...,k\}$ cyclically adjacent. As above, we choose a DDM basis for the n-point amplitudes in which the *minimal* number of amplitudes with a P^2 factorization singularity appear. A natural choice is

$$\{\mathcal{A}_n^{\phi^3}[1,\alpha,n|1,\beta,n] : \alpha,\beta \in \mathcal{P}\,(2,3,...,n-1)\}. \tag{P.0.13}$$

The subset of these amplitudes which have a $P^2 = m^2$ singularity have the form

$$\{\mathcal{A}_n^{\phi^3}[1,\sigma,\rho,n|1,\sigma',\rho',n] : \sigma,\sigma' \in \mathcal{P}\,(2,3,...,k-1,k)\,,\ \rho,\rho' \in \mathcal{P}\,(k+1,k+2,...,n-1,n)\}. \tag{P.0.14}$$

Near the singularity such amplitudes have the form

$$\mathcal{A}_n^{\phi^3}[1,\sigma,\rho,n|1,\sigma',\rho',n] = \frac{\mathcal{A}_{k+1}^{\phi^3}[1,\sigma,-P|1,\sigma',-P]\mathcal{A}_{k+1}^{\phi^3}[P,\rho,n|P,\rho',n]}{P^2+m^2} + \mathcal{O}\left((P^2+m^2)^0\right). \tag{P.0.15}$$

Placing all such amplitudes in the top-left-hand corner of the matrix of biadjoint-scalar amplitudes, we obtain the same result as in Section 5.2.2, that only amplitudes of this form are important on the factorization channel when using the block decomposition inverse formula (P.0.8). Here the associated subspaces are indexed by a *pair* of orderings (σ,ρ) on the left and (σ',ρ') on the right. The required inverse is then given by

$$\begin{aligned}&\left(\mathcal{A}_n^{\phi^3}\right)^{-1}[1,\sigma,\rho,n|1,\sigma',\rho',n]\\ &\quad= (P^2+m^2)\left(\mathcal{A}_{k+1}^{\phi^3}\right)^{-1}[1,\sigma,-P|1,\sigma',-P]\left(\mathcal{A}_{k+1}^{\phi^3}\right)^{-1}[P,\rho,n|P,\rho',n] + \mathcal{O}\left((P^2+m^2)^2\right).\end{aligned} \tag{P.0.16}$$

This is an application of a general result for the so-called *Kronecker* product of matrices

$$(P\otimes Q)^{-1} = P^{-1}\otimes Q^{-1}. \tag{P.0.17}$$

Verifying that this is true is trivial in component form. We label the components as P_{ik} and Q_{jl}, the Kronecker product is then defined component-wise as $(P\otimes Q)_{ijkl} \equiv P_{ik}Q_{jl}$. The right-inverse is defined to satisfy

$$\sum_{m,n}(P\otimes Q)_{ijmn}(P\otimes Q)^{-1}_{mnkl} = \delta_{ik}\delta_{jl}. \tag{P.0.18}$$

It is straightforward to see that this is satisfied by matrices of the form

$$(P\otimes Q)^{-1}_{mnkl} = (P^{-1})_{mk}(Q^{-1})_{nl}, \tag{P.0.19}$$

and similarly for the left-inverse. Using this result, on the neighborhood of the $P^2 = m^2$ pole,

$$
\begin{aligned}
&\mathcal{A}_n^{A\otimes B}(1,2,...,n) \\
&= \sum_{\alpha,\beta} \mathcal{A}_n^A[\alpha] \left(\mathcal{A}_n^{\phi^3}\right)^{-1} [\alpha|\beta] \mathcal{A}_n^B[\beta] \\
&= \sum_{\sigma,\sigma'} \sum_{\rho\rho'} \frac{1}{(P^2+m^2)^2} \Bigg(\mathcal{A}_{k+1}^A[1,\sigma,-P] \mathcal{A}_{n-k+1}^A[P,\rho,n] \times \left(\mathcal{A}_n^{\phi^3}\right)^{-1} [1,\sigma,\rho,n|1,\sigma',\rho',n] \\
&\qquad\qquad \times \mathcal{A}_{k+1}^B[1,\sigma',-P] \mathcal{A}_{n-k+1}^B[P,\rho',n] \Bigg) + \mathcal{O}\left((P^2+m^2)^0\right)
\end{aligned}
$$

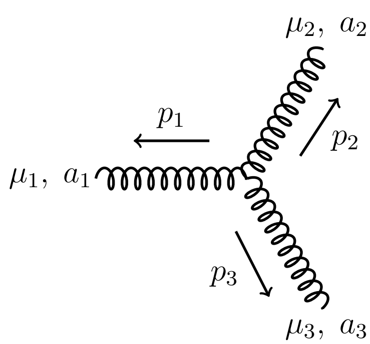
$$
\begin{aligned}
&= \sum_{\sigma,\sigma'} \sum_{\rho\rho'} \frac{1}{P^2+m^2} \Bigg(\mathcal{A}_{k+1}^A[1,\sigma,-P] \mathcal{A}_{n-k+1}^A[P,\rho,n] \left(\mathcal{A}_{k+1}^{\phi^3}\right)^{-1} [1,\sigma,-P|1,\sigma',-P] \\
&\qquad \times \left(\mathcal{A}_{n-k+1}^{\phi^3}\right)^{-1} [P,\rho,n|P,\rho',n] \mathcal{A}_{k+1}^B[1,\sigma',-P] \mathcal{A}_{n-k+1}^B[P,\rho',n] \Bigg) \\
&\qquad + \mathcal{O}\left((P^2+m^2)^0\right) \\
&= \frac{1}{P^2+m^2} \left(\sum_{\sigma,\sigma'} \mathcal{A}_{k+1}^A[1,\sigma,-P] \left(\mathcal{A}_{k+1}^{\phi^3}\right)^{-1} [1,\sigma,-P|1,\sigma',-P] \mathcal{A}_{k+1}^B[1,\sigma',-P] \right) \\
&\qquad \times \left(\sum_{\rho\rho'} \mathcal{A}_{n-k+1}^A[P,\rho,n] \left(\mathcal{A}_{n-k+1}^{\phi^3}\right)^{-1} [P,\rho,n|P,\rho',n] \mathcal{A}_{n-k+1}^B[P,\rho',n] \right) \\
&\qquad + \mathcal{O}\left((P^2+m^2)^0\right) \\
&= \frac{\mathcal{A}_{k+1}^{A\otimes B}(1,2,...,k,-P)\, \mathcal{A}_{n-k+1}^{A\otimes B}(P,k+1,...,n)}{P^2+m^2} + \mathcal{O}\left((P^2+m^2)^0\right).
\end{aligned}
\tag{P.0.20}
$$

So we find that the massive KLT formula generates expressions which factor correctly on all singularities.

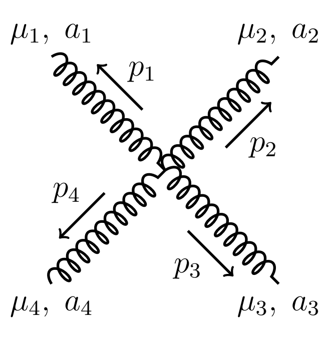
APPENDIX Q

Feynman Rules for Massive Yang-Mills

At low multiplicity it is efficient to calculate the scattering amplitudes of massive Yang-Mills (5.3.6) using Feynman rules. The vertex functions are identical to those of standard non-Abelian gauge theory:

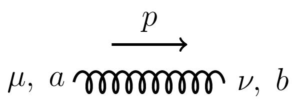


$$= \quad \begin{aligned} g f^{a_1 a_2 a_3} \big[& g^{\mu_1 \mu_2} \left(p_2^{\mu_3} - p_1^{\mu_3}\right) \\ & + g^{\mu_2 \mu_3} \left(p_3^{\mu_1} - p_2^{\mu_1}\right) \\ & + g^{\mu_3 \mu_1} \left(p_1^{\mu_2} - p_3^{\mu_2}\right) \big] \,, \end{aligned}$$



$$= \quad \begin{aligned} g^2 \Big[& f^{a_1 a_2 b} f^{a_3 a_4 b} \left(g^{\mu_1 \mu_3} g^{\mu_2 \mu_4} - g^{\mu_1 \mu_4} g^{\mu_2 \mu_3}\right) \\ & + f^{a_1 a_3 b} f^{a_2 a_4 b} \left(g^{\mu_1 \mu_2} g^{\mu_3 \mu_4} - g^{\mu_1 \mu_4} g^{\mu_2 \mu_3}\right) \\ & + f^{a_1 a_4 b} f^{a_2 a_3 b} \left(g^{\mu_1 \mu_2} g^{\mu_3 \mu_4} - g^{\mu_1 \mu_3} g^{\mu_2 \mu_4}\right) \Big] \,. \end{aligned}$$

Meanwhile the propagator is modified to take the Proca form:



$$= \qquad \frac{\delta^{ab}}{p^2 + m^2}\left(g_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}\right). \tag{Q.0.3}$$

APPENDIX R

4-point Graviton-Dilaton Amplitudes from Double-Copy

The amplitudes given by the double-copy of massive Yang-Mills are given here:

$$\begin{aligned}\mathcal{M}_4^{hhhh} = -\frac{1}{4M_p^2}\Bigg(&\frac{1}{m^2-2p_{12}}\Big(-z_{14}z_{23}m^2+z_{13}z_{24}m^2+2z_{12}z_{34}m^2+2p_{12}z_{14}z_{23}\\&-2p_{12}z_{13}z_{24}-2p_{12}z_{12}z_{34}-4p_{13}z_{12}z_{34}+4z_{34}zp_{13}zp_{21}-4z_{34}zp_{12}zp_{23}\\&+4z_{24}zp_{12}zp_{31}-4z_{14}zp_{21}zp_{31}+4z_{24}zp_{12}zp_{32}-4z_{14}zp_{21}zp_{32}-4z_{23}zp_{12}zp_{41}\\&+4z_{13}zp_{21}zp_{41}+4z_{12}zp_{32}zp_{41}-4z_{23}zp_{12}zp_{42}+4z_{13}zp_{21}zp_{42}-4z_{12}zp_{31}zp_{42}\Big)^2\\&+(2\leftrightarrow 3)+(2\leftrightarrow 4)\Bigg).\end{aligned} \tag{R.0.1}$$

$$\begin{aligned}\mathcal{M}_4^{\phi\phi\phi\phi} = \frac{1}{M_p^2}\Bigg(&-\frac{p_{13}^2\left(75m^2p_{12}+34p_{12}^2+116m^4\right)}{72m^4\left(m^2-2p_{12}\right)}\\&+\frac{3}{64}\left(-24m^2p_{12}+48p_{12}^2+115m^4\right)\left(\frac{1}{2p_{14}+m^2}+\frac{1}{2p_{13}+m^2}\right)\\&+\frac{p_{13}\left(-41m^4p_{12}-41m^2p_{12}^2-34p_{12}^3+116m^6\right)}{72m^4\left(m^2-2p_{12}\right)}\\&-\frac{-4751m^4p_{12}+744m^2p_{12}^2+368p_{12}^3+3696m^6}{288m^2\left(m^2-2p_{12}\right)}\Bigg).\end{aligned} \tag{R.0.2}$$

$$\mathcal{M}_4^{h\phi\phi\phi} = \frac{1}{6\sqrt{3}m^4M_p^2\big(m^2-2p_{12}\big)\big(m^2-2p_{13}\big)\big(m^2-2\big(p_{14}\big)\big)}\big(m^{10}\big(19zp_{12}^2-43zp_{13}zp_{12}
+19zp_{13}^2\big)-m^8\big(p_{13}\big(136zp_{12}^2-53zp_{13}zp_{12}+76zp_{13}^2\big)+p_{12}\big(76zp_{12}^2-53zp_{13}zp_{12}
+136zp_{13}^2\big)\big)+m^6\big(p_{12}^2\big(76zp_{12}^2+61zp_{13}zp_{12}+195zp_{13}^2\big)+3p_{13}p_{12}\big(39zp_{12}^2
-53zp_{13}zp_{12}+39zp_{13}^2\big)+p_{13}^2\big(195zp_{12}^2+61zp_{13}zp_{12}+76zp_{13}^2\big)\big)
+m^4\big(p_{12}^3zp_{13}\big(10zp_{12}+zp_{13}\big)+p_{13}p_{12}^2\big(-41zp_{12}^2+10zp_{13}zp_{12}-37zp_{13}^2\big)
+p_{13}^2p_{12}\big(-37zp_{12}^2+10zp_{13}zp_{12}-41zp_{13}^2\big)+p_{13}^3zp_{12}\big(zp_{12}+10zp_{13}\big)\big)
+2m^2\big(p_{12}^4zp_{13}^2-12p_{13}p_{12}^3zp_{12}zp_{13}-2p_{13}^2p_{12}^2\big(zp_{12}^2+13zp_{13}zp_{12}+zp_{13}^2\big)
-12p_{13}^3p_{12}zp_{12}zp_{13}+p_{13}^4zp_{12}^2\big)-4p_{12}p_{13}\big(p_{12}+p_{13}\big)\big(p_{13}zp_{12}-p_{12}zp_{13}\big)^2\big)\big). \tag{R.0.3}$$

$$\mathcal{M}_4^{hh\phi\phi} = \frac{1}{6m^2M_p^2\big(m^2-2p_{12}\big)\big(m^2-2p_{13}\big)\big(m^2-2p_{14}\big)}\Big(19z_{12}^2m^{10}
-z_{12}\big(62p_{12}z_{12}+92p_{13}z_{12}+zp_{12}zp_{21}-35zp_{13}zp_{21}+17zp_{12}zp_{23}-18zp_{13}zp_{23}\big)m^8
+\big(42p_{12}^2z_{12}^2+156p_{13}^2z_{12}^2+4p_{13}\big(zp_{12}\big(zp_{21}+21zp_{23}\big)-3zp_{13}\big(13zp_{21}+6zp_{23}\big)\big)z_{12}
+p_{12}\big(180p_{13}z_{12}+zp_{12}\big(4zp_{21}+31zp_{23}\big)-zp_{13}\big(121zp_{21}+90zp_{23}\big)\big)z_{12}+34zp_{13}^2zp_{21}^2
+zp_{12}^2zp_{23}^2-33zp_{12}zp_{13}zp_{23}^2+33zp_{13}^2zp_{21}zp_{23}-35zp_{12}zp_{13}zp_{21}zp_{23}\big)m^6
+\big(13z_{12}^2p_{12}^3-4z_{12}\big(14p_{13}z_{12}+zp_{12}\big(zp_{21}-2zp_{23}\big)-zp_{13}\big(25zp_{21}+27zp_{23}\big)\big)p_{12}^2
-\big(200p_{13}^2z_{12}^2+12p_{13}\big(zp_{12}\big(zp_{21}+10zp_{23}\big)-2zp_{13}\big(14zp_{21}+9zp_{23}\big)\big)z_{12}+zp_{12}^2zp_{23}^2
-2zp_{12}zp_{13}zp_{23}\big(34zp_{21}+33zp_{23}\big)+zp_{13}^2\big(100zp_{21}^2+132zp_{23}zp_{21}+33zp_{23}^2\big)\big)p_{12}
-p_{13}\big(128p_{13}^2z_{12}^2+4p_{13}\big(zp_{12}\big(zp_{21}+42zp_{23}\big)-6zp_{13}\big(10zp_{21}+3zp_{23}\big)\big)z_{12}
+37zp_{12}^2zp_{23}^2-2zp_{12}zp_{13}zp_{23}\big(70zp_{21}+33zp_{23}\big)+zp_{13}^2zp_{21}\big(103zp_{21}+66zp_{23}\big)\big)m^4
-2\big(z_{12}^2p_{12}^4+2z_{12}\big(9p_{13}z_{12}-zp_{13}zp_{21}+zp_{12}zp_{23}\big)p_{12}^3+\big(-\big(32zp_{21}^2+66zp_{23}zp_{21}
+33zp_{23}^2\big)zp_{13}^2-2zp_{12}zp_{21}zp_{23}zp_{13}+zp_{12}^2zp_{23}^2+2p_{13}z_{12}\big(zp_{12}\big(7zp_{23}-2zp_{21}\big)
+zp_{13}\big(29zp_{21}+36zp_{23}\big)\big)\big)p_{12}^2-2p_{13}\big(24p_{13}^2z_{12}^2+p_{13}\big(zp_{12}\big(2zp_{21}+27zp_{23}\big)
-9zp_{13}\big(7zp_{21}+4zp_{23}\big)\big)z_{12}+33zp_{13}\big(zp_{21}+zp_{23}\big)\big(zp_{13}zp_{21}-zp_{12}zp_{23}\big)\big)p_{12}
-p_{13}^2\big(32p_{13}^2z_{12}^2-68p_{13}\big(zp_{13}zp_{21}-zp_{12}zp_{23}\big)z_{12}+35\big(zp_{13}zp_{21}-zp_{12}zp_{23}\big)^2\big)\big)m^2
+4p_{12}p_{13}\big(p_{12}+p_{13}\big)\big(p_{12}z_{12}+2p_{13}z_{12}-zp_{13}zp_{21}+zp_{12}zp_{23}\big)^2\big)\Big). \tag{R.0.4}$$

$$
\begin{aligned}
\mathcal{M}_4^{hhh\phi} = & \frac{-1}{2\sqrt{3}M_p^2\big(m^2-2p_{12}\big)\big(m^2-2p_{13}\big)\big(m^2-2(p_{14}\big)}\big(z_{12}z_{13}z_{23}m^8-\big(-14zp_{31}^2z_{12}^2\\
& -2zp_{32}^2z_{12}^2-8zp_{31}zp_{32}z_{12}^2+4p_{12}z_{13}z_{23}z_{12}+4p_{13}z_{13}z_{23}z_{12}-7z_{23}zp_{12}zp_{31}z_{12}\\
& +13z_{23}zp_{13}zp_{31}z_{12}+13z_{13}zp_{21}zp_{31}z_{12}-7z_{13}zp_{23}zp_{31}z_{12}+z_{23}zp_{12}zp_{32}z_{12}\\
& +5z_{23}zp_{13}zp_{32}z_{12}-7z_{13}zp_{21}zp_{32}z_{12}-11z_{13}zp_{23}zp_{32}z_{12}+z_{23}^2zp_{12}^2+z_{23}^2zp_{13}^2\\
& -14z_{13}^2zp_{21}^2-2z_{13}^2zp_{23}^2+10z_{23}^2zp_{12}zp_{13}+13z_{13}z_{23}zp_{12}zp_{21}-7z_{13}z_{23}zp_{13}zp_{21}\\
& +5z_{13}z_{23}zp_{12}zp_{23}+z_{13}z_{23}zp_{13}zp_{23}-8z_{13}^2zp_{21}zp_{23}\big)m^6+\big(4z_{12}z_{13}z_{23}p_{12}^2\\
& +\big(-41zp_{31}^2z_{12}^2-5zp_{32}^2z_{12}^2-14zp_{31}zp_{32}z_{12}^2+12p_{13}z_{13}z_{23}z_{12}+40z_{13}zp_{21}zp_{31}z_{12}\\
& -28z_{13}zp_{23}zp_{31}z_{12}-16z_{13}zp_{21}zp_{32}z_{12}-20z_{13}zp_{23}zp_{32}z_{12}-44z_{13}^2zp_{21}^2-20z_{13}^2zp_{23}^2\\
& +z_{23}^2\big(zp_{12}^2+22zp_{13}zp_{12}-11zp_{13}^2\big)-32z_{13}^2zp_{21}zp_{23}+4z_{23}\big(z_{13}\big(zp_{12}\big(7zp_{21}+2zp_{23}\big)\\
& +zp_{13}\big(4zp_{23}-7zp_{21}\big)\big)+z_{12}\big(zp_{12}\big(zp_{32}-4zp_{31}\big)+zp_{13}\big(13zp_{31}+2zp_{32}\big)\big)\big)\big)p_{12}\\
& +p_{13}\big(-44zp_{31}^2z_{12}^2-20zp_{32}^2z_{12}^2-32zp_{31}zp_{32}z_{12}^2+4p_{13}z_{13}z_{23}z_{12}+40z_{13}zp_{21}zp_{31}z_{12}\\
& -16z_{13}zp_{23}zp_{31}z_{12}-28z_{13}zp_{21}zp_{32}z_{12}-20z_{13}zp_{23}zp_{32}z_{12}-41z_{13}^2zp_{21}^2-5z_{13}^2zp_{23}^2\\
& +z_{23}^2\big(-11zp_{12}^2+22zp_{13}zp_{12}+zp_{13}^2\big)-14z_{13}^2zp_{21}zp_{23}+4z_{23}\big(z_{13}\big(zp_{13}\big(zp_{23}-4zp_{21}\big)\\
& +zp_{12}\big(13zp_{21}+2zp_{23}\big)\big)+z_{12}\big(zp_{13}\big(7zp_{31}+2zp_{32}\big)+zp_{12}\big(4zp_{32}-7zp_{31}\big)\big)\big)\big)\big)m^4\\
& +2\big(\big(13zp_{31}^2z_{12}^2+zp_{32}^2z_{12}^2-2zp_{31}zp_{32}z_{12}^2-4p_{13}z_{13}z_{23}z_{12}-14z_{13}zp_{21}zp_{31}z_{12}\\
& +14z_{13}zp_{23}zp_{31}z_{12}+2z_{13}zp_{21}zp_{32}z_{12}-2z_{13}zp_{23}zp_{32}z_{12}+16z_{13}^2zp_{21}^2+16z_{13}^2zp_{23}^2\\
& -2z_{23}\big(z_{13}\big(zp_{12}-7zp_{13}\big)\big(zp_{21}-zp_{23}\big)+z_{12}zp_{13}\big(13zp_{31}-zp_{32}\big)\\
& +z_{12}zp_{12}\big(zp_{32}-zp_{31}\big)\big)+z_{23}^2\big(zp_{12}^2-2zp_{13}zp_{12}+13zp_{13}^2\big)+16z_{13}^2zp_{21}zp_{23}\big)p_{12}^2\\
& -2p_{13}\big(2p_{13}z_{12}z_{13}z_{23}-3\big(\big(5zp_{31}^2+2zp_{32}zp_{31}+zp_{32}^2\big)z_{12}^2+z_{13}\big(3zp_{23}\big(zp_{31}+zp_{32}\big)\\
& +zp_{21}\big(3zp_{32}-5zp_{31}\big)\big)z_{12}-4z_{23}^2zp_{12}zp_{13}+z_{13}^2\big(5zp_{21}^2+2zp_{23}zp_{21}+zp_{23}^2\big)\\
& -z_{23}\big(z_{13}\big(zp_{13}\big(zp_{23}-3zp_{21}\big)+zp_{12}\big(5zp_{21}+zp_{23}\big)\big)+z_{12}\big(zp_{12}\big(zp_{32}-3zp_{31}\big)\\
& +zp_{13}\big(5zp_{31}+zp_{32}\big)\big)\big)\big)\big)p_{12}+p_{13}^2\big(16\big(zp_{31}^2+zp_{32}zp_{31}+zp_{32}^2\big)z_{12}^2\\
& -2z_{13}\big(7zp_{21}-zp_{23}\big)\big(zp_{31}-zp_{32}\big)z_{12}+z_{23}^2\big(13zp_{12}^2-2zp_{13}zp_{12}+zp_{13}^2\big)\\
& +z_{13}^2\big(13zp_{21}^2-2zp_{23}zp_{21}+zp_{23}^2\big)-2z_{23}\big(z_{13}\big(zp_{12}\big(13zp_{21}-zp_{23}\big)\\
& +zp_{13}\big(zp_{23}-zp_{21}\big)\big)-z_{12}\big(7zp_{12}-zp_{13}\big)\big(zp_{31}-zp_{32}\big)\big)\big)\big)m^2\\
& -4p_{12}p_{13}\big(p_{12}+p_{13}\big)\big(z_{23}\big(zp_{12}-zp_{13}\big)+z_{13}\big(zp_{23}-zp_{21}\big)+z_{12}\big(zp_{31}-zp_{32}\big)\big)^2\big).
\end{aligned}
\tag{R.0.5}
$$

APPENDIX S

BCJ Relations as Null Vectors

The BCJ relations can be obtained as null space relations of the matrix of bi-adjoint scalar amplitudes. To show this, one must first notice a remarkable property about these amplitudes. Just as in the massless case, bi-adjoint scalar theory acts as an identity for the massive double-copy[1], i.e.

$$A \otimes \text{BS} = A\,. \tag{S.0.1}$$

To express this in matrix notation, let us first choose an $(n-2)!$ DDM basis. From this, we choose BCJ-independent $(n-3)!$ sub-bases α, β and γ and use the KLT formula,

$$\mathcal{A}^{\phi^3}[\alpha|\beta]\;\mathcal{A}^{\phi^3}[\beta|\gamma]^{-1}\;\vec{\mathcal{A}}^{A}[\gamma] = \vec{\mathcal{A}}^{A}[\alpha]. \tag{S.0.2}$$

The BCJ relations are consistency conditions that make the KLT formula basis-independent. For example, consider another $(n-3)!$ sub-basis $\tilde{\gamma}$. We can then express a BCJ relation as

$$\mathcal{A}^{\phi^3}[\alpha|\beta]\;\mathcal{A}^{\phi^3}[\beta|\gamma]^{-1}\;\vec{\mathcal{A}}^{A}[\gamma] = \mathcal{A}^{\phi^3}[\alpha|\beta]\;\mathcal{A}^{\phi^3}[\beta|\tilde{\gamma}]^{-1}\;\vec{\mathcal{A}}^{A}[\tilde{\gamma}]\,. \tag{S.0.3}$$

We now embed these matrices in our original $(n-2)!$ DDM basis. The matrix $\mathcal{A}^{\phi^3}[\alpha|\beta]$ is padded with the remaining bi-adjoint scalar amplitudes, while we pad the vector

$$\Big(\mathcal{A}^{\phi^3}[\beta|\gamma]^{-1}\vec{\mathcal{A}}^{A}[\gamma] - \mathcal{A}^{\phi^3}[\beta|\tilde{\gamma}]^{-1}\vec{\mathcal{A}}^{A}[\tilde{\gamma}]\Big) \tag{S.0.4}$$

[1]It is an interesting fact that this is true whether or not the spectral conditions hold.

with zeros. This gives us the following null vector equation,

$$\mathcal{A}^{\phi^3}[\alpha|\beta]\left(\mathcal{A}^{\phi^3}[\beta|\gamma]^{-1}\vec{\mathcal{A}}^A[\gamma]-\mathcal{A}^{\phi^3}[\beta|\tilde{\gamma}]^{-1}\vec{\mathcal{A}}^A[\tilde{\gamma}]\right)=0\,. \tag{S.0.5}$$

To connect this to the BCJ relations of theory A, we consider a double-copy of A with itself,

$$A\otimes A=B. \tag{S.0.6}$$

Choosing the same sub-bases as previously, we can rewrite the KLT formula,

$$\vec{\mathcal{A}}^A[\beta]^T\,\mathcal{A}^{\phi^3}[\beta|\gamma]^{-1}\,\vec{\mathcal{A}}^A[\gamma]=\vec{\mathcal{A}}^B. \tag{S.0.7}$$

Again the BCJ relations are given by demanding basis-independence of this formula,

$$\begin{aligned}\vec{\mathcal{A}}^A[\beta]^T\,\mathcal{A}^{\phi^3}[\beta|\gamma]^{-1}\,\vec{\mathcal{A}}^A[\gamma]&=\vec{\mathcal{A}}^A[\beta]^T\,\mathcal{A}^{\phi^3}[\beta|\tilde{\gamma}]^{-1}\,\vec{\mathcal{A}}^A[\tilde{\gamma}]\\ \Rightarrow\;\vec{\mathcal{A}}^A[\beta]^T\left(\mathcal{A}^{\phi^3}[\beta|\gamma]^{-1}\vec{\mathcal{A}}^A[\gamma]-\mathcal{A}^{\phi^3}[\beta|\tilde{\gamma}]^{-1}\vec{\mathcal{A}}^A[\tilde{\gamma}]\right)&=0.\end{aligned} \tag{S.0.8}$$

We recognize this vector as being a null vector of $\mathcal{A}^{\phi^3}[\alpha|\beta]$. Indeed this equation must hold for all choices of γ and $\tilde{\gamma}$. At 4- and 5-point, we observe that different choices of γ and $\tilde{\gamma}$ span the null space of $\mathcal{A}^{\phi^3}[\alpha|\beta]$, allowing us to generalize this equation to,

$$\vec{\mathcal{A}}^A[\beta]\cdot\vec{n}=0, \tag{S.0.9}$$

where the vector $\vec{n}$ is any null vector of the matrix of bi-adjoint scalar amplitudes.

Thus (S.0.9) is an equivalent representation of the BCJ relations. Since the number of null vectors of $\mathcal{A}^{\phi^3}[\alpha|\beta]$ is exactly the number of independent BCJ relations, we expect this equivalence to continue to any n-point.

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