# Problems in Mathematical Finance Related to Time-inconsistency and Mean Field Games

by

Jingjie Zhang

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Applied and Interdisciplinary Mathematics)
in The University of Michigan
2021

Doctoral Committee:

Professor Erhan Bayraktar, Co-Chair
Assistant Professor Indrajit Mitra, Co-Chair
Assistant Professor Asaf Cohen
Associate Professor Vijay Subramanian
Professor Virginia Young

Jingjie Zhang

jingjiez@umich.edu

ORCID iD: 0000-0003-0401-6950

To my mom and dad

# ACKNOWLEDGEMENTS

I would like to extend my sincere gratitude to my Ph.D. advisor, Professor Erhan Bayraktar, whose expertise in financial mathematics, persistent passion toward research and inspirational mentorship have been invaluable in shaping me into an applied mathematician. Not only has he taught me knowledge and skills of doing research, but he also motivates me to keep challenging myself with interesting and meaningful math problems. I thank him for his enormous support and patience when I suffered setbacks, as well as for encouraging me to dig deeper when we made progress on projects. I am truly fortunate to have such an incredible studying experience with Professor Bayraktar.

I am deeply grateful for my co-advisor, Professor Indrajit Mitra, for introducing me to an interdisciplinary project in mathematics and economics and guiding me through it with insightful ideas and unbounded support. Our discussions have substantially expanded my horizons and provided me with fresh perspectives.

I would like to thank Professor Asaf Cohen, Professor Vijay Subramanian and Professor Virginia Young for serving as my committee members. I greatly appreciate them reading this thesis and providing insightful comments.

This thesis would not have been possible without the help and support of many other people. I would like to thank my talented and diligent collaborator Professor Zhou Zhou for his time to discuss project details with me. I would like to thank Professor Sergey Nadtochiy for his great course in stochastic analysis and patiently answering all of my questions. I would like to thank our warm and caring Graduate Program Coordinator Teresa Stokes for her immense support and assistance. I would like to thank Xueying Hu and Jiaqi Li for all the helpful advice and kind encouragement their gave.

I also want to give special thanks to my friends Lu Yang and Yili Zhang for being there for me during difficult times. I also thank Christina Athanasouli, Ke Feng, Yinqiu He, Claire Lin, Jiaqi Ma, Lequn Geng, Haoyang Guo, Jia Guo, Jia Qi, Siliang Song, Weijing Tang, Guanyang Wang, Yutong Xie, Xiong Xiao, Berkan Yilmaz, Xin Zhang and many other friends, for their company, which has brought so much joy to my life.

Most of all, I would like to express my deepest appreciation to my mother Juan Feng and my father Yong Zhang. Their unconditional love and support have made me who I am today. To them I dedicate this thesis.

# TABLE OF CONTENTS

# LIST OF TABLES

**Table**

# LIST OF FIGURES

Figure

# LIST OF APPENDICES

**Appendix**

# ABSTRACT

This thesis consists of two problems on time inconsistency and one problem on mean field games, all featuring the study of equilibrium and applications in economics and finance.

In Chapter II, we deal with time inconsistency in the infinite horizon mean-variance stopping problem under discrete time setting. In order to determine a proper time-consistent plan, we investigate subgame perfect Nash equilibria among three different types of strategies, pure stopping times, randomized stopping times and liquidation strategies. We show that equilibria among pure stopping times or randomized stopping times may not exist, while an equilibrium liquidation strategy always exists. Furthermore, we argue that the mean-standard deviation variant of this problem makes more sense for this type of strategies in terms of time consistency. The existence and uniqueness of optimal equilibrium liquidation strategies are also analyzed.

In Chapter III, we delve into equilibrium concepts for time inconsistent stopping problems in continuous time. We point out that the two existing notions of equilibrium in the literature, which we call mild equilibrium and weak equilibrium, are inadequate to capture the idea of subgame perfect Nash equilibrium. To characterize it more accurately, we introduce a new notion, strong equilibrium. It is proved that an optimal mild equilibrium is always a strong equilibrium. Moreover, we provide a new iteration method that can directly construct an optimal mild equilibrium and thus also guarantees its existence.

In Chapter IV, we adopt a mean field game (MFG) approach to analyze a costly job search model with incomplete credit and insurance markets. The MFG approach enables us to quantify the impact of a class of countercyclical unemployment benefit policies on labor supply in general equilibrium. Our model provides two interesting predictions. First, the difference between unemployment rates under a countercyclical policy and an acyclical policy is positive and increases rapidly with the size of the aggregate shock. Second, compared with a baseline policy without means test, a means-tested policy which is targeted to provide more generous benefits to liquidity constrained individuals turns out to provide improved consumption insurance to *all* individuals as well as results in a lower equilibrium unemployment rate relative to a comparable non-targeted policy.

# CHAPTER I

# Introduction

This thesis is devoted to three problems in mathematical finance that are related to time inconsistency and mean field games. The concept of Nash equilibrium plays a pivotal role in all of them. The first two problems focus on finding no-regret strategies in the sense of subgame perfect Nash equilibria, for time inconsistent stopping problems with infinite horizon; see Chapter II for discrete time mean-variance stopping problems and see Chapter III for continuous time non-exponential discounting stopping problems. The third problem features a costly job search model with incomplete credit and insurance markets. This model is built and numerically solved using tools developed in mean field game theory and reveals the impact of a class of countercyclical unemployment benefit policies on labor supply in general equilibrium; see Chapter IV.

Dynamic programming is a powerful tool to tackle a wide class of stochastic optimal stopping/optimal control problems. It is based on the famous Bellman's principle of optimality [8], which postulates that an optimal policy computed at the initial stage remains optimal at the later stages [1]. This is also the central idea of time consistency. However such property may fail to hold in many scenarios; see [10, Chap1] for essential factors of time consistency and time inconsistent examples from financial

[1] As in his book, principle of optimality is interpreted as " Any optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision."

economics. Among various strategies to deal with time-inconsistency, only the equilibrium strategy, which is also called "consistent plan", takes the time inconsistency seriously. Thus this is also the approach we adopt in this thesis.

Dating back to the seminal work of Strotz [56], game theoretic approach to time inconsistent control problem has a long history. See [53, 21, 11, 12, 34] and reference therein for more recent development. Time inconsistent stopping problems under different settings have also been discussed in several papers; see [50, 31, 35, 30, 33, 17, 18]. In Chapter II, we study mean-standard deviation stopping problem in discrete time with an infinite time horizon. The main novelty of this study is the determination of the right type of "mixed" equilibrium strategies after we show that the existence of a Markov equilibrium in the class of pure stopping times is not guaranteed in general. The obvious choice of mixing, i.e., a randomized stopping strategy, does not work because when an equilibrium in this class exists it coincides with the pure equilibrium strategy. The right notion of strategies turns out to be the liquidation strategies (see Definition 2.2.9), among which an equilibrium always exists. Moreover, instead of the mean-variance benchmark, the mean-standard deviation criterion plays along well with liquidation strategies and makes more sense in terms of consistent planning. In addition, we show that an optimal equilibrium liquidation strategy in the sense of pointwise dominance may not exist, and may not be unique if it exists. We also establish the existence of a Pareto optimal equilibrium. This chapter is based on [6]. Part of the work has been presented in the Financial/Actuarial Mathematics Seminar at the University of Michigan (March 27, 2018).

While the definition of equilibrium stopping time in discrete time setting is natural with a clear game theoretic interpretation, a direct analogy in continuous time fails to capture the idea of equilibrium for a wide range of state processes. The formulation of equilibrium stopping time is highly nontrivial in continuous time. There are two general notions of equilibrium stopping strategies in continuous time in the literature.

The first notion, which we will call mild equilibrium is proposed in [31] and further studied in [35, 33, 32]. The second notion, which we will call weak equilibrium is proposed in [17] and further studied in [18]. In Chapter III, we first analyze existing notions of equilibrium and their inadequacy in continuous time. A new notion of equilibrium, strong equilibrium is introduced as a more accurate characterization of the equilibrium concept. Next the relations between these different notions are examined thoroughly. In particular, when the state process is a continuous-time Markov chain and the discount function is log sub-additive, we show that an optimal mild equilibrium is also a strong equilibrium, which is far from obvious. Moreover, we provide a new iteration method which directly constructs an optimal mild equilibrium and is much more implementable than the existing method in [30, 33]. Thus it also proves the existence of an optimal mild equilibrium. This chapter is based on [7]. Part of the work has been presented in the Workshop on Optimal Stopping and Free Boundary Problems at the University of Leeds, UK (January 15, 2020) and Student Math Finance Seminar at the University of Michigan (October 19, 2020 ).

Mean field game (MFG) theory is the study of large systems with infinitely many indistinguishable rational players. It arose around 2005 independently in the mathematical community by Lasry and Lions [40, 41, 42, 43] and in the engineering community by P. Caines, Minyi Huang and Roland Malhamé [28, 29]. Featuring the interaction between individual players and the distribution of the whole population, MFG provides a suitable approach to investigating heterogeneous agent models (HAMs) in continuous time, which starts in discrete-time setting in the work of Aiyagari [3], Bewley [9] and Huggett [36], in contrast to the classical representative agent models. In recent years, applications of MFG theory in economic models have gained increasing interest and attention in the literature; see [1, 2, 23, 14]. In Chapter IV, we expand Aiyagari's model by introducing a costly job-search mechanism, which leads to endogenous job-finding rate in equilibrium. On top of that, we add a one-time, unanticipated, negative shock in which a fraction of individuals experience job loss at time $t = 0$. Our goal is to analyze the impact of a class of countercyclical unemployment benefit policies on labor supply. Numerical method for MFG models makes it possible to examine general equilibria of this economy under different unemployment benefit policies. Our model makes two non-trivial predictions. First, it predicts the additional unemployment from countercyclical policies to be a rapidly increasing, non-linear function of the size of the aggregate shock. Second, it predicts that, in equilibrium, countercyclical policies which are targeted to provide more generous benefits to liquidity constrained individuals provide improved consumption insurance to all individuals (even those who are not liquidity constrained) relative to a comparable non-targeted policy. Such a targeted policy costs less and is associated with lower unemployment rate than the comparable non-targeted policy. Part of the work has been presented in Mean Field Games and Related Topics Conference at Levico Terme Italy (September 10, 2019).

# CHAPTER II

# Time Consistent Stopping for the Mean-Standard Deviation Problem — The Discrete Time Case

In this chapter, we formulate the infinite horizon mean-variance stopping problem as a subgame perfect Nash equilibrium in order to determine time consistent strategies with no regret. Equilibria among stopping times or randomized stopping times may not exist. This motivates us to consider the notion of liquidation strategies, which allows the stopping right to be divisible. We then argue that the mean-standard deviation variant of this problem makes more sense for this type of strategies in terms of time consistency. It turns out that an equilibrium liquidation strategy always exists. We then analyze whether optimal equilibrium liquidation strategies exist and whether they are unique and observe that neither may hold.

## 2.1 Introduction

Consider an optimal stopping problem with an infinite time horizon

$$\sup_{\tau} \mathbb{E}_x[g(X_\tau)], \tag{2.1}$$

where $X$ is a Markov process starting from state $x$, and the stopping time $\tau$ is chosen to maximize the expectation of the payoff function $g$. A classical approach to solving this optimal stopping problem is to use dynamic programming. Thanks to the particular form of (2.1), this problem is known to be *time-consistent* in the sense that its optimal stopping strategy does not depend on the initial state $x$. However, such property may fail to hold in some seemingly quite natural problems where the objective function is in a different form. In those cases, a stopping strategy that is optimal from "today's" point of view may not be optimal anymore from "tomorrow's" point of view. Optimal stopping, more generally, optimal control problems with such property are said to be *time-inconsistent*. Typical examples include non-exponential discounting, the optimization criteria used in cumulative prospect theory (e.g. rank based utility), and the mean-variance criterion, which is the focus of this chapter.

There are three ways one could deal with time-inconsistency, dating back to the seminal work of Strotz [56]. The first is to formulate an optimal stopping problem with a given initial state $x$. The optimal stopping time $\tau_x$ is then parametrized by the initial state $x$. Once the state starts at $x$, and optimal policy $\tau_x$ is determined, the player is precommitted to implementing this strategy. This strategy simply does not take the change of future preferences into account. The second is to have the agent repeatedly solve this problem, hence allow for changes in future preferences. [56] called this strategy naive and further repercussions about this strategy are discussed in [19]. [51] discusses these two formulations under the labels *static optimality* and *dynamic optimality*.

The third way is to formulate the problem in game theoretic terms by viewing each state $x$ as a player in a game regarding when to stop the process $X$ and look for equilibrium strategies. Roughly speaking, an equilibrium strategy, which is also called a *consistent plan*, can be viewed as a no-regret strategy since the agent has no incentive to deviate from the strategy at any current state $x$. The third way is the formulation we will follow here to analyze this problem in an infinite-horizon discrete-time setting. In discrete time, infinite horizon is much more challenging than finite horizon because [53]'s backward sequential optimization approach to obtain the consistent plans no longer works.

Recently, there has been a lot of effort in determining the equilibrium strategies in stochastic control problems, see e.g. [11] and the references therein. There are also several papers on the equilibrium strategies for stopping problems. Among them, [50] analyzes the case in discrete time with a finite time horizon, and [24] investigates a particular model in continuous time with an infinite-time horizon. A general treatment for stopping problems in continuous time is considered in [31] in the context of hyperbolic discounting. In particular, [31] proposes a definition of equilibrium in continuous time which avoids using the "first order criteria" as in control problems in the literature. [31] also formulates equilibrium stopping policies as fixed points of an operator and constructs a large class of equilibria by iterating this operator. This effort is continued in [35] when the agents use probability distortions to calculate their criteria. As for the study of how to choose an equilibrium, [30] considers a discrete-time infinite-horizon problem with non-exponential discounting, and investigates optimal equilibria in the sense of pointwise dominance. Apart from establishing the existence of a pure stopping equilibrium, [30] also obtains the existence and uniqueness of an *optimal equilibrium*. Also see [33] which is a continuous-time extension of [30]. Let us also mention the recent work of [17], where the authors consider the equilibrium stopping strategies under the definition associated with first order criteria. They point out that the mean-variance problem is out of the scope of their approach. In another recent paper [18] by the same authors, a continuous time general framework for time-inconsistent stopping problems covering the mean-variance criterion is developed, and a continuous time mean-variance problem is studied, and a mixed stopping strategy for the time-inconsistent stopping problem in continuous time is defined as the first jumping time of a Cox-process associated to the state process.

In this chapter we study time-consistent mean-standard deviation stopping problems in discrete time with an infinite time horizon. We show that while a Markov equilibrium in the class of pure or randomized stopping times may not exist in general, there always exists an equilibrium liquidation strategy. In addition, we show that an optimal equilibrium in the sense of pointwise dominance may not exist, and may not be unique if it exists. We also establish the existence of a Pareto optimal equilibrium.

The main novelty in this project is the determination of the right type of "mixed" equilibrium strategies and the appropriate modification of the criteria to make sense of the consistent planning problem. In particular, we show that the obvious choice of mixing, i.e., a randomized stopping strategy, does not work because when an equilibrium in this class exists it coincides with a pure equilibrium strategy. (One should contrast this to Example 2.6 of [17], where they show that there exists an equilibrium randomized stopping strategy which is not a pure stopping time.) The right notion of strategies turns out to be the *liquidation strategies* that were introduced by [4] (see also [5]) in the context of subhedging American options. The stopping right is taken to be divisible, or rather as a finite resource/fuel that the agent can consume continuously. The differences between randomized and liquidation strategies are highlighted in Remark 2.2.11.

Instead of the mean-variance benchmark, we propose using the mean-standard deviation criterion. We think it is more meaningful in the context of consistent planning. One reason is that the mean and standard deviation are associated with the same units. Moreover, the scaling property of this new objective function plays along well with liquidation strategies — no matter what the history liquidation strategy is, we will face the same problem as soon as we are in the same state, because we can factor out the proportion of the stopping right remaining in the objective function. (For comparison, we also define equilibrium liquidation strategies for the mean-variance problem in a similar way. The consequent results reinforce our concerns about its properness in terms of consistent planning.)

Unlike examples in [50], the method of backward construction for equilibria fails in our setup because the time horizon is infinite. The fixed point approach used in [30] (see e.g., (2.5) in [30]) does not work for our problem either, due to the non-linearity of the criterion. Instead, we provide a characterization of equilibria, from which we are able to calculate explicitly all the equilibria for many examples in this chapter.

The rest of the chapter is organized as follows. In Section 2.2 we introduce the mean-standard deviation problem. We first analyze the equilibrium stopping time, and provide an example to show such equilibrium may not exist. Then we introduce the concepts of randomized stopping strategies and liquidation strategies and analyze the equilibria in these classes. In Section 2.3 we consider the similar concepts for mean-variance problems. In Section 2.4 we compare equilibrium liquidation strategies with statically optimal ones. Some computational details can be found in Appendix A.

## 2.2 Mean-standard deviation problem

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that supports a time-homogeneous discrete-time Markov chain $X = (X_n)_{n\in\mathbb{N}}$, taking values in a finite state space $\mathbb{X} \subset \mathbb{R}$. For each $x \in \mathbb{X}$, if $X_0 = x$, we will write $X$ as $X^x$. The probability, expectation, and variance associated with $X^x$ will be denoted by $\mathbb{P}_x[\cdot]$, $\mathbb{E}_x[\cdot]$ and $\mathrm{Var}_x[\cdot]$, respectively. We assume that the limit $X_\infty := \lim_{n\to\infty} X_n$ exists almost surely[1].

### 2.2.1 Equilibrium stopping times

For any $x \in \mathbb{X}$ and $\tau \in \mathcal{T}$ (where $\mathcal{T}$ is the set of stopping times w.r.t. the filtration generated by the Markov chain), consider the following objective function

[1] This implies there is at least one absorbing state.

in mean-standard deviation problem

$$K_p(x,\tau) := \mathbb{E}_x[X_\tau] - c(\mathrm{Var}_x[X_\tau])^{1/2}, \tag{2.2}$$

where $c > 0$ is a constant and the subscript "$p$" in $K_p(x,\tau)$ stands for "pure Markov stopping time".

As we have discussed in the introduction part, this mean-standard deviation problem is time-inconsistent due to the non-linear term $(\mathrm{Var}_x[X_\tau])^{1/2}$. We treat it as an intra-personal game regarding when to stop the process $X$ between current and future selves whose preferences, identified with the objective function $K_p$, change as the initial state $x$ changes. A reasonable equilibrium strategy should be such that once the agent chooses to follow the equilibrium strategy he will never regret no matter which state he comes into. Furthermore, we only consider the pure Markov stopping times commonly used in game theory; see e.g. [17] and [47].

**Definition 2.2.1.** A stopping time $\tau$ is said to be a pure Markov stopping time, or pure stopping time for short, if $\tau = \inf\{t \geq 0 : X_t \in S\}$ for some measurable set $S \subset \mathbb{X}$ and $S$ is called the stopping region.

*Remark* 2.2.2. Obviously for any stopping region $S$, the value will not change if we add or remove an absorbing state from $S$.[2]Therefore, without loss of generality we may assume a stopping region always contains all the absorbing states. The similar argument applies to the cases when we discuss randomized stopping and liquidation strategies later on.

A pure stopping time governs when the agent should stop. The decision whether to stop or not depends directly on the current state $x$ and not on the past path of process $X$. A corresponding subgame perfect Nash equilibrium based on pure Markov

[2]Consider any fixed path of $(X_n)_{n>0}$, and two stopping regions $S$ which does not contain any absorbing state and $S \cup a$ where $a$ is an absorbing state). If $X$ jumps to $S$ first, then $X_{\tau_S} = X_{\tau_{S\cup a}}$ in this situation. If $X$ jumps to $a$ first, then $\tau_S = \infty$, thus $X_{\tau_S} = X_{\tau_{S\cup a}} = a$ in this situation.

stopping time is defined as the following.

**Definition 2.2.3.** A pure Markov stopping time $\tau$ with stopping region $S$ is said to be an equilibrium stopping time for (2.2) if

$$x \geq K_p(x, \rho(x,S)),\ \forall x \in S \qquad \text{and} \qquad x \leq K_p(x, \rho(x,S)),\ \forall x \notin S, \tag{2.3}$$

where $\rho(x,S) := \inf\{n \geq 1 : X_n^x \in S\} \in \mathcal{T}$.

The next result shows that an equilibrium stopping time may not exist.

**Proposition 2.2.4.** *An equilibrium stopping time does not always exist.*

*Proof.* We will prove this by giving a counterexample. Let $c = 1$. $X$ has state space $\mathbb{X} = \{0, 1, 3, 6, 10\}$ and the following transition matrix.

| | **0** | **1** | **3** | **6** | **10** |
|---|---|---|---|---|---|
| **0** | 1 | 0 | 0 | 0 | 0 |
| **1** | 0.2 | 0 | 0.4 | 0.2 | 0.2 |
| **3** | 0 | 0 | 1 | 0 | 0 |
| **6** | 0 | 0.2 | 0 | 0 | 0.8 |
| **10** | 0 | 0 | 0 | 0 | 1 |

For this Markov chain, $\{0, 3, 10\}$ are absorbing states and $\{1, 6\}$ are transient. Suppose there exists an equilibrium stopping time with stopping region $S \subset \{0, 1, 3, 6, 10\}$ and consider the following four cases.

**Case 1:** $S = \{0, 1, 3, 6, 10\}$. We have that

$$\mathbb{P}_1(X_{\rho(1,S)} = 0) = \mathbb{P}_1(X_{\rho(1,S)} = 6) = \mathbb{P}_1(X_{\rho(1,S)} = 10) = 0.2, \quad \mathbb{P}_1(X_{\rho(1,S)} = 3) = 0.4,$$

then

$$\mathbb{E}_1[X_{\rho(1,S)}] = 4.4,\ \mathbb{E}_1[X^2_{\rho(1,S)}] = 30.8 \Rightarrow K_p(1, \rho(1, S)) = 1.0177 > 1,$$

which yields a contradiction.

**Case 2:** $S = \{0, 3, 6, 10\}$. We have that

$$\begin{aligned}
&\mathbb{P}_6(X_{\rho(6,S)} = 0) = \mathbb{P}_6(X_{\rho(6,S)} = 6) = 0.04,\\
&\mathbb{P}_6(X_{\rho(6,S)} = 3) = 0.08,\\
&\mathbb{P}_6(X_{\rho(6,S)} = 10) = 0.84,
\end{aligned}$$

then

$$\mathbb{E}_6[X_{\rho(6,S)}] = 8.88,\ \mathbb{E}_6[X^2_{\rho(6,S)}] = 86.16 \Rightarrow K_p(6, \rho(6, S)) = 6.1771 > 6,$$

which yields a contradiction.

**Case 3:** $S = \{0, 1, 3, 10\}$. We have that

$$\mathbb{P}_6(X_{\rho(6,S)} = 1) = 0.2, \quad \mathbb{P}_6(X_{\rho(6,S)} = 10) = 0.8,$$

then

$$\mathbb{E}_6[X_{\rho(6,S)}] = 8.2,\ \mathbb{E}_6[X^2_{\rho(6,S)}] = 80.2 \Rightarrow K_p(6, \rho(6, S)) = 4.6 < 6,$$

which yields a contradiction.

**Case 4:** $S = \{0, 3, 10\}$. We have that

$$\mathbb{P}_1(X_{\rho(1,S)} = 0) = \frac{5}{24}, \quad \mathbb{P}_1(X_{\rho(1,S)} = 3) = \frac{5}{12}, \quad \mathbb{P}_1(X_{\rho(1,S)} = 10) = \frac{3}{8},$$

then

$$\mathbb{E}_1[X_{\rho(1,S)}] = 5,\ \mathbb{E}_1[X^2_{\rho(1,S)}] = \frac{165}{4} \Rightarrow K_p(1, \rho(1,S)) = 0.9689 < 1,$$

which yields a contradiction. □

### 2.2.2 Randomized stopping times

In the last section, we observed that there is no guarantee that an equilibrium stopping time exists. Therefore we will now seek an equilibrium among randomized stopping times.

Let us first briefly recall some facts of randomized stopping times, and we refer to [20] for more details. A randomized stopping time (w.r.t. the original space $(\Omega, \mathcal{F})$) is defined to be a stopping time w.r.t. the extended space $(\Omega \times [0,1], \mathcal{F} \otimes \mathcal{B}([0,1]))$. For a randomized stopping time $\gamma : \ \Omega \times [0,1] \mapsto \mathbb{N}$, its $\omega$-distribution is defined by

$$M_k(\omega) := \text{Leb}\{v : \ \gamma(\omega, v) \leq k\}, \quad k \in \mathbb{N},\ \omega \in \Omega,$$

where Leb is Lebesgue measure. Intuitively, $M_k(\omega)$ represents the probability that the underlying process has stopped by time $k$ along the path $\omega$. Moreover, there is a one-to-one correspondence (up to a rearrangement) between $\gamma$ and $M$; see [20]. For the Markov chain $X$ and any randomized stopping time $\gamma$ with the $\omega$-distribution $M$, denote

$$\mathbb{E}[X_\gamma] = \mathbb{E}\left[M_0 X_0 + \sum_{k=1}^{\infty} X_k (M_k - M_{k-1}) + (1 - M_\infty) X_\infty\right],$$

and

$$\text{Var}[X_\gamma] = \mathbb{E}[X_\gamma^2] - (\mathbb{E}[X_\gamma])^2,$$

$M_\infty := \lim_{n\to\infty} M_n$.

**Definition 2.2.5.** We say $\gamma$ is a time-homogeneous randomized stopping time, if there exists $\mathbf{p} : \mathbb{X} \to [0,1]$, such that the $\omega$-distribution of $\gamma$ satisfies

$$M_k(\cdot) = 1 - \prod_{n=0}^{k}(1 - \mathbf{p}(X_k(\cdot))).$$

Here $\mathbf{p}(x)$ represents the probability to stop at state $x$, given the underlying process has not stopped yet. We call $\mathbf{p} : \mathbb{X} \to [0,1]$ a randomized stopping strategy, and denote the set of all of them by $\mathcal{P}$.

Intuitively, given a function $\mathbf{p} : \mathbb{X} \to [0,1]$, we can design $n$ biased coins, where $n$ is the number of states in $\mathbb{X}$. When we are at state $X_k = x$, we will flip the coin with probability $\mathbf{p}(x)$ it comes up heads. If it comes up heads, we will stop. Otherwise, we will continue. In general, we can design more complicated strategies about flipping coins, which will fit in with general randomized stopping times, not just time-homogeneous randomized stopping times.

For any $\mathbf{p}, \mathbf{q} \in \mathcal{P}$, denote $\gamma^{\mathbf{q}\otimes\mathbf{p}}$ as the randomized stopping time with the $\omega$-distribution

$$M_0 = \mathbf{q}(X_0), \quad \text{and} \quad M_k = 1 - (1 - \mathbf{q}(X_0)) \prod_{n=1}^{k}(1 - \mathbf{p}(X_k)), \quad k = 1, 2, \ldots .$$

We sometimes also write $\gamma^{\mathbf{p}}$ instead of $\gamma^{\mathbf{p}\otimes\mathbf{p}}$ for short. With a bit abuse of notation, we use $\mathbb{E}[X_{\mathbf{q}\otimes\mathbf{p}}]$ to represent $\mathbb{E}[X_{\gamma^{\mathbf{q}\otimes\mathbf{p}}}]$, and $\mathrm{Var}[X_{\mathbf{q}\otimes\mathbf{p}}]$ to represent $\mathrm{Var}[X_{\gamma^{\mathbf{q}\otimes\mathbf{p}}}]$.

In Definition 2.2.3, an equilibrium stopping time is a subgame perfect Nash equilibrium in which all players use pure Markov stopping times. Now we propose to consider an equilibrium randomized stopping strategy which is a subgame perfect Nash equilibrium in the game where all players use time-homogeneous randomized stopping times and their preferences are identified with the following objective function

$$K_r(x, \mathbf{p}) := \mathbb{E}_x[X_{\mathbf{p}}] - c(\mathrm{Var}_x[X_{\mathbf{p}}])^{1/2}, \tag{2.4}$$

where $\mathbf{p}$ is a randomized stopping strategy and the subscript "$r$" in $K_r$ stands for "randomized stopping strategy".

**Definition 2.2.6.** $\mathbf{p} \in \mathcal{P}$ is said to be an equilibrium randomized stopping strategy for (2.4), if for any mapping $\mathbf{q} : \mathbb{X} \to [0, 1]$,

$$K_r(x, \mathbf{q} \otimes \mathbf{p}) \leq K_r(x, \mathbf{p} \otimes \mathbf{p}), \quad \forall x \in \mathbb{X}, \tag{2.5}$$

where $K_r(x, \mathbf{q} \otimes \mathbf{p})$ is from (2.4) by replacing $\mathbf{p}$ with $\mathbf{p} \otimes \mathbf{q}$.

The randomized strategy is also called "mixed strategy" in game theory, i.e., an assignment of a probability to each pure strategy. In our context, we assign probability $\mathbf{p}(x)$ to the pure strategy "to stop at state $x$" and probability $1 - \mathbf{p}(x)$ to the pure strategy "not to stop at state $x$".

If the randomized stopping strategy $\mathbf{p} \in \mathcal{P}$ satisfies that $\mathbf{p}(x) \in \{0, 1\}$ for any $x \in \mathbb{X}$, then it is actually a pure strategy, which is to stop at state $x$ if $\mathbf{p}(x) = 1$ and not to stop at state $x$ if $\mathbf{p}(x) = 0$. In this case, it simply gives us a pure stopping time with stopping region $\{x \in \mathbb{X} : \ \mathbf{p}(x) = 1\}$. We have the following result, which together with Proposition 2.2.4 implies that an equilibrium randomized stopping strategy does not always exist.

**Proposition 2.2.7.** *If $\mathbf{p} \in \mathcal{P}$ is an equilibrium randomized stopping strategy for (2.4), then $\mathbf{p}(x) = 0$ or $1$ for any (transient state) $x \in \mathbb{X}$. Conversely, if there is an equilibrium stopping time with stopping region $S$, then $\mathbf{p} \in \mathcal{P}$ defined by*

$$\mathbf{p}(x) = \begin{cases} 1, & x \in S, \\ 0, & x \notin S, \end{cases} \tag{2.6}$$

*is an equilibrium randomized stopping strategy. Consequently, an equilibrium randomized stopping strategy does not always exist.*

For the proof of this proposition we will need the following result:

**Lemma 2.2.8.** *Let* $\gamma_1, \gamma_2, \gamma$ *be randomized stopping times and* $\lambda \in (0,1)$*, such that*

$$\mathbb{P}(\gamma = \gamma_1) = 1 - \mathbb{P}(\gamma = \gamma_2) = \lambda.$$

*(Denote* $\gamma = \lambda\gamma_1 \oplus (1-\lambda)\gamma_2$*.) Then*

$$\begin{aligned}\mathrm{Var}[X_{\lambda\gamma_1 \oplus (1-\lambda)\gamma_2}] \geq & \lambda \mathrm{Var}[X_{\gamma_1}] + (1-\lambda)\mathrm{Var}[X_{\gamma_2}] \\ & \geq \left(\lambda(\mathrm{Var}[X_{\gamma_1}])^{1/2} + (1-\lambda)(\mathrm{Var}[X_{\gamma_2}])^{1/2}\right)^2.\end{aligned}$$

*Moreover, the first equality holds if and only if* $\mathbb{E}[X_{\gamma_1}] = \mathbb{E}[X_{\gamma_2}]$*, and the second equality holds if and only if* $\mathrm{Var}[X_{\gamma_1}] = \mathrm{Var}[X_{\gamma_2}]$.

*Proof.* We have that

$$\begin{aligned}\mathrm{Var}[X_{\lambda\gamma_1 \oplus (1-\lambda)\gamma_2}] &= \mathbb{E}[X^2_{\lambda\gamma_1 \oplus (1-\lambda)\gamma_2}] - (\mathbb{E}[X_{\lambda\gamma_1 \oplus (1-\lambda)\gamma_2}])^2 \\ &= \lambda\mathbb{E}[X^2_{\gamma_1}] + (1-\lambda)\mathbb{E}[X^2_{\gamma_2}] - (\lambda\mathbb{E}[X_{\gamma_1}] + (1-\lambda)\mathbb{E}[X_{\gamma_2}])^2 \\ &\geq \lambda\mathbb{E}[X^2_{\gamma_1}] + (1-\lambda)\mathbb{E}[X^2_{\gamma_2}] - \left(\lambda(\mathbb{E}[X_{\gamma_1}])^2 + (1-\lambda)(\mathbb{E}[X_{\gamma_2}])^2\right) \\ &= \lambda\mathrm{Var}[X_{\gamma_1}] + (1-\lambda)\mathrm{Var}[X_{\gamma_2}].\end{aligned}$$

We obtain the inequality using Jensen's inequality. The rest of the result is easy to check. □

*Proof of Proposition 2.2.7.* Let $\mathbf{p} \in \mathcal{P}$ be an equilibrium randomized stopping strategy for (2.4). Suppose there exists a transient state $x \in \mathbb{X}$ such that $0 < \lambda := \mathbf{p}(x) < 1$. Denote

$$\alpha := \mathbf{1} \otimes \mathbf{p} \quad \text{and} \quad \beta := \mathbf{0} \otimes \mathbf{p}, \tag{2.7}$$

where $\mathbf{1} \in \mathcal{P}$ (resp. $\mathbf{0} \in \mathcal{P}$) is the strategy with all components 1 (resp. 0). We have the following.

$$
\begin{aligned}
K_r(x, \mathbf{p} \otimes \mathbf{p}) &= K_r(x, \lambda\alpha \oplus (1-\lambda)\beta) \\
&= \mathbb{E}\left[X_{\lambda\alpha\oplus(1-\lambda)\beta}\right] - c\left(\mathrm{Var}[X_{\lambda\alpha\oplus(1-\lambda)\beta}]\right)^{1/2} \\
&\leq \lambda\mathbb{E}[X_\alpha] + (1-\lambda)\mathbb{E}[X_\beta] - c\left(\lambda\left(\mathrm{Var}[X_\alpha]\right)^{1/2} + (1-\lambda)\left(\mathrm{Var}[X_\beta]\right)^{1/2}\right) \qquad (2.8) \\
&= \lambda K_r(x, \alpha) + (1-\lambda)K_r(x, \beta) \\
&\leq K_r(x, \mathbf{p} \otimes \mathbf{p}), \qquad (2.9)
\end{aligned}
$$

where (3.9) follows from Lemma 2.2.8 and (2.9) follows from (3.3). This implies that equality holds for (3.9). By Lemma 2.2.8

$$x = X_\alpha = X_\beta.$$

Since the state $x$ is transient, there is a positive probability that the Markov chain never returns back to $x$. As a result, it is not possible that $X_\beta = x$ with probability 1.

Conversely, assume there is an equilibrium stopping time with stopping region $S$, and define $\mathbf{p} \in \mathcal{P}$ as in (2.6). Let $\mathbf{q} \in \mathcal{P}$ and $x \in \mathbb{X}$. Denote $\lambda' := \mathbf{q}(x)$, and define $\alpha$ and $\beta$ as in (2.7). We consider two cases:

(i) $p(x) = 1$: Then

$$K_r(x, \beta) = K_p(x, \rho(x, S)) \leq x = K_r(x, \alpha).$$

Then by a similar argument as above, we have that

$$K_r(x, \mathbf{q} \otimes \mathbf{p}) \leq \lambda' K_r(x, \alpha) + (1-\lambda')K_r(x, \beta) \leq K_r(x, \alpha) = K_r(x, \mathbf{p} \otimes \mathbf{p}).$$

(ii) $p(x) = 0$: $K_r(x, \beta) = K_r(x, \mathbf{p} \otimes \mathbf{p}) \geq K_r(x, \alpha) = x$, and thus $K_r(x, \mathbf{q} \otimes \mathbf{p}) \leq K_r(x, \beta) = K_r(x, \mathbf{p} \otimes \mathbf{p})$. $\square$

### 2.2.3 Equilibrium liquidation strategies

**Definition 2.2.9.** An adapted nondecreasing process $\theta = (\theta_n)_{n \in \mathbb{N}}$ is said to be a liquidation strategy, if $\theta_0 \geq 0$, and

$$\lim_{n \to \infty} \theta_n \leq 1, \ a.s..$$

A liquidation strategy $\theta$ is said to be time homogeneous, if there exists $\eta : \mathbb{X} \mapsto [0, 1]$, such that along any path $(x_n)_{n \in \mathbb{N}} \in \mathbb{X}^\infty$,

$$\theta_n(x_0, \ldots, x_n) = 1 - \prod_{i=0}^{n} (1 - \eta(x_i)).$$

Denote by $\mathcal{L}$ the collection of all time-homogeneous liquidation strategies.

Consider the objective function

$$K_l(x, \theta) := \mathbb{E}_x[\theta(X)] - c(\mathrm{Var}_x[\theta(X)])^{1/2}, \tag{2.10}$$

where the subscript "$l$" in $K_l(x, \theta)$ stands for "liquidation strategy" and $\theta(X)$ is the payoff under liquidation strategy $\theta$

$$\theta(X) = X_0\theta_0 + \sum_{n=1}^{\infty} X_n(\theta_n - \theta_{n-1}) + X_\infty(1 - \theta_\infty).$$

If $\theta = \theta^\eta \in \mathcal{L}$ is a time-homogeneous liquidation strategy, then

$$\begin{aligned}\theta^\eta(X) = &\eta(X_0)X_0 + (1-\eta(X_0))\Big[\eta(X_1)X_1 \\ &+ \sum_{k=2}^{\infty}(1-\eta(X_1))\cdots(1-\eta(X_{k-1}))\eta(X_k)X_k + \prod_{k=1}^{\infty}(1-\eta(X_k))X_\infty\Big].\end{aligned} \tag{2.11}$$

Intuitively liquidation strategy means to liquidate the asset at several periods instead of at one time. Such strategy is very common in practice. For instance, when an investor has a large amount of identical asset, e.g., 10000 shares of American option, she may excise these shares at different times instead of once. In the following, we use an example to illustrate the motivation to consider liquidation strategies. In particular, we will show that if $X$ is divisible, then it is possible that the optimal value for pure stopping time $\sup_\tau K_p(x,\tau)$ is strictly less than $K_l(x,\theta)$ for some liquidation strategy $\theta$ and some $x \in \mathbb{X}$.

**Example 2.2.10.** Let $c = 1/(\sqrt{44}-5)$. $X$ has the following transition matrix.

| | **0** | **1** | **2** | **3** |
|---|---|---|---|---|
| **0** | 1 | 0 | 0 | 0 |
| **1** | $\frac{1}{6}$ | 0 | $\frac{1}{2}$ | $\frac{1}{3}$ |
| **2** | $\frac{1}{5}$ | 0 | 0 | $\frac{4}{5}$ |
| **3** | 0 | 0 | 0 | 1 |

Then it is easy to see that the optimal stopping value for $K_p(x,\tau)$ is given by

$$\sup_\tau K_p(1,\tau) = K_p(1,\tau') = K_p(1,\tau'') = 2 - c = 1.3877,$$

where

$$\tau' := \inf\{n \geq 0 : \; X_n = 0, 2, 3\},$$

and

$$\tau'' := \inf\{n \geq 0 : \ X_n = 0, 3\}.$$

Now consider the liquidation strategy $\theta'$ given by

$$\theta'_0(\cdot) = 0, \quad \theta'_1(\cdot, 2) = 1/2, \quad \text{and } \theta'_n = 0 \text{ for all other cases.}$$

Then it is easy to see that

$$\theta'(X^1) = \frac{1}{2} X^1_{\tau'} + \frac{1}{2} X^1_{\tau''}.$$

Then the distribution of $\theta(X^1)$ is given by

$$\mathbb{P}(\theta'(X^1) = 0) = \frac{1}{6}, \quad \mathbb{P}(\theta'(X^1) = 1) = \frac{1}{10},$$
$$\mathbb{P}(\theta'(X^1) = 5/2) = \frac{2}{5}, \quad \mathbb{P}(\theta'(X^1) = 3) = \frac{1}{3}.$$

Therefore, we have that

$$\sup_{\theta} K_l(1, \theta) \geq K_l(1, \theta') = \frac{21}{10} - \frac{\sqrt{119}}{10} c = 1.4321 > \sup_{\tau} K_p(1, \tau).$$

*Remark* 2.2.11. As discussed in Remark 3.1 in Bayraktar and Zhou's paper [4], there is a one-to-one correspondence between the set of time-homogeneous liquidation strategies $\mathcal{L}$ and the set of time-homogeneous randomized stopping times $\mathcal{P}$. But the paths of a liquidation strategy and a randomized stopping time are quite different. First of all, in terms of behavior, when using a randomized stopping time, we flip a coin at each period to decide whether we stop or not, and we still liquidate the whole unit asset over a single period. Second, in terms of variance, randomized stopping time will result in a larger variance, since the overall variance will include the part from randomization of the stopping time, while liquidation strategy results in a smaller variance, since averaging random variable leads to a smaller variance. This can also be seen from Lemma 2.2.8 and the following result.

**Lemma 2.2.12.** *Let $\theta_1, \theta_2$ be two liquidation strategies and $\lambda \in (0,1)$. Then $\lambda\theta_1 + (1-\lambda)\theta_2$ is also a liquidation strategy and*

$$\begin{aligned}\mathrm{Var}[(\lambda\theta_1 + (1-\lambda)\theta_2)(X)] &\le \left(\lambda(\mathrm{Var}[\theta_1(X)])^{1/2} + (1-\lambda)(\mathrm{Var}[\theta_2(X)])^{1/2}\right)^2 \\ &\le \lambda\mathrm{Var}[\theta_1(X)] + (1-\lambda)\mathrm{Var}[\theta_2(X)].\end{aligned}$$

*Proof.* We have that

$$\begin{aligned}&\mathrm{Var}[(\lambda\theta_1 + (1-\lambda)\theta_2)(X)] = \mathrm{Var}[\lambda\theta_1(X) + (1-\lambda)\theta_2(X)] \\ &= \lambda^2\mathrm{Var}[\theta_1(X)] + (1-\lambda)^2\mathrm{Var}[\theta_2(X)] + 2\lambda(1-\lambda)\mathrm{Cov}[\theta_1(X), \theta_2(X)] \\ &\le \lambda^2\mathrm{Var}[\theta_1(X)] + (1-\lambda)^2\mathrm{Var}[\theta_2(X)] + 2\lambda(1-\lambda)(\mathrm{Var}[\theta_1(X)])^{1/2}(\mathrm{Var}[\theta_1(X)])^{1/2} \\ &= \left(\lambda(\mathrm{Var}[\theta_1(X)])^{1/2} + (1-\lambda)(\mathrm{Var}[\theta_2(X)])^{1/2}\right)^2.\end{aligned}$$

The second inequality is easy to check. □

Our next goal is to analyze the subgame perfect Nash equilibrium in the game where all players use time-homogeneous liquidation strategies. Notice that each time-homogeneous liquidation strategy is characterized by a function $\eta(x)$ that represents the proportion of the remaining asset we will liquidate when the Markov chain moves to position $x$. $\eta(x)$ is independent of time and the history of the paths. For simplicity of notation, we use $K_l(x, \eta)$ instead of $K_l(x, \theta^\eta)$ for $\theta = \theta^\eta \in \mathcal{L}$.

**Definition 2.2.13.** A liquidation strategy $\theta = \theta^\eta \in \mathcal{L}$ is said to be an equilibrium liquidation strategy for (2.10) if for any mapping $\xi : \mathbb{X} \to [0,1]$, we have

$$K_l(x, \xi \otimes \eta) \le K_l(x, \eta \otimes \eta), \ \ \forall x \in \mathbb{X},$$

where $\theta^{\xi\otimes\eta}$ is a perturbation of strategy $\theta^{\eta}$ in which we liquidate $\xi(\cdot)$ at time 0 and then from time 1 we liquidate $\eta(\cdot)$ proportion of the remaining asset at each period.

*Remark* 2.2.14. Notice that this definition looks similar to the definition of equilibrium randomized stopping time. But as we mentioned earlier, unlike selling the whole unit of asset in one period which is random, the liquidation strategy will leave us with different proportions of asset at different periods, so the objective function might change as time goes on. Thanks to the square root term in (2.10), mean-standard deviation problem has the scaling effect which allows Definition 2.2.13 to make perfect sense since we essentially face the same problem (with the same parameter $c$) at every period. However, we will see in Section 2.3, a similar definition of equilibrium liquidation strategy in mean-variance problem is not a proper definition.

Also note that a liquidation strategy should be considered as a pure strategy from the game theory point of view, with the added assumption that partial selling the asset over time is possible. In contrast, a randomized stopping strategy should be considered as a mixed strategy.

### 2.2.4 Existence of an equilibrium liquidation strategy

In this section we will prove that in contrast to the equilibrium stopping time for (2.2) and equilibrium randomized stopping strategy for (2.4), an equilibrium liquidation strategy for (2.10) always exists.

**Lemma 2.2.15.** *For $\eta_n, \eta \in \mathcal{L}, n \in \mathbb{N}$, if $\eta_n \to \eta$ as $n \to \infty$, then*

$$\theta^{\eta_n}(X) \to \theta^{\eta}(X), \quad a.s..$$

*Proof.* For a.e. $\omega \in \Omega$, there exists $N = N(\omega)$ such that for any $k \geq N$, $X_N(\omega) =$

$X_\infty(\omega)$. Then along $\omega$, we have that

$$\begin{aligned}
\theta^{\eta_n}(X) &= \sum_{k=0}^{\infty} \left((1-\eta_n(X_0))\dots(1-\eta_n(X_{k-1}))\right)\eta_n(X_k)X_k + \prod_{k=0}^{\infty}(1-\eta_n(X_k))X_\infty \\
&= \sum_{k=0}^{N} \left((1-\eta_n(X_0))\dots(1-\eta_n(X_{k-1}))\right)\eta_n(X_k)X_k + \prod_{k=0}^{N}(1-\eta_n(X_k))X_\infty \\
&\xrightarrow{n\to\infty} \sum_{k=0}^{N} \left((1-\eta(X_0))\dots(1-\eta(X_{k-1}))\right)\eta(X_k)X_k + \prod_{k=0}^{N}(1-\eta(X_k))X_\infty \\
&= \sum_{k=0}^{\infty} \left((1-\eta(X_0))\dots(1-\eta(X_{k-1}))\right)\eta(X_k)X_k + \prod_{k=0}^{\infty}(1-\eta(X_k))X_\infty \\
&= \theta^{\eta}(X).
\end{aligned}$$

□

**Lemma 2.2.16.** *For $\xi_n, \eta_n, \xi, \eta \in \mathcal{L}, n \in \mathbb{N}$, if $\xi_n \to \xi$ and $\eta_n \to \eta$ as $n \to \infty$, then*

$$K_l(x, \xi_n \otimes \eta_n) \to K_l(x, \xi \otimes \eta), \quad \forall\, x \in \mathbb{X}.$$

*Proof.* As

$$|K_l(x,\xi_n\otimes\eta_n) - K_l(x,\xi\otimes\eta)| \leq |K_l(x,\xi_n\otimes\eta_n) - K_l(x,\xi\otimes\eta_n)| + |K_l(x,\xi\otimes\eta_n) - K_l(x,\xi\otimes\eta)|,$$

it suffices to show that

$$\left|\mathbb{E}_x\left[\theta^{\xi_n\otimes\eta_n}(X)\right] - \mathbb{E}_x\left[\theta^{\xi\otimes\eta_n}(X)\right]\right| \to 0, \quad n\to\infty; \tag{2.12}$$

$$\left|\mathbb{E}_x\left[\left(\theta^{\xi_n\otimes\eta_n}(X)\right)^2\right] - \mathbb{E}_x\left[\left(\theta^{\xi\otimes\eta_n}(X)\right)^2\right]\right| \to 0, \quad n\to\infty; \tag{2.13}$$

$$\left|\mathbb{E}_x\left[\theta^{\xi\otimes\eta_n}(X)\right] - \mathbb{E}_x\left[\theta^{\xi\otimes\eta}(X)\right]\right| \to 0, \quad n\to\infty; \tag{2.14}$$

$$\left|\mathbb{E}_x\left[\left(\theta^{\xi\otimes\eta_n}(X)\right)^2\right] - \mathbb{E}_x\left[\left(\theta^{\xi\otimes\eta}(X)\right)^2\right]\right| \to 0, \quad n\to\infty. \tag{2.15}$$

We have that

$$
\begin{aligned}
&\left|\mathbb{E}_x\left[\theta^{\xi_n\otimes\eta_n}(X)\right]-\mathbb{E}_x\left[\theta^{\xi\otimes\eta_n}(X)\right]\right| \\
=&\Bigg|\left(\xi_n(x)x+(1-\xi_n(x))\sum_{y\in\mathbb{X}}p(x,y)\mathbb{E}_y\left[\theta^{\eta_n}(X)\right]\right) \\
&-\left(\xi(x)x+(1-\xi(x))\sum_{y\in\mathbb{X}}p(x,y)\mathbb{E}_y\left[\theta^{\eta_n}(X)\right]\right)\Bigg| \\
\le&|x|\cdot|\xi_n(x)-\xi(x)|+\left|\sum_{y\in\mathbb{X}}p(x,y)\mathbb{E}_y\left[\theta^{\eta_n}(X)\right]\right|\cdot|\xi_n(x)-\xi(x)| \\
\le&(\alpha+|x|)|\xi_n(x)-\xi(x)|\to 0,\text{as } n\to\infty,
\end{aligned}
$$

where $\alpha:=\sup\{|y|:\ y\in\mathbb{X}\}$. Hence, we have (3.12) holds.

Noticing that

$$
\theta^{\xi_n\otimes\eta_n}(X)=\xi_n(x)x+(1-\xi_n(x))\theta^{\eta_n}(X_{\cdot+1}),
$$

we have that

$$
\begin{aligned}
&\left|\mathbb{E}_x\left[\left(\theta^{\xi_n\otimes\eta_n}(X)\right)^2\right]-\mathbb{E}_x\left[\left(\theta^{\xi\otimes\eta_n}(X)\right)^2\right]\right| \\
=&\Big|\mathbb{E}_x\left[(\xi_n(x))^2x^2+2\xi_n(x)x(1-\xi_n(x))\theta^{\eta_n}(X_{\cdot+1})+(1-\xi_n(x))^2(\theta^{\eta_n}(X_{\cdot+1}))^2\right] \\
&-\mathbb{E}_x\left[(\xi(x))^2x^2+2\xi(x)x(1-\xi(x))\theta^{\eta_n}(X_{\cdot+1})+(1-\xi(x))^2(\theta^{\eta_n}(X_{\cdot+1}))^2\right]\Big| \\
\le&\left|(\xi_n(x))^2x^2-(\xi(x))^2x^2\right|+2|\xi_n(x)x(1-\xi_n(x))-\xi(x)x(1-\xi(x))|\cdot|\mathbb{E}_x\left[\theta^{\eta_n}(X_{\cdot+1})\right]| \\
&+\left|(1-\xi_n(x))^2-(1-\xi(x))^2\right|\cdot\left|\mathbb{E}_x\left[(\theta^{\eta_n}(X_{\cdot+1}))^2\right]\right| \\
\le&\left|(\xi_n(x))^2x^2-(\xi(x))^2x^2\right|+2|\xi_n(x)x(1-\xi_n(x))-\xi(x)x(1-\xi(x))|\cdot\alpha \\
&+\left|(1-\xi_n(x))^2-(1-\xi(x))^2\right|\cdot\alpha^2\to 0,\text{as } n\to\infty,
\end{aligned}
$$

and thus (3.11) follows.

Moreover,

$$
\begin{aligned}
&\left|\mathbb{E}_x\left[\theta^{\xi\otimes\eta_n}(X)\right]-\mathbb{E}_x\left[\theta^{\xi\otimes\eta}(X)\right]\right|\\
=&\Bigg|\left(\xi(x)x+(1-\xi(x))\sum_{y\in\mathbb{X}}p(x,y)\mathbb{E}_y\left[\theta^{\eta_n}(X)\right]\right)\\
&-\left(\xi(x)x+(1-\xi(x))\sum_{y\in\mathbb{X}}p(x,y)\mathbb{E}_y\left[\theta^{\eta}(X)\right]\right)\Bigg|\\
\leq&(1-\xi(x))\sum_{y\in\mathbb{X}}p(x,y)\left|\mathbb{E}_y\left[\theta^{\eta_n}(X)\right]-\mathbb{E}_y\left[\theta^{\eta}(X)\right]\right|\to 0,\text{as } n\to\infty,
\end{aligned}
$$

where the last line follows from Lemma 2.2.15, and thus (3.14) holds.

Finally,

$$
\begin{aligned}
&\left|\mathbb{E}_x\left[\left(\theta^{\xi\otimes\eta_n}(X)\right)^2\right]-\mathbb{E}_x\left[\left(\theta^{\xi\otimes\eta}(X)\right)^2\right]\right|\\
=&\Big|\mathbb{E}_x\left[(\xi(x))^2x^2+2\xi(x)x(1-\xi(x))\theta^{\eta_n}(X_{\cdot+1})+(1-\xi(x))^2(\theta^{\eta_n}(X_{\cdot+1}))^2\right]\\
&-\mathbb{E}_x\left[(\xi(x))^2x^2+2\xi(x)x(1-\xi(x))\theta^{\eta}(X_{\cdot+1})+(1-\xi(x))^2(\theta^{\eta}(X_{\cdot+1}))^2\right]\Big|\\
\leq&\ 2\xi(x)|x|(1-\xi(x))\left|\mathbb{E}_x\left[\theta^{\eta_n}(X_{\cdot+1})\right]-\mathbb{E}_x\left[\theta^{\eta}(X_{\cdot+1})\right]\right|\\
&+(1-\xi(x))^2\left|\mathbb{E}_x\left[(\theta^{\eta_n}(X_{\cdot+1}))^2\right]-\mathbb{E}_x\left[(\theta^{\eta}(X_{\cdot+1}))^2\right]\right|\to 0,\text{as } n\to\infty,
\end{aligned}
$$

which implies (3.15). □

**Theorem 2.2.17.** *There exists an equilibrium liquidation strategy for the mean-standard deviation problem (2.10).*

*Proof.* For $\eta\in\mathcal{L}$, define the set valued map

$$\Phi(\eta):=\{\xi^*\in\mathcal{L}:\ K_l(x,\xi^*\otimes\eta)\geq K_l(x,\xi\otimes\eta),\ \forall\, x\in\mathbb{X},\forall\,\xi\in\mathcal{L}\}.$$

First, we have that for any $\eta\in\mathcal{L}$, $\Phi(\eta)$ is not empty. Indeed, since $K_l(x,\xi\otimes\eta)$ depends on $\xi$ only through $\xi(x)$, we can choose $\xi^*(x)$ to be a maximizer for each $x$

fixed.

For $\xi_1, \xi_2, \eta \in \mathcal{L}$ and $\lambda \in (0,1)$, we have that $\theta^{\eta \otimes \eta} = \theta^{\eta}$ and

$$\theta^{(\lambda\xi_1 + (1-\lambda)\xi_2)\otimes\eta} = \lambda\theta^{\xi_1\otimes\eta} + (1-\lambda)\theta^{\xi_2\otimes\eta}.$$

Moreover, thanks to Lemma 2.2.12, we obtain that $\Phi(\eta)$ is a convex set for any $\eta \in \mathcal{L}$. In addition, by Lemma 2.2.16, the map $\Phi$ is u.s.c.. That is, for $\eta_n, \xi_n^*, \eta, \xi^* \in \mathcal{L}$ with $\eta_n \to \eta$ and $\xi_n^* \to \xi^*$, if $\xi_n^* \in \Phi(\eta_n)$, then $\xi^* \in \Phi(\eta)$.

Applying [22, Theorem 1], we obtain the desired result. □

*Remark* 2.2.18. As we can see in this proof, the assumption that $X$ have finite state space and the limit $X_\infty$ exists a.s. is necessary to obtain some key estimations which are hard to achieve when the state space is infinite. Despite this, the concepts introduced in this chapter do not rely on this assumption and a future work can focus on extending certain results to the case with infinite state space.

### 2.2.5 Optimal equilibrium liquidation strategies

According to consistent planning in Strotz [56], finding equilibria is only the first step and the agent should choose the best one among all equilibria. We then formulate the definition of optimal equilibrium liquidation strategy as the following.

**Definition 2.2.19.** Let $\mathcal{E} \subset \mathcal{L}$ be the collection of equilibrium liquidation strategies. We say an equilibrium liquidation strategy $\eta^* \in \mathcal{E}$ is optimal if

$$K_l(x, \eta^*) \geq K_l(x, \eta), \quad \forall\, x \in \mathbb{X}, \forall\, \eta \in \mathcal{E}.$$

To find an optimal equilibrium liquidation strategy, we need to study the set $\mathcal{E}$. We will provide a characterization of equilibrium liquidation strategies in the following proposition.

**Proposition 2.2.20.** *$\eta$ is an equilibrium liquidation strategy if and only if the following holds:*

*(i) $\eta(x) = 0$ for all $x \in \mathbb{X}$ such that $\mathbb{E}_x[Y_\eta] - c(\mathrm{Var}_x[Y_\eta])^{1/2} > x$, and*

*(ii) $\eta(x) = 1$ for all $x \in \mathbb{X}$ such that $\mathbb{E}_x[Y_\eta] - c(\mathrm{Var}_x[Y_\eta])^{1/2} < x$,*

*where $Y_\eta = \theta^\eta(X_{\cdot+1}) := \eta(X_1)X_1 + \sum_{k=2}^\infty(1-\eta(X_1))\cdots(1-\eta(X_{k-1}))\eta(X_k)X_k + \prod_{k=1}^\infty(1-\eta(X_k))X_\infty$.*

*Remark* 2.2.21. The term $\mathbb{E}_x[Y_\eta] - c(\mathrm{Var}_x[Y_\eta])^{1/2}$ is interpreted as the continuation value, i.e., the value we expect to get if we choose not to stop at the current state $x$. Then (i) and (ii) tell us whether we should liquidate the whole unit or not liquidate at all by comparing the current value $x$ and the continuation value. This is in fact in the same vein as (2.3).

*Proof of Proposition 2.2.20.* By (2.11), we have

$$\begin{aligned}\theta^\eta(X) =& \eta(X_0)X_0 + (1-\eta(X_0))Y_\eta,\\ \theta^{\xi\otimes\eta}(X) =& \xi(X_0)X_0 + (1-\xi(X_0))Y_\eta,\end{aligned}$$

and

$$\begin{aligned}K_l(x,\xi\otimes\eta) &= x\xi(x) + (1-\xi(x))\mathbb{E}_x[Y_\eta] - c(1-\xi(x))(\mathrm{Var}_x[Y_\eta])^{1/2}\\ &= (x - (\mathbb{E}_x[Y_\eta] - c(\mathrm{Var}_x[Y_\eta])^{1/2}))\xi(x) + \mathbb{E}_x[Y_\eta] - c(\mathrm{Var}_x[Y_\eta])^{1/2}, \quad (2.16)\end{aligned}$$

which implies that when $\eta$ is fixed, $K_l(x,\xi\otimes\eta)$ is a linear function of $\xi(x)$.

If $\eta$ is an equilibrium liquidation strategy, then according to Definition 2.2.13,

$$K_l(x,\eta) = \sup_{\xi\in\mathcal{L}} K_l(x,\xi\otimes\eta).$$

Therefore, if $x - (\mathbb{E}_x[Y_\eta] - c(\mathrm{Var}_x[Y_\eta])^{1/2}) > 0$, then $\eta(x) = 1$. If $x - (\mathbb{E}_x[Y_\eta] -$

$c(\mathrm{Var}_x[Y_\eta])^{1/2}) < 0$, then $\eta(x) = 0$. $\eta(x) \in (0,1)$ only when $\eta(x)$ is a solution to the equation $x - (\mathbb{E}_x[Y_\eta] - c(\mathrm{Var}_x[Y_\eta])^{1/2}) = 0$. □

**Corollary 2.2.22.** *If there exists an equilibrium stopping time $\tau$ with stopping region $S$, then $\eta(x) = \mathbf{1}_S(x), x \in \mathbb{X}$ is an equilibrium liquidation strategy.*

By Proposition 2.2.20, we can find an equilibrium liquidation strategy by solving a system of equations $\mathbb{E}_x[Y_\eta] - c(\mathrm{Var}_x[Y_\eta])^{1/2} = x, \forall x \in \mathbb{X}$. The solution $\{\eta(x) \in [0,1] : x \in \mathbb{X}\}$ must be an equilibrium liquidation strategy if it exists. Other candidates of equilibrium liquidation strategies can be found by checking conditions (i) and (ii) in Proposition 2.2.20 when $\mathbb{E}_x[Y_\eta] - c(\mathrm{Var}_x[Y_\eta])^{1/2} = x$ does not hold for some $x \in \mathbb{X}$. Here are some examples of mean-standard deviation problems with different sets of equilibrium liquidation strategies $\mathcal{E}$.

**Example 2.2.23.** In this example, a unique equilibrium liquidation strategy exists, which is also an equilibrium stopping time.

Let $c = 1/4$. $X$ has state space $\mathbb{X} = \{0, 1, 2\}$ and the following transition matrix.

| | **0** | **1** | **2** |
|---|---|---|---|
| **0** | 1 | 0 | 0 |
| **1** | 0.2 | 0.4 | 0.4 |
| **2** | 0 | 0 | 1 |

Note that $\{0, 2\}$ are absorbing states. For any pure stopping time, its stopping region must contain the absorbing states $\{0, 2\}$. Likewise, any liquidation strategy must satisfy $\eta(0) = \eta(2) = 1$.

One can easily check that the pure stopping time with stopping region $S = \{0, 2\}$ is an equilibrium stopping time with

$$\mathbb{E}_1[X_{\rho(1,S)}] - c(\mathrm{Var}_x[X_{\rho(1,S)}])^{1/2} = \frac{4}{3} - \frac{\sqrt{2}}{6} > 1.$$

Moreover, this is the only equilibrium stopping time. If $S = \{0, 1, 2\}$, then

$$\mathbb{E}_1[X_{\rho(1,S)}] - c(\mathrm{Var}_x[X_{\rho(1,S)}])^{1/2} = \frac{6}{5} - \frac{\sqrt{14}}{20} > 1,$$

which means the pure stopping time with stopping region $\{0, 1, 2\}$ is not an equilibrium stopping time.

Now we want to find all equilibrium liquidation strategies for this problem. The only parameter remains to be determined is $a := \eta(1)$.

By analyzing the behavior of this Markov chain we have

$$\begin{aligned}
&\mathbb{P}_1(X_1 = 0 = X_n, n \geq 1) = 0.2, \\
&\mathbb{P}_1(X_1 = 2 = X_n, n \geq 1) = 0.4, \\
&\mathbb{P}_1(X_1 = 1, X_2 = 0 = X_n, n \geq 2) = 0.4 \cdot 0.2, \\
&\mathbb{P}_1(X_1 = 1, X_2 = 2 = X_n, n \geq 2) = 0.4 \cdot 0.4, \\
&\cdots \\
&\mathbb{P}_1(X_k = 1, k = 1, 2, \cdots, m, X_n = 0, n > m) = 0.4^m \cdot 0.2, \\
&\mathbb{P}_1(X_k = 1, k = 1, 2, \cdots, m, X_n = 2, n > m) = 0.4^m \cdot 0.4,
\end{aligned}$$

and

$$X_k = \begin{cases} 1, & k = 1, 2, \cdots, m, \\ 0, & k \geq m + 1, \end{cases} \qquad \Rightarrow Y_\eta = \eta(1)(\sum_{i=0}^{m-1} (1 - \eta(1))^i),$$

$$X_k = \begin{cases} 1, & k = 1, 2, \cdots, m, \\ 2, & n \geq m + 1, \end{cases} \qquad \Rightarrow Y_\eta = \eta(1)(\sum_{i=0}^{m-1} (1 - \eta(1))^i) + 2(1 - \eta(1))^m.$$

In conclusion, the random variable $Y_\eta$ has the following distribution

$$\mathbb{P}_1(Y_\eta = 1 - (1-\eta(1))^n) = 0.4^n \cdot 0.2, \quad n = 0, 1, 2, \cdots ,$$
$$\mathbb{P}_1(Y_\eta = 1 + (1-\eta(1))^n) = 0.4^n \cdot 0.4, \quad n = 0, 1, 2, \cdots .$$

It can be derived that

$$\mathbb{E}_1[Y_\eta] = \frac{0.8+0.4a}{0.6+0.4a},$$
$$\mathrm{Var}_1[Y_\eta] = \frac{0.112a^2 + 0.256a + 0.192}{(1-0.4(1-a)^2)(0.6+0.4a)^2}.$$

Let $h(a) := \mathbb{E}_1[Y_\eta] - \frac{1}{4}(\mathrm{Var}_1[Y_\eta])^{1/2}$. By computation we have $h(a) > 1$ for all $a \in [0,1]$. So there is only one equilibrium liquidation strategy, $\eta(1) = 0$, which also coincides with the unique equilibrium stopping time mentioned above.

**Example 2.2.24.** In this example, a unique equilibrium liquidation strategy exists, which is not a pure stopping time.

Consider the example in the proof of Proposition 2.2.4. Since $\{0, 3, 10\}$ are absorbing states, we have $\eta(0) = \eta(3) = \eta(10) = 1$. The only parameters remain to be determined are $a := \eta(1)$ and $b := \eta(6)$. By the proof of Proposition 2.2.4. we know that there is no equilibrium stopping time in this example. However we will see that it does have an equilibrium liquidation strategy.

Let $g_i(a,b) := \mathbb{E}_i[Y_\eta] - (\mathbb{E}_i[Y_\eta^2] - \mathbb{E}_i[Y_\eta]^2)^{1/2}$ for $i = 1, 6$. They have explicit expressions as shown in Appendix A.1. We obtain the graph of sets $\{(a,b) \in [0,1] \times [0,1] : g_1(a,b) = 1\}$ and $\{(a,b) \in [0,1] \times [0,1] : g_6(a,b) = 6\}$ as following.

From Figure 2.1 we observe that there exists a unique intersection of the curve $g_1(a,b) = 1$ and $g_6(a,b) = 6$, denoted by $(a_0, b_0)$. Then $\eta(1) = a_0, \eta(6) = b_0$ is an equilibrium liquidation strategy which is not a pure stopping time. There could be other equilibrium liquidation strategies in the following cases.

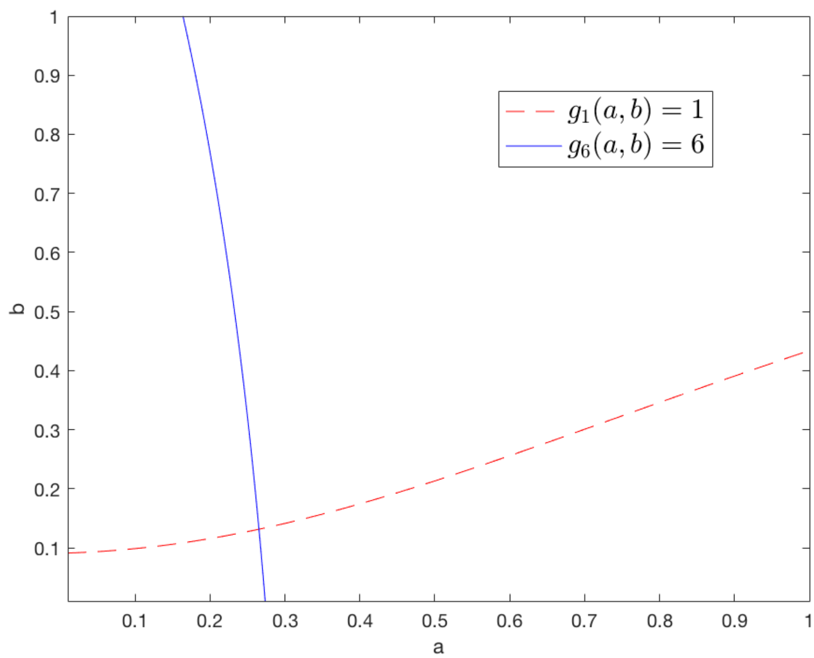


Figure 2.1: Graph for Example 2.2.24

(i) If $g_1(a,0) = 1$ and $g_6(a,0) > 6$, then $\eta(1) = a, \eta(6) = 0$ is an equilibrium liquidation strategy;

(ii) If $g_1(a,1) = 1$ and $g_6(a,1) < 6$, then $\eta(1) = a, \eta(6) = 1$ is an equilibrium liquidation strategy;

(iii) If $g_6(0,b) = 6$ and $g_1(0,b) > 1$, then $\eta(1) = 0, \eta(6) = b$ is an equilibrium liquidation strategy;

(iv) If $g_6(1,b) = 6$ and $g_1(1,b) < 1$, then $\eta(1) = 1, \eta(6) = b$ is an equilibrium liquidation strategy.

However, from the above graph, we conclude that there are no solutions for $g_1(a,0) = 1$, $g_1(a,1) = 1$, $g_6(0,b) = 6$ or $g_6(1,b) = 6$. So there is only one equilibrium liquidation strategy in this example.

The above two examples illustrate that an equilibrium liquidation strategy exists regardless of the existence of equilibrium stopping times. Since the equilibrium liquidation strategies are unique in Example 2.2.23 and Example 2.2.24, they are also optimal. However, uniqueness of an equilibrium liquidation strategy and existence of an optimal equilibrium liquidation strategy are not guaranteed in general as we will show later.

From Proposition 2.2.20 and (2.16), we have that

$$K_l(x,\eta) = \max\{x,\ \mathbb{E}_x[Y_\eta] - c\mathrm{Var}_x[Y_\eta]\} = \begin{cases} x, & 0 < \eta(x) \leq 1, \\ \mathbb{E}_x[Y_\eta] - c\mathrm{Var}_x[Y_\eta], & 0 \leq \eta(x) < 1. \end{cases} \tag{2.17}$$

A necessary condition for $\eta^* \in \mathcal{E}$ to be an optimal equilibrium liquidation strategy is given in the following proposition.

**Proposition 2.2.25.** *If an equilibrium liquidation strategy $\eta^* \in \mathcal{E}$ is optimal then*

$$\mathcal{C}(\eta^*) = \bigcup_{\eta\in\mathcal{E}} \mathcal{C}(\eta),$$

*where $\mathcal{C}(\eta) := \{x \in \mathbb{X} : x < \mathbb{E}_x[Y_\eta] - c(Var_x[Y_\eta])^{1/2}\}$.*

*Proof.* Suppose $\mathcal{C}(\eta^*) = \bigcup_{\eta\in\mathcal{E}} \mathcal{C}(\eta)$ does not hold. Then there exists some $\eta \in \mathcal{E}$ and some $x \in \mathbb{X}$ such that $x \in \mathcal{C}(\eta)$ and $x \notin \mathcal{C}(\eta^*)$. Then by (2.17) we have that

$$K_l(x,\eta^*) = x < \mathbb{E}_x[Y_\eta] - C(\mathrm{Var}_x[Y_\eta])^{1/2} = K_l(x,\eta),$$

which contradicts that $\eta^*$ is optimal. □

**Corollary 2.2.26.** *If $\bigcup_{\eta\in\mathcal{E}} \mathcal{C}(\eta) = \emptyset$, then any $\eta \in \mathcal{E}$ is an optimal equilibrium liquidation strategy.*

*Proof.* If $\bigcup_{\eta\in\mathcal{E}} \mathcal{C}(\eta) = \emptyset$, then for all $\eta \in \mathcal{E}$ and all $x \in \mathbb{X}$, we have $K_l(x,\eta) = x$. By definition, they are all optimal. □

The next proposition shows that the existence and uniqueness of an optimal equilibrium liquidation strategy are not guaranteed.

**Proposition 2.2.27.** *The optimal equilibrium liquidation strategy may not exist. When it does exist, it may not be unique.*

*Proof.* (i) We will give an example in which there exist multiple equilibrium liquidation strategies and one of them is optimal.

Let $c = 0.4$. $X$ has state space $\mathbb{X} = \{0, 1, 2, 7, 9\}$ and the following transition matrix.

| | **0** | **1** | **2** | **7** | **9** |
|---|---|---|---|---|---|
| **0** | 1 | 0 | 0 | 0 | 0 |
| **1** | 0.2 | 0 | 0.4 | 0.2 | 0.2 |
| **2** | 0 | 0 | 1 | 0 | 0 |
| **7** | 0 | 0.2 | 0 | 0 | 0.8 |
| **9** | 0 | 0 | 0 | 0 | 1 |

Since $\{0, 2, 9\}$ are absorbing states, we have $\eta(0) = \eta(2) = \eta(9) = 1$ for all $\eta \in \mathcal{L}$. Let $a = \eta(1)$ and $b = \eta(7)$. Let $g_i(a, b) := \mathbb{E}_i[Y_\eta] - (\mathbb{E}_i[Y_\eta^2] - \mathbb{E}_i[Y_\eta]^2)^{1/2}$ for $i = 1, 7$. By analysis shown in Appendix A.2, we obtain the following results.

(1) $g_1(a, b) > 1$ for all $(a, b) \in [0, 1] \times [0, 1]$.

(2) There exists a unique $b_0 \in (0, 1)$ such that $g_7(0, b_0) = 7$.

(3) $g_7(0, 0) > 7$ and $g_7(0, 1) < 7$.

So there are three equilibrium liquidation strategies in total: $(a, b) = (0, 0)$, $(a, b) = (0, b_0)$ and $(a, b) = (0, 1)$. By graphs of $g_1(0, b)$ and $g_7(0, b)$ in Appendix A.2, we observe that $g_1(0, 0) > g_1(0, b_0) > g_1(0, 1)$ and $g_7(0, 0) > g_7(0, b_0) > g_7(0, 1)$, which implies that $K(x, (0, 0)) = \max_{\eta \in \mathcal{E}} K(x, \eta)$ for all $x \in \mathbb{X}$. By Definition 2.2.19 $(a, b) = (0, 0)$ is the optimal equilibrium liquidation strategy.

(ii) We will give an example in which there exist multiple equilibrium liquidation strategies but none of them are optimal.

Let $c = 0.1$. $X$ has state space $\mathbb{X} = \{0, 11, 17, 18\}$ and the following transition matrix.

| | **0** | **11** | **17** | **18** |
|---|---|---|---|---|
| **0** | 1 | 0 | 0 | 0 |
| **11** | 0.1 | 0.7 | 0 | 0.2 |
| **17** | 0 | 0.1 | 0.1 | 0.8 |
| **18** | 0 | 0 | 0 | 1 |

Since $\{0, 18\}$ are absorbing states, we have $\eta(0) = \eta(18) = 1$ for all $\eta \in \mathcal{L}$. Let $a = \eta(11)$ and $b = \eta(17)$. Let $g_i(a, b) := \mathbb{E}_i[Y_\eta] - (\mathbb{E}_i[Y_\eta^2] - \mathbb{E}_i[Y_\eta]^2)^{1/2}$ for $i = 11, 17$. By analysis shown in Appendix A.3, we obtain the following results.

(1) There is not intersection of the curve $g_{11}(a, b) = 11$ and $g_{17}(a, b) = 17$.

(2) There exist $0 < a_1 < a_2 < a_3 < a_4 < 1$ and $0 < b_0 < 1$ such that $g_{17}(a_1, 0) = 17, g_{17}(a_2, 1) = 17, g_{17}(a_4, 1) = 17, g_{11}(a_3, 0) = g_{11}(a_3, 1) = 11$, and $g_{17}(1, b_0) = 17$.

(3) $g_{11}(a_1, 0) \neq 11, g_{11}(a_2, 1) \neq 11$ and $g_{11}(a_4, 1) \neq 11$.

(4) $g_{17}(a_3, 0) > 17, g_{17}(a_3, 1) > 17$ and $g_{11}(1, b_0) < 11$.

So there are five equilibrium liquidation strategies in total. The following table 2.1summarises the values of objective functions under these equilibrium liquidation strategies.

| $\eta : (a, b)$ | $K_l(11, \eta)$ | $K_l(17, \eta)$ |
|---|---|---|
| $(1, 1)$ | 11 | 17 |
| $(0, 1)$ | 11.1515 | 17 |
| $(1, 0)$ | 11 | 17.0022 |
| $(a_3, 0)$ | 11 | $\approx 17.0212$ |
| $(1, b_0)$ | 11 | 17 |

Table 2.1: An example of multiple equilibrium liquidation strategies without an optimal one

The table shows that there is no optimal equilibrium liquidation strategy.

(iii) We will give an example in which there are two equilibrium liquidation strategies and both are optimal.

Let $c = 0.5$. $X$ has state space $\mathbb{X} = \{0, 1, 4\}$ and the following transition matrix.

| | **0** | **1** | **4** |
|---|---|---|---|
| **0** | 1 | 0 | 0 |
| **1** | 0.1 | 0.8 | 0.1 |
| **4** | 0 | 0 | 1 |

Since $\{0, 4\}$ are absorbing states, we have $\eta(0) = \eta(4) = 1$ for all $\eta$. Let $a = \eta(1)$ and we obtain function $h(a) = \mathbb{E}_1[Y_\eta] - c\mathrm{Var}_x[Y_\eta]$ as shown in Appendix A.4. $h(a)$ is decreasing on the interval $[0, 1]$ and $h(0) = 1, h(1) = 0.7101 < 1$. By definition, there are two equilibrium liquidation strategies in total $\eta(1) = 0$ and $\eta(1) = 1$. By Corollary 2.2.26, $K_l(1, \eta) = 1$ for both equilibrium liquidation strategies, so they are both optimal. □

Since the existence of optimal equilibrium liquidation strategy is not guaranteed, we naturally turn to the concept of Pareto optimality.

**Definition 2.2.28.** $\eta^* \in \mathcal{E}$ is called a Pareto optimal equilibrium liquidation strategy if there is no $\eta \in \mathcal{E}$ such that

(i) $\forall\, x \in \mathbb{X}, K_l(x, \eta) \geq K_l(x, \eta^*)$;

(ii) $\exists\, x \in \mathbb{X}, K_l(x, \eta) > K_l(x, \eta^*)$

*Remark* 2.2.29. In the second example in proof of Proposition 2.2.27, the optimal equilibrium liquidation strategy does not exist, but $(0, 1)$ and $(a_3, 0)$ are both Pareto optimal equilibrium liquidation strategies.

**Proposition 2.2.30.** *A Pareto optimal equilibrium liquidation strategy always exists.*

*Proof.* Consider the following optimization problem

$$\sup_{\eta \in \mathcal{E}} \sum_{x \in \mathbb{X}} K_l(x, \eta). \tag{2.18}$$

Then any maximizer $\eta^*$ of this problem is a Pareto optimal equilibrium liquidation strategy. Otherwise, there exists $\eta' \in \mathcal{E}$ such that

$$K_l(x, \eta') \geq K_l(x, \eta^*), \forall x \in \mathbb{X};\ K_l(x_0, \eta') > K_l(x_0, \eta^*), \exists x_0 \in \mathbb{X};$$

then $\sum_{x \in \mathbb{X}} K_l(x, \eta') > \sum_{x \in \mathbb{X}} K_l(x, \eta^*)$, which contradicts that $\eta^*$ is the maximizer of the problem.

Next we will show such maximizer exists and is in $\mathcal{E}$. Let $\eta^n$ be the $\frac{1}{n}$-optimizer of (2.18). Then there exists $\eta^* \in \mathcal{L}$ such that up to a subsequence $\eta^n \to \eta^*$. Since $\{\eta^n\}_{n \in \mathbb{N}}$ are all equilibrium liquidation strategies, $K_l(x, \xi \otimes \eta^n) \leq K_l(x, \eta^n)$ holds for all $x \in \mathbb{X}$ and all $\xi \in \mathcal{L}$. By the continuity of the mappings $\eta \to K_l(x, \xi \otimes \eta)$ and $\eta \to K_l(x, \eta)$, $K_l(x, \xi \otimes \eta^*) \leq K_l(x, \eta)$ also holds for all $x \in \mathbb{X}$ and all $\xi \in \mathcal{L}$, i.e., $\eta^*$ is also an equilibrium liquidation strategy. Again by the continuity of mapping $\eta \to K_l(x, \eta)$, $\eta^*$ is the maximizer of (2.18). □

## 2.3 Mean-variance problem

As what we have done in mean-standard deviation problem, we will analyze different types of subgame perfect Nash equilibrium in mean-variance problems. More specifically, we define equilibrium stopping time, equilibrium randomized stopping strategy and equilibrium liquidation strategy for mean-variance problems as the following.

**Definition 2.3.1.** A pure Markov stopping time $\tau$ with stopping region $S$ is said to be an equilibrium stopping time for the mean-variance problem if

$$x \geq J_p(x, \rho(x, S)),\ \forall x \in S \qquad \textit{and} \qquad x \leq J_p(x, \rho(x, S)),\ \forall x \notin S,$$

where $\rho(x, S) := \inf\{n \geq 1 : X_n^x \in S\}$ and $J_p(x, \rho(x, S)) := \mathbb{E}_x[X_{\rho(x,S)}] - c\mathrm{Var}_x[X_{\rho(x,S)}]$.

**Definition 2.3.2.** A randomized stopping time $\mathbf{p} \in \mathcal{P}$ is said to be an equilibrium randomized stopping time for the mean-variance problem, if for any mapping $\mathbf{q} : \mathbb{X} \to [0, 1]$,

$$J_r(x, \mathbf{q} \otimes \mathbf{p}) \leq J_r(x, \mathbf{p} \otimes \mathbf{p}), \ \ \forall x \in \mathbb{X},$$

where $J_r(x, \mathbf{q} \otimes \mathbf{p}) := \mathbb{E}_x[X_{\mathbf{q} \otimes \mathbf{p}}] - c\text{Var}_x[X_{\mathbf{q} \otimes \mathbf{p}}]$.

**Definition 2.3.3.** A liquidation strategy $\theta = \theta^\eta \in \mathcal{L}$ is said to be an equilibrium liquidation strategy for the mean-variance problem if for any mapping $\xi : \mathbb{X} \to [0, 1]$, we have

$$J_l(x, \xi \otimes \eta) \leq J_l(x, \eta \otimes \eta), \ \ \forall x \in \mathbb{X},$$

where $J_l(x, \xi \otimes \eta) := \mathbb{E}_x[\theta^{\xi \otimes \eta}(X)] - c\text{Var}_x[\theta^{\xi \otimes \eta}(X)]$.

**Proposition 2.3.4.** *An equilibrium stopping time for mean-variance problem may not exist.*

*Proof.* We will prove this by giving a counterexample. Let $c = \frac{21}{50}$ and $X$ has the following transition matrix.

| | **0** | **1** | **2** | **3** |
|---|---|---|---|---|
| **0** | 1 | 0 | 0 | 0 |
| **1** | $\frac{1}{3}$ | 0 | $\frac{1}{3}$ | $\frac{1}{3}$ |
| **2** | 0 | $\frac{1}{3}$ | 0 | $\frac{2}{3}$ |
| **3** | 0 | 0 | 0 | 1 |

Suppose there exists an equilibrium stopping time with stopping region $S \subset \{0, 1, 2, 3\}$. Denote $H(\cdot, S) = J_p(\cdot, \rho(\cdot, S))$. We consider the following four cases.

**Case 1:** $1, 2 \in S$**.** We have

$$\mathbb{P}_1(X_{\rho(1,S)} = 0) = \mathbb{P}_1(X_{\rho(1,S)} = 2) = \mathbb{P}_1(X_{\rho(1,S)} = 3) = \frac{1}{3},$$

and

$$\mathbb{E}_1\left[X_{\rho(1,S)}\right] = \frac{5}{3}, \quad \mathbb{E}_1\left[X^2_{\rho(1,S)}\right] = \frac{13}{3}, \quad \text{and thus} \quad H(1,S) = \frac{76}{75} > 1,$$

that yields a contradiction.

**Case 2:** $1 \notin S$ **and** $2 \in S$**.** We have

$$\mathbb{P}_2(X_{\rho(2,S)} = 0) = \mathbb{P}_2(X_{\rho(2,S)} = 2) = \frac{1}{9}, \quad \mathbb{P}_2(X_{\rho(2,S)} = 3) = \frac{7}{9},$$

and

$$\mathbb{E}_2\left[X_{\rho(2,S)}\right] = \frac{23}{9}, \mathbb{E}_2\left[X^2_{\rho(2,S)}\right] = \frac{67}{9},$$

and thus $H(2,S) = \frac{1466}{675} > 2$, that yields a contradiction.

**Case 3:** $1 \in S$ **and** $2 \notin S$**.** We have that

$$\mathbb{P}_2(X_{\rho(2,S)} = 1) = \frac{1}{3}, \quad \mathbb{P}_2(X_{\rho(2,S)} = 3) = \frac{2}{3},$$

and

$$\mathbb{E}_2\left[X_{\rho(2,S)}\right] = \frac{7}{3}, \quad \mathbb{E}_2\left[X^2_{\rho(2,S)}\right] = \frac{19}{3}, \quad \text{and thus} \quad H(2,S) = \frac{147}{75} < 2,$$

that yields a contradiction.

**Case 4:** $1, 2 \notin S$**.** We have that

$$\mathbb{P}_1(X_{\rho(1,S)} = 0) = \frac{3}{8}, \quad \mathbb{P}_1(X_{\rho(1,S)} = 3) = \frac{5}{8},$$

and

$$\mathbb{E}_1\left[X_{\rho(1,S)}\right] = \frac{15}{8}, \quad \mathbb{E}_1\left[X^2_{\rho(1,S)}\right] = \frac{45}{8}, \quad \text{and thus} \quad H(1,S) = \frac{633}{640} < 1,$$

that yields a contradiction. ◻

Besides, using a proof similar to the case of mean-standard deviation problem, we can show that an equilibrium randomized stopping time for mean-variance problem exists if and only if it is an equilibrium stopping time.

Now we will focus on the equilibrium liquidation strategy for mean-variance problem as defined in Definition 2.3.3, although it is not a proper definition as pointed out in Remark 2.3.6. Following an argument similar to the one in the proof of Theorem 2.2.17, we can prove that there exists an equilibrium liquidation strategy in mean-variance problem. The next proposition shows that in contrast to mean-standard deviation problem, the equilibrium liquidation strategy is not a generalization of equilibrium stopping time in mean-variance problem, although equilibrium liquidation strategies can be thought of as a relaxation of equilibrium stopping times.

**Proposition 2.3.5.** *An equilibrium stopping time may not be an equilibrium liquidation strategy.*

*Proof.* Consider the Markov process in Example 2.2.23 and let $c = 0.25$. It can be shown that there is only one equilibrium stopping time with stopping region $S = \{0, 2\}$ and $\mathbb{E}_x[X_S] - c\mathrm{Var}_x[X_S] = \frac{10}{9}$. Next we will show that the corresponding liquidation strategy $\eta(1) = 0$ is not an equilibrium liquidation strategy. Since $\mathbb{E}_1[Y_\eta] = \frac{4}{3}$ and $\mathrm{Var}_1[Y_\eta] = \frac{8}{9}$, we have

$$J_l(1, \xi \otimes \eta) = \xi(1) + \frac{4}{3}(1 - \xi(1)) - \frac{2}{9}(1 - \xi(1))^2.$$

It is easy to check that $\max_\xi J_l(1, \xi \otimes \eta) = \frac{9}{8} > \frac{10}{9}$ and the maximum is attained at $\xi(1) = \frac{1}{4}$ instead of 0. By definition $\eta$ is not an equilibrium liquidation strategy. ◻

This result illustrates that the equilibrium liquidation strategy for mean-variance problem is not a proper definition as we briefly discuss next.

*Remark* 2.3.6. Definition 2.3.3 seems to be reasonable as an analogy of Definition 2.2.13. However, since there is no scaling effect in the mean-variance problem, this definition has deviated from the concept of subgame perfect Nash equilibrium. For example, at time $t = 0$, the objective function is

$$\mathbb{E}_{X_0}[\theta(X)] - c\text{Var}_{X_0}[\theta(X)].$$

If we liquidate $1-\alpha$ proportion of the asset, then the objective function at time $t = 1$ would become

$$\mathbb{E}_{X_1}[\alpha\theta(X)] - c\text{Var}_{X_1}[\alpha\theta(X)] = \alpha(\mathbb{E}_{X_1}[\theta(X)] - c\alpha\text{Var}_{X_1}[\theta(X)]),$$

i.e., the preference of player "$X_1$" becomes $\mathbb{E}_{X_1}[\theta(X)] - c\alpha\text{Var}_{X_1}[\theta(X)]$ instead of $\mathbb{E}_{X_1}[\theta(X)] - c\text{Var}_{X_1}[\theta(X)]$. Generally, the proportion of asset remaining, $\alpha$, is decreasing as time goes on, therefore we are faced with different problems with different parameter $c$ at different time, even if the initial state remains the same. Definitions of equilibrium liquidation strategies in Definition 2.3.3 and Definition 2.2.13 only make sense when the objective function remains the same for the same initial state $x$.

A possible improved definition is incorporating the remaining component, i.e., to enlarge the strategy set such that it depends on the state as well as the remaining component of the asset. However, this expansion makes the set of players an uncountable set (instead of identifying the players with the states of the Markov chain, we will need to use an additional variable which is not discrete). This is an intergenerational problem with exhaustible resources. Such a problem is beyond the scope of this project and will be left for future research. But we should emphasize that one of the main messages of this chapter is that mean-standard deviation problem is more appropriate and for this criterion such an extension of the state space is not necessary.

## 2.4 Comparison with static optimal stopping time

In this section, we want to compare the pre-commitment strategy with the equilibrium liquidation strategy. It is obvious that given any current state $x$, the static optimality from pre-commitment strategy is no less than the value of $K_p(x,\tau)$ where $\tau$ is an equilibrium stopping time. However this may not be the case when we compare static optimal stopping times with equilibrium liquidation strategies. As we have discussed in Example 2.2.10, a liquidation strategy may produce larger value than the static optimal stopping time does. However in Example 2.2.10, the liquidation strategy is not an equilibrium. The following examples show that an equilibrium liquidation strategy may produce larger value than the static optimal stopping time does in both mean-standard deviation problem and mean-variance problem. The intuitive reason is that a liquidation strategy allows for selling parts of an asset over time, while the static optimal stopping time problem relies on the assumption that the whole asset must be sold at exactly one point in time.

**Example 2.4.1.** Consider the first example in the proof of Proposition 2.2.27.

For mean-standard deviation problem,

$$\sup_{\tau\in\mathcal{T}} K_p(1,\tau) = K_p(1,\tau') = 2.6940, \quad \sup_{\tau\in\mathcal{T}} K_p(7,\tau) = K_p(7,\tau') = 7.0187.$$

where $\tau' = \inf\{n \geq 0 : X_n^1 \in \{0,2,9\}\}$.

The optimal equilibrium liquidation strategy $\eta$ is the same as $\tau'$. We have

$$K_l(1,\eta) = \sup_{\tau\in\mathcal{T}} K_p(1,\tau), \quad K_l(7,\eta) = \sup_{\tau\in\mathcal{T}} K_p(7,\tau).$$

For mean-variance problem,

$$\sup_{\tau} J_p(1,\tau) = J_p(1,\tau'') = 1, \ \sup_{\tau} J_p(7,\tau) = J_p(7,\tau'') = 7.$$

where $\tau'' = 0$.

The unique equilibrium liquidation strategy is $\eta(1) = a' \in (0,1), \eta(7) = b' \in (0,1)$ where $a' \approx 0.6778$ and $b' \approx 0.9089$. Details on finding the equilibrium liquidation strategy can be found in Appendix A.5. We have

$$J_l(1,\eta) = 2.6438 > \sup_{\tau\in\mathcal{T}} J_p(1,\tau), \quad J_l(7,\eta) = 6.4521 < \sup_{\tau\in\mathcal{T}} J_p(7,\tau).$$

**Example 2.4.2.** Consider the second example in the proof of Proposition 2.2.27.

For mean-standard deviation problem,

$$\sup_{\tau\in\mathcal{T}} K_p(11,\tau) = K_p(11,\tau') = K_p(11,\tau'') = 11.1515,$$

where $\tau' = \inf\{n \geq 0 : X_n^1 \in \{0, 18\}\}, \tau'' = \inf\{n \geq 0 : X_n^1 \in \{0, 17, 18\}\}$, and

$$\sup_{\tau\in\mathcal{T}} K_p(17,\tau) = K_p(17,\tau''') = 17.0022,$$

where $\tau''' = \inf\{n \geq 0 : X_n^1 \in \{0, 11, 18\}\}$.

There is an equilibrium liquidation strategy $\eta(11) = a_3 \in (0,1), \eta(17) = 0$. We have

$$K_l(11,\eta) = 11 < \sup_{\tau\in\mathcal{T}} K_p(11,\tau), \quad K_l(17,\eta) = 17.0212 > \sup_{\tau\in\mathcal{T}} K_p(17,\tau).$$

For mean-variance problem,

$$\sup_{\tau\in\mathcal{T}} J_p(11,\tau) = J_p(11,\tau') = 11, \quad \sup_{\tau\in\mathcal{T}} J_p(17,\tau) = J_p(17,\tau') = 17,$$

where $\tau' = 0$.

The unique equilibrium liquidation strategy is $\eta(11) = a' \in (0,1), \eta(17) = b' \in (0,1)$ where $a' \approx 0.9312$ and $b' \approx 0.7629$. Details on finding the equilibrium liquidation strategy can be found in Appendix A.5. We have

$$J_l(11, \eta) = 10.8365 < \sup_{\tau \in \mathcal{T}} J_p(11, \tau), \quad J_l(17, \eta) = 16.9981 < \sup_{\tau \in \mathcal{T}} J_p(17, \tau).$$

# CHAPTER III

# Equilibrium Concepts for Time-inconsistent Stopping Problems in Continuous Time

In this chapter, a *new* notion of equilibrium, which we call *strong equilibrium*, is introduced for time-inconsistent stopping problems in continuous time. Compared to the existing notions introduced in [31] and [17], which in this chapter are called *mild equilibrium* and *weak equilibrium* respectively, a strong equilibrium captures the idea of subgame perfect Nash equilibrium more accurately. When the state process is a continuous-time Markov chain and the discount function is log sub-additive, we show that an optimal mild equilibrium is always a strong equilibrium. Moreover, we provide a new iteration method that can directly construct an optimal mild equilibrium and thus also prove its existence.

## 3.1 Introduction

On a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\infty)}, \mathbb{P})$ consider an optimal stopping problem in continuous time

$$\sup_{\tau\in\mathcal{T}} \mathbb{E}_x[\delta(\tau)X_\tau], \tag{3.1}$$

where $X = (X_t)_{t\in[0,\infty)}$ is a time-homogeneous Markov process taking values in some space $\mathbb{X} \subset \mathbb{R}$, $\mathcal{T}$ is a set of stopping times, $\delta$ is a discount function, and $\mathbb{E}_x$ is the expectation given $X_0 = x$. It is well known that when $\delta$ is not exponential, the problem (3.1) may be time-inconsistent. That is, the optimal stopping strategy obtained today may not be optimal in the eyes of future selves. There are mainly three ways to approach such time inconsistency: pre-committed strategy, naive strategy and consistent planning strategy, dating back to [56]. Our project focuses on consistent planning, which is formulated as a subgame perfect Nash equilibrium: once an equilibrium strategy is enforced over the planning horizon, the current self has no incentive to deviate from it, given all future selves will follow the equilibrium strategy. For discussions on different approaches, see [53], [21], [11], [12], [17], [35], [31] and references therein.

There are two general notions of equilibrium stopping strategies in continuous time in the literature. The first notion is proposed in [31] and further studied in [35, 33, 32], which we will call mild equilibrium in this chapter. Following [31, Definition 3.3] and [33, Definition 2.2], we have the following definition of mild equilibrium.

**Definition 3.1.1.** A measurable set $S \subset \mathbb{X}$ is said to be a mild equilibrium, if

$$\begin{cases} x \le \mathbb{E}_x[\delta(\tau_S)X_{\tau_S}], \quad \forall\, x \notin S, & (3.2) \\ x \ge \mathbb{E}_x[\delta(\tau_S^+)X_{\tau_S}], \quad \forall\, x \in S, & (3.3) \end{cases}$$

where

$$\tau_S := \inf\{t \ge 0 : X_t \in S\}, \quad \text{and} \quad \tau_S^+ := \inf\{t > 0 : X_t \in S\}. \tag{3.4}$$

and $S$ is the stopping region. The economic interpretation for Definition 3.1.1 is clear – there is no incentive to deviate from an equilibrium stopping strategy. That is, in (3.2) when $x \notin S$, it is better to continue and get $\mathbb{E}_x[\delta(\tau_S)X_{\tau_S}]$, rather than to stop and get $x$; *on the surface* a similar statement applies to (3.3). However, when the time of return for $X$ is 0 (i.e., $\mathbb{P}(\tau_{\{x\}}^+ = 0 \,|\, X_0 = x) = 1$ ), which is satisfied for continuous-time Markov chains and many one-dimension diffusion processes, $\tau_S = \tau_S^+$ and thus (3.3) trivially holds. In other words, when the time of return is 0, there is no actual deviation captured by (3.3) from stopping to continuing, and Definition 3.1.1 is equivalent to the following.

**Definition 3.1.2.** A measurable set $S \subset \mathbb{X}$ is said to be a mild equilibrium, if

$$x \leq \mathbb{E}_x[\delta(\tau_S)X_{\tau_S}] =: J(x, S), \quad \forall\, x \notin S. \tag{3.5}$$

Consequently, with the time of return being 0 the notion of mild equilibrium cannot fully capture the economic meaning of equilibrium. It is easy to see that $S = \mathbb{X}$ is always a mild equilibrium, and it is not clear why always stopping immediately is a reasonable strategy. Notice that in discrete time there is no such degeneration issue for equilibrium since $\tau_S^+ = \inf\{t \geq 1 : X_t \in S\}$ in discrete time setting. See [30, Remark 2.3] and [6, Definition 2.2].

As can be seen from [31, 35, 33, 32], there is often a continuum of mild equilibria in many natural models, which naturally leads to the question of equilibrium selection. In [33], optimal mild equilibrium in the sense of point-wise dominance is considered. In particular, from [33, Definition 2.3] we have the following definition.

**Definition 3.1.3.** A mild equilibrium $S^*$ is said to be optimal, if for any other mild equilibrium $S$,

$$x \vee J(x, S^*) \geq x \vee J(x, S) (\Longleftrightarrow J(x, S^*) \geq J(x, S)), \quad \forall\, x \in \mathbb{X}.$$

Note that $x \vee J(x, S)$ represents the value associated with the stopping region/strategy $S$. In [33] the existence of optimal equilibrium is established. A discrete-time version is in [30].

The second notion of equilibrium, which we call weak equilibrium in this chapter is proposed in [17] and further investigated in [18]. Following [17], we have the definition of weak equilibrium (we adapt the definition slightly for our setting).

**Definition 3.1.4.** A measurable set $S \subset \mathbb{X}$ is said to be a weak equilibrium, if

$$\begin{cases} x \leq \mathbb{E}_x[\delta(\tau_S)X_{\tau_S}], \quad \forall x \notin S, \\ \liminf_{\varepsilon \searrow 0} \dfrac{x - \mathbb{E}_x[\delta(\tau_S^\varepsilon)X_{\tau_S^\varepsilon}]}{\varepsilon} \geq 0, \quad \forall x \in S, \end{cases} \tag{3.6}$$

where

$$\tau_S^\varepsilon = \inf\{t \geq \varepsilon : X_t \in S\}. \tag{3.7}$$

Compared to (3.3), the first-order condition (3.6) does capture the deviation from stopping to continuing. However, similar to that for time-inconsistent control (see e.g., [12, Remark 3.5] and [34]), the first-order criterion does not correspond to the equilibrium concept perfectly: when the limit in (3.6) equals zero, it is possible that for all $\varepsilon > 0$, $x < \mathbb{E}_x[\delta(\tau_S^\varepsilon)X_{\tau_S^\varepsilon}]$, in which case there is an incentive to deviate.

To sum up, the economic interpretation of being "equilibrium" for mild and weak ones is inadequate. There are similar issues in continuous-time time-inconsistent stochastic control problems as mentioned in [12, Remark 3.5]. In response to [12, Remark 3.5], a new definition of continuous-time equilibrium control is introduced in [34]. In time-inconsistent optimal stopping problems, we introduce the following concept of strong equilibrium, which is inspired by [34].

**Definition 3.1.5.** A measurable set $S \subset \mathbb{X}$ is said to be a strong equilibrium, if

$$\begin{cases} x \leq \mathbb{E}_x[\delta(\tau_S)X_{\tau_S}], \quad \forall x \notin S, \\ \exists\, \varepsilon = \varepsilon(x) > 0, \text{s.t. } \forall\, \varepsilon' \in (0, \varepsilon),\ x \geq \mathbb{E}_x[\delta(\tau_S^{\varepsilon'})X_{\tau_S^{\varepsilon'}}], \quad \forall x \in S. \end{cases} \tag{3.8}$$

Compared to (3.3) and (3.6), condition (3.8) not only captures the deviation from stopping to continuing, but also more precisely indicates the disincentive of such deviation. Consequently, a strong equilibrium delivers better economic meaning as being an "equilibrium".

In this project, when $X$ is a Markov chain we show that an optimal mild equilibrium is a strong equilibrium (see Theorem 3.2.4). (Obviously, a strong equilibrium is also weak, and a weak equilibrium is also mild.) We also provide examples showing that a strong equilibrium may not be an optimal mild equilibrium, and a weak equilibrium may not be strong. Therefore, we thoroughly obtain the relation between mild, weak, strong, and optimal mild (and thus optimal weak, optimal strong) equilibria. Moreover, we provide a new iteration method which directly constructs an optimal mild equilibrium and thus also establish its existence (see Thoerem 3.2.5). In [30, 33], an optimal equilibrium is constructed by the intersection of all (mild) equilibriums. In principle, this requires us to first find all (mild) equilibria in order to get the optimal one, which may not be implementable in many cases. The new iteration method proposed in this chaper is much easier and more efficient to implement. Examples are provided to demonstrate the application of the new iteration method (see Example 3.1 and Example 3.2). It would be interesting to see whether such results can be extended to diffusion models, which we will leave for future research.

As in reality people often discount non-exponentially (see Remark 3.2.3), our results can be applied to stopping problems in finance ad economics. Generally we can use $X = f(Y)$ for some nonnegative measurable payoff function $f$ and some price process of underlying asset $Y$. Our results still hold and the proofs still work when replacing $X$ with $f(Y)$. For instance, in Example 3.2, $Y$ is a stock price process and $X = f(Y)$ is the payoff of an American put option. This can be viewed as an example of exercising an American option when the investor tries to maximize the expected payoff yet subject to hyperbolic discounting. We refer to [30, Section 5] and [33, Section 6.3] for more examples that satisfy this condition. The two-state example provided in Section 4 of this chapter can also be thought of as an application of stopping (e.g., selling a house) when the economy (e.g., property market) is good/bad. Our project is inline with the work [24], where equilibrium stopping strategies are considered in an entrepreneur's investment-timing problem under time-inconsistent preferences due to quasi-hyperbolic discounting.

People have considered incorporating non-exponential discounting into decision making including optimal stopping. However, this leads to time inconsistency as argued by Grenafier and Wang in [24]. They proposed the time-consistent modeling framework and our result can be seen as making advances on proposing better equilibrium concepts in this line of work. Our notion of strong equilibria also applies to other types of time-inconsistent stopping, such as mean-variance stopping problems and stopping under probability distortion. Consider $G(x,\tau)$, where $x$ is the initial position for the underlying process $X$, and $G$ is payoff utility. For example, $G(x,\tau) = \mathbb{E}_x[\delta(\tau)f(X_\tau)]$ for stopping with non-exponential discounting; $G(x,\tau) = \mathbb{E}_x[f(X_\tau)] - c\text{Var}_x[f(X_\tau)]$ for mean-variance stopping; and $G(x,\tau) = \int_0^\infty w(\mathbb{P}[f(X_\tau^x) > y])dy$ for stopping under probability distortion. Note that $G(x,0) = f(x)$. Then in general strong equilibria can be formulated accordingly as: $S \subset \mathbb{X}$ is said to be a strong equilibrium, if

$$\begin{cases} f(x) \leq G(x,\tau_S), \quad \forall x \notin S, \\ \exists\, \varepsilon = \varepsilon(x) > 0, \text{s.t. } \forall\, \varepsilon' \in (0,\varepsilon),\; f(x) \geq G(x,\tau_S^{\varepsilon'}), \quad \forall x \in S. \end{cases}$$

The mild and weak equilibria can also be defined accordingly, and they still suffer from being short of economic meaning.

This chapter provides novel and conceptual contributions in the topic of time-inconsistent stopping. First, we analyze existing notions of equilibrium and their inadequacy in continuous time. A new notion of equilibrium, strong equilibrium is introduced. It captures the economic meaning of being an “equilibrium” more accurately. Second, we show that an optimal mild equilibrium is also a strong equilibrium, which is far from obvious. This result together with the examples in this chapter completely shows the relations between mild, weak, strong, optimal mild/weak/strong equilibria. No such result has been obtained before. Moreover, we completely obtain the existence and (non)uniqueness results of these equilibria. Third, we provide an iteration method, which directly constructs an optimal equilibrium and is much more implementable than the existing method in [30, 33]. Moreover, the key ideas in these proofs provide some novel proof approaches in the literature of time-inconsistent control/stopping. The recent work of [26] discusses notions of equilibrium control based on condition (1.3) in [34]. The focus of their paper is to distinguish between weak and strong (and regular) equilibrium controls. It is intuitively like distinguishing between local maxima and critical points. In our results we not only distinguish between strong, weak and mild equilibrium stopping times, but also obtain the property that an optimal mild equilibrium is a strong equilibrium. The notions of mild equilibrium and optimal mild equilibrium only make sense for stopping problems not control problems. Thus the focus of our project is different from [26] and our intuitively unexpected result makes a novel contribution to the literature.

The rest of the chapter is organized as follows. Section 2 collects the main results of the chapter. An optimal mild equilibrium is proved to be a strong equilibrium, and can be directly constructed via a new iteration method. Section 3 provides examples to illustrate the iteration method in Theorem 3.2.5. Section 4 focuses on a concrete two-state model, which demonstrates the differences between these equilibria.

## 3.2 The Main Results

In this section, we apply the concepts in Section 1 to a continuous-time Markov chain and present our main results under this setting. Let $X = (X_t)_{t\geq 0}$ be a time-homogeneous continuous-time Markov chain. It has a finite or countably infinite state space $\mathbb{X} \subset [0, \infty)$. Let $\lambda_x$ be the transition rate out of the state $x \in \mathbb{X}$, and $q_{xy}$ be the transition rate from state $x$ to $y$ for $y \neq x$. Then we have that $\lambda_x = \sum_{y\neq x} q_{xy}$. The discount function $t \mapsto \delta(t)$ is assumed to be non-exponential and decreasing, with $\delta(0) = 1$ and $\lim_{t\to\infty} \delta(t) = 0$. Let the filtration $(\mathcal{F}_t)_{t\in[0,\infty)}$ be generated by $X$.

Furthermore, we make the following assumptions on $X$ and $\delta(\cdot)$.

**Assumption 3.2.1.** (i) $C := \sup \mathbb{X} < \infty$ and $\lambda := \sup_{x \in \mathbb{X}} \lambda_x < \infty$.

(ii) $X$ is irreducible, i.e., for any $x, y \in \mathbb{X}$, $\inf\{t \geq 0 : X_t = y \,|\, X_0 = x\} < \infty$, a.s..

**Assumption 3.2.2.** (i) $\delta$ is log-subadditive, i.e.,

$$\delta(s)\delta(t) \leq \delta(s+t), \quad \forall\, s, t > 0. \tag{3.9}$$

(ii) $t \mapsto \delta(t)$ is differentiable at $t = 0$, and $\delta'(0) < 0$.

*Remark* 3.2.3. Assumption A.2 (i) is closely related to *decreasing impatience* (*DI*) in Behavioral Finance and Economics. [1] Following [54, Definition 1] and [49], the discount function $\delta$ induces *DI* if

$$s \mapsto \frac{\delta(s+t)}{\delta(s)} \text{ is strictly increasing}, \quad \forall\, t > 0. \tag{3.10}$$

Observe that (3.10) implies (3.9), since $\delta(s+t)/\delta(s) \geq \delta(t)/\delta(0) = \delta(t)$ for all $s, t \geq 0$.

Note that hyperbolic, generalized hyperbolic, quasi-hyperboic, pseudo-exponential discount functions all induce DI, and thus satisfy Assumption A.2 (i). Consequently, (3.9) is often used when studying problems involving non-exponential discounting; see e.g., [31, 30, 33].

The following is the first main result of this project which shows that an optimal mild equilibrium is a strong equilibrium. The proof is provided in Section 2.1.

**Theorem 3.2.4.** *Let Assumptions A.1 and A.2 hold. If $S$ is an optimal mild equilibrium, then it is a strong equilibrium.*

[1] As mentioned in [31]: "It is well-documented in empirical studies, e.g. [45, 46, 57], that people admit *DI*: when choosing between two rewards, people are more willing to wait for the larger reward (more patient) when these two rewards are further away in time. For instance, in the two scenarios (i) getting \$100 today or \$110 tomorrow, and (ii) getting \$100 in 100 days or \$110 in 101 days, people tend to choose \$100 in (i), but \$110 in (ii)."

Since all strong equilibria are mild equilibria, an optimal mild equilibrium will generate larger values than any strong equilibrium as well. With Theorem 3.2.4, we can conclude that any optimal mild equilibrium is a strong equilibrium and in fact is an optimal strong equilibrium.

The following is the second main result of this chapter. It provides an iteration method which directly constructs an optimal mild equilibrium, and thus also establishes the existence of weak, strong, and optimal mild equilibria. The proof of this result is presented in Section 2.2.

**Theorem 3.2.5.** *Let $S_0 := \emptyset$, and*

$$S_{n+1} := S_n \cup \left\{ x \in \mathbb{X} \setminus S_n : x > \sup_{S: S_n \subset S \subset \mathbb{X} \setminus \{x\}} J(x, S) \right\}. \tag{3.11}$$

*Let*

$$S_\infty := \cup_{n=0}^{\infty} S_n. \tag{3.12}$$

*If Assumptions A.1 (i) and A.2 (i) hold, then $S_\infty$ is an optimal mild equilibrium. If in addition Assumption A.2 (ii) holds, then $S_\infty$ is a strong equilibrium.*

### 3.2.1 Proof of Theorem 3.2.4

Recall $\tau_S, \tau_S^\varepsilon, J(\cdot, \cdot)$ defined in (3.4),(3.7),(3.5) respectively. We have the following characterization of (3.6) in Definition 3.1.2.

**Proposition 3.2.6.** *Let Assumptions A.1 and A.2 (ii) hold. Then $S \subset \mathbb{X}$ is a weak equilibrium if and only if $S$ is a mild equilibrium and for all $x \in S$,*

$$x(\lambda_x - \delta'(0)) \geq \sum_{y \in S \setminus \{x\}} y q_{xy} + \sum_{y \in S^c} J(y, S) q_{xy}.$$

*Proof.* By definition, we only need to check condition (3.6) in Definition 3.1.2 is equivalent to the above inequality.

Denote $T_x := \inf\{t \geq 0 : X_t \neq x, X_0 = x\}$ as the holding time at state $x$, which has exponential distribution with parameter $\lambda_x$. Then for all $\varepsilon > 0$

$$
\begin{aligned}
\mathbb{E}_x[\delta(\tau_S^\varepsilon)X_{\tau_S^\varepsilon}] &= \mathbb{E}_x[\delta(\tau_S^\varepsilon)X_{\tau_S^\varepsilon}\mathbf{1}_{\{T_x>\varepsilon\}}] + \sum_{y\in\mathbb{X}\setminus\{x\}} \mathbb{E}_x[\delta(\tau_S^\varepsilon)X_{\tau_S^\varepsilon}\mathbf{1}_{\{T_x\leq\varepsilon, X_{T_x}=y, T_y+T_x>\varepsilon\}}] + O(\varepsilon^2) \\
&= \delta(\varepsilon)xe^{-\lambda_x\varepsilon} \\
&\quad + \left\{\sum_{y\in S\setminus\{x\}} \delta(\varepsilon)y\frac{q_{xy}}{\lambda_x} + \sum_{y\in S^c} \mathbb{E}_y[\delta(\varepsilon+\tau_S)X_{\tau_S}]\frac{q_{xy}}{\lambda_x}\right\}(\lambda_x\varepsilon + O(\varepsilon^2)) \\
&\quad + O(\varepsilon^2).
\end{aligned}
$$

Notice that $\delta(\varepsilon) = 1 + \delta'(0)\varepsilon + o(\varepsilon)$. Therefore we have

$$
\mathbb{E}_x[\delta(\tau_S^\varepsilon)X_{\tau_S^\varepsilon}] = x + \left\{-x(\lambda_x - \delta'(0)) + \sum_{y\in S\setminus\{x\}} yq_{xy} + \sum_{y\in S^c} q_{xy}\mathbb{E}_y[\delta(\varepsilon+\tau_S)X_{\tau_S}]\right\}\varepsilon + o(\varepsilon).
$$

Therefore, (3.6) is equivalent to

$$
x(\lambda_x - \delta'(0)) \geq \sum_{y\in S\setminus\{x\}} yq_{xy} + \sum_{y\in S^c} \mathbb{E}_y[\delta(\tau_S)X_{\tau_S}]q_{xy}.
$$

□

**Corollary 3.2.7.** *Let Assumptions A.1 and A.2 (ii) hold. If $S$ is a mild equilibrium and satisfies*

$$
x(\lambda_x - \delta'(0)) > \sum_{y\in S\setminus\{x\}} yq_{xy} + \sum_{y\in S^c} \mathbb{E}_y[\delta(\tau_S)X_{\tau_S}]q_{xy},
$$

*then it is a strong equilibrium.*

For the rest of the chapter we will sometimes use the notation

$$
\rho(x, S) := \inf\{t \geq 0 : X_t^x \in S\}
$$

in the place of $\tau_S$ to emphasize the initial state $X_0 = x$ ($X^x$ here is the Markov chain starting at $x$).

**Lemma 3.2.8.** *Let Assumption A.2 (i) hold. For $x \in S$, denote $\hat{S} = S \backslash \{x\}$. If $S$ is an optimal mild equilibrium, then for any $y \notin S$,*

$$J(y, \hat{S}) - J(y, S) \geq \mathbb{E}_y[\delta(\tau_S) \mathbf{1}_{\{X_{\tau_S}=x\}}](J(x, \hat{S}) - x).$$

*Proof.* Since $\hat{S} \subset S$, we have $\rho(y, S) \leq \rho(y, \hat{S})$. Then

$$\begin{aligned}
&J(y, \hat{S}) - J(y, S) \\
&= \mathbb{E}_y[\delta(\rho(y, \hat{S})) X_{\rho(y,\hat{S})} \mathbf{1}_{\{X_{\rho(y,S)}=x\}}] + \mathbb{E}_y[\delta(\rho(y, \hat{S})) X_{\rho(y,\hat{S})} \mathbf{1}_{\{X_{\rho(y,S)} \in \hat{S}\}}] \\
&\quad - \mathbb{E}_y[\delta(\rho(y, S)) X_{\rho(y,S)}] \\
&= \mathbb{E}_y[\delta(\rho(y, \hat{S})) X_{\rho(y,\hat{S})} \mathbf{1}_{\{X_{\rho(y,S)}=x\}}] + \mathbb{E}_y[\delta(\rho(y, S)) X_{\rho(y,S)} \mathbf{1}_{\{X_{\rho(y,S)} \in \hat{S}\}}] \\
&\quad - \mathbb{E}_y[\delta(\rho(y, S)) X_{\rho(y,S)}] \\
&= \mathbb{E}_y[\delta(\rho(y, \hat{S})) X_{\rho(y,\hat{S})} \mathbf{1}_{\{X_{\rho(y,S)}=x\}}] - x \mathbb{E}_y[\delta(\rho(y, S)) \mathbf{1}_{\{X_{\rho(y,S)}=x\}}] \\
&\geq \mathbb{E}_y[\delta(\rho(y, S)) \mathbf{1}_{\{X_{\rho(y,S)}=x\}} \mathbb{E}[\delta(\rho(y, \hat{S}) - \rho(y, S)) X_{\rho(x,\hat{S})} | \mathcal{F}_{\rho(y,S)}]] \\
&\quad - x \mathbb{E}_y[\delta(\rho(y, S)) \mathbf{1}_{\{X_{\rho(y,S)}=x\}}] \\
&= \mathbb{E}_y[\delta(\tau_S) \mathbf{1}_{\{X_{\tau_S}=x\}}](\mathbb{E}_x[\delta(\tau_{\hat{S}}) X_{\tau_{\hat{S}}}] - x),
\end{aligned}$$

where we use (3.9) for the inequality above. □

**Lemma 3.2.9.** *Let Assumption A.2 (i) hold. If $S$ is an optimal mild equilibrium, then for any $x \in S$ we have that*

$$x \geq J(x, \hat{S}), \quad \textit{where} \quad \hat{S} = S \backslash \{x\}.$$

*As a result, $0 \notin S$ and $J(y, S) > 0$ for all $y \in \mathbb{X}$.*

*Proof.* If $\hat{S}$ is also a mild equilibrium, then

$$x \leq J(x, \hat{S}) \leq J(x, S) = x,$$

and thus $x = J(x, \hat{S})$.

If $\hat{S}$ is not a mild equilibrium, then there exists $y \notin \hat{S}$ such that $J(y, \hat{S}) < y \leq J(y, S)$. By Lemma 3.2.8,

$$0 > J(y, \hat{S}) - J(y, S) \geq \mathbb{E}_y[\delta(\tau_S)\mathbb{I}_{\{X_{\tau_S}=x\}}](J(x, \hat{S}) - x),$$

which implies that

$$x > J(x, \hat{S}). \tag{3.13}$$

Now suppose $0 \in S$. By the above result, we have $0 \geq J(0, S\backslash\{0\})$. Since $X_{\tau_{S\backslash\{0\}}} > 0$, $J(0, S\backslash\{0\}) > 0$, which is a contraction. As a result, $0 \notin S$ and $J(y, S) > 0$ for all $y \in \mathbb{X}$. □

***Proof of Theorem 3.2.4***. By Assumption A.2, $\delta(t) \geq e^{\delta'(0)t}$ for all $t \geq 0$. Moreover, there exist $t_0 > 0$ such that for $t > t_0$, $\delta(t) > e^{\delta'(0)t}$ since $\delta$ is non-exponential. As a result, for any $x \in \mathbb{X}$,

$$\mathbb{E}_x[\delta(T_x)] = \int_0^\infty \lambda_x \delta(t) e^{-\lambda_x t} dt > \int_0^\infty \lambda_x e^{(\delta'(0)-\lambda_x)t} dt = \frac{\lambda_x}{\lambda_x - \delta'(0)}.$$

Denote $c_x := \frac{\lambda_x}{\lambda_x - \delta'(0)}$.

If $S = \{x\}$, then as $x \neq 0$ by Lemma 3.2.9 we have that

$$\sum_{y \neq x} J(y, S) q_{xy} \leq x \sum_{y \neq x} \mathbb{E}_y[\delta(T_y)] q_{xy} < x\lambda_x < x(\lambda_x - \delta'(0)),$$

which implies that $S$ is a strong equilibrium.

For the rest of the proof, we assume $S$ contains at least two points. Fix any $x \in S$, we have

$$J(x, \hat{S}) = \sum_{y \in S \setminus \{x\}} \frac{q_{xy}}{\lambda_x} \mathbb{E}_x[\delta(\tau_{\hat{S}}) X_{\tau_{\hat{S}}} | X_{T_x} = y] + \sum_{y \notin S} \frac{q_{xy}}{\lambda_x} \mathbb{E}_x[\delta(\tau_{\hat{S}}) X_{\tau_{\hat{S}}} | X_{T_x} = y].$$

Since for $y \in S \setminus \{x\}$,

$$\mathbb{E}_x[\delta(\tau_{\hat{S}}) X_{\tau_{\hat{S}}} | X_{T_x} = y] = y \mathbb{E}_x[\delta(\tau_{\hat{S}}) | X_{T_x} = y] = y \mathbb{E}_x[\delta(T_x) | X_{T_x} = y] = y \mathbb{E}_x[\delta(T_x)],$$

and for $y \in S^c$,

$$\begin{aligned}
&\mathbb{E}_x[\delta(\tau_{\hat{S}}) X_{\tau_{\hat{S}}} | X_{T_x} = y] \\
\geq &\mathbb{E}_x[\delta(T_x) \delta(\tau_{\hat{S}} - T_x) X_{\tau_{\hat{S}}} | X_{T_x} = y] \\
= &\mathbb{E}_x[\delta(T_x) | X_{T_x} = y] \cdot \mathbb{E}_x[\delta(\tau_{\hat{S}} - T_x) X_{\tau_{\hat{S}}} | X_{T_x} = y] = \mathbb{E}_x[\delta(T_x)] \cdot J(y, \hat{S}),
\end{aligned}$$

we have that

$$J(x, \hat{S}) \geq \left( \sum_{y \in S \setminus \{x\}} \frac{q_{xy}}{\lambda_x} y + \sum_{y \notin S} \frac{q_{xy}}{\lambda_x} J(y, \hat{S}) \right) \cdot \mathbb{E}_x[\delta(T_x)]. \tag{3.14}$$

Denote

$$\mathrm{I} := \sum_{y \in S \setminus \{x\}} \frac{q_{xy}}{\lambda_x} y, \quad \mathrm{II} := \sum_{y \notin S} \frac{q_{xy}}{\lambda_x} J(y, S), \quad \hat{\mathrm{II}} := \sum_{y \notin S} \frac{q_{xy}}{\lambda_x} J(y, \hat{S}).$$

By Lemma 3.2.9, $y > 0$ for all $y \in \hat{S}$ and $J(y, \hat{S}) > 0$ for all $y \notin \hat{S}$, thus $\mathrm{I} + \hat{\mathrm{II}} > 0$. This together with $\mathbb{E}_x[\delta(T_x)] > c_x$ implies that

$$J(x, \hat{S}) > (\mathrm{I} + \hat{\mathrm{II}}) c_x.$$

Then

$$
\begin{aligned}
x - J(x,\hat{S}) &< x - (\mathrm{I} \ + \hat{\mathrm{II}})c_x \\
&= x - (\mathrm{I} \ + \mathrm{II})c_x + (\mathrm{II} - \hat{\mathrm{II}})c_x \\
&= x - (\mathrm{I} \ + \mathrm{II})c_x + c_x \sum_{y \notin S} \frac{q_{xy}}{\lambda_x}(J(y,S) - J(y,\hat{S})) \\
&\leq x - (\mathrm{I} \ + \mathrm{II})c_x + c_x \sum_{y \notin S} \frac{q_{xy}}{\lambda_x}(\mathbb{E}_y[\delta(\tau_S)\mathbf{1}_{\{X_{\tau_S}=x\}}](x - J(x,\hat{S})),
\end{aligned}
$$

where the last line follows from Lemma 3.2.8. Thus

$$
\left(1 - c_x \sum_{y \notin S} \frac{q_{xy}}{\lambda_x}(\mathbb{E}_y[\delta(\tau_S)\mathbf{1}_{\{X_{\tau_S}=x\}}])\right)(x - J(x,\hat{S})) < x - (\mathrm{I} + \mathrm{II})c_x. \tag{3.15}
$$

Notice that

$$
c_x \sum_{y \notin S} \frac{q_{xy}}{\lambda_x}(\mathbb{E}_y[\delta(\tau_S)\mathbf{1}_{\{X_{\tau_S}=x\}}] \leq c_x \sum_{y \notin S} \frac{q_{xy}}{\lambda_x} \leq c_x < 1.
$$

Then by Lemma 3.2.9,

$$
x - (\mathrm{I} + \mathrm{II})c_x > 0, \quad \forall\, x \in \mathbb{X},
$$

which implies $S$ is a strong equilibrium. □

### 3.2.2 Proof of Theorem 3.2.5

We start with the following lemma, which in particular indicates that a mild equilibrium with smaller stopping region generates larger values.

**Lemma 3.2.10.** *Let Assumption A.2 (i) hold. If $S$ is a mild equilibrium, then for any subset $R \subset \mathbb{X}$ with $S \subset R$, we have*

$$
J(x,S) \geq J(x,R), \quad \forall x \in \mathbb{X}.
$$

*Proof.* Since $S \subset R$, $\rho(x,S) \geq \rho(x,R)$ for all $x \in \mathbb{X}$.

$$\begin{aligned}J(x,S) &= \mathbb{E}_x[\delta(\rho(x,S))X_{\rho(x,S)}]\\ &= \mathbb{E}_x[\mathbb{E}_x[\delta(\rho(x,S))X_{\rho(x,S)}|\mathcal{F}_{\rho(x,R)}]]\\ &\geq \mathbb{E}_x[\delta(\rho(x,R))\mathbb{E}_x[\delta(\rho(x,S)-\rho(x,R))X_{\rho(x,S)}|\mathcal{F}_{\rho(x,R)}]]\\ &= \mathbb{E}_x[\delta(\rho(x,R))\mathbb{E}_{X_{\rho(x,R)}}[\delta(\rho(X_{\rho(x,R)},S))X_{\rho(x,S)}]]\\ &\geq \mathbb{E}_x[\delta(\rho(x,R))X_{\rho(x,R)}] = J(x,R).\end{aligned}$$

The last inequality holds because $S$ is a mild equilibrium and by definition,

$$\mathbb{E}_{X_{\rho(x,R)}}[\delta(\rho(X_{\rho(x,R)},S))X_{\rho(x,S)}] \geq X_{\rho(x,R)}.$$

□

**Corollary 3.2.11.** *Let Assumption A.2 (i) hold. If $S$ is the smallest mild equilibrium, i.e. $S \subset \widetilde{S}$ for any mild equilibrium $\widetilde{S}$, then $S$ is an optimal mild equilibrium.*

Thanks to this corollary, in order to show $S_\infty$ defined in (3.12) is an optimal mild equilibrium, it suffices to show that $S_\infty$ is the smallest one. [2]

Recall $S_n$ defined in (3.11). We have the following lemma.

**Lemma 3.2.12.** *For any mild equilibrium $R$, we have that $S_n \subset R$ for all $n \in \mathbb{N}$.*

*Proof.* We prove this lemma by induction. First $S_0 \subset R$. Suppose $S_n \subset R$ for $n \geq 0$. Since $R$ is a mild equilibrium, for any $x \notin R$,

$$x \leq J(x,R) \leq \sup_{S: S_n \subset S \subset \mathbb{X}\setminus\{x\}} J(x,S).$$

[2]This implies the uniqueness of optimal mild equilibrium. If $S^*$ is an optimal mild equilibrium, then $S^*$ is a mild equilibrium, thus contains the smallest mild equilibrium. By Lemma 3.2.10, optimality of S* implies S* is contained in any mild equilibrium. So S* is the smallest mild equilibrium, which is unique.

Therefore $x \notin S_{n+1}$. As a result, $S_{n+1} \subset R$ for all $n \in \mathbb{N}$. $\square$

**Lemma 3.2.13.** *Let Assumption A.1 (i) hold. For $y \notin S_\infty$, denote*

$$V_n := \sup_{S: S_n \subset S \subset \mathbb{X}\setminus\{y\}} J(y,S), \qquad V_\infty := \sup_{S: S_\infty \subset S \subset \mathbb{X}\setminus\{y\}} J(y,S),$$

*then we have $V_n \searrow V_\infty, n \to \infty$.*

*Proof.* Since $S_\infty = \bigcup_{n\geq 1} S_n$, we have $\rho(y, S_\infty \backslash S_n) \to \infty, n \to \infty$. Then for any $\varepsilon > 0$, there exists $N = N(\varepsilon, y)$ such that for $n > N$, $\mathbb{E}_y[\delta(\tau_{S_\infty\backslash S_n})] < \varepsilon$ since $\lim_{t\to\infty} \delta(t) = 0$.

For any $R_n$ such that $S_n \subset R_n \subset \mathbb{X}\backslash\{y\}$, denote $\overline{R_n} := R_n \bigcup S_\infty$, then we have,

$$\begin{aligned}
J(y,R_n) - J(y,\overline{R_n}) &= \mathbb{E}_y[(\delta(\tau_{R_n})X_{\tau_{R_n}} - \delta(\tau_{\overline{R_n}})X_{\tau_{\overline{R_n}}})\mathbf{1}_{\{X_{\tau_{\overline{R_n}}} \in S_\infty\backslash R_n\}}] \\
&\leq C\mathbb{E}_y[\delta(\tau_{R_n})\mathbf{1}_{\{X_{\tau_{\overline{R_n}}} \in S_\infty\backslash R_n\}}] \\
&\leq C\mathbb{E}_y[\delta(\tau_{S_\infty\backslash R_n})\mathbf{1}_{\{X_{\tau_{\overline{R_n}}} \in S_\infty\backslash R_n\}}] \\
&\leq C\varepsilon
\end{aligned}$$

Since $S_\infty \subset \overline{R_n} \subset \mathbb{X}\backslash\{y\}$, by definition, $J(y, \overline{R_n}) \leq V_\infty$. Therefore we have that for any $\varepsilon > 0$, there exists $N$ such that for any $n \geq N$,

$$V_n = \sup_{R_n: S_n \subset R_n \subset \mathbb{X}\backslash\{y\}} J(y, R_n) \leq V_\infty + C\varepsilon.$$

Clearly $S_n \subset S_{n+1}$ implies that $V_n$ is non-increasing and $V_n \geq V_\infty$ for all $n$. This completes the proof that $V_n \searrow V_\infty, n \to \infty$. $\square$

***Proof of Theorem 3.2.5.*** By Corollary 3.2.11 and Lemma 3.2.12, to show that $S_\infty$ is an optimal mild equilibrium, it suffices to show $S_\infty$ is a mild equilibrium.

Suppose $S_\infty$ is not a mild equilibrium. Then

$$\alpha := \sup_{x\in\mathbb{X}}\{x - J(x, S_\infty)\} > 0.$$

For any $\varepsilon > 0$, there exists $y \notin S_\infty$ such that $y - J(y, S_\infty) \geq \alpha - \varepsilon$. Since $y \notin S_n$ for all $n \geq 0$, we have

$$y \leq \sup_{S: S_n \subset S \subset \mathbb{X} \setminus \{y\}} J(y, S), \quad \forall n \geq 0.$$

By Lemma 3.2.13,

$$y \leq \sup_{S: S_\infty \subset S \subset \mathbb{X} \setminus \{y\}} J(y, S).$$

Thus, there exists subset $R$ with $S_\infty \subset R \subset \mathbb{X} \setminus \{y\}$ such that

$$y \leq J(y, R) + \varepsilon.$$

Then we have $J(y, R) - J(y, S_\infty) \geq y - \varepsilon + \alpha - \varepsilon - y = \alpha - 2\varepsilon$. Since $S_\infty \subset R$, $\rho(y, S_\infty) \geq \rho(y, R)$. It follows that

$$\begin{aligned}
J(y,R) - J(y,S_\infty) &= \mathbb{E}_y[\delta(\rho(y,R))X_{\rho(y,R)}] - \mathbb{E}_y[\mathbb{E}_y[\delta(\rho(y,S_\infty))X_{\rho(y,S_\infty)}|\mathcal{F}_{\rho(y,R)}]] \\
&\leq \mathbb{E}_y[\delta(\rho(y,R))X_{\rho(y,R)}] \\
&\quad - \mathbb{E}_y[\delta(\rho(y,R))\mathbb{E}_y[\delta(\rho(y,S_\infty) - \rho(y,R))X_{\rho(y,S_\infty)}|\mathcal{F}_{\rho(y,R)}]] \\
&= \mathbb{E}_y[\delta(\rho(y,R))(X_{\rho(y,R)} - \mathbb{E}_{X_{\rho(y,R)}}[\delta(\rho(X_{\rho(y,R)}, S_\infty))X_{\rho(X_{\rho(y,R)},S_\infty)}]) \\
&\leq \mathbb{E}_y[\delta(\rho(y,R))]\alpha \\
&\leq \mathbb{E}_y[\delta(T_y)]\alpha.
\end{aligned}$$

By Assumption A.2 (i), $\lambda = \sup_{x \in \mathbb{X}} \lambda_x < \infty$ and since $y \notin R$, we have $0 < \mathbb{E}_y[\delta(T_y)] < c < 1$ where $c = \int_0^\infty \delta(t)\lambda e^{-\lambda t} dt$. By choosing $0 < \varepsilon \leq \frac{\alpha(1-c)}{2}$, we obtain a contradiction.

Next let us prove $S_\infty$ is a strong equilibrium. If $X$ is irreducible, then $S_\infty$ is a strong equilibrium by Theorem 3.2.4. In general, following the proof for Proposition 3.2.6, to show $S_\infty$ is a strong equilibrium, it suffices to show that for any $x \in S_\infty$ with

$\lambda_x > 0$,

$$x(\lambda_x - \delta'(0)) > \sum_{y \in S_\infty \setminus \{x\}} y q_{xy} + \sum_{y \in S_\infty^c} \mathbb{E}_y[\delta(\tau_S) X_{\tau_S}] q_{xy}. \tag{3.16}$$

Take $x \in S_\infty$ with $\lambda_x > 0$. Following the argument for (3.14), we have that

$$J(x, \hat{S}_\infty) \geq \left( \sum_{y \in S_\infty \setminus \{x\}} \frac{q_{xy}}{\lambda_x} y + \sum_{y \notin S_\infty} \frac{q_{xy}}{\lambda_x} J(y, \hat{S}_\infty) \right) \cdot \mathbb{E}_x[\delta(T_x)],$$

where $\hat{S}_\infty = S_\infty \setminus \{x\}$. Using an argument similar to that for (3.15), we have that

$$\left( 1 - c_x \sum_{y \notin S_\infty} \frac{q_{xy}}{\lambda_x} (\mathbb{E}_y[\delta(\tau_{S_\infty}) \mathbf{1}_{\{X_{\tau_{S_\infty}} = x\}}]) \right) (x - J(x, \hat{S}_\infty)) \leq x - (\mathrm{I}_\infty + \mathrm{II}_\infty) c_x,$$

where

$$\mathrm{I}_\infty := \sum_{y \in S_\infty \setminus \{x\}} \frac{q_{xy}}{\lambda_x} y \quad \text{and} \quad \mathrm{II}_\infty := \sum_{y \notin S_\infty} \frac{q_{xy}}{\lambda_x} J(y, S_\infty).$$

Since $S_\infty$ is the smallest mild equilibrium, $\hat{S}_\infty$ is not a mild equilibrium. Then $x > J(x, \hat{S}_\infty)$ by (3.13). Therefore,

$$x - (\mathrm{I}_\infty + \mathrm{II}_\infty) c_x > 0,$$

which implies (3.16). □

## 3.3 Examples illustrating the iteration method in Theorem 3.2.5

In this section, we provide examples to demonstrate the iteration method in Theorem 3.2.5.

The next proposition shows that the iteration method in Theorem 3.2.5 will terminate within one step in the case of time consistency and leads to an optimal stopping time.

**Proposition 3.3.1.** *If $\delta(s)\delta(t) = \delta(s+t)$ for all $t, s \geq 0$ and Assumptions A.1 (i) holds, then $S_1 = S_n$ for all $n \geq 2$ and $S_1$ is an optimal stopping strategy.*

*Proof.* By definition, $S_0 = \emptyset$ and

$$S_1 = \{x \in \mathbb{X} : x > \sup_{S \subset \mathbb{X} \backslash \{x\}} J(x, S)\}.$$

We show that for any $x \notin S_1$ and any set $R \subset \mathbb{X} \backslash \{x\}$, we have $J(x, R) \leq J(x, \tilde{R})$ where $\tilde{R} = R \cup S_1 = R \cup (S_1 \backslash R)$.

For $S_1 \subset R$, $R = \tilde{R}$ and $J(x, R) \leq J(x, \tilde{R})$ holds trivially.

For $S_1 \not\subset R$, denote $\gamma = \tau_R$ and $\tilde{\gamma} = \tau_{\tilde{R}}$. Then a.s. $\gamma \geq \tilde{\gamma}$. We have

$$\begin{aligned}
J(x, R) - J(x, \tilde{R}) &= \mathbb{E}_x[\delta(\gamma) X_\gamma] - \mathbb{E}_x[\delta(\tilde{\gamma}) X_{\tilde{\gamma}}] \\
&= \mathbb{E}_x[\delta(\gamma) X_\gamma (\mathbf{1}_{\{X_{\tilde{\gamma}} \in R\}} + \mathbf{1}_{\{X_{\tilde{\gamma}} \notin R\}})] - \mathbb{E}_x[\delta(\tilde{\gamma}) X_{\tilde{\gamma}}] \\
&= \mathbb{E}_x[(\delta(\gamma) X_\gamma - \delta(\tilde{\gamma}) X_{\tilde{\gamma}}) \mathbf{1}_{\{X_{\tilde{\gamma}} \notin R\}}] \\
&= \mathbb{E}_x[\mathbb{E}[(\delta(\gamma) X_\gamma - \delta(\tilde{\gamma}) X_{\tilde{\gamma}}) \mathbf{1}_{\{X_{\tilde{\gamma}} \notin R\}} | \mathcal{F}_{\tilde{\gamma}}]] \\
&= \mathbb{E}_x[\mathbb{E}_{X_{\tilde{\gamma}}}[\delta(\gamma) X_\gamma] \delta(\tilde{\gamma}) \mathbf{1}_{\{X_{\tilde{\gamma}} \notin R\}} - \delta(\tilde{\gamma}) X_{\tilde{\gamma}} \mathbf{1}_{\{X_{\tilde{\gamma}} \notin R\}}] \\
&= \mathbb{E}_x[\delta(\tilde{\gamma}) \mathbf{1}_{\{X_{\tilde{\gamma}} \notin R\}} (\mathbb{E}_{X_{\tilde{\gamma}}}[\delta(\gamma) X_\gamma] - X_{\tilde{\gamma}})] \\
&\leq 0,
\end{aligned}$$

since on $\{X_{\tilde{\gamma}} \notin R\}$, $X_{\tilde{\gamma}} \in S_1$ and $X_{\tilde{\gamma}} > \mathbb{E}_{X_{\tilde{\gamma}}}[\delta(\gamma) X_\gamma]$.

As a result, $J(x, R) \leq \sup_{S: S_1 \subset S \subset \mathbb{X} \backslash \{x\}} J(x, S)$ for all $R \subset \mathbb{X} \backslash \{x\}$. Thus for any $x \notin S_1$, $x \leq \sup_{R: R \subset \mathbb{X} \backslash \{x\}} J(x, R) \leq \sup_{S: S_1 \subset S \subset \mathbb{X} \backslash \{x\}} J(x, S)$, which implies $S_1 = S_2 = S_\infty$.

Next we show that for all $x \in \mathbb{X}$,

$$J(x, S_1) \geq J(x, S), \qquad \forall S \subset \mathbb{X}.$$

For $x \in S_1$, by definition of $S_1$, $x > \sup_{S \subset \mathbb{X} \setminus \{x\}} J(x, S)$. Therefore $x \geq \sup_{S \subset \mathbb{X}} J(x, S)$.

For $x \notin S_1$, for any $S \subset \mathbb{X}$, let $\tilde{S} = S \cup S_1$. Since $S \subset \tilde{S}$ and $S_1 \subset \tilde{S}$, we have a.s. $\tau_S \geq \tau_{\tilde{S}}$ and $\tau_{S_1} \geq \tau_{\tilde{S}}$. By similar arguments as above, we obtain

$$J(x, \tilde{S}) - J(x, S) = \mathbb{E}_x[\delta(\tau_{\tilde{S}}) \mathbf{1}_{\{X_{\tilde{S}} \notin S\}} (X_{\tau_{\tilde{S}}} - \mathbb{E}_{X_{\tau_{\tilde{S}}}}[\delta(\tau_S) X_{\tau_S}])] \geq 0,$$

and

$$J(x, S_1) - J(x, \tilde{S}) = \mathbb{E}_x[\delta(\tau_{\tilde{S}}) \mathbf{1}_{\{X_{\tilde{S}} \notin S_1\}} (\mathbb{E}_{X_{\tau_{\tilde{S}}}}[\delta(\tau_{S_1}) X_{\tau_{S_1}}] - X_{\tau_{\tilde{S}}})] \geq 0.$$

Therefore $J(x, S_1) \geq J(x, \tilde{S}) \geq J(x, S)$. □

In the case of time inconsistency, the above result generally does not hold. The next example demonstrates an application of the iteration method in Theorem 3.2.5.

**Example 3.3.2.** Consider hyperbolic discount function $\delta(t) = \frac{1}{1+\beta t}$ for $\beta > 0$ and $\mathbb{X} = \{x_1, x_2, x_3, x_4\}$, whose generator is given by

$$Q = \begin{bmatrix} -\lambda_1 & q_{12} & q_{13} & q_{14} \\ q_{21} & -\lambda_2 & q_{23} & q_{24} \\ q_{31} & q_{23} & -\lambda_3 & q_{34} \\ q_{41} & q_{42} & q_{43} & -\lambda_4 \end{bmatrix} = \begin{bmatrix} -3 & 1 & 1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0.4 & -2 & 1.6 \\ 1 & 1 & 1 & -3 \end{bmatrix}.$$

Let $\beta = 3$, $x_1 = 10, x_2 = 40, x_3 = 46, x_4 = 100$.

Next we show that by applying the iteration method, we have $S_0 = \emptyset, S_1 = \{x_2, x_4\}, S_2 = \{x_2, x_3, x_4\} = S_\infty$.

Denote $T_i := \inf\{t \geq 0 : X_t \neq x_i | X_0 = x_i\}$.

(i) Since $100 = x_4 > x_3 = 46 \geq \sup_{S \subset \mathbb{X} \setminus \{x_4\}} J(x_4, S)$, we have that $x_4 \in S_1$.

(ii) For $x_3$, consider $S = \{x_4\}$.

$$\begin{aligned}
J(x_3, \{x_4\}) &= \mathbb{E}_{x_3}[\delta(\tau_{\{x_4\}})X_{\tau_{\{x_4\}}}] \\
&= x_4(\frac{q_{34}}{\lambda_3}\mathbb{E}_{x_3}[\delta(T_3)] + \frac{q_{32}}{\lambda_3}\mathbb{E}_{x_3}[\mathbb{E}[\delta(T_2+T_3)|X_{T_3} = x_2]]) \\
&= 100(0.8\int_0^\infty \frac{2}{1+3t}e^{-2t}dt + 0.2\int_0^\infty\int_0^\infty \frac{2}{1+3(t+s)}e^{-t-2s}dtds) \\
&\doteq 100(0.8 \times 0.5173 + 0.2 \times 0.2539) = 46.46.
\end{aligned}$$

Therefore $x_3 = 46 < J(x_3, \{x_4\}) \leq \sup_{S \subset \mathbb{X} \setminus \{x_3\}} J(x_3, S)$ and $x_3 \notin S_1$.

(iii) Note that $\sup_{S \subset \mathbb{X} \setminus \{x_2\}} J(x_2, S) \leq x_4 \mathbb{E}_{x_2}[\delta(T_2)]$. We have

$$\mathbb{E}_{x_2}[\delta(T_2)] = \int_0^\infty \frac{1}{1+3t}e^{-t}dt \doteq 0.3856.$$

Therefore $x_2 = 40 > 0.3856 \times 100 = x_4\mathbb{E}_{x_2}[\delta(T_2)] \geq \sup_{S \subset \mathbb{X} \setminus \{x_2\}} J(x_2, S)$ and $x_2 \in S_1$.

(iv) For $x_1$, consider $S = \{x_2, x_3, x_4\}$.

$$\begin{aligned}
J(x_1, \{x_2, x_3, x_4\}) &= \mathbb{E}_{x_1}[\delta(T_1)X_{\tau_{\{x_2,x_3,x_4\}}}] \\
&= \frac{1}{3}(x_2 + x_3 + x_4)\mathbb{E}_{x_1}[\delta(T_1)] \\
&= 62 \times \int_0^\infty \frac{3}{1+3t}e^{-3t}dt \doteq 62 \times 0.5963.
\end{aligned}$$

Thus $x_1 = 10 \leq J(x_1, \{x_2, x_3, x_4\}) \leq \sup_{S \subset \mathbb{X} \setminus \{x_1\}} J(x_1, S)$ and $x_1 \notin S_1$.

(v) By (iv), $x_1 \notin S_2$ given that $S_1 = \{x_2, x_4\}$.

(vi) To show that $x_3 \in S_2$, we only need to show that $J(x_3, \{x_2, x_4\}) < x_3$ since

$S_1 = \{x_2, x_4\}$ and $q_{31} = 0$.

$$\begin{aligned} J(x_3, \{x_2, x_4\}) &= (\frac{q_{32}}{\lambda_3} x_2 + \frac{q_{34}}{\lambda_3} x_4) \mathbb{E}_{x_3}[\delta(T_3)] \\ &= (0.2 \times 40 + 0.8 \times 100) \int_0^\infty \frac{2}{1+3t} e^{-2t} dt \\ &\doteq 88 \times 0.5173 = 45.52 \end{aligned}$$

Thus $x_3 \in S_2$.

(vii) Again by (iv), $x_1 \notin S_3$ given that $S_2 = \{x_2, x_3, x_4\}$. Therefore $S_n = S_2 = \{x_2, x_3, x_4\}$ for $n \geq 2$.

### 3.3.1 Example 3.2

In this example, process $X$ has infinite state space and can be viewed as the payoff of some American option. Consider a stock price process $Y$ that takes values in $\mathbb{Y} := \{u^i : i \in \mathbb{Z}\}$ for some fixed $u > 1$. There exists $\lambda > 0$ and $p \in [\frac{1}{1+u}, 1)$ such that

$$q_{u^i u^{i+1}} = p\lambda, \quad q_{u^i u^{i-1}} = (1-p)\lambda, \quad \forall i \in \mathbb{Z}.$$

Let the discount function be $\delta(t) = \frac{1}{1+\beta t}$ for some constant $\beta > 0$ and let the payoff process be $X = f(Y)$ for some payoff function $f(y) = (K - y)^+$, where $K$ is a positive constant. Since $f$ is bounded and nonnegative, our results still holds when we have $X = f(Y)$. Next we will show how to use the iteration method to find an optimal mild equilibrium in this problem.

**Lemma 3.3.3.** $S_1 = \{u^i \in (0, K) : K - u^i > J(u^i, \{u^m\}), \forall m < i, m \in \mathbb{Z}, i \in \mathbb{Z}\}$.

*Proof.* Since for any $u^i \geq K$, $f(u^i) = (K - u^i)^+ = 0 \leq J(u^i, (0, K) \cap \mathbb{Y})$, we obtain that $S_1 \subset (0, K)$. Thus we only consider $u^i \in (0, K)$ and we show that $u^i \in S_1$ if and only if $u^i \in (0, K)$ and $K - u^i > J(u^i, \{u^l\})$ for all $l < i$.

"$\Longrightarrow$": Take $u^i \in S_1$. Then obviously $K - u^i > J(u^i, \{u^l\})$ for all $l < i$.

"$\Longleftarrow$": Take $u^i \in (0, K)$. For any nonempty set $S \subset \mathbb{Y}\backslash\{u^i\}$, there are three cases.

Case 1: $S \in A := \{\tilde{S} \subset \mathbb{Y}\backslash\{u^i\} : \tilde{S} \cap (0, u^i) = \emptyset \text{ and } \tilde{S} \cap (u^i, \infty) \neq \emptyset\}$. Let $u^r = \min(S \cap (u^i, \infty))$. Then $J(u^i, S) = J(u^i, u^r)$. Since $u^r > u^i$, we have $f(u^i) = K - u^i > J(u^i, \{u^r\}) = (K - u^r)^+ \mathbb{E}_{u^i}[\delta(\tau_{\{u^r\}})]$. Then obviously we have that

$$f(u^i) > \sup_{S \in A} J(u^i, S).$$

Case 2: $S \in B := \{\tilde{S} \subset \mathbb{Y}\backslash\{u^i\} : \tilde{S} \cap (0, u^i) \neq \emptyset \text{ and } \tilde{S} \cap (u^i, \infty) = \emptyset\}$. Note that for any $n \in \mathbb{Z}$ such that $n < i$,

$$K - u^i > \sup_{n \leq k \leq i-1} J(u^i, \{u^k\}).$$

Moreover, $\lim_{n \to \infty} J(u^i, \{u^n\}) = 0$. Thus

$$K - u^i > \sup_{k \leq i-1} J(u^i, \{u^k\}). \tag{3.17}$$

Now let $u^l = \max(S \cap (0, u^i))$. Then $J(u^i, S) = J(u^i, \{u^l\})$. Thus by (3.17),

$$f(u^i) > \sup_{S \in B} J(u^i, S).$$

Case 3: $S \in C := \{\tilde{S} \subset \mathbb{Y}\backslash\{u^i\} : \tilde{S} \cap (0, u^i) \neq \emptyset \text{ and } \tilde{S} \cap (u^i, \infty) \neq \emptyset\}$. Let $u^l = \max(S \cap (0, u^i))$ and $u^r = \min(S \cap (u^i, \infty))$.

If $u^r \leq K$, observe that $Y$ is a submartingale, so $u^i \leq \mathbb{E}_{u^i}[Y_{\tau_{\{u^l, u^r\}}}] = \mathbb{E}_{u^i}[Y_{\tau_S}]$. Thus $f(u^i) = K - u^i \geq \mathbb{E}_{u^i}[(K - Y_{\tau_S})^+] = \mathbb{E}_{u^i}[(K - Y_{\tau_S})^+] > \mathbb{E}_{u^i}[\delta(\tau_{\{u^l, u^r\}})(K - Y_{\tau_S})^+]$.

If $u^r > K$, then

$$
\begin{aligned}
J(u^i, S) &= \mathbb{E}_{u^i}[\delta(\tau_{\{u^l\}})(K-u^l)^+ \mathbf{1}_{\{\tau_{\{u^l\}}<\tau_{\{u^r\}}\}}] + \mathbb{E}_{u^i}[\delta(\tau_{\{u^r\}})(K-u^r)^+ \mathbf{1}_{\{\tau_{\{u^r\}}<\tau_{\{u^l\}}\}}] \\
&= \mathbb{E}_{u^i}[\delta(\tau_{\{u^l\}})(K-u^l) \mathbf{1}_{\{\tau_{\{u^l\}}<\tau_{\{u^r\}}\}}] \\
&\le \mathbb{E}_{u^i}[\delta(\tau_{\{u^l\}})(K-u^l)] \\
&= J(u^i, \{u^l\}) \\
&\le \sup_{S\in B} J(u^i, S).
\end{aligned}
$$

Therefore,

$$f(u^i) > \sup_{S\in C} J(u^i, S).$$

This completes the proof. □

Fix $m, i \in \mathbb{Z}$ such that $m < i < \log_u K$. $J(u^i, \{u^m\}) = (K-u^m)\mathbb{E}_{u^i}[\delta(\tau_{\{u^m\}})]$. Since $(q_{u^iu^j})_{j\neq i}$ are the same for each $i \in \mathbb{Z}$, we have $\mathbb{E}_{u^i}[\delta(\tau_{\{u^m\}})] = \mathbb{E}_{u^{i-k}}[\delta(\tau_{\{u^{m-k}\}})]$ for any $k \in \mathbb{N}$. Therefore denote $\alpha_{i-m} := \mathbb{E}_{u^i}[\delta(\tau_{\{u^m\}})]$. Note that $\alpha_n, n \in \mathbb{N}$ can be computed explicitly. For example,

$$\alpha_1 = \sum_{k=1}^{\infty} \frac{\binom{2k-1}{k} p^{k-1}(1-p)^k}{2k-1} \cdot \int_0^{\infty} \frac{1}{1+\beta t} g(t, 2k-1)dt,$$

where $g(t,n) = \frac{\lambda^n}{(n-1)!} t^{n-1} e^{-\lambda t}$ is the density function of gamma distribution with shape parameter $n$ and rate parameter $\lambda$.

**Proposition 3.3.4.** $S_\infty = \{u^i : i \le n_0\}$ *where* $n_0 = \lceil \log_u(\frac{1-\alpha_1}{u-\alpha_1}K)\rceil$.

*Proof.* Since for any $u^i \ge K$ and any $S \subset \mathbb{Y}\backslash\{u^i\}$, $f(u^i) = 0 \le \sup J(u^i, S)$, $S_\infty \subset (0, K)\cap\mathbb{Y}$. In the following we only consider $u^i$ with $i \le \lfloor \log_u K \rfloor$. Consider sequence $\{\frac{K-u^m}{K}\}_{m\le\lfloor\log_u K\rfloor}$. It is easy to check that

$$\frac{K-u^{m-1}}{K} > \frac{K-u^m}{K} > 0, \quad \forall m \le \lfloor \log_u K \rfloor,$$

and $\lim_{m\to-\infty}\frac{K-u^m}{K}=1$. Then there exists $m_0 \le \lfloor \log_u K \rfloor$ such that

$$\frac{K-u^{m_0}}{K} > \alpha_1 \ge \frac{K-u^{m_0+1}}{K}.$$

Then $K-u^{m_0} > K\alpha_1 > (K-u^m)\alpha_{m_0-m},\ \forall m < m_0$. By Lemma 3.3.3, $u^{m_0} \in S_1$. Since $K-u^m > K-u^{m_0} > K\alpha_1$ for all $m < m_0$, by similar argument, $u^m \in S_1, \forall m < m_0$. Therefore

$$\{u^m : m \le m_0\} \subset S_1.$$

Consider the sequence $\{\frac{K-u^n}{K-u^{n-1}}\}_{n\le\lfloor \log_u K\rfloor}$. It is easy to check that

$$\frac{K-u^{n-1}}{K-u^{n-2}} > \frac{K-u^n}{K-u^{n-1}} \ge 0, \quad \forall n \le \lfloor \log_u K \rfloor,$$

and $\lim_{n\to-\infty}\frac{K-u^n}{K-u^{n-1}}=1$. Then there exists $n_0 \le \lfloor \log_u K \rfloor$ such that

$$\frac{K-u^{n_0}}{K-u^{n_0-1}} > \alpha_1 \ge \frac{K-u^{n_0+1}}{K-u^{n_0}}. \tag{3.18}$$

Then for any $n \ge n_0+1$, $(K-u^n)^+ \le (K-u^{n-1})^+\alpha_1$. Thus $u^n \notin S_1,\ \forall n > n_0$. That is

$$\{u^m : m \le m_0\} \subset S_1 \subset \{u^m : m \le n_0\}$$

Next we claim that for all $n \in \mathbb{N}$, $S_n \subset \{u^m : m \le n_0\}$. We will prove this claim by induction. By the above discussion, this claim holds for $n = 1$. Suppose $S_n \subset \{u^m : m \le n_0\}$ for $n \ge 1$. Then for any $m > n_0$,

$$(K-u^m)^+ \le (K-u^{m-1})^+\alpha_1 \le \sup_{S:S_n\subset S\subset \mathbb{Y}\setminus\{u^m\}} J(u^m, S),$$

which implies $u^m \notin S_{n+1}$ for all $m > n_0$ and $S_{n+1} \subset \{u^m : m \le n_0\}$. As a result, $S_\infty \subset \{u^m : m \le n_0\}$.

If $m_0 = n_0$, then we have $S_1 = \{u^m : m \le n_0\} \subset S_\infty$. Thus $S_\infty = \{u^m : m \le n_0\}$.

If $m_0 < n_0$, let $k = n_0 - m_0$. Consider $u^{m_0+i}, i \in \{0, 1, 2, \cdots, k\}$. We claim that $u^{m_0+i} \in S_{i+1}$. Then we obtain $\{u^m : m \le n_0\} \subset S_\infty$.

Next we will prove this claim by induction. The claim holds for $i = 0$. Suppose $u^{m_0+i} \in S_{i+1}$. Then $\{u^m : m \le m_0 + i\} \subset S_{i+1}$. Consider the case when $u^{m_0+i+1} \notin S_{i+1}$. Note that $\sup_{S:S_{i+1}\subset S\subset \mathbb{Y}\setminus\{u^{m_0+i+1}\}} J(u^{m_0+i+1}, S) \le$

$$\max_{m_0+i+2\le k\le n_0} J(u^{m_0+i+1}, \{u^{m_0+i}, u^k\}) \vee J(u^{m_0+i+1}, \{u^{m_0+i}\})$$

As $Y$ is a submartingale, $K - u^{m_0+i+1} > J(u^{m_0+i+1}, \{u^{m_0+i}, u^k\})$ for any $k$ satisfying $m_0 + i + 2 \le k \le n_0$. This together with (3.18) implies that

$$K - u^{m_0+i+1} > \sup_{S:S_{i+1}\subset S\subset \mathbb{Y}\setminus\{u^{m_0+i+1}\}} J(u^{m_0+i+1}, S).$$

Thus, $u^{m_0+i+1} \in S_{i+2}$.

Therefore the iteration method will terminate within $n_0 - m_0 + 1$ steps and we obtain $S_\infty = \{u^m : m \le n_0\}$ where $n_0$ satisfies $\frac{K-u^{n_0}}{K-u^{n_0-1}} > \alpha_1 \ge \frac{K-u^{n_0+1}}{K-u^{n_0}}$. Equivalently,

$$S_\infty = \left\{u^i : i \le \left\lceil \log_u \left(\frac{1-\alpha_1}{u-\alpha_1}K\right)\right\rceil\right\}.$$

□

#### 3.3.1.1 Discussion on how non-standard discounting affects the value of the option

Consider the optimal stopping problem

$$U(y) := \sup_{\tau\in\mathcal{T}} \mathbb{E}_y \left[\frac{1}{1+\beta\tau}(K - Y_\tau)^+\right]. \tag{3.19}$$

Let

$$\tau^* := \inf\left\{t \geq 0: \ \delta(t)Y_t \geq \sup_{\tau\in\mathcal{T}_t} \mathbb{E}\left[\delta(\tau)(K - Y_\tau)^+ \middle| \mathcal{F}_t\right]\right\}, \tag{3.20}$$

where $\mathcal{T}_t$ is the set of stopping times taking values in $[t, \infty]$. From the classical theory of optimal stopping we know that $\tau^*$ is an optimal solution for the problem (3.19). Recall that

$$\tau_{S_\infty} := \inf\{t \geq 0: \ Y_t \in S_\infty\}$$

is the stopping time corresponding to the optimal mild equilibrium $S_\infty$, where $S_\infty$ is obtained from the iteration in Proposition 3.3.4. We have the following.

**Proposition 3.3.5.** *Suppose* $\log_u(\frac{1-\alpha_1}{u-\alpha_1}K)$ *is not an integer. Then*

$$\tau_{S_\infty} \leq \tau^*. \tag{3.21}$$

Note that $\tau^*$ is an optimal pre-commitment strategy. That is, it is a strategy which is carried out based on the initial preference, and the agent commits to this strategy over the whole planning horizon and ignores the change of her future preference. On the other hand, $\tau_{S_\infty}$ is an equilibrium strategy (sophisticated strategy) which incorporates the change of preference. To be more specific, by using strategy $\tau_{S_\infty}$ the agent seriously takes the possible change of her future preference into consideration, and works on consistent planning: a strategy such that once it is enforced over time, all her future selves have no incentive to deviate from it. Proposition 3.3.5 indicates that with the recognition of the change of preference, the agent would actually expedite the exercise of the American put option.

As $\tau_{S_\infty}$ may not be optimal for the problem (3.19), the use of the equilibrium strategy $\tau_{S_\infty}$ will lower the expected payoff, if such evaluation is based on the initial preference. However, when the change of future preference is considered, there is no unique/proper way to define the dynamically optimal expected payoff over time. In this case, the equilibrium strategy is carried out such that the agent's future selves will not regret the decision.

***Proof of Proposition 3.3.5***. By the Markov property of $Y$, we can rewrite (3.20) as

$$\tau^* = \inf\{t \geq 0 : \ Y_t \in A_t\}, \tag{3.22}$$

where

$$A_t := \left\{ y \in \mathbb{Y} : \ y \geq \sup_{\tau \in \mathcal{T}} \mathbb{E}_y \left[ \frac{1+\beta t}{1+\beta(t+\tau)} (K - Y_\tau)^+ \right] \right\}.$$

It is easy to see that $A_t \subset A_0$ for any $t \geq 0$. We claim that $A_0 \subset S_\infty$, which further implies that $A_t \subset S_\infty$ for $t \geq 0$ and thus (3.21).

Indeed, take $u^n \in A_0$. Obviously $u^n \in (0, K)$. Then we have that

$$K - u^n = U(u^n) \geq J(u^n, \{u^{n-1}\}) = \alpha_1(K - u^{n-1}),$$

which implies that

$$n \leq \log_u \left( \frac{1-\alpha_1}{u-\alpha_1} K \right) + 1.$$

By assumption, we have that

$$n \leq n_0,$$

which implies $u^n \in S_\infty$. □

*Remark* 3.3.6. The assumption in Proposition 3.3.5, i.e., $\log_u \left( \frac{1-\alpha_1}{u-\alpha_1} K \right)$ not being an integer, is very weak, since for it holds for a.e. $u$ and $K$.

*Remark* 3.3.7. Here $\tau^*$ considered in (3.20) is the smallest optimal solution for the problem (3.19). If it is replaced by the largest optimal solution, then the assumption in Proposition 3.3.5 is not needed.

## 3.4 Exact Containment of Equilibria: Optimal Mild $\subsetneqq$ Strong $\subsetneqq$ Weak $\subsetneqq$ Mild

In this section, we will use an example to illustrate that a mild equilibrium may not be a weak equilibrium, a weak equilibrium may not be a strong equilibrium and a strong equilibrium may not be an optimal mild equilibrium.

Consider a two-state continuous-time Markov chain $X_t \in \{a, b\}$ for $t \geq 0$. Assume $a > 0, b > 0$ and without loss of generality we assume $a > b$. The generator is

$$Q = \begin{bmatrix} -\lambda_a & \lambda_a \\ \lambda_b & -\lambda_b \end{bmatrix},$$

where $\lambda_a > 0$ and $\lambda_b > 0$.

There are four subsets of $\{a, b\}$. Clearly $S = \emptyset$ and $S = \{b\}$ cannot be mild equilibria and $S = \{a, b\}$ is a mild equilibrium. Next, let's check when $S = \{a\}$ is a mild equilibrium.

By definition $S = \{a\}$ is a mild equilibrium if and only if

$$b \leq a\mathbb{E}_b[\delta(T_b)] = a \int_0^\infty \delta(t)\lambda_b e^{-\lambda_b t} dt.$$

Consider the following cases.

(i) If $\frac{b}{a} = \int_0^\infty \delta(t)\lambda_b e^{-\lambda_b t} dt < 1$, then both $\{a\}$ and $\{a, b\}$ are optimal mild equilibria and thus both are strong equilibria.

(ii) If $\frac{b}{a} < \int_0^\infty \delta(t)\lambda_b e^{-\lambda_b t} dt < 1$, then $\{a\}$ is the only optimal mild equilibrium, which is also a strong equilibrium. But the mild equilibrium $\{a, b\}$ may not be a weak equilibrium. For example, when $\frac{b}{a} < \frac{\lambda_b}{\lambda_b - \delta'(0)} < 1$, the second condition for weak equilibrium is violated at state $b$, thus it is not a weak equilibrium.

(iii) $\frac{\lambda_a}{\lambda_a - \delta'(0)} < 1 < \frac{a}{b}$ holds automatically since $a > b$ and $\delta'(0) < 0$. If $\frac{\lambda_b}{\lambda_b - \delta'(0)} <$

$\frac{b}{a} < \int_0^\infty \delta(t)\lambda_b e^{-\lambda_b t} dt < 1$, then $\{a, b\}$ is not an optimal mild equilibrium, but it is a weak equilibrium and also a strong equilibrium.

(iv) If $\frac{\lambda_b}{\lambda_b - \delta'(0)} = \frac{b}{a} < 1$, $\{a, b\}$ is a weak equilibrium, but it may not be a strong equilibrium, i.e. condition (3.8) on strong equilibrium may not hold at state $b$. This can be shown by computing the related term of order $\varepsilon^2$.

Since

$$\mathbb{P}(X_\varepsilon = a | X_0 = b) = \lambda_b \varepsilon - \frac{\lambda_b^2 + \lambda_a \lambda_b}{2}\varepsilon^2 + o(\varepsilon^2),$$

and

$$\mathbb{P}(X_\varepsilon = b | X_0 = b) = 1 - \lambda_b \varepsilon + \frac{\lambda_b^2 + \lambda_a \lambda_b}{2}\varepsilon^2 + o(\varepsilon^2),$$

we have $\quad b - \mathbb{E}_b[\delta(\varepsilon) X_\varepsilon]$

$$\begin{aligned}
&= b - \delta(\varepsilon)[a\mathbb{P}(X_\varepsilon = a | X_0 = b) + b\mathbb{P}(X_\varepsilon = b | X_0 = b)] \\
&= b - (1 + \delta'(0)\varepsilon + \frac{\delta''(0)}{2}\varepsilon^2 + o(\varepsilon^2))[a(\lambda_b\varepsilon - \frac{\lambda_b^2 + \lambda_a\lambda_b}{2}\varepsilon^2 + o(\varepsilon^2)) \\
&\quad + b(1 - \lambda_b\varepsilon + \frac{\lambda_b^2 + \lambda_a\lambda_b}{2}\varepsilon^2 + o(\varepsilon^2))] \\
&= (b\lambda_b - a\lambda_b - b\delta'(0))\varepsilon \\
&\quad + [b(\lambda_b\delta'(0) - \frac{\lambda_b^2 + \lambda_a\lambda_b}{2} - \frac{\delta''(0)}{2}) - a(\delta'(0)\lambda_b - \frac{\lambda_b^2 + \lambda_a\lambda_b}{2})]\varepsilon^2 + o(\varepsilon^2)
\end{aligned}$$

Therefore when the first order term and the second order term respectively satisfy

$$b(\lambda_b - \delta'(0)) - a\lambda_b = 0, \tag{3.23}$$

and

$$b(\lambda_b\delta'(0) - \frac{\lambda_b^2 + \lambda_a\lambda_b}{2} - \frac{\delta''(0)}{2}) - a(\delta'(0)\lambda_b - \frac{\lambda_b^2 + \lambda_a\lambda_b}{2}) < 0, \tag{3.24}$$

$\{a, b\}$ is a weak equilibrium but not a strong equilibrium. Using (3.23), (3.24) can be simplified to

$$\lambda_a + \lambda_b < \frac{\delta''(0) - 2(\delta'(0)^2)}{-\delta'(0)}. \tag{3.25}$$

An interesting case is when $\delta(t) = \frac{1}{1+\beta t}$. Then (3.25) does not hold: $\delta'(0) = -\beta$ and $\delta''(0) = 2\beta^2$. In this case $\frac{\delta''(0)-2(\delta'(0)^2)}{-\delta'(0)} = 0$, which contradicts $\lambda_a + \lambda_b > 0$. That means if we have hyperbolic discount function, a weak equilibrium is always a strong equilibrium in the two-state setting.

But when $\delta(t) = (1+\beta t)^{-\frac{1}{2}}$, then it can easily be seen that (3.25) holds: $\delta'(0) = -\frac{\beta}{2}, \delta''(0) = \frac{3}{4}\beta^2$ implies that when $0 < \lambda_a + \lambda_b < \frac{\beta}{2}$ and $\frac{b}{a} = \frac{2\lambda_b}{2\lambda_b+\beta}$, {a, b} is a weak equilibrium but not a strong equilibrium. In this case, $\{a, b\}$ is not an optimal mild equilibrium.

# CHAPTER IV

# Countercyclical Unemployment Benefits in Disasters: A Quantitative General Equilibrium Analysis

## 4.1 Introduction

Unemployment insurance (UI) in the United States increases in recessions. It takes the form of either an increase in the level of benefits or an extension of the duration for which an unemployed individual receives benefits.[1] Historically, the size of the increase in UI benefits has been positively related to the level of the unemployment rate prevailing at the time the policy is implemented. The increase in benefits over the past two recessions has been substantial. The CARES Act added $600 per week to the regular UI benefits received (the maximum regular benefits varied from $200 per week to $600 per week across states) and extended the duration of benefits by 13 weeks (the usual duration for most states is 26 weeks); while UI benefits following the Great Recession were extended by 99 weeks from the usual 26 weeks. Since such increases in the generosity of benefits involve an enormous expansion of total UI

[1] More recently, in response to the Coronavirus pandemic of 2020, a third form of increase in benefits was implemented. Namely, UI benefits were extended to include individuals who were ineligible for regular UI benefits. In fact, the Coronavirus Aid, Relief, and Economic Security (CARES) Act of 2020 featured all of the three types of increases mentioned above.

expenditures, they are usually accompanied by a debate on whether the benefits of such UI expansion outweighs its costs. A key argument for increasing benefits in a crisis is that it prevents large consumption declines of individuals who suffer job loss during the crisis. A key argument against such UI expansion is that it reduces the incentives for unemployed individuals to seek employment, which therefore reduces labor supply.[2]

In this paper we analyze and quantify the general equilibrium implications of UI policy changes following an aggregate shock. This shock increases the unemployment rate in the economy. We consider two types of UI policy changes both of which feature an increase in UI benefit payments following the shock. Under the first type of policy, all unemployed individuals experience the same increase in benefits. We call this our baseline policy (BL). Our key finding for this policy is that the additional unemployment rate under a countercyclical UI policy relative to an acyclical policy (for which UI benefits do not increase following the shock) is a sharply increasing, nonlinear function of the size of the aggregate shock. The second type of policy that we consider is one in which UI payments increase for individuals with capital below a threshold, with the magnitude of the increase being larger for individuals with lower capital. We call this a "means tested" policy (MT), since, in our model, this policy results in higher UI payments for individuals with lower total income. We compare the effects of the MT and the BL policies. In making this comparison, we choose the two policies such that they have the same time-0 cost. Our key finding is that the MT policy provides substantially better consumption insurance against income loss to its intended beneficiaries (i.e., individuals who experience an increase in UI payments following the aggregate shock), without a substantial increase in the equilibrium unemployment rate.

Our setting features an incomplete market with a continuum of individuals, that

[2] "...paying people not to work is dampening what should be a stronger jobs market." Neil Bradley, Chief Policy Officer, U.S. Chamber of Commerce, *Wall Street Journal*, May 8, 2021.

are either employed or unemployed. They make saving decisions and earn a return on their savings. In addition to capital gains, employed individuals earn wages and the unemployed receive UI benefits. Employed individuals face idiosyncratic unemployment risk. They are unable to hedge this risk due to incomplete credit and insurance markets. Unemployed individuals choose their intensity of finding jobs; this search is costly. In addition to the constant idiosyncratic unemployment risk, we add a one-time unanticipated aggregate shock at time $t = 0$ that causes job loss to a fraction of individuals in the economy. At the same time, an unanticipated policy change in UI benefits (either BL or MT) is implemented. We use the numerical method from mean field game theory to compute the transition dynamics of this economy. We examine the effect of countercyclical policies in the cross-section as well as on aggregate quantities such as the unemployment rate.

Our first result is that the excess unemployment rate, defined as the difference between the equilibrium unemployment rates under a countercyclical policy and that under an acyclical policy, is positive and increases nonlinearly with the size of the aggregate shock. The excess unemployment rate is positive because unemployed individuals optimally exert less effort at finding a job under a countercyclical policy than under an acyclical policy, since the level of benefits are higher under the former policy. The nonlinear dependence on the size of the aggregate shock is mainly explained by two feedback effects in general equilibrium. First, the excess unemployment rate at some small time $\Delta t$ is approximately proportional to the product of the size of aggregate shock and the difference between aggregate job finding rates under a countercyclical policy and an acyclical policy. The latter is an increasing function of the size of the aggregate shock, since individuals search harder to find a job under an acyclical policy relative to a countercyclical policy (since benefits under the countercyclical policy is an increasing function of the unemployment rate). As a result, the approximated excess unemployment rate at $\Delta t$ increases superlinearly with the size of aggregate shock. Second, this superlinearity is further enhanced as time advances since the flow rate out of unemployment under the countercyclical policy is dampened by the larger stock of unemployed workers relative to the acyclical policy. Taken together, these two effects generate a path of the excess unemployment rate that is a super linear function of the size of the aggregate shock.

Next, we quantify the effectiveness of the MT policy, and in particular, we compare this policy with the BL policy. We find that the MT policy provides substantially better consumption insurance against income loss for individuals who experience an increase in UI payments following the aggregate shock. While this result is perhaps expected (since higher payments allow individuals with relatively lower $k$ to better smooth consumption), our quantitative analysis establishes that this benefit is not associated with a noticeably slower recovery in labor markets. Quantitatively, the time-0 consumption decline following job loss of low income individuals is more than 12% smaller under the MT policy than under the BL policy. However, the unemployment rate under the MT is less than 0.7% higher than that under the BL policies.

We contribute to the literature that uses dynamic stochastic general equilibrium models to quantify the effect of UI policies on welfare and on dynamics of aggregate quantities. Within this class of models, our paper is most closely related to [25] (HI) who explicitly account for the role of individual's asset accumulation as a means to self-insure. There are two key differences between our paper and HI. First, we analyze the effect of means-tested UI policies. In contrast, the analysis in HI does not examine the effect of allowing for benefits to depend on individual income. Second, we consider the transition dynamics following an (unanticipated) aggregate shock, while the analysis in HI is in steady-state without any aggregate shocks.

Our general equilibrium model builds on the partial equilibrium models that analyze the trade-off between consumption insurance and the provision of sufficient incentives for the unemployed to search for jobs. Recent examples include [44, 16, 37].

In contrast to the partial equilibrium analysis in these papers, our general equilibrium model captures the externality of changes in job search intensity of one individual on another (through level and duration of benefits and wages).

While we take the UI policy as given and analyze its implications, there is an important strand of the literature that addresses the optimal design of UI policies by taking a contract theory approach. The classical paper by [55] examining the optimal time-schedule of UI benefits and the important extension to allow for a wage tax upon employment by [27] are important examples. The analyses in this literature are in partial equilibrium and are qualitative.

There is also a recent literature that uses dynamic, general equilibrium models to examine the effect of UI on welfare and the dynamics of aggregate quantities in the presence of aggregate shocks. Examples include [48, 39, 52]. In contrast to our paper, these papers do not allow for self-insurance by individuals through savings.

## 4.2 The Model

We construct a general equilibrium model to analyze the effects of a class of counter-cyclical unemployment insurance (UI) benefit policies in an economy with imperfect risk-sharing. Our model features the classical trade-off between consumption smoothing and moral hazard due to unobservable job-search effort. The focus of our analysis is the transitional dynamics following an unanticipated, transitory, negative shock at $t = 0$ which increases the unemployment rate, and an unanticipated policy response following this shock. We analyze different forms of such UI policies.

### The Environment

The economy is populated by a continuum of infinitely lived individuals of measure one with identical preferences:

$$\mathbb{E}_t \int_t^{\infty} e^{-\rho(\tau-t)} u(c_\tau) d\tau \,. \tag{4.1}$$

where $u(c)$ is the individual's running utility function, $c_\tau$ is her/his consumption trajectory, and $\rho$ is her/his time preference parameter. At any point in time, an individual is in one of two possible employment states $\epsilon = \{\epsilon_1, \epsilon_2\}$, where $\epsilon_1$ and $\epsilon_2$ are the unemployment and employment states, respectively.

Each individual owns capital and is endowed with a single, indivisible unit of labor. Individuals rent capital to a representative firm and earn rental income at a rate $r_t k_t$, where $r_t$ is the market-wide rental rate for capital, and $k_t$ is the individual's time-$t$ capital stock. They also earn labor income $y_t = \{y_{1t}, y_{2t}\}$, that is either equal to unemployment insurance (UI) benefit $y_{1t}$ (for unemployed individuals), or after-tax wages $y_{2t} = (1-\theta_t)w_t$ (for employed individuals) where $w_t$ is the market-wide wage and $\theta_t$ is the tax rate. We describe UI benefits in detail below.

Each individual uses her/his total income partly for consumption and saves the remainder:

$$i_t = r_t k_t + y_t - c_t, \tag{4.2}$$

where $i_t$ is the savings (investment) rate. The law of motion for an individual's capital $k_t$ is:

$$dk_t = -\delta k_t dt + i_t dt \,, \tag{4.3}$$

where $\delta$ is the depreciation rate. Individuals face a liquidity limit on capital

$$k_t \geq \underline{k}, \tag{4.4}$$

for some constant $\underline{k} \geq 0$.

We assume that the transition intensity from employment to unemployment $\lambda_1$ is an exogenously specified constant. The transition intensity from unemployment to employment $\lambda_2(s)$, on the other hand, depends on the search intensity $s$ of an unemployed individual. We model the tradeoffs faced by an unemployed individual using an off-the-shelf costly job-search model (see e.g., [15], [44], [13], [16], and [38]). Increasing search intensity $s$ results in a linear increase in the transition intensity from unemployment to employment:

$$\lambda_2(s) = s\,, \tag{4.5}$$

where we have normalized the proportionality constant between $\lambda_2$ and $s$ to one. Searching is costly, incurring a flow cost

$$\psi(s) = \frac{\phi s^{1+\kappa}}{1+\kappa}\,, \tag{4.6}$$

where $\phi > 0$ and $\kappa > 0$ are constant.

We will denote the distributions of capital of unemployed and employed individuals by $g_1(k,t)$ and $g_2(k,t)$, respectively. These densities satisfy $\int_{\underline{k}}^{\infty} g_1(k,t)dk + \int_{\underline{k}}^{\infty} g_2(k,t)dk = 1$ for all $t$. The first term on the left-hand side of this equation corresponds to the aggregate unemployment rate, that is: $U_t = \int_{\underline{k}}^{\infty} g_1(k,t)dk$.

The production side of the economy consists of a representative firm which produces output with a Cobb-Douglas technology: $Y_t = K_t^{\alpha} L_t^{1-\alpha}$, where $K_t$ and $L_t$ are capital and labor inputs, respectively, and $0 < \alpha < 1$ is the capital share parameter. For simplicity, we assume that total factor productivity stays constant, and we normalize this to one. The firm rents capital and labor in competitive spot markets, taking the rental rate for capital $r$ and the wage $w$ as given. There are no adjustment costs for factor inputs and the firm chooses $K_t$ and $L_t$ to maximize the profit flow

$\Pi_t = Y_t - r_t K_t - w_t L_t$. According to the first order optimality considiton, the rental rate $r_t = \alpha\big(K_t/L_t\big)^{\alpha-1}$ and the wage $w_t = (1-\alpha)\big(K_t/L_t\big)^{\alpha}$.

Finally, the income-tax rate $\theta_t$ is determined by requiring total UI expenditure equal total tax collected from employed individuals:

$$\int_{\underline{k}}^{\infty} g_1(k,t) y_{1t} dk = \theta_t \int_{\underline{k}}^{\infty} w_t g_2(k,t) dk,$$

where $y_{1t}$, in general, depends on $k$.

**The Shock.** For the period $t < 0$, the economy is in the steady-state with time-invariant density functions $g_1^\star(k)$ and $g_2^\star(k)$ of unemployed and employed individuals, respectively, and a constant unemployment rate $U^\star = \int_{\underline{k}}^{\infty} g_1^\star(k) dk$. We call this the precrisis period. At $t = 0$, an unanticipated negative shock is realized. This results in a fraction of employed individuals becoming unemployed, that is, the distributions $g_1$ and $g_2$ change discontinuously at $t = 0$. The size of the aggregate shock is characterized by the change in the unemployment rate $U_0 - U^\star$, where $U_0 = \int_{\underline{k}}^{\infty} g_1(k,0) dk$ is the time-0 unemployment rate immediately after the shock is realized. For simplicity we assume that the probability of job loss is independent of individual characteristics. This implies that: $g_1(k,0) = \frac{U_0}{U^\star} g_1^\star(k)$.

**The policy response.** We focus on UI payments $y_{1t}$ of the form:

$$y_{1t} = \min\left(a + b(k_t - \underline{k}), \overline{y}\right) + \eta\left(U_t - U^\star\right) h(k_t)\,, \tag{4.7}$$

where $a$, $b$, $\overline{y}$, and $\eta \geq 0$ are four constants which represent the minimum UI benefit, the sensitivity of UI benefits to $k$, the maximum UI benefit, and the sensitivity of changes in UI benefits to the unemployment rate, respectively. The function $h(\cdot)$ allows for this change in benefits to depend on the agent's total income by allowing for $h$ to depend on $k$. This is because, an unemployed individual's total income $r_t k_t + y_{1t}$ depends on $k_t$.

We assume that in the precrisis period, $\eta = 0$. During this period, the UI payments are a piece-wise linear function of $k$, increasing linearly with slope $b$ for individuals with $k \leq k_H = \underline{k} + \frac{\overline{y}-a}{b}$. All unemployed individuals with $k \geq k_H$, receive the same amount $\overline{y}$.

When the shock is realized at $t = 0$, policy-makers announce a change in the UI policy. Throughout the paper, we assume that all policy changes are unanticipated by agents in the economy. Policies which continue to adopt $\eta = 0$ are acyclical, since changes in the aggregate unemployment rate $U_t$ are not accompanied by changes in UI benefits (see equation (4.7)). In contrast, $\eta > 0$ provide countercyclical UI benefits, since UI payments are a (weakly) increasing function of $U_t$. In this paper we focus on two types of countercyclical policies:

i. Baseline: These are policies for which $h(k) = 1$. Under this set of policies, all unemployed agents experience the same increase in UI benefits when $U_t > U^\star$.

ii. Means-tested: Under this set of policies, unemployed individuals with lower $k$ experience a (weakly) greater increase in UI benefits when $U_t > U^\star$. In order to preserve the piece-wise linear form of UI payments as a function of $k$ in the precrisis state, we choose $h(k) = \max\left(1 - \frac{b}{\overline{y}-a}(k - \underline{k}), 0\right)$.[3] We call this class of policies "means-tested", since the increase in UI benefits following an increase in the unemployment rate $U_t > U^\star$, depends on total income.

**The individual's problem.** Each individual chooses consumption $c_t$ and, if unemployed, job search intensity $s(k,t)$ to maximize (4.1) subject to: the constraints (4.3) and (4.4), the benefit and cost of job search (4.5) and (4.6), and taking prices $r_t$ and $w_t$, the UI policy $y_{1t}$, the evolution of $g_1$ and $g_2$, and the value from future

[3] Note that this specification of $h(k)$ does not introduce any additional parameters.

potential employment $v_2$, as given.

An unemployed individual's Hamilton-Jacobi-Bellman (HJB) equation is

$$\begin{aligned} \rho v_1(k,t) &= \max_{c_1,s} u\big(c_1\big) + \big(y_{1t} + (r(t)-\delta)k - c_1\big)\partial_k v_1(k,t) \\ &+ \big(v_2(k,t) - v_1(k,t)\big)\lambda_2(s) - \psi(s) + \partial_t v_1(k,t)\,, \end{aligned} \tag{4.8}$$

where $c_1$ is the consumption of an unemployed individual, $v_1$ and $v_2$ are the individual value functions in the unemployed and employed states, respectively. The first order condition for job-search effort $s = s(k,t)$ is

$$\psi'(s(k,t)) = \lambda_2'(s(k,t))\big(v_2(k,t) - v_1(k,t)\big)\,. \tag{4.9}$$

Note that the optimal $s$ depends on individuals' current savings and the aggregate conditions. The envelope condition is

$$u'\big(c_1(k,t)\big) = \frac{d}{dk} v_1(k,t) \tag{4.10}$$

An employed individual's HJB equation is

$$\begin{aligned} \rho v_2(k,t) &= \max_{c_2} u\big(c_2\big) + \big(y_{2t} + (r(t)-\delta)k - c_2\big)\partial_k v_2(k,t) \\ &+ \big(v_1(k,t) - v_2(k,t)\big)\lambda_1 + \partial_t v_2(k,t)\,. \end{aligned} \tag{4.11}$$

where $c_2$ is the consumption of an employed individual. The envelope condition is

$$u'\big(c_2(k,t)\big) = \frac{d}{dk} v_2(k,t) \tag{4.12}$$

The Kolmogorov forward equation for the evolution of the distributions $g_1$ and $g_2$

are:

$$
\begin{aligned}
\partial_t g_1(k,t) = & - \frac{d}{dk}\big(g_1(k,t)(y_{1t} + (r_t - \delta)k - c_1(k,t))\big) \\
& - \lambda_2(s(k,t))g_1(k,t) + \lambda_1 g_2(k,t) \\
\partial_t g_2(k,t) = & - \frac{d}{dk}\big(g_2(k,t)(y_{2t} + (r_t - \delta)k - c_2(k,t))\big) \\
& - \lambda_1 g_2(k,t) + \lambda_2(s(k,t))g_1(k,t)\,. \qquad (4.13)
\end{aligned}
$$

**Equilibrium.** The competitive equilibrium consists of consumption and job search policies of unemployed individuals $c_1(k,t)$ and $s(k,t)$, respectively, the consumption policy of employed individuals $c_2(k,t)$, the densities of capital for the unemployed and employed $g_1(k,t)$ and $g_2(k,t)$, aggregate capital $K_t$ and labor $L_t$, the rental rate $r_t$ and the wage $w_t$, and the UI policy $y_{1t}$, given the initial distributions $g_1(k,0)$ and $g_2(k,0)$ immediately after realization of the shock, such that: (i) unemployed individuals choose consumption and job search to maximize (4.8), (ii) employed individuals choose consumption to maximize (4.11), (iii) the densities $g_1(k,t)$ and $g_2(k,t)$ satisfy (4.13), (iv) the firm chooses capital $K$ and labor $L$ to maximize firm profit $\Pi_t$, (v) the capital market clears: $K_t = \int_{\underline{k}}^{\infty} k\big(g_1(k,t) + g_2(k,t)\big)dk$, (vi) the labor market clears: $L_t = \int_{\underline{k}}^{\infty} g_2(k,t)dk$, and (vii) the goods market clears: $C_t = Y_t - I_t$, where the aggregate investment $I_t = \int_{\underline{k}}^{\infty} (i_1(k,t)g_1(k,t) + i_2(k,t)g_2(k,t))\,dk$.

## 4.3 Quantitative Results

### 4.3.1 Calibration

We use the parameters in Table 4.1. All values are annual. We describe our computational approach in solving for the steady-state equilibrium in section B.1 and in solving for the transition path following the shock in section B.2.

We choose commonly-used values for the preference and technology parameters.

Table 4.1: **Parameter values.** All values are annual.

| Parameter | Symbol | Model |
|---|---|---|
| Capital share | $\alpha$ | 0.36 |
| Depreciation rate | $\delta$ | 0.10 |
| Time-preference parameter | $\rho$ | 0.03 |
| Risk aversion | $\gamma$ | 2 |
| Job-separation intensity | $\lambda_1$ | 0.035 |
| Job-search cost parameters | $\phi$ | 4.5 |
| | $\kappa$ | 0.3 |
| UI benefit parameters | $a$ | 0.06 |
| | $k_H$ | 4.3 |
| | $\overline{y}$ | 0.3 |

We choose the capital share parameter $\alpha = 0.36$, the depreciation rate $\delta = 0.10$, and the liquidity limit $\underline{k} = 0$. We choose the time preference parameter $\rho = 0.03$, and we assume that individuals have a constant relative risk aversion $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, with relative risk aversion $\gamma = 2$.

We choose the job loss intensity $\lambda_1 = 0.035$ to match the job-separation rate of non-farm payroll workers between 2000M1 - 2019M12 as reported by the U.S. Bureau of Labor Statistics.[4]

We calibrate the UI benefit function (see equation (4.7)) in the precrisis state. We choose the values of the 3 parameters: the minimum and maximum benefits $a = 0.06$ and $\overline{y} = 0.3$, respectively, and the threshold level of capital $k_H = 4.3$ at which the UI benefit reaches its maximum value relative to the cross-sectional mean of $k$ to approximately match: (i) the slope of the UI schedule with respect to income (over the increasing part of the UI payment schedule), (ii) the location of the threshold income in the income distribution at which the UI benefit reaches its maximum value, and (iii) the ratio of the maximum to minimum UI benefits. To this end we rely on recent estimates of UI insurance payments documented by **?** ]

[4] We retrieved this series from the Federal Reserve Economic Data (FRED) series.

Table 4.2: **Aggregate Quantities and Prices in pre-crisis state.** Steady state values for $t < 0$. We use the parameter values shown in Table 4.1.

| | $L^\star$ | $K^\star$ | $Y^\star$ | $C^\star$ | $w^\star$ | $r^\star$ |
|---|---|---|---|---|---|---|
| Pre-crisis values | 0.949 | 4.796 | 1.701 | 1.221 | 1.147 | 0.128 |

(GNV). In particular, GNV find that the slope of the UI schedule with respect to income is 0.5. In comparison, our model-implied estimate is 0.44. GNV find that UI benefits reach the maximum value for individuals whose income is close to the mean of the income distribution.[5] In our calibration, UI benefits reach their maximum for individuals with total income (rent from capital plus wages) equal to 1.70, while the mean income is equal to 1.68. Finally, **?** ] find the average value of the ratio of the maximum to minimum UI benefits across states to be 0.2. Our model-implied value for $a/\overline{y}$ is also 0.2.

We choose the values of the job-search cost parameters $\kappa = 0.3$ and $\phi = 3$ to approximately match the unemployment rate and the elasticity of unemployment duration with respect to the benefit level for the median $k$ individual. Our model-implied unemployment rate in the precrisis state is 5.5% compared to the U.S. unemployment rate of 5.7% over the period 1948M1—2019M12, where the latter is computed from the seasonally adjusted monthly rate series. Our model-implied average elasticity of unemployment duration with respect to unemployment benefits is 0.4, while [16] estimates this value to be 0.5 in the data.

Table 4.2 shows the values of aggregate quantities and prices in the pre-crisis state. The solid lines in panels A and B of Figure 4.1 show the pre-crisis state stationary distributions $g_1^\star(k)$ and $g_2^\star(k)$ of the unemployed and employed, respectively.

[5] For instance, **?** ] report that, in Nevada, weekly UI benefits reach their maximum for individuals with weekly income above \$902. This is close to the mean income of \$886 per week in that state.

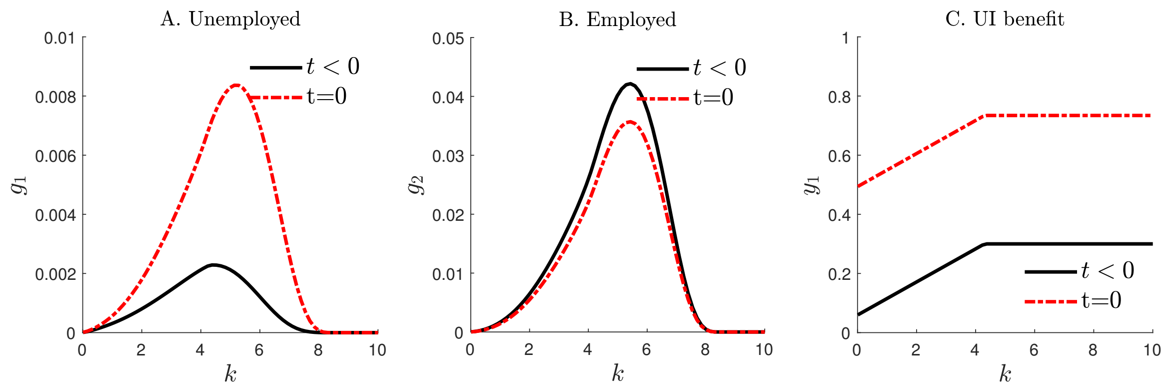


Figure 4.1: **The Shock and UI policy.** Panels A and B show the distributions of the unemployed and the employed, respectively, before the shock $t < 0$, and immediately after the shock is realized at $t = 0$. Panel C shows the UI payments $y_1$ as a function of individual capital $k$. The dot-dash line in panel C shows UI payments in equation (4.7) with $\eta = 3$.

### 4.3.2 Baseline Policy

In this subsection we consider a shock which increases the unemployment rate from its steady-state value $U^\star = 5.5\%$ to $U_0 = 20\%$ at $t = 0$. Panels A and B of Figure 4.1 show the distributions of the unemployed $g_1$ and the employed $g_2$, respectively, before the shock $t < 0$, and immediately after the shock is realized at $t = 0$. For our baseline analysis, we assume that $h(k) = 1$ in equation (4.7). With this choice, all unemployed individuals experience the same increase in UI benefits for $U_t > U^\star$.

We begin by comparing the effect of two policies: an acyclical UI benefit policy ($\eta = 0$) with that of a countercyclical policy ($\eta = 3$). Under the acyclical policy, the wage-replacement ratio does not change after the shock. Under the countercyclical policy, UI benefits increase for all unemployed individuals by the same amount; as shown in Panel C of Figure 4.1, at $t = 0$, this increase is equal to 0.44. Thereafter, its future evolution is determined by the equilibrium path of $U_t$ though equation (4.7), reverting back to its steady-state value as $t \to \infty$.

#### 4.3.2.1 Individual utilities

In this section we quantify the cost and benefit of the UI policies as measured by their effects on the job-search intensity and consumption smoothing, respectively.

Panel A of Figure 4.2 compares the time-0 job-search intensity $s$ for the acyclical UI policy (solid line) and the countercyclical policy (dot-dash line). We see that, in equilibrium, higher UI benefits under the countercyclical policy result in lower effort compared to that under the acyclical policy. The magnitude of the reduction in job-search intensity under the countercyclical policy compared to the acyclical one is large. For instance, an individual in the 50th percentile of the distribution of $g_u(k, 0)$ (with $k = 4.2$) chooses $s = 0.56$ under the acyclical UI policy, but $s = 0.36$ under the countercyclical policy. As a result, there is a substantial reduction in the equilibrium transition intensity from unemployment to employment $\lambda_2$ under the countercyclical policy compared to that under the acyclical policy.[6]

While Panel A shows the cost of higher benefits under the countercyclical policy, Panel B shows its benefit. In this figure, we plot the fractional drop in consumption $\Delta c/c$ of individuals who become unemployed at $t = 0$. We focus on individuals in the bottom 25% of $g_u(k, 0)$, since they suffer the largest decline in $\Delta c/c$ . We see that the countercyclical UI policy provides substantially more consumption smoothing to individuals who lose their job at $t = 0$, especially those with low $k$. Consider, for example, an individual who suffers job-loss at $t = 0$ and with $k$ in the lowest 1-percentile of $g_u(k, 0)$ (with $k = 0.7$). This individual's fractional drop in consumption $\Delta c/c$ is 33% under the acyclical policy but is only 19% under the countercyclical policy. The consumption smoothing benefit is smaller for relatively wealthier individuals. For an individual with $k$ in the 10th percentile of $g_u(k, 0)$ (with $k = 2.1$), $\Delta c/c$ is 15% under the acyclical policy but is 13% under the countercyclical policy.

In Panel C, we compare the value functions of unemployed individuals in the

[6] The search intensity $s$ is equal to $\lambda_2$ in our specification (4.5).

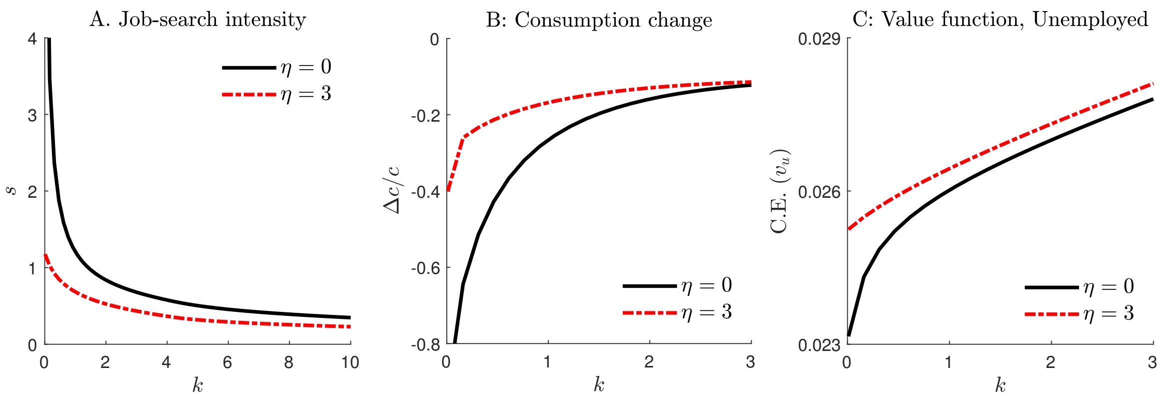


Figure 4.2: **Costs and benefits of acyclical and countercyclical UI policies.** Panel A shows the time-0 job-search intensity $s$ under the acyclical and countercyclical UI policies, in response to an unanticipated aggregate shock followed by the UI policy response with $h(k) = 1$ in (4.7). Panel B compares the fractional drop in consumption of those individuals who become unemployed at $t = 0$ following the shock for these two UI policies. Panel C compares the time-0 value functions for these two policies (measured in certainty equivalents (C.E.)). Both panels B and C show results for unemployed individuals in the bottom 25 percentile of $g_u$.

bottom 25% of $g_u(k, 0)$ under the two UI policies. While Panel B compared the immediate effect of these policies following the shock, the value functions compare the long-run effect of these policies. In presenting results, we convert the value function to consumption units using the utility function (i.e., we plot $u^{-1}(v_u)$ on the y-axis of the figure). We see that unemployed individuals have higher welfare under the countercyclical policy compared to the acyclical policy.[7] However, we see that the difference in the value function across the two policies is relatively small. In other words, a countercyclical UI policy's benefit is front-loaded in that the biggest benefit of this policy is that it prevents a large decline in consumption immediately when the shock is realized. This result suggests a role for short-lived countercyclical policies with a high level of benefits in the initial period following a negative shock, but which decline more rapidly compared to the policy in equation (4.7).

[7] Employed individuals are worse off under the countercyclical policy compared to the acyclical policy (not shown in the figure) because they pay higher taxes necessary to fund the more generous countercyclical UI policy.

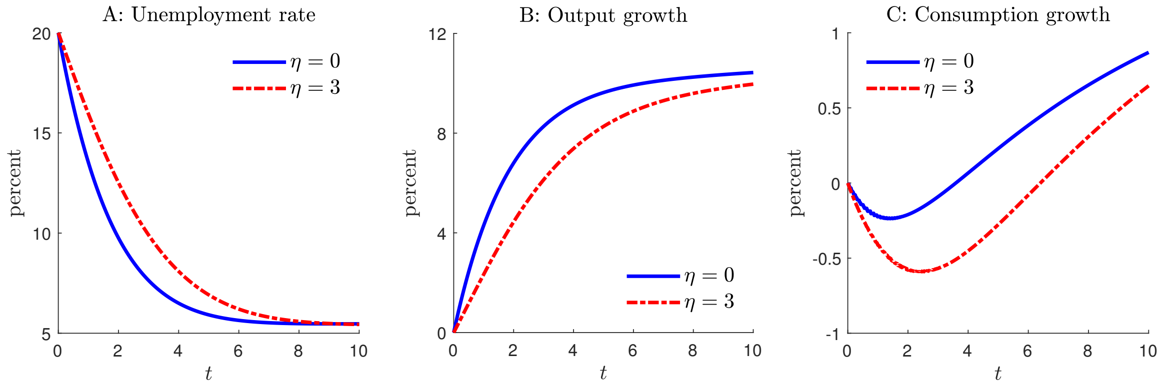


Figure 4.3: **Aggregate results.** Panel A shows the path of the aggregate unemployment rate. Panels B and C show the growth in output and consumption, respectively, from their values immediately after realization of the shock at $t = 0$. In these figures, time is measured in years.

#### 4.3.2.2 Response of Aggregate Quantities

In this section, we compare the response of the aggregate unemployment rate, output, and consumption, under the two UI policies. Our key finding in this section is that aggregate quantities, especially the unemployment rate, recovers substantially slower under the countercyclical policy than under the acyclical policy.

In Panel A of Figure 4.3, we compare the paths of the unemployment rate under the two UI policies. We see that the unemployment rate is slower to revert to its pre-crisis level under the countercyclical policy (dot-dash line) than under the acyclical policy (solid line). We quantify this slowdown by through the *excess unemployment rate* under a given countercyclical UI policy. We define this quantity at time $t$ as the time-$t$ difference between the unemployment rate under the countercyclical policy and the unemployment rate under an acyclical policy. From panel A of Figure 4.3, we see that the maximum excess unemployment rate in this figure is 2.8%, occurring at $t = 1.7$ years.

Labor markets recover more slowly under the countercyclical policy because higher UI benefits reduce the incentives for job-search effort, so that unemployed individuals optimally choose lower job-search effort relative to that under an acyclical policy.

There is a general equilibrium feedback effect here. Lower job-search effort under a countercyclical policy relative to an acyclical policy, results in a higher equilibrium unemployment rate (both current and future). This, in turn, leads to higher current and future benefits according to equation (4.7), which increases the value of unemployment. This further reduces equilibrium job-search effort (see equation (4.9)).

The lower employment level in response to the countercylical benefit policy results in slower recovery of output and aggregate consumption. Panel B shows that under the acyclical policy, output increases by 4.3% from its value at the trough (at $t = 0$) in one year. The corresponding growth is only 2.3% under the countercyclical policy. The difference in the growth rate of aggregate consumption under the two UI policies is smaller, and it shows up with a lag. Panel C shows that in the one year following the shock, aggregate consumption declines by 0.4% and 0.2% under the countercyclical and acyclical policies, respectively.

#### 4.3.2.3 Countercyclical policy and shock size

In this section we show that the severity of the effect of countercyclical UI policies on labor supply is a rapidly increasing, non-linear function of the size of the aggregate shock $u_0$. We establish this result through a comparative static exercise in which we vary $u_0$ while holding all other model parameters fixed. Throughout this section, we fix $\eta = 3$ for countercyclical policies.

Panel A of Figure 4.4 shows the time-0 total UI expenditure normalized by output as a function of $u_0$. The dashed line corresponds to the acyclical policy $\eta = 0$. This line is linear because individual UI payments do not change with $U_0$, but the measure of recipients increase linearly with $U_0$. The solid line corresponds to the countercyclical policy $\eta = 3$. Total UI expenses increase faster than under the acyclical policy because individual payments are higher under the policy with $\eta = 3$. The difference is quantitatively large when unemployment $U_0$ is high. For $U_0 = 20\%$, the difference is 5.66% of GDP.

Panel B of Figure 4.4 shows the excess unemployment rate as a function of $U_0$, two years following the realization of the aggregate shock. We see the excess unemployment rate increases non-linearly with the shock size $U_0$. For instance, this rate is only 0.4% if $U_0 = 10\%$ (the maximum unemployment rate following the Great Recession), but it is seven times higher at 2.8% if $U_0 = 20\%$. Recent findings in the empirical literature suggest that the adoption of extended unemployment benefits following the Great Recession likely had a small effect on the unemployment rate ([**?** ] estimate a 0.3% increase in the unemployment rate). Our result of a non-linear increase in the excess unemployment rate as a function of $u_0$ implies that naively extrapolating these estimates to large disasters could significantly understate the effect.

The super-linear dependence of excess unemployment on the size of the aggregate shock is mainly explained by two feedback effects in general equilibrium: (i) the interplay between job search intensity and unemployment rate and (ii) the time aggregation on the stock of unemployed individuals.

Notice that the unemployment rate $U$ can be viewed as a function of time $t$, shock size $U_0$ and UI policy parameter $\eta$. What we are interested in is how different policies and the size of aggregate shock affect $U$. The answer to this question lies in our first result: (a) with a fixed aggregate shock $U_0$, the unemployment rate under a countercyclical policy is larger than that under an acyclical policy (i.e., the excess unemployment rate is positive); (b) the excess unemployment rate increases super-linearly with respect to the size of aggregate shock.

To see this, denote the unemployment rate at time $t$ under countercylical policy and acyclical policy as $U_t^C$ and $U_t^A$ respectively, and denote the excess unemployment as

$$\bar{U}_t = U_t^C - U_t^A.$$

Since the employment/unemployment state of each individual is a continuous time

Markov chain with two states, we have that

$$\frac{dU_t}{dt} = \lambda_1(1 - U_t) - \int_{\underline{k}}^{\infty} \lambda_2(k,t) g_1(k,t) dk$$
$$= \lambda_1 - \int_{\underline{k}}^{\infty} [\lambda_1 + \lambda_2(k,t)] g_1(k,t) dk.$$

Thus

$$\frac{d\bar{U}_t}{dt} = -\lambda_1 \bar{U}_t - \int_{\underline{k}}^{\infty} [\lambda_2^C(k,t) g_1^C(k,t) - \lambda_2^A(k,t) g_1^A(k,t)] dk.$$

First consider some fixed small time $t$ and the linear approximation of $\bar{U}_t$ as

$$\bar{U}_t \approx \bar{U}_0 + \bar{U}'(0)t.$$

Since $\bar{U}_0 = 0$, we only need to consider

$$\bar{U}'(0) = -\int_{\underline{k}}^{\infty} [\lambda_2^C(k,0) g_1^C(k,0) - \lambda_2^A(k,0) g_1^A(k,0)] dk.$$

Since the initial distribution $g_1^C(k,0) = g_1^A(k,0) = g_1(k,0)$ are the same under these two policies,

$$\bar{U}'(0) = -\int_{\underline{k}}^{\infty} [\lambda_2^C(k,0) - \lambda_2^A(k,0)] g_1(k,0)] dk. \tag{4.14}$$

Intuitively, under a countercyclical policy, unemployed individuals are less motivated to search a job due to the higher level of benefits. Then $\lambda_2^C(k,0) - \lambda_2^A(k,0) < 0$ and $\bar{U}'(0) > 0$, which is consistent with the positiveness of excess unemployment rate.

Next we examine how the difference in job search intensities change with respect to the size of aggregate shock. Using the assumption that the job loss happens with uniform distribution to individuals with different total income, we derive that for each $k$, $g_1(k,0) = \frac{U_0}{U^\star} g_1^\star(k)$ is a linear increasing function of the size of aggregate shock. Moreover, we obtain that the total capital does not change at initial time : $K_0 = K^\star$. In equilibrium the wage increases at time 0: $w^\star < w_0$. Intuitively, unemployed individuals choose their optimal job serach intensity based on how much extra utility value they will gain if they are employed rather than unemployed. This extra utility value is positively related to the extra income from wage rather than benifits (i.e., $y_2 - y_1 = (1-\theta)w - y_1$). That is, the difference between job search intensities under an acyclical policy and a countercyclical policy $\lambda_2^A(k,0) - \lambda_2^C(k,0)$ is an increasing function of

$$
\begin{aligned}
&(1-\theta^A(0))w(0) - y_1^A(k,0) - (1-\theta^C(0))w(0) + y_1^C(k,0) \\
=& y_1^C(k,0) - y_1^A(k,0) + \theta^C(0)w(0) - \theta^A(0)w(0) \\
=& y_1^C(k,0) - y_1^A(k,0) + \frac{1}{1-U_0}\int_{\underline{k}}^{\infty} (y_1^C(k,0) - y_1^A(k,0))g(k,0)dk \\
=& \eta(U_0 - U^\star)\left(1 + \frac{U_0}{1-U_0}\right) \\
=& \eta\frac{U_0 - U^\star}{1-U_0}
\end{aligned}
$$

which is an increasing function of the size of aggregate shock. Therefore the approximated excess unemployment $\bar{U}'(0)t$ as in (4.14) is a superlinear increasing function of the size of aggregate shock. This is consistent with the nonlinear dependence of the excess unemployment rate on the size of aggregate shock.

To see the time aggregation effect, consider a simplified setting where job search intensity is independent of individuals' total income and is just a decreasing function of the level of benefits at time 0, thus it is a decreasing function of shock size $U_0$. Then

$$\frac{d\bar{U}_t}{dt} = -(\lambda_1 + \lambda_2(U_0))\bar{U}_t, \quad \bar{U}_0 = 0$$

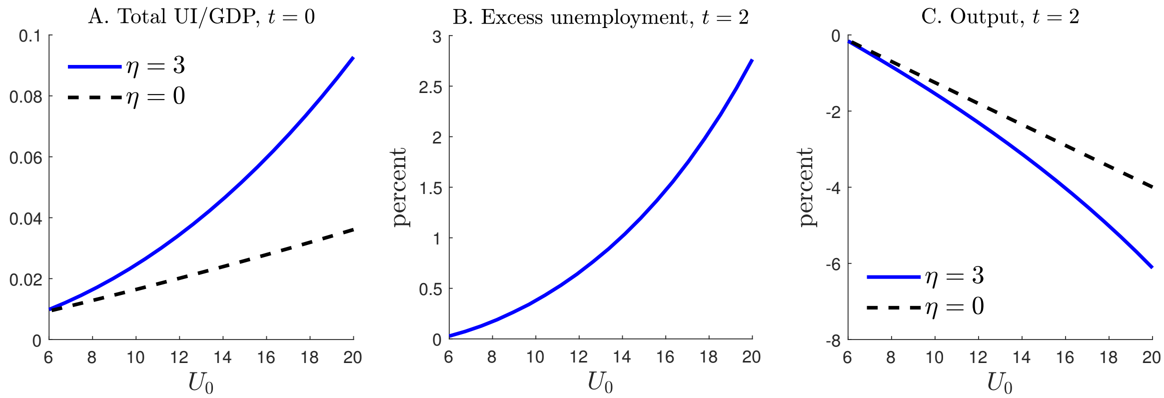


Figure 4.4: **Countercyclical policies and shock size.** Panels A through C compare total UI expense normalized by output, the excess unemployment rate at $t = 2$ years, and the output relative to its precrisis level also at $t = 2$ years for the baseline policy with $\eta = 3$.

implies

$$\bar{U}_t = e^{-(\lambda_1+\lambda_2(U_0))t} - 1.$$

Therefore for any $t$, $\bar{U}_t$ is a non-linear function of $U_0$.

Panel C of Figure 4.4 shows aggregate output two years following the shock relative to its pre-crisis level. The solid and dash lines refer to countercyclical and acyclical policies, respectively. This figure also highlights the non-linear effect of countercyclical UI policies on the economic recovery. For instance, for $U_0 = 10\%$, output growth is quite similar under the countercyclical and acyclical policies—output is 1.54% and 1.25% below pre-crisis levels under these policies, respectively. However, the difference in output growth is significantly large under the two policies for $U_0 = 20\%$; they are 6.1% and 4% under the countercyclical and acyclical policies, respectively.

### 4.3.3 Means-tested countercyclical policies

In this section we quantitatively analyze a form of "means-tested" (MT) policy. Under this policy, lower income individuals experience a greater increase in UI benefit following the realization of the aggregate at $t = 0$. In particular, we analyze "means-

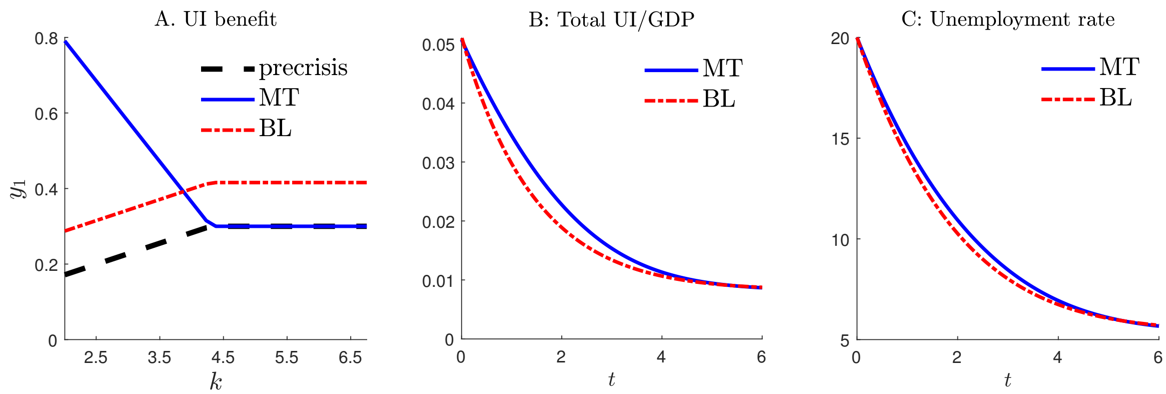


Figure 4.5: **Means-tested UI policy.** Panel A shows UI benefit paid as a function of the recipient's capital $k$. The heavy, dash line shows this policy in the precrisis period $t < 0$. The thin, dot-dash line corresponds to the baseline policy (BL) with $h(k) = 1$ in (4.7), and the thin, solid line corresponds to the means-tested (MT) policy shown in equation (4.15); both at $t = 0$. The range of the x-axis is over the 5th to 95th percentiles of $g_1(k, 0)$. Panel B shows total UI benefits paid as a fraction of output for the MT policy (solid line) and the baseline policy BL (dot-dash line). Panel C shows the path of the equilibrium unemployment rate under the two policies.

tested" (MT) policies with

$$h(k) = \max\left(1 - \frac{b}{\overline{y} - a}(k - \underline{k}), 0\right) \tag{4.15}$$

and $\eta = 8$, in equation (4.7).

We compare the equilibrium under this policy to that under the baseline policy (BL) that we analyzed in Section 4.3.2 for which $h(k) = 1$. For this comparison, we choose $\eta = 0.8$ for the BL policy because this choice equates the time-0 total UI expenditures under the MT and BL policies. The aggregate shock is the same as in Section 4.3.2, namely, $U_0 = 20\%$, and the likelihood of job loss at $t = 0$ is independent of the individual's capital $k$. Our key finding in this section is that the MT policy provides substantially better consumption insurance against income loss to its intended beneficiaries (i.e. individuals with $k < k_H$), without a substantial increase in the equilibrium unemployment rate.

Panel A of Figure 4.5 shows the time-0 total UI expense under the two policies.

The heavy, dashed line corresponds to the UI payments in the precrisis state. The thin, dot-dash line corresponds to the BL policy. We see that the increase in UI payments from the precrisis state is the same for all individuals. The thin, solid line corresponds to the MT policy. We see that UI payments increase for individuals with $k < k_H$, with this increase being larger for lower $k$ individuals. In comparing the UI payments under the two policies, we see that individuals with time-0 capital $k < 3.9$ receive higher UI benefit under the MT policy than under the BL policy. This threshold corresponds to the 30th percentile of $g_1(k, 0)$. Panel B of Figure 4.5 shows the total UI payments. By construction, the total time-0 UI payments are the same under both policies. However, the total payments under the BL policy declines faster than under the MT policy; at $t = 2$ years, the UI expenditure under the MT policy costs 0.5% of GDP more than the BL policy. This is because, in equilibrium, the unemployment rate under the MT policy declines slower than under the BL policy (see panel C). However, this difference is quantitatively small—along the transition path, the maximum difference in the unemployment rate under the MT and BL policies is only 0.7%.

Labor markets recover slower under the MT policy because of relatively lower job-search effort for most individuals in comparison to that under the BL policy. We show this in panel A of Figure 4.6. The difference in job-search intensity is especially large for low income individuals. This is because the MT policy is associated with more generous UI benefits for relatively low income individuals. Part of the higher benefits is due to the choice of the UI benefit function at $t = 0$. Part of it is also due to the following general equilibrium feedback effect. Lower job-search effort under the MT policy relative to the BL policy, results in a higher equilibrium unemployment rate (both current and future). This, in turn, leads to higher current and future benefits according to equation (4.7), which increases the value of unemployment. This further reduces equilibrium job-search effort (see equation (4.9)).

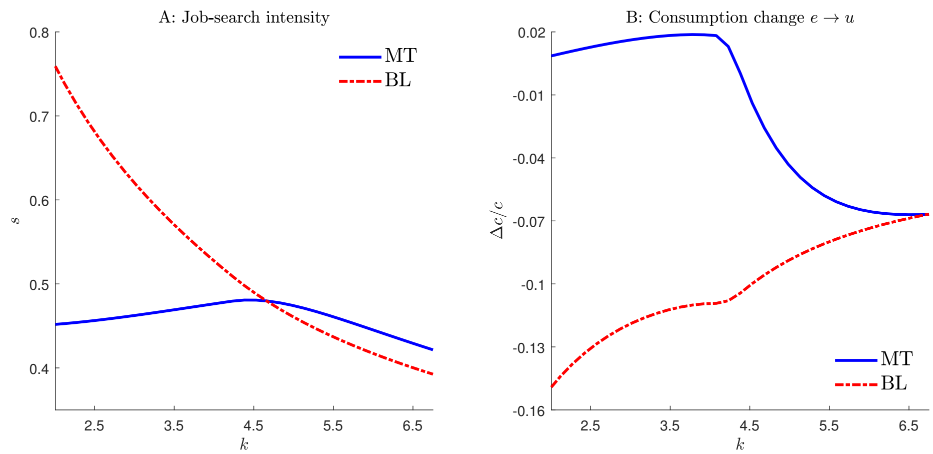


Figure 4.6: **Cross-sectional effects.** Panels A and B show the job-search intensity and consumption insurance (as measured by the fractional drop in consumption of those individuals who become unemployed at t = 0) under the means-tested counterycyclical policy (MT) and the baseline countercyclical policy (BL), respectively. The range of the x-axis in these figures is over the 5th to 95th percentiles of $g_1(k, 0)$.

Panel B of Figure 4.6 shows that the MT policy provides substantially better consumption insurance for individuals with $k < k_H$. The time-0 fractional decline in consumption $\Delta c/c$ for individuals who becomes unemployed from the aggregate shock and with capital $k < k_H$ is more than 12% smaller under the MT policy relative to the BL policy.

## 4.4 Conclusion

In this paper we quantify the impact of countercyclical UI policies on labor supply when general equilibrium effect is taken into consideration. A baseline countercyclical policy without means test and a means-tested countercyclical policy are incorporated into our costly job search model. By employing numerical method from mean field game theory, we are able to examine their impact quantitatively. Firstly, from our baseline model, we find that the additional unemployment generated by countercyclical policies relative to an acyclical policy is a sharply increasing, nonlinear function of the size of the aggregate shock. Secondly, we find that the means-tested policy provides substantially better consumption insurance against income loss to low income individuals, without a substantial increase in the equilibrium unemployment rate.

In addition to the above non-trivial findings, our novel model provides some highly interesting problems to investigate, including the existence and uniqueness of equilibrium solution, the convergence of the numerical algorithm, etc. Moreover, the setting and methodology in this paper are promising for quantifying and comparing the effect of other policies. For instance, it would be interesting to ask if extending UI is more or less effective than a one-time stimulus payment, where the total stimulus expenditure is equal to the present discounted value of UI payments. Another useful exercise would be to study UI benefits with finite life which would allow the model to make closer contact with the data and therefore provide more informed inferences about the tradeoffs of consumption insurance against insufficient job search effort by the unemployed. We leave them as future projects.

# APPENDICES

# APPENDIX A

# Computation details in Chapter II

## A.1 Example 2.2.24

We first analyze all the possible trajectories of Markov chain $X$ when starting from 1 and 6.

**Case 1:** $X : 1 \to 6 \to 1 \to 6 \to 1 \to \cdots \to 6 \to 1 \to 0$.

$\Rightarrow Y_\eta = 0$ for $k = 0$ and $Y_\eta = a\sum_{i=1}^{k}(1-a)^{i-1}(1-b)^i + 6b\sum_{i=0}^{k-1}(1-a)^i(1-b)^i$ for $k \geq 1$. Then $Y_\eta = c(1-(1-a)^k(1-b)^k)$ with probability $0.2^{2k+1}$ for $k \geq 0$ where $c = \frac{a+6b-ab}{1-(1-a)(1-b)}$.

**Case 2:** $X : 1 \to 6 \to 1 \to 6 \to 1 \to \cdots \to 6 \to 1 \to 3$.

$\Rightarrow Y_\eta = 3$ for $k = 0$ and $Y_\eta = a\sum_{i=1}^{k}(1-a)^{i-1}(1-b)^i + 6b\sum_{i=0}^{k-1}(1-a)^i(1-b)^i + 3(1-a)^k(1-b)^k$ for $k \geq 1$. Then $Y_\eta = c(1-(1-a)^k(1-b)^k) + 3(1-a)^k(1-b)^k$ with probability $2 \times 0.2^{2k+1}$ for $k \geq 0$.

**Case 3:** $X : 1 \to 6 \to 1 \to 6 \to 1 \to \cdots \to 6 \to 1 \to 10$.

$\Rightarrow Y_\eta = 10$ for $k = 0$ and $Y_\eta = a\sum_{i=1}^{k}(1-a)^{i-1}(1-b)^i + 6b\sum_{i=0}^{k-1}(1-a)^i(1-b)^i + 10(1-a)^k(1-b)^k$ for $k \geq 1$. Then $Y_\eta = c(1-(1-a)^k(1-b)^k) + 10(1-a)^k(1-b)^k$ with probability $0.2^{2k+1}$ for $k \geq 0$.

**Case 4:** $X : 1 \to 6 \to 1 \to 6 \to 1 \to \cdots \to 6 \to 10$.

$\Rightarrow Y_\eta = a\sum_{i=1}^{k}(1-a)^{i-1}(1-b)^i + 6b\sum_{i=0}^{k}(1-a)^i(1-b)^i + 10(1-a)^k(1-b)^{k+1}$ for $k \geq 0$. Then $Y_\eta = c(1-(1-a)^k(1-b)^k) + (10-4b)(1-a)^k(1-b)^k$ with probability

$0.8 \times 0.2^{2k+1}$ for $k \geq 0$.

**Case 5:** $X : 6 \to 1 \to 6 \to 1 \to \cdots \to 6 \to 1 \to 0$ for $k \geq 0$.

$\Rightarrow Y_\eta = a$ for $k = 0$ and $Y_\eta = a\sum_{i=0}^{k}(1-a)^i(1-b)^i + 6b\sum_{i=0}^{k-1}(1-a)^{i+1}(1-b)^i$ for $k \geq 1$. Then $Y_\eta = d(1-(1-a)^k(1-b)^k) + a(1-a)^k(1-b)^k$ with probability $0.2^{2k+2}$ for $k \geq 0$ where $d = \frac{a+6b-6ab}{1-(1-a)(1-b)}$.

**Case 6:** $X : 6 \to 1 \to 6 \to 1 \to \cdots \to 6 \to 1 \to 3$.

$\Rightarrow Y_\eta = a + 3(1-a)$ for $k = 0$ and $Y_\eta = a\sum_{i=0}^{k}(1-a)^i(1-b)^i + 6b\sum_{i=0}^{k-1}(1-a)^{i+1}(1-b)^i + 3(1-a)^{k+1}(1-b)^k$ for $k \geq 1$. Then $Y_\eta = d(1-(1-a)^k(1-b)^k) + (3-2a)(1-a)^k(1-b)^k$ with probability $2 \times 0.2^{2k+2}$ for $k \geq 0$.

**Case 7:** $X : 6 \to 1 \to 6 \to 1 \to \cdots \to 6 \to 1 \to 10$.

$\Rightarrow Y_\eta = a + 10(1-a)$ for $k = 0$ and $Y_\eta = a\sum_{i=0}^{k}(1-a)^i(1-b)^i + 6b\sum_{i=0}^{k-1}(1-a)^{i+1}(1-b)^i + 10(1-a)^{k+1}(1-b)^k$ for $k \geq 1$. Then $Y_\eta = d(1-(1-a)^k(1-b)^k) + (10-9a)(1-a)^k(1-b)^k$ with probability $0.2^{2k+2}$ for $k \geq 0$.

**Case 8:** $X : 6 \to 1 \to 6 \to 1 \to \cdots \to 6 \to 10$.

$\Rightarrow Y_\eta = 10$ for $k = 0$ and $Y_\eta = a\sum_{i=0}^{k-1}(1-a)^i(1-b)^i + 6b\sum_{i=0}^{k-1}(1-a)^{i+1}(1-b)^i + 10(1-a)^k(1-b)^k$ for $k \geq 1$. Then $Y_\eta = d(1-(1-a)^k(1-b)^k) + 10(1-a)^k(1-b)^k$ with probability $0.8 \times 0.2^{2k}$ for $k \geq 0$.

From the above, we can conclude that

(1) When $X_0 = 1$,

$$\mathbb{P}_1(Y_\eta = c - ct^k) = 0.2^{2k+1}, k \geq 0,$$

$$\mathbb{P}_1(Y_\eta = c + (3-c)t^k) = 2 \times 0.2^{2k+1}, k \geq 0,$$

$$\mathbb{P}_1(Y_\eta = c + (10-c)t^k) = 0.2^{2k+1}, k \geq 0,$$

$$\mathbb{P}_1(Y_\eta = c + (10-4b-c)t^k) = 0.8 \times 0.2^{2k+1}, k \geq 0,$$

where $t = (1-a)(1-b)$, and

$$\begin{aligned}\mathbb{E}_1[Y_\eta] &= c + \frac{4.8 - 0.96c - 0.64b}{1 - 0.04t}, \\ \mathbb{E}_1[Y_\eta^2] &= c^2 + \frac{-1.92c^2 + 9.6c - 1.28cb}{1 - 0.04t} \\ &\quad + \frac{0.96c^2 - 9.6c + 39.6 - 12.8b + 1.28bc + 2.56b^2}{1 - 0.04t^2}.\end{aligned}$$

(2) When $X_0 = 6$,

$$\mathbb{P}_6(Y_\eta = d + (a-d)t^k) = 0.2^{2k+2}, k \geq 0,$$

$$\mathbb{P}_6(Y_\eta = d + (3 - 2a - d)t^k) = 2 \times 0.2^{2k+2}, k \geq 0,$$

$$\mathbb{P}_6(Y_\eta = d + (10 - 9a - d)t^k) = 0.2^{2k+2}, k \geq 0,$$

$$\mathbb{P}_6(Y_\eta = d + (10 - d)t^k) = 0.8 \times 0.2^{2k}, k \geq 0,$$

where $t = (1-a)(1-b)$, and

$$\begin{aligned}\mathbb{E}_6[Y_\eta] &= d + \frac{8.64 - 0.96d - 0.48a}{1 - 0.04t}, \\ \mathbb{E}_6[Y_\eta^2] &= d^2 + \frac{-1.92d^2 + 17.28d - 0.96ad}{1 - 0.04t} \\ &\quad + \frac{0.96d^2 - 17.28d + 84.72 + 3.6a^2 + 0.96ad - 8.16a}{1 - 0.04t^2}.\end{aligned}$$

Then we will obtain the result in Example 2.2.24.

## A.2 The first example in Proposition 2.2.27

Since the transition matrix in this example is the same as Example 2.2.24, by following a similar analysis of $X$'s trajectories, we have

(1) When $X_0 = 1$,

$$\mathbb{P}_1(Y_\eta = c - ct^k) = 0.2^{2k+1}, k \geq 0,$$

$$\mathbb{P}_1(Y_\eta = c + (2-c)t^k) = 2 \cdot 0.2^{2k+1}, k \geq 0,$$

$$\mathbb{P}_1(Y_\eta = c + (9-c)t^k) = 0.2^{2k+1}, k \geq 0,$$

$$\mathbb{P}_1(Y_\eta = c + (9-2b-c)t^k) = 0.8 \cdot 0.2^{2k+1}, k \geq 0,$$

where $c = \frac{a+7b-ab}{1-(1-a)(1-b)}$ and $t = (1-a)(1-b)$. Then we have

$$\mathbb{E}_1[Y_\eta] = c + \frac{0.2(20.2 - 4.8c - 1.6b)}{1-0.04t},$$

$$\mathbb{E}_1[Y_\eta^2] = c^2 + \frac{0.4(-4.8c^2 + 20.2c - 1.6cb)}{1-0.04t} + \frac{0.2(c^2 + 2(2-c)^2 + (9-c)^2 + 0.8(9-2b-c)^2)}{1-0.04t^2}.$$

(2) When $X_0 = 7$,

$$\mathbb{P}_7(Y_\eta = d + (a-d)t^k) = 0.2^{2k+2}, k \geq 0,$$

$$\mathbb{P}_7(Y_\eta = d + (2-a-d)t^k) = 2 \times 0.2^{2k+2}, k \geq 0,$$

$$\mathbb{P}_7(Y_\eta = d + (9-8a-d)t^k) = 0.2^{2k+2}, k \geq 0,$$

$$\mathbb{P}_7(Y_\eta = d + (9-d)t^k) = 0.8 \times 0.2^{2k}, k \geq 0,$$

where $d = \frac{a+7b-7ab}{1-(1-a)(1-b)}$ and $t = (1-a)(1-b)$. Then we have

$$\mathbb{E}_7[Y_\eta] = d + \frac{0.04(193 - 24d - 9a)}{1-0.04t},$$

$$\mathbb{E}_7[Y_\eta^2] = d^2 + \frac{0.08(-24d^2 + 193d - 9ad)}{1-0.04t} + \frac{0.04((a-d)^2 + 2(2-a-d)^2 + (9-8a-d)^2 + 20(9-d)^2)}{1-0.04t^2}.$$

Furthermore, we can find that for any $a, b \in [0,1]$, $g_1(a,b) > 1$, which implies that

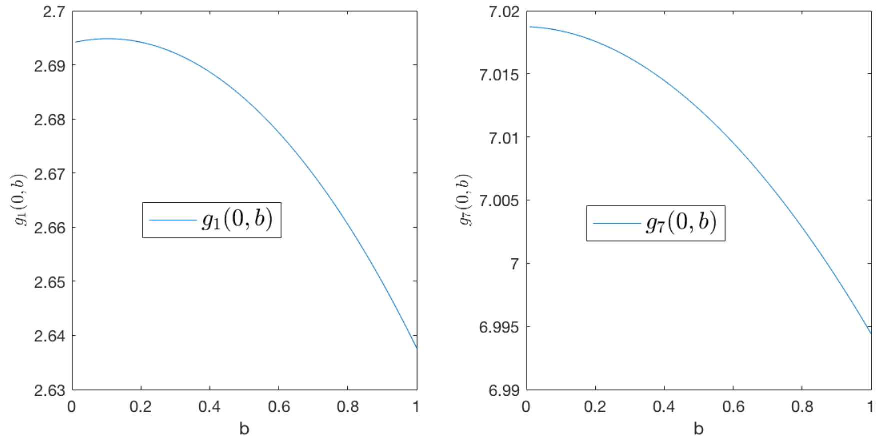


Figure A.1: Graphs for the first example in Proposition 2.2.27

$a = 0$. By plotting the graphs of $g_1(0,b)$ and $g_7(0,b)$ as functions of $b \in [0,1]$, we obtain Figure A.1, which follows the discussion in part (i) of the proof of Proposition 2.2.27.

## A.3 The second example in Proposition 2.2.27

It can be computed directly that there are three equilibrium stopping times. They can be written in the form of liquidation strategies: $(a,b) = (1,1)$, $(a,b) = (0,1)$, and $(a,b) = (1,0)$. To check they are indeed equilibria, compute

$$g_{11}(1,1) = 10.8330 < 11, \;\; g_{17}(1,1) = 16.9912 < 17,$$
$$g_{11}(0,1) = 11.1515 > 11, \;\; g_{17}(0,1) = 16.9774 < 17,$$
$$g_{11}(1,0) = 10.8330 < 11, \;\; g_{17}(1,0) = 17.0022 > 17,$$

where $g_{11}(a,b) = \mathbb{E}_{11}[Y_\eta] - 0.1(\mathbb{E}_{11}[Y_\eta^2] - \mathbb{E}_{11}[Y_\eta]^2)^{1/2}$ and $g_{17}(a,b) = \mathbb{E}_{17}[Y_\eta] - 0.1(\mathbb{E}_{17}[Y_\eta^2] - \mathbb{E}_{17}[Y_\eta]^2)^{1/2}$.

Also notice that $(a,b) = (0,0)$ is not a equilibrium liquidation strategy since

$g_{17}(0,0) = 16.9934 < 17$.

To find all equilibrium liquidation strategies, we need to analyze all the possible trajectories of this Markov chain when starting from 11 and 17.

**Case 1:** $X : 11 \to 11 \to \cdots \to 11 \to 0$.

Then $Y_\eta = 0$ with probability 0.1 and $Y_\eta = 11a\sum_{i=0}^{k-1}(1-a)^i$ with probability $0.1 \cdot 0.7^k$ for $k \geq 1$.

**Case 2:** $X : 11 \to 11 \to \cdots \to 11 \to 18$.

Then $Y_\eta = 18$ with probability 0.2 and $Y_\eta = 11a\sum_{i=0}^{k-1}(1-a)^i + 18(1-a)^k$ with probability $0.2 \cdot 0.7^k$ for $k \geq 1$.

**Case 3:** $X : 17 \to 17 \to \cdots \to 17 \to 18$.

Then $Y_\eta = 18$ with probability 0.8 and $Y_\eta = 17b\sum_{i=0}^{k-1}(1-b)^i + 18(1-b)^k$ with probability $0.8 \cdot 0.1^k$ for $k \geq 1$.

**Case 4:** $X : 17 \to 17 \to \cdots \to 17 \to 11 \to 11 \to \cdots \to 11 \to 0$.

Then $Y_\eta = 11a\sum_{j=0}^{m}(1-a)^j$ with probability $0.01 \cdot 0.7^m$ for $m \geq 0$ and $Y_\eta = 17b\sum_{i=0}^{k-1}(1-b)^i + (1-b)^k 11a\sum_{j=0}^{m}(1-a)^j$ with probability $0.01 \cdot 0.7^m \cdot 0.1^k$ for $k \geq 1, m \geq 0$.

**Case 5:** $X : 17 \to 17 \to \cdots \to 17 \to 11 \to 11 \to \cdots \to 11 \to 18$.

Then $Y_\eta = 11a\sum_{j=0}^{m}(1-a)^j + 18(1-a)^{m+1}$ with probability $0.02 \cdot 0.7^m$ for $m \geq 0$ and $Y_\eta = 17b\sum_{i=0}^{k-1}(1-b)^i + (1-b)^k(11a\sum_{j=0}^{m}(1-a)^j + 18(1-a)^{m+1})$ with probability $0.02 \cdot 0.7^m \cdot 0.1^k$ for $k \geq 1, m \geq 0$.

From the above, we can conclude that

(1) When $X_0 = 11$,

$$\mathbb{P}_{11}(Y_\eta = 11 - 11(1-a)^k) = 0.1 \cdot 0.7^k, \quad k \geq 0,$$
$$\mathbb{P}_{11}(Y_\eta = 11 + 7(1-a)^k) = 0.2 \cdot 0.7^k, \quad k \geq 0,$$

and

$$\mathbb{E}_{11}[Y_\eta] = 11 + \frac{0.3}{0.3 + 0.7a},$$
$$\mathbb{E}_{11}[Y_\eta^2] = 11^2 + \frac{6.6}{0.3 + 0.7a} + \frac{21.9}{1 - 0.7(1-a)^2}.$$

(2) When $X_0 = 17$,

$$\mathbb{P}_{17}(Y_\eta = 17 + (1-b)^k) = 0.8 \cdot 0.1^k, \quad k \geq 0$$
$$\mathbb{P}_{17}(Y_\eta = 17 - (6 + 11(1-a)^{m+1})(1-b)^k) = 0.01 \cdot 0.7^m \cdot 0.1^k, \quad k \geq 0, m \geq 0,$$
$$\mathbb{P}_{17}(Y_\eta = 17 - (6 - 7(1-a)^{m+1})(1-b)^k) = 0.02 \cdot 0.7^m \cdot 0.1^k, \quad k \geq 0, m \geq 0,$$

and

$$\mathbb{E}_{17}[Y_\eta] = 17 + (0.2 + \frac{0.03(1-a)}{0.3 + 0.7a})\frac{1}{0.9 + 0.1b},$$
$$\mathbb{E}_{17}[Y_\eta^2] = 17^2 + (0.2 + \frac{0.03(1-a)}{0.3 + 0.7a})\frac{34}{0.9 + 0.1b} +$$
$$(4.4 - \frac{0.36(1-a)}{0.3 + 0.7a} + \frac{2.19(1-a)^2}{1 - 0.7(1-a)^2})\frac{1}{1 - 0.1(1-b)^2}.$$

The sets $\{(a,b) \in [0,1] \times [0,1] : g_{11}(a,b) = 11\}$ and $\{(a,b) \in [0,1] \times [0,1] : g_{17}(a,b) = 17\}$ are shown as the following. Figure A.2 shows that the curves $g_{11}(a,b) = 11$ and $g_{17}(a,b) = 17$ do not intersect. So the candidates for equilibrium liquidation strategies only lie on the boundary of $[0,1] \times [0,1]$. From Figure A.2 we can observe that there exist $0 < a_1 < a_2 < a_3 < a_4 < 1$ and $0 < b_0 < 1$ such that

$$g_{17}(a_1, 0) = 17, \quad g_{17}(a_2, 1) = 17, \quad g_{17}(a_4, 1) = 17;$$
$$g_{11}(a_3, 0) = g_{11}(a_3, 1) = 11;$$
$$g_{17}(1, b_0) = 17.$$

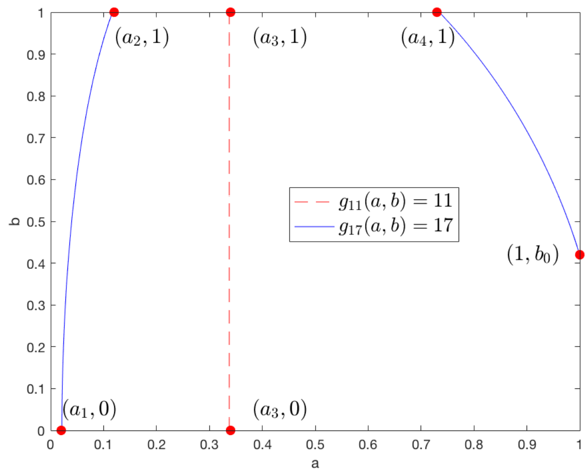


Figure A.2: Graphs for the second example in Proposition 2.2.27: curves $g_{11}(a, b) = 11$ and $g_{17}(a, b) = 17$

Also from Figure A.2 we know that $g_{11}(a_1, 0) \neq 11, g_{11}(a_2, 1) \neq 11$ and $g_{11}(a_4, 1) \neq 11$, so they cannot be equilibrium liquidation strategies. To find out whether $(a_3, 0), (a_3, 1)$ and $(1, b_0)$ are equilibrium liquidation strategies. We plot the graphs of $g_{11}(a, b_0), g_{17}(a, 0)$ and $g_{17}(a, 1)$ as functions of $a \in [0, 1]$.

These graphs show that $g_{11}(1, b_0) < 11, g_{17}(a_3, 0) > 17$, and $g_{17}(a_3, 1) > 17$. So there are five equilibrium liquidation strategies as discussed in part (ii) of the proof of Proposition 2.2.27.

## A.4 The third example in Proposition 2.2.27

We first analyze all the possible trajectories of Markov chain $X$ when starting from 1.

**Case 1:** $X : 1 \to 1 \to \cdots \to 1 \to 1 \to 0$.

$\Rightarrow Y_\eta = 0$ for $k = 0$ and $Y_\eta = a \sum_{i=1}^{k-1} (1-a)^i$ for $k \geq 1$. Then $Y_\eta = 1 - (1-a)^k$ with probability $0.1 \times 0.8^k$ for $k \geq 0$.

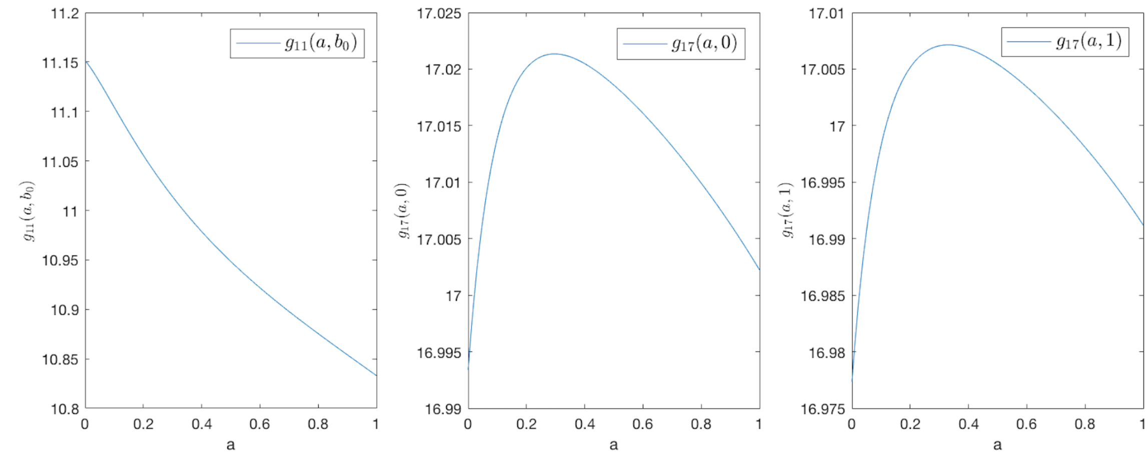


Figure A.3: Graphs for the second example in Proposition 2.2.27: $g_{11}(a,b_0), g_{17}(a,0)$ and $g_{17}(a,1)$ as functions of $a \in [0,1]$

**Case 2:** $X : 1 \to 1 \to \cdots \to 1 \to 1 \to 4$.

$\Rightarrow Y_\eta = 4$ for $k = 0$ and $Y_\eta = a\sum_{i=1}^{k-1}(1-a)^i + 4(1-a)^k$ for $k \geq 1$. Then $Y_\eta = 1 + 3(1-a)^k$ with probability $0.1 \times 0.8^k$ for $k \geq 0$.

By computation, we have

$$\mathbb{E}_1[Y_\eta] = 1 + \frac{0.2}{0.2+0.8a}, \qquad \mathbb{E}_1[Y_\eta^2] = 1 + \frac{0.4}{0.2+0.8a} + \frac{1}{1-0.8(1-a)^2}$$

Then the explicit expression for $h(a) = \mathbb{E}_1[Y_\eta] - c\mathrm{Var}_x[Y_\eta]$ can be obtained and we have the results in part (iii) of the proof of Proposition 2.2.27.

## A.5 Equilibrium liquidation strategies for the mean-variance problems in Examples 2.4.1 and 2.4.2

If $\eta$ is an equilibrium liquidation strategy in the mean-variance problem, then

$$J_l(x,\eta) = \sup_{\xi \in \mathcal{L}} J_l(x, \xi \otimes \eta), \ \ \forall x \in \mathbb{X}.$$

Recall that $\mathbb{E}_x[\theta^{\xi\otimes\eta}(X)] = x\xi(x) + (1-\xi(x))\mathbb{E}_x[Y_\eta]$ and $\mathrm{Var}_x[\theta^{\xi\otimes\eta}(X)] = (1-\xi(x))^2\mathrm{Var}_x[Y_\eta]$. Therefore,

$$J_l(x,\xi\otimes\eta) = -c\mathrm{Var}_x[Y_\eta]\xi(x)^2 + (2c\mathrm{Var}_x[Y_\eta] - \mathbb{E}_x[Y_\eta] + x)\xi(x) + \mathbb{E}_x[Y_\eta] - c\mathrm{Var}_x[Y_\eta],$$

is a quadratic function of $\xi(x)$ when $\eta$ is fixed. We then have

$$\eta(x) = \begin{cases} 1, & if\ h_x(\eta) \in [1,\infty); \\ h_x(\eta), & if\ h_x(\eta) \in (0,1); \\ 0, & if\ h_x(\eta) \in (-\infty,0]. \end{cases}$$

where $h_x(\eta) = \frac{2c\mathrm{Var}[Y_\eta] - \mathbb{E}_x[Y_\eta] + x}{2c\mathrm{Var}_x[Y_\eta]}$.

In Example 2.4.1, $\mathbb{E}_i[Y_\eta]$ and $\mathbb{E}_i[Y_\eta^2]$ for $i = 1,7$ have been computed in Appendix A.2, so we obtain the explicit expressions of $h_1(\eta)$ and $h_7(\eta)$ as functions of $a := \eta(1)$ and $b =: \eta(7)$. Then we observe that $h_i(a,b) \in (0,1)$, for all $(a,b) \in [0,1]\times[0,1]$, $i = 1,7$, and there is exactly one intersection of the curve $\{(a,b) : h_1(a,b) = a\}$ and the curve $\{(a,b) : h_7(a,b) = 7\}$, which is the equilibrium liquidation strategy for mean-variance problem in Example 2.4.1. Similarly we can find the equilibrium liquidation strategy for mean-variance problem in Example 2.4.2. The corresponding graphs are shown below.

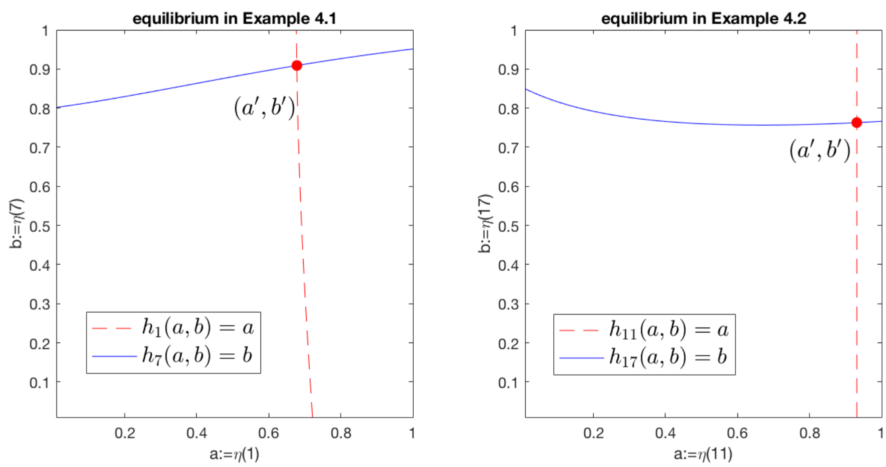


Figure A.4: Equilibrium liquidation strategy for mean-variance problem in Example 2.4.2

# APPENDIX B

# Algorithms in Chapter IV

In the following two sections, we take the model with a baseline countercyclical policy as an example to describe algorithms for computing the equilibrium solutions in steady state and transitional dynamics respectively. The algorithm for models with a means-tested policy is very similar.

## B.1 Algorithm for stationary equilibrium

For the infinite horizon model with a constant level of UI benefits $b^\star$, we consider the stationary equilibrium, which can be summarized with a coupled HJB-KF equation system. For $k \in [\underline{k}, \infty)$,

$$\rho v_1(k) = \max_{c_1 s} u(c_1) + (b^\star w^\star + (r^\star - \delta)k - c_1)\, \dot{v}_1(k) + (v_2(k) - v_1(k))\, \lambda_2(s) - \phi(s) \tag{B.1}$$

$$\rho v_2(k) = \max_{c_2} u(c_2) + ((1-\tau^\star)w^\star + (r^\star - \delta)k - c_2)\, \dot{v}_2(k) + (v_1(k) - v_2(k))\, \lambda_1 \tag{B.2}$$

$$0 = -\frac{d}{dk}\left(g_1(k)\left(b^\star w^\star + (r^\star - \delta)k - c_1^\star\right)\right) - g_1(k)\lambda_2(s^\star) + g_2(k)\lambda_1 \tag{B.3}$$

$$0 = -\frac{d}{dk}\left(g_2(k)\left((1-\tau^\star)w^\star + (r^\star - \delta)k - c_2^\star\right)\right) - g_2(k)\lambda_1 + g_1(k)\lambda_2(s^\star) \tag{B.4}$$

where $c_1^\star, s^\star$ are such that the right hand side of (B.1) attains maximum, $c_2^\star$ is such that the right hand side of (B.2) attains maximum, more specifically,

$$
\begin{aligned}
c_1^\star(k) &= (\dot{v}_1(k))^{-1/\gamma} \\
c_2^\star(k) &= (\dot{v}_2(k))^{-1/\gamma} \\
s^\star(k) &= \left(\frac{v_2(k) - v_1(k)}{\phi}\right)^{1/\kappa}
\end{aligned}
$$

and $r^\star, w^\star, \tau^\star$ satisfy

$$
\begin{aligned}
r^\star &= \alpha\,(K^\star/L^\star)^{\alpha-1} \\
w^\star &= (1-\alpha)\,(K^\star/L^\star)^{\alpha} \\
\tau^\star &= b^\star(1-L^\star)/L^\star
\end{aligned}
$$

with $K^\star = \int_{\underline{k}}^\infty k(g_1(k) + g_2(k))dk$ and $L^\star = \int_{\underline{k}}^\infty g_2(k)dk$.

Suppose we have a solution of equation system (B.1, B.2, B.3, B.4), and denote it as $v_1^\star, v_2^\star, g_1^\star, g_2^\star$. Then if we start from the stationary distribution with density function $g_1^\star, g_2^\star$, the distribution will remain the same as each individual uses the optimal $c_2^\star, c_1^\star, s^\star$.

Thanks to the work of [2], for fixed $r^\star, L^\star$, thus fixed $w^\star, \tau^\star$, if $\lambda_2(s)$ is a constant, we can use finite difference method to solve HJB equations (B.1, B.2). Then given the solution of (B.1, B.2), we are able to obtain all coefficients in KF equations (B.3, B.4) and the solution of (B.3, B.4) will be obtained by solving a linear equation system. [2] uses a so-called "upwind-scheme" and the discretized equation system can be conveniently written in matrix notation.

Instead of finding a fixed point of $r^\star$ in [2], we need to find a fixed point of $(r^\star, L^\star)$ in order to solve for the equilibrium in our problem. Here is the sketched algorithm for solving (B.1, B.2, B.3, B.4).

**Algorithm 1:** Algorithms for stationary equilibrium

**Data:** $\alpha, b^\star, \gamma, \phi, \lambda_1, \kappa, \epsilon, N, r_{\max}, r_{\min}$

**Result:** $v_1^\star, v_2^\star, g_1^\star, g_2^\star, c_2^\star, c_1^\star, s^\star, r^\star, L^\star, w^\star, K^\star$

Take an initial guess of $r^\star := r^{(0)} \in [r_{\max}, r_{\min}]$;

Let $\Delta_K^{(0)} = \infty, n = 0$;

**while** $n < N$ **do**

  Compute the corresponding $w^\star = w(r^{(n)})$;

  Take an initial guess of $L^\star := L^{(0)}$. Let $\Delta_L^{(0)} = \infty, m = 0$;

  **while** $\Delta_L^{(m)} > \epsilon$ **do**

    Compute the corresponding $\tau^\star := \tau(L^{(m)})$ ;

    Use $r^{(n)}, w(r^{(n)}), L^{(m)}, \tau(L^{(m)})$ as given coefficients in (B.1, B.2);

    Take an initial guess of $v_2^{(0)}, v_1^{(0)}$ ;

    Let $\Delta_v^{(0)} = \infty, l = 0$;

    **while** $\Delta_v^{(l)} > \epsilon$ **do**

      Compute the optimal $c_2^\star, c_1^\star, s^\star$ using $v_2^{(l)}, v_1^{(l)}$;

      Solve for $v_2^{(l+1)}, v_1^{(l+1)}$ using by solving a system of matrix equations;

      Compute $\Delta_v^{(l+1)} = \|(v_2^{(l)}, v_1^{(l)}) - (v_2^{(l+1)}, v_1^{(l+1)})\|$;

      Update $v_2^{(l+1)} = v_2^{(l)}, v_1^{(l+1)} = v_1^{(l)}, l = l + 1$;

    With the above solution $v_2^\star, v_1^\star$ for the current $r^{(n)}, L^{(m)}$, solve for the linear differential equation (B.3, B.4) and denote the solution as $g_2^\star, g_1^\star$;

    Compute $L^{(m+1)} = \int_{\underline{k}} g_2^\star(k)dk$ and $\Delta_L^{(m+1)} = \|L^{(m+1)} - L^{(m)}\|$;

    Update $L^{(m)} = L^{(m+1)}$;

  Compute $K^{demand} = L^{(m)} \left(\alpha A^\star / r^{(n)}\right)^{1/(1-\alpha)}$;

  Compute $K^{supply} = \int_{\underline{k}} k \left(g_2^\star(k) + g_1^\star(k)\right) dk$;

  **if** $K^{supply} - K^{demand} > \epsilon$ **then**

    $r_{\max} = r^{(n)}, r^{(n+1)} = \frac{1}{2}(r_{\min} + r^{(n)}), n = n + 1$;

  **else**

    **if** $K^{supply} - K^{demand} < -\epsilon$ **then**

      $r_{\min} = r^{(n)}, r^{(n+1)} = \frac{1}{2}(r_{\max} + r^{(n)}), n = n + 1$;

    **else**

      **Return** current values of $r^{(n)}, L^{(m)}, w(r^{(n)}), v_u^\star, v_2^\star, g_1^\star, g_2^\star, c_2^\star, c_1^\star, s^\star, K^{demand}$;

  **Print:** "No equilibrium founded";

  **Return**

## B.2 Algorithm for time dependent equilibrium

Although we have infinite horizon in our problem, we do not start from the steady state due to the unanticipated shock at time $t = 0$. Therefore we need solve for a time-dependent equilibrium. For the convenience of numerical computation, we assign a large time $T$ as the terminal time and assume that with this long enough time $T$, the economy has converged to the steady state with constant $b^\star$. We use the equilibrium of this finite horizon problem to approximate the time-dependent equilibrium of the original infinite horizon problem, which can be summarized with a coupled HJB-KF equation system. For $k \in [\underline{k}, \infty), t \in [0, T]$,

$$\begin{aligned}
\rho v_1(k,t) &= \max_{c_1, s} u(c_1) + (b_t w_t + (r_t - \delta)k - c_1)\, \partial_k v_1(k,t) \\
&\quad + (v_2(k,t) - v_1(k,t))\, \lambda_2(s) - \phi(s) + \partial_t v_1(k,t) \qquad \text{(B.5)} \\
\rho v_2(k,t) &= \max_{c_2} u(c_2) + ((1-\tau_t) w_t + (r_t - \delta)k - c_2)\, \partial_k v_2(k,t) \\
&\quad + (v_1(k,t) - v_2(k,t))\, \lambda_1 + \partial_t v_2(k,t) \qquad \text{(B.6)} \\
\partial_t g_1(k,t) &= -\partial_k \left( g_1(k,t) \left( b_t w_t + (r_t - \delta)k - c_1(k,t) \right) \right) \\
&\quad - g_1(k,t)\lambda_2(s(k,t)) + g_2(k,t)\lambda_1 \qquad \text{(B.7)} \\
\partial_t g_2(k,t) &= -\partial_k \left( g_2(k,t) \left( (1-\tau_t) w_t + (r_t - \delta)k - c_2(k,t) \right) \right) \\
&\quad - g_2(k,t)\lambda_1 + g_1(k,t)\lambda_2(s(k,t)) \qquad \text{(B.8)}
\end{aligned}$$

where in (B.7, B.8)

$$\begin{aligned}
c_1(k,t) &= (\partial_k v_1(k,t))^{-1/\gamma} \\
c_2(k,t) &= (\partial_k v_2(k,t))^{-1/\gamma} \\
s(k,t) &= \left( \frac{v_2(k,t) - v_1(k,t)}{\phi} \right)^{1/\kappa}
\end{aligned}$$

and $r_t, w_t, \tau_t$ satisfy

$$r_t = \alpha \left(K_t/L_t\right)^{\alpha-1}$$

$$w_t = (1-\alpha)\left(K_t/L_t\right)^{\alpha}$$

$$\tau_t = b_t(1-L_t)/L_t \text{ where } b_t = b^\star + \eta(L_t - L^\star)$$

with $K_t = \int_{\underline{k}}^{\infty} k(g_1(k,t) + g_2(k,t))dk$ and $L_t = \int_{\underline{k}}^{\infty} g_2(k,t)dk$.

The terminal condition is that $v_2(k,T) = v_2^\star(k), v_1(k,T) = v_1^\star(k)$ where $v_2^\star(k), v_1^\star(k)$ are the solution of the stationary equilibrium (B.1,B.2,B.3,B.4). To obtain the initial condition $g_1(k,0), g_2(k,0)$, we make some adjustment on the density function in the stationary equilibrium $g_1^\star(k), g_2^\star(k)$ such that the employment rate drops from $L^\star$ to an exogenous constant $L_0$. The initial distributions right after the aggregate shock $g_1(k,0), g_2(k,0)$are given.

With the above initial and terminal conditions, we have the following sketched algorithm for solving (B.5,B.6, B.7, B.8).

**Algorithm 2:** Algorithms for time dependent equilibrium

**Data:** $\alpha, \gamma, \phi, \lambda_1, \kappa, \eta, b_\eta, \epsilon, N, T, g_1(\cdot, 0), g_2(\cdot, 0), v_1^\star(\cdot), v_2^\star(\cdot)$

**Result:** $\{v_1(\cdot,t), v_2(\cdot,t), g_1(\cdot,t), g_2(\cdot,t), c_2(\cdot,t), c_1(\cdot,t), s(\cdot,t), r_t L_t, w_t, K_t\}_{t\in[0,T]}$

Take an initial guess of $K_t^{(0)}, L_t^{(0)}$ for $t \in [0,T]$;

Let $\Delta^{(0)} = \infty, n = 0$;

**while** $n < N$ **do**

  Using $\left\{K_t^{(0)}, L_t^{(0)}\right\}_{t\in[0,T]}$, compute the corresponding $r_t^{(n)}, w_t^{(n)}, b_t^{(n)}, \tau^{(n)}$ for $t \in [0,T]$;

  Using the terminal condition $v_1^\star(\cdot), v_2^\star(\cdot)$, apply finite difference method to compute $(v_u^{(n)}, v_2^{(n)})$ backward in time;

  $\left\{c_e^{(n)}(\cdot,t), c_u^{(n)}(\cdot,t), s^{(n)}(\cdot,t)\right\}_{t\in[0,T]}$ are obtained when computing $\left\{v_e^{(n)}(\cdot,t), v_u^{(n)}(\cdot,t)\right\}_{t\in[0,T]}$;

  Using the initial condition $g_1(\cdot,0), g_2(\cdot,0)$ and $\left\{c_e^{(n)}(\cdot,t), c_u^{(n)}(\cdot,t), s^{(n)}(\cdot,t)\right\}_{t\in[0,T]}$ to solve the linear equation system (B.7, B.8) forward in time and denote the results as $(g_u^{(n)}, g_2^{(n)})$;

  Compute $K_t^{(n+1)} = \int_{\underline{k}}^\infty k\left(g_1^{(n)}(k) + g_2^{(n)}(k)\right)dk, L_t^{(n+1)} = \int_{\underline{k}}^\infty g_2^{(n)}(k)dk$;

  **if** $\max\left\{\|K^{(n+1)}(\cdot) - K^{(n)}(\cdot)\|, \|L^{(n+1)}(\cdot) - L^{(n)(\cdot)}\|\right\} < \epsilon$ **then**

    **Return** $\{v_1(\cdot,t), v_2(\cdot,t), g_1(\cdot,t), g_2(\cdot,t), c_2(\cdot,t), c_1(\cdot,t), s(\cdot,t), r_t L_t, w_t, K_t\}_{t\in[0,T]}$;

  **else**

    Update $K_t^{(n)} = K_t^{(n+1)}, L_t^{(n)} = L_t^{(n+1)}$ for $t \in [0,T]$;

    $n = n + 1$;

  **Print:** "No equilibrium founded";

  **Return**

# BIBLIOGRAPHY

# BIBLIOGRAPHY


[1] Achdou, Y., F. J. Buera, J.-M. Lasry, P.-L. Lions, and B. Moll (2014), Partial differential equation models in macroeconomics, *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, *372*(2028), 20130,397, doi:10.1098/rsta.2013.0397.

[2] Achdou, Y., J. Han, J.-M. Lasry, P.-L. Lions, and B. Moll (2017), Income and wealth distribution in macroeconomics: A continuous-time approach, *Working Paper 23732*, National Bureau of Economic Research, doi:10.3386/w23732.

[3] Aiyagari, S. R. (1994), Uninsured Idiosyncratic Risk and Aggregate Saving, *The Quarterly Journal of Economics*, *109*(3), 659–684.

[4] Bayraktar, E., and Z. Zhou (2016), Arbitrage, hedging and utility maximization using semi-static trading strategies with american options, *Ann. Appl. Probab.*, *26*(6), 3531–3558, doi:10.1214/16-AAP1184.

[5] Bayraktar, E., and Z. Zhou (2018), No-arbitrage and hedging with liquid american options, *Mathematics of Operations Research*, *Published online*, doi:10.1287/moor.2018.0932.

[6] Bayraktar, E., J. Zhang, and Z. Zhou (2019), Time consistent stopping for the mean-standard deviation problem—the discrete time case, *SIAM Journal on Financial Mathematics*, *10*(3), 667–697, doi:10.1137/18M1216432.

[7] Bayraktar, E., J. Zhang, and Z. Zhou (2021), Equilibrium concepts for time-inconsistent stopping problems in continuous time, *Mathematical Finance*, *31*(1), 508–530, doi:https://doi.org/10.1111/mafi.12293.

[8] Bellman, R. (1957), *Dynamic Programming*, 1 ed., Princeton University Press, Princeton, NJ, USA.

[9] Bewley, T. (1983), A difficulty with the optimum quantity of money, *Econometrica*, *51*(5), 1485–1504.

[10] Bjork, T., and A. Murgoci (2010), A general theory of markovian time inconsistent stochastic control problems, *SSRN Electronic Journal*, doi:10.2139/ssrn.1694759.

[11] Björk, T., and A. Murgoci (2014), A theory of markovian time-inconsistent stochastic control in discrete time, *Finance and Stochastics*, *18*(3), 545–592, doi:10.1007/s00780-014-0234-y.

[12] Björk, T., M. Khapko, and A. Murgoci (2017), On time-inconsistent stochastic control in continuous time, *Finance and Stochastics*, *21*(2), 331–360, doi:10.1007/s00780-017-0327-5.

[13] Card, D., R. Chetty, and A. Weber (2007), Cash-on-Hand and Competing Models of Intertemporal Behavior: New Evidence from the Labor Market, *Quarterly Journal of Economics*, *122*(4), 1511–60.

[14] Carmona, R. (2020), Applications of mean field games in financial engineering and economic theory., *arXiv: General Finance*.

[15] Chetty, R. (2003), Consumption commitments, unemployment durations, and local risk aversion, *Working Paper 10211*, National Bureau of Economic Research, doi:10.3386/w23732.

[16] Chetty, R. (2008), Moral hazard versus liquidity and optimal unemployment insurance, *Journal of Political Economy*, *116*(2), 173–234.

[17] Christensen, S., and K. Lindensjö (2018), On finding equilibrium stopping times for time-inconsistent markovian problems, *SIAM Journal on Control and Optimization*, *56*(6), 4228–4255, doi:10.1137/17M1153029.

[18] Christensen, S., and K. Lindensjö (2020), On time-inconsistent stopping problems and mixed strategy stopping times, *Stochastic Process. Appl.*, *130*(5), 2886–2917, doi:10.1016/j.spa.2019.08.010.

[19] Ebert, S., and P. Strack (2015), Until the bitter end: On prospect theory in a dynamic context, *American Economic Review*, *105*(4), 1618–33, doi:10.1257/aer.20130896.

[20] Edgar, G. A., A. Millet, and L. Sucheston (1982), On compactness and optimality of stopping times, in *Martingale theory in harmonic analysis and Banach spaces (Cleveland, Ohio, 1981)*, *Lecture Notes in Math.*, vol. 939, pp. 36–61, Springer, Berlin-New York.

[21] Ekeland, I., and A. Lazrak (2006), Being serious about non-commitment: Subgame perfect equilibrium in continuous time.

[22] Fan, K. (1952), Fixed-point and minimax theorems in locally convex topological linear spaces, *Proc. Nat. Acad. Sci. U. S. A.*, *38*, 121–126.

[23] Gomes, D., L. Nurbekyan, and E. Pimentel (2015), *Economic models and mean-field games theory*.

[24] Grenadier, S. R., and N. Wang (2007), Investment under uncertainty and time-inconsistent preferences, *Journal of Financial Economics*, *84*(1), 2–39.

[25] Hansen, G. D., and A. Imrohoroglu (1992), The role of unemployment insurance in an economy with liquidity constraints and moral hazard, *Journal of Political Economy*, *100*(1), 118–142, doi:10.1086/261809.

[26] He, X., and Z. Jiang (2021), On the Equilibrium Strategies for Time-Inconsistent Problems in Continuous Time, available at SSRN: https://dx.doi.org/10.2139/ssrn.3308274.

[27] Hopenhayn, H., and J. P. Nicolini (1997), Optimal unemployment insurance, *Journal of Political Economy*, *105*(2), 412–38.

[28] Huang, M., R. P. Malhamé, and P. E. Caines (2006), Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle, *Communications in Information & Systems*, *6*(3), 221 – 252, doi:cis/1183728987.

[29] Huang, M., P. E. Caines, and R. P. Malhame (2007), Large-population cost-coupled lqg problems with nonuniform agents: Individual-mass behavior and decentralized $\varepsilon$-nash equilibria, *IEEE Transactions on Automatic Control*, *52*(9), 1560–1571, doi:10.1109/TAC.2007.904450.

[30] Huang, Y., and Z. Zhou (2019), The optimal equilibrium for time-inconsistent stopping problems—the discrete-time case, *SIAM Journal on Control and Optimization*, *57*(1), 590–609, doi:10.1137/17M1139187.

[31] Huang, Y.-J., and A. Nguyen-Huu (2018), Time-consistent stopping under decreasing impatience, *Finance and Stochastics*, *22*(1), 69–95, doi:10.1007/s00780-017-0350-6.

[32] Huang, Y.-J., and X. Yu (2021), Optimal stopping under model ambiguity: A time-consistent equilibrium approach, *Mathematical Finance*, *31*(3), 979–1012, doi:https://doi.org/10.1111/mafi.12312.

[33] Huang, Y.-J., and Z. Zhou (2020), Optimal equilibria for time-inconsistent stopping problems in continuous time, *Math. Finance*, *30*(3), 1103–1134, doi:10.1111/mafi.12251.

[34] Huang, Y.-J., and Z. Zhou (2021), Strong and weak equilibria for time-inconsistent stochastic control in continuous time, *Mathematics of Operations Research*, *46*(2), 428–451, doi:10.1287/moor.2020.1066.

[35] Huang, Y.-J., A. Nguyen-Huu, and X. Y. Zhou (2020), General stopping behaviors of naïve and noncommitted sophisticated agents, with application to probability distortion, *Mathematical Finance*, *30*(1), 310–340, doi:https://doi.org/10.1111/mafi.12224.

[36] Huggett, M. (1993), The risk-free rate in heterogeneous-agent incomplete-insurance economies, *Journal of Economic Dynamics and Control*, *17*(5), 953–969, doi:https://doi.org/10.1016/0165-1889(93)90024-M.

[37] Kroft, K., and M. J. Notowidigdo (2016), Should Unemployment Insurance Vary with the Unemployment Rate? Theory and Evidence, *Review of Economic Studies*, *83*(3), 1092–1124.

[38] Kroft, K., and M. J. Notowidigdo (2016), Should Unemployment Insurance Vary with the Unemployment Rate? Theory and Evidence, *Review of Economic Studies*, *83*(2), 1092–1124.

[39] Landais, C., P. Michaillat, and E. Saez (2018), A Macroeconomic Approach to Optimal Unemployment Insurance: Theory, *American Economic Journal: Economic Policy*, *10*(2), 152–181.

[40] Lasry, J.-M., and P.-L. Lions (2006), Jeux à champ moyen. i – le cas stationnaire, *Comptes Rendus Mathematique*, *343*(9), 619–625, doi:https://doi.org/10.1016/j.crma.2006.09.019.

[41] Lasry, J.-M., and P.-L. Lions (2006), Jeux à champ moyen. ii – horizon fini et contrôle optimal, *Comptes Rendus. Mathématique. Académie des Sciences, Paris*, *10*, doi:10.1016/j.crma.2006.09.018.

[42] Lasry, J.-M., and P.-L. Lions (2007), Mean field games, *Japanese Journal of Mathematics*, *2*, 229–260, doi:10.1007/s11537-007-0657-8.

[43] Lasry, J.-M., and P.-L. Lions (2007), Mean field games, *Cahiers de la Chaire Finance et Développement Durable*, *2*.

[44] Lentz, R., and T. Tranaes (2005), Job Search and Savings: Wealth Effects and Duration Dependence, *Journal of Labor Economics*, *23*, 467–89.

[45] Loewenstein, G., and D. Prelec (1992), Anomalies in intertemporal choice: evidence and an interpretation, *Q. J. Econ.*, *57*, 573–598.

[46] Loewenstein, G., and R. Thaler (1989), Anomalies: Intertemporal choice, *Journal of Economic Perspectives*, *3*, 181–193.

[47] Maskin, E., and J. Tirole (2001), Markov perfect equilibrium: I. observable actions, *Journal of Economic Theory*, *100*(2), 191 – 219, doi:https://doi.org/10.1006/jeth.2000.2785.

[48] Mitman, K., and S. Rabinovich (2014), Unemployment benefit extensions caused jobless recoveries!?, *Pier working paper archive*, Penn Institute for Economic Research, Department of Economics, University of Pennsylvania.

[49] Noor, J. (2009), Decreasing impatience and the magnitude effect jointly contradict exponential discounting, *Journal of Economics Theory*, *144*(2), 869–875.

[50] O'Donoghue, T., and M. Rabin (1999), Doing it now or later, *American Economic Review*, *89*(1), 103–124, doi:10.1257/aer.89.1.103.

[51] Pedersen, J., and G. Peskir (2016), Optimal mean–variance selling strategies, *Mathematics and Financial Economics*, *10*, doi:10.1007/s11579-015-0156-2.

[52] Pei, Y., and Z. Xie (2021), A quantitative theory of time-consistent unemployment insurance, *Journal of Monetary Economics*, *117*, 848–870, doi:https://doi.org/10.1016/j.jmoneco.2020.06.003.

[53] Pollak, R. A. (1968), Consistent planning, *The Review of Economic Studies*, *35*(2), 201–208.

[54] Prelec, D. (2004), Decreasing impatience: A criterion for non-stationary time preference and "hyperbolic" discounting, *The Scandinavian Journal of Economics*, *106*(3), 511–532.

[55] Shavell, S., and L. Weiss (1979), The optimal payment of unemployment insurance benefits over time, *Journal of Political Economy*, *87*(6), 1347–62.

[56] Strotz, R. H. (1955), Myopia and inconsistency in dynamic utility maximization, *Review of Economic Studies*, *23*(3), 165–180.

[57] Thaler, R. (1981), Some empirical evidence on dynamic inconsistency, *Econ. Lett.*, *8*, 201–207.