Counterexamples in the Theory of Ulrich Modules

by

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To A.T. Grey and T.T. Grey

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My point in saying all this is that I was fortunate to have come to Michigan, where I could meet and learn from some of the best commutative algebraists in the world. And it was certainly Mel's mathematical philosophy (at least from my perspective), the elegance and clarity of his arguments, and his unparalleled ability to see the big picture that compelled me to ask him to be my advisor. However, I was doubly fortunate because it is Mel's generosity of spirit and his humanity that has gotten me across the finish line. In all my – albeit short – time as a mathematician, I have never met anyone else who was so generous with his ideas, time, and support. Moreover, Mel has an uncanny ability to meet people where they are mathematically – be they high school students or undergraduates or graduates – and engage them as equals in mathematical dialogue. Mel is the kind of mathematician and mentor that I one day hope to become. I am deeply grateful for Mel's mentorship and guidance these past four years.

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Abstract

The theory of Ulrich modules has many powerful and broad applications ranging from the original purpose of giving a criterion for when a local Cohen-Macaulay ring is Gorenstein to new methods of finding Chow forms of a variety to longstanding open conjectures in multiplicity theory. For example, the existence of Ulrich modules and Ulrich-like objects has been the main approach to Lech's conjecture, which has been open for over 60 years. However, existence results have been very difficult to establish and for over thirty years, it was unknown whether (complete) local domains always have Ulrich modules. Recently, Ma introduced the weaker notion of (weakly) lim Ulrich sequences and showed that their existence for (complete) local domains implies Lech's conjecture. Ma then asks if (weakly) lim Ulrich sequences always exist for complete local domains.

In this thesis, we answer the question of existence for both Ulrich modules and weakly lim Ulrich sequences in the negative by constructing (complete) local domains that do not have any Ulrich modules or weakly lim Ulrich sequences. A key insight in our proofs is the classification of MCM modules over a ring R via the S_2-ification of R. Moreover, for local domains of dimension 2, we show that the existence of weakly lim Ulrich sequences implies the existence of lim Ulrich sequences. Finally, our counterexamples are not standard-graded or Cohen–Macaulay. As such, we construct candidate counterexample rings that are standard-graded and/or Cohen–Macaulay from our original counterexamples.

Chapter 1

Introduction

This thesis explores the existence of Ulrich modules, lim Ulrich sequences, and weakly lim Ulrich sequences. Ulrich modules were introduced by Bernd Ulrich in 1984 as a means to study the Gorenstein property of Cohen-Macaulay rings [U]. Since then, the theory of Ulrich modules has become a very active area of research in both commutative algebra and algebraic geometry [B].

Commutative algebra is the study of commutative rings, ideals, and modules. Commutative rings arise naturally in the field of algebraic geometry, which is the study of algebraic varieties. Algebraic varieties are geometric shapes determined by solutions to systems of polynomials. An example of such a geometric shape is a circle, which is the set of points (a, b) that are solutions to the equation $x^2 + y^2 - 1 = 0$. To each algebraic variety, we can associate a commutative ring, which is called a coordinate ring. Conversely, given a coordinate ring, we can recover the geometric shape. By studying the algebraic properties of commutative rings, we can understand the geometric properties of algebraic varieties.

One such algebraic property is the Hilbert-Samuel multiplicity (or multiplicity for short), which can be thought of as a measure of the complexity of a ring. Geometrically, the Hilbert-Samuel multiplicity measures the singularity of a point on an algebraic variety. For example, the multiplicity of the local ring at the origin of the parabola $y = x^2$ is 1 while the multiplicity of the local ring at the origin of the cusp $y^3 = x^2$ is 2. In general, larger multiplicities correspond to worse singularities.

1.1 Motivation and History

A major motivation for studying Ulrich modules is that the existence of Ulrich modules for (complete) local domains implies Lech's conjecture, which has been open for over 60 years.

Conjecture 1.1.1 (Lech's Conjecture [L60]). *Let $\varphi : (R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ be a flat local map between local rings. Then $e_{\mathfrak{m}}(R) \leq e_{\mathfrak{n}}(S)$.*

Intuitively, this conjecture says that if there is a nice map between R and S, the ring S is at least as complex as R. While Lech's conjecture is simple to state and philosophically reasonable to expect, it was wide open for virtually all cases until very recently. In the 1960 paper where he first introduced the conjecture [L60], Lech proved the conjecture for rings of dimension 2. For next three decades or so, this was the only known major case of Lech's conjecture.

The major breakthroughs that followed utilized either Ulrich modules or sequences of modules $\{M_n\}$ that better approximate the Ulrich condition as n gets larger. In the 1990s, Herzog, Ulrich, and Backelin proved that strict complete intersection rings had Ulrich modules – thus proving Lech's conjecture for strict complete intersections. The next major case was proven in Hanes's thesis [Ha99] where he proved Lech's conjecture for standard-graded rings of dimension 3 over a perfect field k of characteristic $p > 0$. Hanes's proof involved constructing a sequence of maximal Cohen–Macaulay (or MCM) modules $\{M_n\}$ that approximated the Ulrich condition.

Almost two decades later, Ma proved Lech's conjecture for rings of dimension 3 in the equicharacteristic case in [Ma1]. Then in [Ma2], Ma introduced the notion of weakly lim Ulrich sequences which is a much more general notion of a sequence of modules that approximate the Ulrich condition than the one used in Hanes's thesis. For example, while Hanes's sequence consisted of maximal Cohen–Macaulay modules, the sequences in Ma's construction need not be Cohen–Macaulay, much less maximal Cohen–Macaulay. See Chapter 2 for definitions. In [Ma2], Ma shows that the existence of weakly lim Ulrich sequences implies Lech's conjecture. He then proves Lech's conjecture for all standard graded rings over perfect fields by constructing weakly lim Ulrich sequences for all standard graded domains over perfect fields of characteristic $p > 0$. The characteristic 0 case of Lech's conjecture follows by a reduction modulo p argument.

Ulrich modules entered the sphere of algebraic geometry in a paper by Eisenbud and Schreyer, where they defined and used Ulrich bundles to give new methods for computing Chow forms of a variety [ES]. A major open question in algebraic geometry is the following:

Question 1.1.2 ([ES], [B]). *Does every smooth projective variety $X \subseteq P^n$ have an Ulrich bundle, i.e., a vector bundle whose associated graded module $\Gamma_*(E) = \bigoplus_{m \in Z} H^0(X, E(m))$ is an Ulrich module?*

In [ES], Eisenbud and Schreyer proved new existence results for Ulrich modules including for Veronese subrings of degree d of a polynomial ring in n variables.

Very recently, Iyengar, Ma, and Walker introduced two new multiplicity conjectures:

Conjecture 1.1.3 ([IMW]). *For a local ring R, every nonzero R-module M of finite projective dimension satisfies $\ell_R(M) \geq e(R)$.*

Conjecture 1.1.4 ([IMW]). *For a local ring R, every short complex F supported on the maximal ideal satisfies $\chi_\infty(F) \geq e(R)$, where $\chi_\infty(F)$ is the Dutta multiplicity.*

Remark 1.1.5. A short complex supported on the maximal ideal is a non-exact complex F of finite free R-modules that has the form

$$F : 0 \to F_{\dim(R)} \to \ldots \to F_1 \to F_0 \to 0$$

such that the homology modules $H_i(F)$ have finite length for all i.

The first conjecture implies Lech's conjecture for Cohen–Macaulay rings. The second conjecture implies Lech's conjecture. Many of the cases established for these two conjectures in [IMW] utilize Ulrich modules and lim Ulrich sequences.

Historically, the existence of Ulrich modules has been a very difficult and elusive question. The existing literature is rather sparse and has mainly explored positive existence results, i.e., classes of rings for which Ulrich modules exist. The major existence results are that Ulrich modules exist for the following classes of rings:

1. two-dimensional, standard graded Cohen-Macaulay domains [BHU]
2. strict complete intersection rings [HUB]
3. generic determinantal rings [BRW]
4. some Veronese subrings of degree d of a polynomial ring in n variables:
    - $n = 3$ with d arbitrary and $n = 4$ with $d = 2^\ell$ in arbitrary characteristic [Ha99]

- arbitrary n and d in characteristic 0 [ES]
- arbitrary n and d for characteristic $p \geq (d-1)n + (n+1)$ [Sa]

Beyond these results, there has been limited progress. In particular, for over thirty years, it has been unknown whether or not (complete) local domains always have Ulrich modules.

1.2 Main Theorems

Given the strength of the implications of existence, one may expect that Ulrich modules do not exist for all local domains. A major obstruction to finding counterexamples is that there are essentially no good criteria to test whether or not a ring has an Ulrich module. Directly proving that a ring has no Ulrich modules involves classifying all of its MCM modules – yet another incredibly difficult problem – and showing that none of the MCM modules are Ulrich. Furthermore, positive existence results for large classes of two-dimensional rings seemed to indicate that a counterexample would be fairly complex. As such, finding a counterexample was considered to be a very difficult – if not intractable – problem.

In this thesis, we resolve the question of existence of Ulrich modules in the negative.

Theorem A. *Ulrich modules do not always exist for complete local domains. More explicitly, the local domain*

$$R = k[x^n, x^{n+1}, x^n y, y^n, y^{n+1}, xy^n, xy]_{\mathfrak{m}},$$

where $\mathfrak{m}$ *is the maximal ideal* $(x^n, x^{n+1}, x^n y, y^n, y^{n+1}, xy^n, xy)$*, and its completion*

$$\widehat{R} = k[\![x^n, x^{n+1}, x^n y, y^n, y^{n+1}, xy^n, xy]\!]$$

do not have Ulrich modules for $n \geq 2$*.*

The key ingredient to proving that R does not have Ulrich modules is Lemma 3.1.1, which states that any MCM module over R is an MCM module over its S_2-ification S. When S is regular, any MCM module over S has the form $S^{\oplus h}$. This yields the following intermediary theorem:

Theorem B. *Let* $(R, \mathfrak{m}, k)$ *be a local domain. Suppose* R *has an* S_2*-ification* S *such that* S *is a regular local ring. Then every MCM module of* R *has the form* $S^{\oplus h}$*. Consequently* R *has Ulrich modules if and only if* S *is an Ulrich module of* R*.*

Then the idea behind the counterexample in Theorem A is to construct a ring R such that its S_2-ification S is a regular ring, but S is not an Ulrich module of R. Now, Theorem B holds in greater generality than just dimension 2. A natural question to ask is if we can construct a more general class of rings using the same idea. The answer is yes! In fact, we can construct a counterexample in any dimension $d \geq 2$.

Theorem C. *Let* $S = k[\![\underline{x}]\!] = k[\![x_1, \ldots, x_n]\!]$ *where* $n \geq 2$*. Let* $\underline{u} = u_1, \ldots, u_n$ *be a system for parameters of* S *such that* $I = (\underline{u})S$ *is not integrally closed. Let* $\overline{I}$ *be the integral closure of* I *in* S*. Let* $\{g_\lambda\}_{\lambda \in \Lambda}$ *be an arbitrary collection of elements in* $\overline{I}$ *and* $f \in \overline{I} - I$*. For* $1 \leq j \leq n$*, let* v_j, w_j *be elements of the maximal ideal of* $k[\![\underline{u}]\!]$ *that generate a height 2 ideal in* R *(e.g. one can take powers of distinct elements in* $\{u_1, \ldots, u_n\}$*). Define* R *to be the domain*

$$R \coloneqq k[\![\underline{u}]\!][f][v_j x_j, w_j x_j]_{1 \leq j \leq n}[g_\lambda]_{\lambda \in \Lambda}.$$

Then R has no Ulrich modules.

The counterexamples in Theorems A and C can be taken to be fairly nice monomial algebras that have low dimension. This suggests that counterexamples may be quite ubiquitous and thus, having an Ulrich module may be a special property that characterizes certain classes of rings. The ubiquity of counterexamples remains completely unexplored. In general, the question of existence still remains wide open in many cases of interest. For example, the counterexamples in this thesis are not Cohen–Macaulay and the original question posed by Ulrich in [U] remains open:

Question 1.2.1 ([U])**.** *Does every local Cohen–Macaulay ring have an Ulrich module?*

The second major contribution of this thesis answers a question posed by Ma in [Ma2]:

Question 1.2.2 ([Ma2])**.** *Does every complete local domain of characteristic $p > 0$ with an F-finite residue field admit a lim Ulrich sequence, or at least a weakly lim Ulirch sequence?*

A positive answer to Question 1.2.2 in conjunction with the argument for reduction modulo $p > 0$ in [Ma1] would have resolved Lech's conjecture in the equicharacteristic case. However, in this paper, we answer the question in the negative.

Theorem D. *Weakly lim Ulrich sequences do not always exist for complete local domains.*

To prove Theorem D, we establish important characterizations of (weakly) lim Ulrich sequences for local domains of dimension 2. In particular, we first show:

Theorem E. *Let $(R, \mathfrak{m}, k)$ be a local domain of dimension 2. If R has a weakly lim Ulrich sequence, then R has a lim Ulrich sequence consisting of torsion-free modules.*

Moreover, in the case where $R \subseteq S$ is a local module-finite extension of local domains such that S/R has finite length and $S \subseteq frac(R)$, there exists a lim Ulrich sequence of R-modules that is also a lim Cohen–Macaulay sequence of S-modules.

Then we prove the following theorem:

Theorem F. *Let $(R, \mathfrak{m}, k)$ be a local domain of dimension 2. Suppose R has an S_2-ification S that is a regular local ring. If R has a weakly lim Ulrich sequence, then R has an Ulrich module. In particular, by Theorem B, if R has a weakly lim Ulrich sequence, then S is an Ulrich module of R.*

We want to make two important observations with regard to our results on the existence of weakly lim Ulrich sequences.

First, in the proofs of the statements involving (weakly) lim Ulrich sequences, we heavily use the fact that the rings have dimension 2, in addition to the fact that the S_2-ifications of the counterexample rings are regular. On the other hand, in the proofs of our theorems about Ulrich modules, the dimension did not matter as long as the dimension was at least 2.

Second, the idea of constructing sequences that approximate the Ulrich condition arose, in part, because of the difficulty of constructing Ulrich modules. Weakly lim Ulrich sequences are a weaker notion than Ulrich modules. For example, the modules in the sequence need not be Cohen–Macaulay. Ma uses this flexibility to great effect to prove Lech's conjecture for standard-graded rings over perfect fields. While our counterexample to the existence of weakly lim Ulrich sequences indicates that there are some limitations to this approach, it also highlights the mysterious nature of the Ulrich property. In particular, lim Ulrich and weakly lim Ulrich sequences are special kinds of lim Cohen–Macaulay and respectively, weakly lim Cohen–Macaulay sequences. In [BHM], Bhatt, Hochster, and Ma prove that every complete local domain of characteristic $p > 0$ with an F-finite residue field admits a lim Cohen–Macaulay sequence. (See comment in [Ma2].)

1.3 Structure of the Thesis

The structure of this thesis is as follows: In Chapter 2, we review the basic definitions and properties that will be used throughout the thesis including the definitions and properties of Ulrich modules and (weakly) lim Ulrich sequences. In Chapter 3, we prove our results about rings with regular S_2-ifications as well as the main theorem that Ulrich modules do not always exist for complete local domains. In Chapter 4, we construct a candidate class of Cohen–Macaulay rings from the counterexample rings R_n from Chapter 3 that possibly could yield a counterexample in the Cohen–Macaulay case. In Chapter 5, we prove our theorems characterizing weakly lim Ulrich sequences in dimension 2 as well as the theorem that weakly lim Ulrich sequences do not always exist for complete local domains.

Chapter 2

Preliminaries

In this chapter, we review the definitions and properties as well as the notation that will be used throughout the thesis. Many of the results in this chapter are standard in the literature and can be found in [Mat], [SH], [S]. As such, we omit most of the proofs. Ma's paper [Ma2] is the main reference for the definitions and properties of (weakly) lim Cohen–Macaulay sequences and (weakly) lim Ulrich sequences. All rings in this thesis are commutative with multiplicative identity 1 and Noetherian. In particular, all local rings $(R, \mathfrak{m}, k)$ include the Noetherian property. For simplicity, we will assume that k is infinite unless explicitly stated otherwise.

2.1 Notation

Let $(R, \mathfrak{m}, k)$ be a local ring of dimension d. Let M be a finitely generated module over R. For simplicity, we assume that k is infinite unless explicitly stated otherwise. Throughout the thesis, we use the following notation:

- $\underline{x} = x_1, \ldots, x_d$

- $\ell_R(M)$ is the length of M as a module over R. We write $\ell(M)$ when it is clear from the context which R is being used.

- $H_i(\underline{x}; M)$ is the i-th Koszul homology of the module M with respect to $\underline{x}$.

- $h_i^R(\underline{x}; M) = \ell_R(H_i(\underline{x}; M))$

- $\chi(\underline{x}; M) = \sum_{i=0}^{d}(-1)^i \ell(H_i(\underline{x}; M)) = \sum_{i=0}^{d}(-1)^i h_i(\underline{x}; M)$.

- $\chi_1(\underline{x}; M) = \sum_{i=1}^{d}(-1)^{i-1} \ell(H_i(\underline{x}; M)) = \sum_{i=1}^{d}(-1)^{i-1} h_i(\underline{x}; M)$.

- $\nu_R(M)$ is the minimal number of generators of M.

- $e_R(M)$ is the multiplicity of M with respect to the maximal ideal $\mathfrak{m}$. When $M = R$, we write $e(R)$.

2.2 Definitions and Properties

2.2.1 Integral Closure of Rings and Ideals

Definition 2.2.1. Let $R \subseteq S$ be an extension of rings. Then an element $s \in S$ is integral over R if s satisfies a monic polynomial with coefficients in R. That is, there exists a positive integer n such that

$$s^n = r_{n-1}s^{n-1} + r_{n-2}s^{n-2} + \ldots + r_1 s + r_0.$$

Definition 2.2.2. Let $R \subset S$ be an extension of rings. Then the integral closure of R in S is the subring of S generated by all the elements of S that are integral over R. If R is a domain, we say that R is normal if the integral closure of R in $\text{frac}(R)$ is R.

A key ingredient to the main results in this thesis is a module-finite extension of a domain called the S_2-ification. The following definition and proposition is adapted from [HH].

Definition 2.2.3. (S_2-ification [HH]) Let R be an domain. Let frac(R) be the fraction field of R. The S_2-ification of R, denoted S, is a ring extension $R \subseteq S \subseteq \text{frac}(R)$ that satisfies the following conditions:

1. S is module-finite over R.

2. S satisfies S_2 as an R-module.

3. For any $f \in S - R$, the ideal $R :_R f = \{a \in R \mid \ af \in R\}$ has height ≥ 2.

Proposition 2.2.4 ([HH])**.** *Let $(R, \mathfrak{m}, k)$ be a local domain. Let S be the subring of frac(R) consisting of all elements $f \in$ frac(R) such that the ideal $R :_R f$ has height ≥ 2. Then R has an S_2-ification if and only if S is module-finite over R, in which case S is the unique S_2-ification of R.*

Definition 2.2.5. Let R be a ring and let $I \subseteq R$ an ideal. Then an element $r \in R$ is integral over I if it satisfies a monic polynomial $f(z)$ of degree n such that the coefficient of z^{d-t} is an element of I^t for $1 \leq t \leq n$.

Definition 2.2.6. Let R be a ring and let $I \subseteq R$ be an ideal. Then the integral closure of I, denoted $\overline{I}$, is the ideal consisting of all elements $r \in R$ that are integral over I.

A proposition that we will be use frequently in this thesis is the following:

Proposition 2.2.7. *Let R be a ring and let $I \subseteq R$ be an ideal. Let $R \subseteq S$ be an integral extension of rings. Then $\overline{IS} \cap R = \overline{I}$.*

Definition 2.2.8. Let R be a ring. Let $I \subseteq J$. Then I is a reduction of J if J is integral over I.

Proposition 2.2.9. *Let $(R, \mathfrak{m}, k)$ be a local ring of dimension d. Then any reduction of $\mathfrak{m}$ is generated by at least d elements. If k is infinite, then $\mathfrak{m}$ has a reduction that is generated by d elements $r_1, \ldots, r_d$. The elements $r_1, \ldots, r_d$ form a system of parameters for $\mathfrak{m}$.*

Definition 2.2.10. Let $(R, \mathfrak{m}, k)$ be a local ring of dimension d with infinite residue field k. Then we say that an ideal $I \subseteq \mathfrak{m}$ is a minimal reduction of $\mathfrak{m}$ if I is a reduction of $\mathfrak{m}$ generated by d elements.

Remark 2.2.11. In [SH], a minimal reduction of $\mathfrak{m}$ (more generally, for an ideal I) is defined to be a reduction of $\mathfrak{m}$ (resp. I) that is minimal with respect to inclusion. In the case where the residue field k is infinite, this definition coincides with Definition 2.2.10.

2.2.2 The Hilbert-Samuel Multiplicity

Definition 2.2.12. Let $(R, \mathfrak{m}, k)$ be a local ring of dimension d. Let M be a finitely generated module over R. Let $I \subset R$ be an $\mathfrak{m}$-primary ideal. Then the length $\ell_R(M/I^nM)$ eventually agrees with a polynomial of degree c in the variable n where c is

$$\dim(M) = \dim(R/\mathrm{Ann}_R(M)).$$

If $\dim(M) = \dim(R)$, then the leading term of the polynomial has the form $\frac{e}{d!}n^d$ where e is a positive integer.

Definition 2.2.13 (Hilbert-Samuel Multiplicity)**.** Let $(R, \mathfrak{m}, k)$ be a local ring of dimension d. Let $I \subseteq R$ be an $\mathfrak{m}$-primary ideal. Let M be a finitely generated module over R.

The *Hilbert-Samuel multiplicity* of R with respect to I is

$$e_I(M) \coloneqq d! \lim_{n\to\infty} \frac{\ell(M/I^nM)}{n^d}$$

where $\dim(R) = d$. If the $\dim(M) < d$, then $e_I(M) = 0$. If $I = \mathfrak{m}$, then we call $e_\mathfrak{m}(M)$ the multiplicity of M and denote $e_\mathfrak{m}(M)$ as $e_R(M)$ or $e(M)$.

Proposition 2.2.14. *Let $(R, \mathfrak{m}, k)$ be a local ring of dimension d. Let M be a finitely generated R-module. Let $I \subset J$ be $\mathfrak{m}$-primary ideals of R. Then we have the following properties:*

(a) $e_J(M) \leq e_I(M)$

(b) Let $\overline{I}$ and $\overline{J}$ be the integral closures of I and J respectively. Then if $\overline{I} = \overline{J}$, we have $e_I(M) = e_J(M)$.

Theorem 2.2.15 (Rees)**.** *Let $(R, \mathfrak{m}, k)$ be a formally equidimensional local ring. Let $I \subseteq J$ be $\mathfrak{m}$-primary ideals. Then $\overline{I} = \overline{J}$ if and only if $e_I(R) = e_J(R)$.*

Proposition 2.2.16. *Let $(R, \mathfrak{m}, k)$ be a local domain, $I \subset R$ an $\mathfrak{m}$-primary ideal, and M a finitely generated R-module. Then $e_I(M) = rank_R(M) \cdot e_I(R)$.*

Theorem 2.2.17. *(Serre [S]) Let $(R, \mathfrak{m}, k)$ be a local ring of dimension d. Let M be a finitely generated R-module. Let $x_1, \ldots, x_d$ be a system of parameters for R and let $I = (x_1, \ldots, x_d)R$. Then*

$$\chi(\underline{x}; M) = \sum_{i=0}^{d} (-1)^i h_i(\underline{x}; M)$$

is equal to $e_I(M)$.

Theorem 2.2.18. *(Serre [S]) Let $(R, \mathfrak{m}, k)$ be a local ring of dimension d and M a finitely generated R-module. Let $x_1, \ldots, x_d$ be a system of parameters of R. Then*

$$\chi_1(\underline{x}; M) = \sum_{i=1}^{d} (-1)^{i-1} h_i(\underline{x}; M) \geq 0.$$

Proposition 2.2.19. *Let $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ be a flat local map of local rings such that $\mathfrak{m}S$ is $\mathfrak{n}$-primary. Let M be a finitely generated R-module. Let I be an $\mathfrak{m}$-primary ideal.*

(a) Then $dim(S \otimes_R M) = dim(M)$.

(b) If $\mathfrak{n} = \mathfrak{m}S$, then $e_I(M) = e_{IS}(S \otimes_R M)$.

2.3 Ulrich modules and (weakly) lim Ulrich sequences

Definition 2.3.1 (MCM)**.** Let M be a finitely generated module over (R, m, k). Then M is *maximal Cohen-Macaulay* (or MCM) module of R if $\operatorname{depth}_R(M) = \dim(R)$.

Proposition 2.3.2. *Let $(R, \mathfrak{m}, k)$ be a local ring of dimension d. Let M be a maximal Cohen–Macaulay module of R. Then there exists a system of parameters $r_1, \ldots, r_d$ that is a regular sequence on M. Equivalently, every (part of a) system of parameters is (part of) a regular sequence on M.*

Proposition 2.3.3. *Let $(R, \mathfrak{m}, k)$ be a local domain of dimension d. Let M be an MCM module over R. Let $\nu_R(M)$ be the minimal number of generators of M. Then we have*

(a) $e_R(M) = \ell(M/IM)$, where $I \subseteq \mathfrak{m}$ is a minimal reduction of $\mathfrak{m}$, and

(b) $e_R(M) \geq \nu_R(M)$.

Definition 2.3.4. Let $(R, \mathfrak{m}, k)$ is a local ring of dimension d. Let M be an MCM module over R. Then M is an *Ulrich module* if $e_R(M) = \nu_R(M)$. Equivalently, M is an Ulrich module if $\mathfrak{m}M = IM$ for any minimal reduction $I \subseteq \mathfrak{m}$.

Lemma 2.3.5. *Let $(R, \mathfrak{m}, k)$ be a local domain containing k. Let L be a finite algebraic extension of k. Then $S = L \otimes_k R$ is a local ring with maximal ideal $\mathfrak{m}S$ and S has an Ulrich module if and only if R has an Ulrich module.*

The proof of Lemma 2.3.5 is standard. We include it below for completeness.

Proof. Observe that S is a free module-finite extension of R and that $\mathfrak{m}S$ is the maximal ideal of S. So any system of parameters for R is a system of parameters for S.

Now suppose N is an Ulrich module over S. It is clear that any MCM module over S is an MCM module over R. We have $e(R) = e(S)$ because the length of $S/(\mathfrak{m}S)^t = L \otimes_R (R/\mathfrak{m}^t)$ over S is the same as the length of $R/\mathfrak{m}^t$ over R. Let $[L : k]$ be the degree of the field extension. Then $\nu_R(N) = [L : k]\nu_S(N)$ and we have

$$e_R(N) = \text{rank}_R(N)e(R) = [L : k]\text{rank}_S(N)e(R) = [L : k]\text{rank}_S(N)e(S) = [L : k]e_S(N).$$

Then

$$\frac{e_R(N)}{\nu_R(N)} = \frac{e_S(N)}{\nu_S(N)} = 1.$$

So N is an Ulrich module of R.

On the other hand, if M is an MCM module of R, then $S \otimes_R M$ is an MCM module of S, and we have $e_R(M) = e_S(S \otimes_R M)$ and $\nu_R(M) = \nu_S(S \otimes_R M)$. So if M is an Ulrich module of R, then $S \otimes_R M$ is an Ulrich module of S.

□

Remark 2.3.6. Let $(R, \mathfrak{m}, k)$ be a local ring of dimension d. Throughout this thesis, we assume that k is infinite for simplicity. However, what we really want to assume is the existence of a minimal reduction of $\mathfrak{m}$ –that is, a reduction of $\mathfrak{m}$ that has d generators. If k is infinite, then such a minimal reduction always exists. If k is finite, such a minimal reduction may not exist. But this is easily remedied without loss of generality. In particular, in Lemma 2.3.5, one need not restrict to the case where the local ring $(R, \mathfrak{m}, k)$ contains k. There always exists a flat local ring extension $(S, \mathfrak{n}, l)$ where $\mathfrak{n} = \mathfrak{m}S$, the field extension $k \subseteq l$ is finite, and a minimal reduction with d generators exists for $\mathfrak{n}$. Then by proposition 2.2.19 and a virtually identical argument as Lemma 2.3.5, we see that R has an Ulrich module if and only if S has an Ulrich module.

Definition 2.3.7. Let $(R, \mathfrak{m}, k)$ be a local ring of dimension d. A sequence of finitely generated R-modules $\{M_n\}$ of dimension d is called *lim Cohen–Macaulay* if there exists a system of parameters $\underline{x}$ such that for all $i \geq 1$, we have

$$\lim_{n\to\infty} \frac{\ell(H_i(\underline{x}; M_n))}{\nu_R(M_n)} = 0.$$

A sequence of finitely generated R-modules $\{M_n\}$ of dimension d is called *weaky lim Cohen–Macaulay* if there exists a system of parameters $\underline{x}$ such that

$$\lim_{n\to\infty} \frac{\chi_1(\underline{x}; M_n)}{\nu_R(M_n)} = 0.$$

Definition 2.3.8. Let $(R, \mathfrak{m}, k)$ be a local ring of dimension d. A sequence of finitely generated R-modules $\{M_n\}$ of dimension d is called *lim Ulrich* (respectively, *weakly lim Ulrich*) if $\{M_n\}$ is lim Cohen–Macaulay (respectively, weakly lim Cohen–Macaulay) and

$$\lim_{n\to\infty} \frac{e_R(M_n)}{\nu_R(M_n)} = 1.$$

Proposition 2.3.9 ([BHM][Ma2]). *Let $(R, \mathfrak{m}, k)$ be a local ring of dimension d.*

(a) [BHM] If $\{M_n\}$ is a lim Cohen–Macaulay sequence of R, then for every system of parameters $\underline{x} = x_1, \ldots, x_d$, we have

$$\lim_{n\to\infty} \frac{h_i(\underline{x}; M_n)}{\nu_R(M_n)} = 0$$

where $i \geq 1$.

(b) [Ma2] If $\{M_n\}$ is a weakly lim Cohen–Macaulay sequence of R, then for every system of parameters $\underline{x} = x_1, \ldots, x_d$, we have

$$\lim_{n\to\infty} \frac{\chi_1(\underline{x}; M_n)}{\nu_R(M_n)} = 0.$$

2.3.1 Lech's Conjecture

Recall the statement of Lech's Conjecture:

Conjecture 2.3.10 (Lech's Conjecture [L60]). *Let $\varphi : (R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ be a flat local map between local rings. Then $e_{\mathfrak{m}}(R) \leq e_{\mathfrak{n}}(S)$.*

The existence of Ulrich modules for complete local domains implies Lech's conjecture. It is known that we can reduce Lech's conjecture to the case where:

1. R is a complete local domain.

2. dim(R) = dim(S) = d, equivalently $\mathfrak{m}S$ is primary to $\mathfrak{n}S$.

See [Ma1], [Ha99].

Proposition 2.3.11. *Let $(R, \mathfrak{m}, k)$ be a complete local domain. Suppose R has an Ulrich module. Then Lech's conjecture holds for R.*

Proof. We can reduce to the case where $\mathfrak{m}S$ is primary to $\mathfrak{n}S$. Let M be an Ulrich module over R. Then $S \otimes_R M$ is a MCM module over S and we have

$$\begin{aligned} e_{\mathfrak{n}}(S) = \frac{1}{\operatorname{rank}_S(S \otimes_R M)} \cdot e_{\mathfrak{n}}(S \otimes_R M) &\geq \frac{1}{\operatorname{rank}_R(M)} \cdot \nu_S(S \otimes_R M) \\ &= \frac{1}{\operatorname{rank}_R(M)} \cdot \nu_R(S) \\ &= e_{\mathfrak{m}}(R). \end{aligned}$$

□

In [Ma2], Ma proves that the existence of a weakly lim Ulrich sequence for complete local domains implies Lech's conjecture. The proof of this is similar to the one above once some key facts about (weakly) lim Cohen–Macaulay sequences and (weakly) lim Ulrich sequences are established.

Chapter 3

Counterexamples Concerning Ulrich Modules

3.1 Ulrich modules do not always exist for local domains

Let $(R, \mathfrak{m}, k)$ be a local ring. For simplicity, we assume that the residue field k is infinite.

Lemma 3.1.1. *Let $(R, \mathfrak{m}, k)$ be a local domain. If R has an S_2-ification S that is a local ring, then any MCM module M of R is an MCM module of S.*

Proof. Let M be an MCM module over R. We want to show that for any $f \in S - R$ and any $m \in M$, there is a well-defined element $f \cdot m \in M$. Let $W = R - \{0\}$. Since M is MCM, it is torsion-free over R and embeds in $W^{-1}M$. It suffices to show that $f \cdot (m/1) \in M$. Since the height of the ideal $R :_R f$ is at least two, there exist u and v in $R :_R f$ such that the sequence u, v is a part of a system of parameters for R. Since M

is MCM, the sequence u, v is a regular sequence on M. Then

$$v \cdot ((uf) \cdot (m/1)) = u \cdot ((vf) \cdot (m/1)) \in vM$$

implies that $(vf) \cdot (m/1) \in vM$. Since M is torsion-free over R, we have $f \cdot (m/1) \in M$. □

Theorem 3.1.2. *Let $(R, \mathfrak{m}, k)$ be a local domain with k infinite. Suppose R has an S_2-ification S such that S is a regular local ring. Then every MCM module of R has the form $S^{\oplus h}$. Consequently R has Ulrich modules if and only if S is an Ulrich module of R if and only if $IS = \mathfrak{m}S$ for any minimal reduction I of $\mathfrak{m}$.*

Proof. By Lemma 3.1.1, any MCM module M over R is MCM over S. But S is regular. Hence $M \cong S^{\oplus h}$. The second statement follows immediately because $S^{\oplus h}$ is an Ulrich module of R if and only if S is an Ulrich module of R. □

Theorem 3.1.3. *The local domain*

$$R = k[x^n, x^{n+1}, x^n y, y^n, y^{n+1}, xy^n, xy]_{\mathfrak{m}},$$

where $\mathfrak{m}$ is the maximal ideal $(x^n, x^{n+1}, x^n y, y^n, y^{n+1}, xy^n, xy)$, and its completion

$$\widehat{R} = k[\![x^n, x^{n+1}, x^n y, y^n, y^{n+1}, xy^n, xy]\!]$$

do not have Ulrich modules for $n \geq 2$.

Proof of Theorem 3.1.3. The proof is essentially the same for R and $\widehat{R}$. We will work with R. We claim that the S_2-ification S of R is $k[x, y]_{(x,y)}$. Both x^n and y^n multiply x and y into R. The ideal $(x^n, y^n)R$ has height 2 in R; so the S_2-ification S of R must contain x and y by Proposition 2.2.4. Since $k[x,y]_{(x,y)}$ is normal and contains S, it must be the S_2-ification S of R.

Since S is a regular local ring, it suffices to check that S is not an Ulrich module over R by Theorem 3.1.2. One can compute that the ideal $(xy, x^n - y^n)R$ is a minimal reduction for $\mathfrak{m}$. But $(xy, x^n - y^n)S \neq (x^n, x^{n+1}, x^n y, y^n, y^{n+1}, xy^n, xy)S$ as ideals in S. □

We can use R to give a new counterexample to the localization of Ulrich modules – that is, a local ring $(T, \mathfrak{n}, \ell)$ that has an Ulrich module M and a prime ideal $\mathfrak{p} \subseteq T$ such that $M_{\mathfrak{p}}$ is not an Ulrich module over $T_{\mathfrak{p}}$. While a counterexample to localization was first given by Hanes in [Ha99], the following counterexample is stronger in the sense that T localizes to a ring that has no Ulrich modules whereas Hanes's counterexample localizes to a ring that does have an Ulrich module.

Counterexample 3.1.4 (Localization)**.** Consider the ring

$$T = k[s^{n+1}, sx^n, x^{n+1}, x^n y, sy^n, y^{n+1}, xy^n, s^{n-1}xy]_{\mathfrak{n}}$$

where $\mathfrak{n} = (s^{n+1}, sx^n, x^{n+1}, x^n y, sy^n, y^{n+1}, xy^n, s^{n-1}xy)$ and $n \geq 2$. Let

$$\varphi : T \hookrightarrow k[s,x,y]_{(s,x,y)}$$

be the inclusion map and $\mathfrak{p} = \varphi^{-1}((x,y))$. Then the localization $T_{\mathfrak{p}}$ is the ring

$$k(s^{n+1})\Big[\Big(\frac{x}{s}\Big)^n, \Big(\frac{x}{s}\Big)^{n+1}, \Big(\frac{x}{s}\Big)^n\Big(\frac{y}{s}\Big), \Big(\frac{y}{s}\Big)^n, \Big(\frac{y}{s}\Big)^{n+1}, \Big(\frac{x}{s}\Big)\Big(\frac{y}{s}\Big)^n, \Big(\frac{x}{s}\Big)\Big(\frac{y}{s}\Big)\Big]$$

localized at the obvious maximal ideal. But $T_{\mathfrak{p}}$ has no Ulrich modules by Theorem 3.1.3.

It remains to show that T has an Ulrich module. Let $S = k[s,x,y]^{(n+1)}_{(s,x,y)}$ be the Veronese subring of degree $n+1$ with maximal ideal $\mathfrak{a}$. One can compute $e_T(T) = (n+1)^2 = e_S(S)$. The rings T and S have the same fraction field and so $\operatorname{rank}_T(S) = 1$. Now S has an Ulrich module M by Proposition 3.6 in [Ha04]. Then M is MCM over T and $\operatorname{rank}_T(M) = \operatorname{rank}_S(M)$. Then

$$(n+1) \cdot \operatorname{rank}_T(M) = e_T(M) \geq \nu_T(M) \geq \nu_S(M) = e_S(M) = (n+1) \cdot \operatorname{rank}_T(M).$$

Thus $e_T(M) = \nu_T(M)$ and M is Ulrich over T.

We can extend the ideas in Theorem 3.1.3 to construct a more general class of counterexamples.

Theorem 3.1.5. *Let $S = k[\![\underline{x}]\!] = k[\![x_1, \ldots, x_n]\!]$ where $n \geq 2$. Let $\underline{u} = u_1, \ldots, u_n$ be a system for parameters of S such that $I = (\underline{u})S$ is not integrally closed. Let $\overline{I}$ be the integral closure of I in S. Let $\{g_\lambda\}_{\lambda \in \Lambda}$ be an arbitrary collection of elements in $\overline{I}$ and $f \in \overline{I} - I$. For $1 \leq j \leq n$, let v_j, w_j be elements of the maximal ideal of $k[\![\underline{u}]\!]$ that generate a height 2 ideal in R (e.g. one can take powers of distinct elements in $\{u_1, \ldots, u_n\}$). Define R to be the domain*

$$R \coloneqq k[\![\underline{u}]\!][f][v_j x_j, w_j x_j]_{1 \leq j \leq n}[g_\lambda]_{\lambda \in \Lambda}.$$

Then R has no Ulrich modules.

Proof. First, notice that $k[\![\underline{u}]\!] \subset k[\![\underline{x}]\!]$ is a module-finite extension. Then R is (Noetherian) local and $R \subset k[\![\underline{x}]\!]$ is a module-finite extension.

Let $\mathfrak{m}_R$ be the maximal ideal of R. From the construction of R, it is clear that $\underline{u} = u_1, \ldots, u_n$ is a system for parameters for R and in fact, a minimal reduction of $\mathfrak{m}_R$ because all the other adjoined elements are integral over $(\underline{u})S$ in S and thus integral over $(\underline{u})R$ in R. Then for all $1 \leq j \leq n$, the element x_j is multiplied into R by v_j and w_j which generate a height 2 ideal in R. Thus x_j is in the S_2-ification of R for all $1 \leq j \leq n$. But this means that $S = k[\![\underline{x}]\!]$ is the S_2-ification of R.

By Theorem 3.1.2, it suffices to show that S is not an Ulrich module of R. But $(\underline{u})S \neq \mathfrak{m}_R S$ because $f \notin (\underline{u})S$. Thus R has no Ulrich modules. □

Remark 3.1.6. In [IMW], Iyengar, Ma, and Walker consider rings of the form $T = k + J$ where $S = k[\![x, y]\!]$ and $J \subseteq S$ is an ideal primary to $(x, y)S$. If J has a minimal reduction $I = (u, v)S$, then the rings T have the form in Theorem 3.1.5. Thus $T = k + J$ has no Ulrich modules if $J \neq IS$.

In the case where J does not have a minimal reduction, we can reduce to the previous case by taking a finite algebraic field extension of k so that J has a minimal reduction and then applying Lemma 2.3.5.

Chapter 4

A Candidate Class of Cohen-Macaulay Rings

In Chapter 3, we constructed counterexamples to the existence of Ulrich modules for (complete) local domains in all dimensions. The key idea was to find rings R such that the S_2-ification S of R is a regular local ring that is not an Ulrich module of R. Our counterexamples were necessarily not Cohen-Macaulay because a Cohen-Macaulay ring is S_2 and a regular local ring is Ulrich module over itself. The following difficult question remains open:

Question 4.1. *Let* $(R, \mathfrak{m}, k)$ *be a local Cohen-Macaulay domain. Does* R *have an Ulrich module?*

Moreover, the counterexample rings in Chapter 3 are not standard-graded. We will show in Chapter 5 that the two-dimensional rings R_n have no weakly lim Ulrich sequences, which is a weaker notion than that of an Ulrich module. On the other hand, in [Ma2], Ma showed that standard-graded rings over a perfect field of characteristic $p > 0$ have weakly lim Ulrich sequences. Then a natural question to ask is the following:

Question 4.2. *Let R be a standard-graded k-algebra. Does R have an Ulrich module?*

In this Chapter, we construct a candidate class of rings. Recall that for $n \geq 2$, the rings

$$R_n = k[x^n, x^{n+1}, x^n y, y^n, y^{n+1}, xy^n, xy]$$

do not have Ulrich modules.

In Chapter 3, we "homogenized" the generators of R_n and added the element s^{n+1} to construct the rings

$$T_n = k[s^{n+1}, x^n s, x^{n+1}, x^n y, y^n s, y^{n+1}, xy^n, xys^{n-1}].$$

These rings have Ulrich modules but localize to rings that have no Ulrich modules.

In this chapter, we construct candidate classes of standard-graded and Cohen-Macaulay rings for counterexamples to existence of Ulrich modules. More specifically, we construct the "homogenized" rings U_n that do not contain s^{n+1} in the hopes that the "pathologies" of the nonexistence of Ulrich modules from the original counterexamples R_n are preserved.

Define U_n as follows:

$$U_n \coloneqq k[sx^n, x^{n+1}, x^n y, sy^n, y^{n+1}, xy^n, s^{n-1}xy].$$

Moreover, we determine the integral closure V_n of U_n. The integral closure of a monomial algebra is a monomial algebra as well. Surprisingly V_n is also generated in degree $n+1$ and thus, also standard-graded. By a famous result of Hochster [Hoc72], a normal monomial algebra is Cohen–Macaulay. We also compute the integral closure W_n of U_n in $k[x, y, s]$. We compute multiplicities and minimal reductions for U_n, V_n, and W_n and discuss the relationships between these rings and their Ulrich modules. Finally, we have not yet been able to determine whether the rings U_n, V_n, and W_n have Ulrich modules for $n \geq 3$. As a V_n-module, the ring W_n splits into a direct sum of rank one MCM modules over V_n (and over U_n.) We show that these rank one MCM modules are not Ulrich modules of V_n or U_n

4.1 Computing V_n and W_n

One can check the following lemma:

Lemma 4.1.1. *The fraction field of U_n is the same as the fraction field of the Veronese subring $k[x, y, s]^{n+1} \subseteq k[x, y, s]$, which is $k(x^{n+1}, y/x, s/x)$.*

Theorem 4.1.2. *Let $\mathcal{L}_d$ be the monomials of degree d in the d-th graded piece $k[x, y]_d$ of the polynomial ring $k[x, y]$. Let $n \geq 2$. Then V_n is generated by monomials of degree $n+1$ and has the form*

$$V_n = k[\mathcal{L}_{n+1}, \mathcal{L}_n s, xy\mathcal{L}_{n-3}s^2, xy\mathcal{L}_{n-4}s^3, \ldots, xy\mathcal{L}_1 s^{n-2}, xy\mathcal{L}_0 s^{n-1}]$$

where $\alpha\mathcal{L}_d\beta = \{\alpha\mu\beta \mid \mu \in \mathcal{L}_d\}$.

Proof. The integral closure of a monomial algebra is a monomial algebra. So it is enough to find all the monomials of degree $n+1$ in V_n and show that any higher degree monomial in V_n can be written as a product of the monomials of degree $n+1$.

Recall that

$$U_n \coloneqq k[x^n s, x^{n+1}, x^n y, y^n s, y^{n+1}, xy^n, xys^{n-1}].$$

Any monomial in the integral closure V_n must have a power in U_n. So we want to determine all monomials $x^a y^b s^j$ such that there exists a positive integer N such that

$(x^a y^b s^j)^N \in U_n$.

Notice that V_n contains all monomials of the form $x^a y^b$ where a and b are non-negative integers such that $a + b = n + 1$ because

$$(x^a y^b)^{n+1} = (x^{n+1})^a (y^{n+1})^b.$$

We also have all monomials of the form $x^a y^b s$ where a and b are non-negative integers such that $a + b = n$ because

$$(x^a y^b s)^n = (x^n s)^a (y^n s)^b.$$

Therefore, a monomial $x^a y^b s^j$ is in V_n if and only if for the corresponding exponent vector (a, b, j), there exists a positive integer N such that (Na, Nb, Nj) is in the $\mathbb{Z}_{\geq 0}$-linear sum of vectors of the following types:

- $(i, n - i, 1)$ corresponding to monomials of the form $x^i y^{n-i} s$ where $0 \leq i \leq n$,
- $(1, 1, n - 1)$ corresponding to the monomial xys^{n-1}, and
- $(i, n+1-i, 0)$ corresponding to monomials of the form $x^i y^{n+1-i}$ where $0 \leq i \leq n+1$.

We want to characterize all such exponent vectors (a, b, j). We consider the case where $n = 2$ and the case where $n \geq 3$ separately.

If $n = 2$, then xys is a monomial of type $(i, 2-i, 1)$ where $1 \leq i \leq n$. Then a monomial $\mu = x^a y^b s^j$ of degree $a + b + j = 3m$ is in V_2 if and only if there exists a positive integer N such that (Na, Nb, Nj) can be written as a sum of

- h vectors of type $(i, 2 - i, 1)$ where $0 \leq i \leq 2$, and
- $Nm - h$ vectors of type $(i, 3 - i, 0)$ where $0 \leq i \leq 3$.

Since $\mathcal{L}_3 \subseteq V_2$, this is equivalent to N, a, b, j and h satisfying

- $h = Nj$,
- $a + b = 3m - j$,
- $2h \leq Na + Nb$, and
- $h \leq Nm$.

Since $h = Nj$ and $a + b = 3m - j$, the two inequalities are both

$$Nj \leq Nm.$$

This inequality does not depend on N. Dividing by N yields $j \leq m$. So $x^a y^b s^j \in V_2$ if and only if $a + b + j = 3m$ and $j \leq m$.

We want to determine the monomials of degree $n + 1 = 3$ in V_2. But this means $j \leq m = 1$. We have already shown that all such monomials are in V_n. Now suppose that $m \geq 2$. Then $a + b = 3m - j \geq 2j$ because $j \leq m$. If $a, b \geq j$, then $x^a y^b s^j = (xys)^j (x^{a-j} y^{b-j})$. So we are done. If we have the case where $b < j$, then we also have the case where $a < j$ because of the symmetry of x and y in U_2. So without loss of generality, suppose $b < j$. Then $a - b \geq 2(j - b)$. So $x^a y^b s^j = (xys)^b (x^2 s)^{j-b} x^{a-b-2(j-b)}$. Thus, V_2 is generated in degree $n + 1 = 3$ and

$$V_2 = k[x^3, x^2y, xy^2, y^3, x^2s, y^2s, xys].$$

Notice that in this case $U_2 = V_2$.

Now assume that $n \geq 3$. We already showed that V_n contains all monomials in x, y, s where $j = 0$ and $j = 1$ in V_n. So we may assume that $j \geq 2$. Let $x^a y^b s^j$ be a monomial of degree $m(n+1)$ in V_n where m is a positive integer. Then there is a positive integer N such that $(x^a y^b s^j)^N$ is in U_n and there are non-negative integers h and k such that the vector (Na, Nb, Nj) can be written with

- h vectors of type $(i, n-i, 1)$ where $0 \leq i \leq n$,
- t vectors of type $(1, 1, n-1)$, and
- $Nm - h - t$ vectors of type $(i, n+1-i, 0)$ where $0 \leq i \leq n+1$.

Let (p, q, h) be the exponent vector corresponding to the product of the h monomials of type $(i, n-i, 1)$. Then since $\mathcal{L}_{n+1} \subseteq V_n$, we have $x^a y^b x^j \in V_n$ if and only if the following conditions are satisfied:

(a) $h + t \leq Nm$.

(b) $h + t(n-1) = Nj$.

(c) $p + q = hn$.

(d) $p + t \leq Na$.

(e) $q + t \leq Nb$.

Rewriting (b) as $h = Nj - t(n-1)$, we can replace (a) with

$$Nj - t(n-1) + t = Nj - t(n-2) \leq Nm \tag{4.1}$$

and we can replace (c) with

$$p + q = (Nj - t(n-1))n. \tag{4.2}$$

Then the conditions that we need to satisfy are:

(a′) $Nj - t(n-2) \le Nm$.

(b′) $h = Nj - t(n-1)$.

(c′) $p + q = (Nj - t(n-1))n$.

(d′) $p + t \le Na$.

(e′) $q + t \le Nb$.

Now suppose that $n \ge 3$ and $j \ge 2$. Then we can rewrite (a′) as

$$t \ge \frac{N(j-m)}{n-2}.$$

Since $h \ge 0$, we can rewrite (b′) as

$$t \le \frac{Nj}{n-1}$$

and (d′) and (e′) as

$$t \le Na \quad \text{and} \quad t \le Nb.$$

Finally, using $t \le Na$ and $t \le Nb$, we can rewrite (c′) as

$$(Nj - t(n-1))n + 2t \le N(a+b)$$

which then yields

$$t \ge \frac{N(jn - (a+b))}{(n-2)(n+1)}. \tag{4.3}$$

Now $a+b=m(n+1)-j$. So we have

$$N(jn-(a+b)) = N(jn-m(n+1)+j) = N(n+1)(j-m).$$

Then the inequality in 4.3 is just

$$t \geq \frac{N(j-m)}{n-2}.$$

Combining the inequalities above, we have

$$\frac{N(j-m)}{n-2} \leq t \leq \min\Big\{\frac{Nj}{n-1}, Na, Nb\Big\}. \tag{4.4}$$

Thus, a monomial $x^a y^b s^j$ is in V_n if and only if $a+b+j=m(n+1)$ for some m, and there exists a positive integer N and non-negative integers t and h such that $h = Nj - t(n-1)$ and t is in the interval

$$\Big[\frac{N(j-m)}{n-2}, \min\Big\{\frac{Nj}{n-1}, Na, Nb\Big\}\Big].$$

Now, the inequality in 4.4 does not depend on N and holds for all sufficiently large and sufficiently divisible N. So we may divide 4.4 by N to get an equivalent condition involving only j. That is, we have

$$\frac{j-m}{n-2} \leq \min\Big\{\frac{j}{n-1}, a, b\Big\}. \tag{4.5}$$

Claim 4.1.2.1. *Let $j \geq 2$. Let $\mu = x^a y^b s^j$ be a monomial of degree $n+1$. The $\mu \in V_n$ if and only if $j \leq n-1$ and $\mu \in xy\mathcal{L}_{n-j-1}s^j$.*

Proof. This is the case where $m = 1$. Then from 4.5, we have

$$\frac{j-1}{n-2} \leq \frac{j}{n-1}$$

which simplifies to

$$j \leq n - 1.$$

Since $j \geq 2$ and $n \geq 3$, we know that

$$0 < \frac{j-1}{n-2} \leq a, b.$$

Since a and b are integers, we have $a, b \geq 1$. Moreover, for any $j \leq n-1$, we have

$$0 < \frac{j-1}{n-2} \leq 1 \leq a, b.$$

So $\mu = x^a y^b s^j \in xy\mathcal{L}_{n-j-1}s^j$ where $j \leq n-1$. □

It remains to show that V_n is generated in degree $n+1$. Let $m \geq 2$. Let $\mu = x^a y^b s^j$ be a monomial of degree $m(n+1)$ in V_n. We may assume that $j \geq 2$. Then (a, b, j) satisfies

$$\frac{j-m}{n-2} \leq \min\left\{\frac{j}{n-1}, a, b\right\}.$$

This condition is equivalent to the following three inequalities:

(a) $j \leq m(n-1)$.

(b) $\frac{j-m}{n-2} \leq a$.

(c) $\frac{j-m}{n-2} \leq b$.

Claim 4.1.2.2. *If $a > 0$, $b > 0$, and $j \geq n-1$, then we can factor μ as*

$$\mu = x^a y^b s^j = (xys^{n-1})x^{a-1}y^{b-1}s^{j-(n-1)}$$

and the monomial $x^{a-1}y^{b-1}s^{j-(n-1)}$ is an element of V_n of degree $(m-1)(n+1)$.

Proof. We just need to show that $x^{a-1}y^{b-1}s^{j-(n-1)}$ satisfies the inequalities

(a′) $j-(n-1) \leq (m-1)(n-1)$

(b′) $\frac{j-(n-1)-(m-1)}{n-2} \leq a-1$

(c′) $\frac{j-(n-1)-(m-1)}{n-2} \leq b-1$

Now

$$\frac{j-(n-1)-(m-1)}{n-2} = \frac{j-m}{n-2} - 1.$$

But this means that (a′), (b′), and (c′) are equivalent to

(a) $j \leq m(n-1)$,

(b) $\frac{j-m}{n-2} \leq a$,

(c) $\frac{j-m}{n-2} \leq b$.

So we are done. □

Using Claim 4.1.2.2, we can reduce the proof to the case where $a = 0$ or $b = 0$ or $j < n-1$.

<u>Case 1:</u> ($a = 0$ or $b = 0$) The case where $a = 0$ is the same as the case $b = 0$ by symmetry. Without loss of generality, we assume $b = 0$. In this case, we have $x^a s^j$. Since

$\frac{j-m}{n-2} \leq b = 0$, we have $j \leq m$. Now $j + a = m(n+1)$ and so

$$a = m(n+1) - j \geq mn \geq jn.$$

Then $x^a s^j = (x^n s)^j x^{a-jn}$ and $a - jn = (m-j)(n+1)$. So we are done.

<u>Case 2:</u> $(j < n-1)$ In this case, we must have $a, b > 0$. Then there exists $\mu_1 \in xy\mathcal{L}_{n-j-1}s^j$ and $\mu_2 = x^c y^d$ such that $x^a y^b s^j = \mu_1 \mu_2$. So we are done.

This completes the proof that

$$V_n = k[\mathcal{L}_{n+1}, \mathcal{L}_n s, xy\mathcal{L}_{n-3}s^2, xy\mathcal{L}_{n-4}s^3, \ldots, xy\mathcal{L}_1 s^{n-2}, xy\mathcal{L}_0 s^{n-1}]. \qquad \square$$

Lemma 4.1.3. *Let W_n be the integral closure of U_n in $k[x, y, s]$. Then*

$$V_n = W_n \cap k[x, y, s]^{(n+1)}.$$

Proof. Since W_n is the integral closure of U_n in $k[x, y, s]$ and V_n is the integral closure of U_n in $k[x, y, s]^{n+1} \subseteq k[x, y, s]$, it follows that W_n contains V_n. So V_n is contained in the intersection $W_n \cap k[x, y, s]^{(n+1)}$.

On other hand, if $u \in W_n \cap k[x, y, s]^{(n+1)}$, then u is integral over V_n because W_n is integral over V_n. Now u is also an element of $k[x, y, s]^{(n+1)}$ which has the same fraction field as V_n. So u must be in the fraction field of V_n and therefore, u must be an element of V_n. $\square$

Theorem 4.1.4. *Let W_n be the integral closure of U_n in $k[x, y, s]$. Then W_n is normal and*

$$W_n \coloneqq k[x, y, x^n s, y^n s, xys, xys^2, \ldots, xys^{n-1}].$$

Proof. Let $W_n' = k[x, y, x^n s, y^n s, xys, xys^2, \dots, xys^{n-1}]$. We first check that the generators of W_n' are integral over U_n. It is clear that x and y are integral over U_n because $x^{n+1}, y^{n+1} \in U_n$. Now consider $s^j xy$ where $1 \le j \le n-1$. Then

$$(s^j xy)^{n-1} = (s^{n-1}xy)^j x^{n-1-j} y^{n-1-j}.$$

So W_n' is contained in W_n. It remains to show that $W_n' = W_n$.

A monomial $x^a y^b s^j \in W_n$ if and only if $(x^a y^b s^j)^{n+1} \in V_n$. Then $x^a y^b s^j \in W_n$ if and only if the exponent vector $(a(n+1), b(n+1), j(n+1))$ satisfies

$$\frac{j(n+1) - m}{n-2} \le \min\Big\{\frac{j(n+1)}{n-1}, a(n+1), b(n+1)\Big\}. \tag{4.6}$$

where $m = a + b + j$. We want to show that $x^a y^b s^j \in W_n'$. Since $x, y \in W_n'$, we already have the case where $j = 0$. We may assume that $j > 0$. The inequality in 4.6 is equivalent to the following inequalities:

(a) $2j \le (a+b)(n-1)$,

(b) $j(n+1) - (a+b+j) \le a(n+1)(n-2)$,

(c) $j(n+1) - (a+b+j) \le b(n+1)(n-2)$.

We claim that we can factor out xys^{n-1}. More precisely, the monomial $x^{a-1}y^{b-1}s^{j-n+1}$ is also an element of W_n'. We need to check that $(a-1, b-1, j-(n-1))$ satisfies

(a′) $2(j-(n-1)) \le (a-1+b-1)(n-1)$,

(b′) $(j-(n-1))(n+1) - (a-1+b-1+j-(n-1)) \le (a-1)(n+1)(n-2)$, and

(c′) $(j-(n-1))(n+1) - (a-1+b-1+j-(n-1)) \le (b-1)(n+1)(n-2)$.

But (a$'$), (b$'$), and (c$'$) are equivalent to (a), (b), and (c). So $x^{a-1}y^{b-1}s^{j-n+1} \in W_n$. This means that we can factor out as many xys^{n-1} as possible and reduce to the case where $a = 0$ or $b = 0$ or $j < n - 1$. Notice that if $j > 1$, then from condition (a), we have $0 < 2j/(n-1) < a + b$. Then we must have $a > 0$ or $b > 0$.

Case 1: ($a > 0$, $b > 0$, $j < n-1$) First notice that if $j < n-1$, we have $x^a y^b s^j = (xys^j)\mu$ where μ is a monomial in x and y. So we are done.

Case 2: ($a = 0$ or $b = 0$) Without loss of generality, we may assume that $b = 0$ because of the symmetry of x and y in W_n'. We have $x^a s^j \in W_n'$. Then the three inequalities are

(a$''$) $2j \leq a(n-1)$,

(b$''$) $j(n+1) - (a+j) \leq a(n+1)(n-2)$, and

(c$''$) $j(n+1) - (a+j) \leq 0$.

The last inequality yields $jn \leq a$. Then $x^a s^j = (x^n s)^j x^{a-jn} \in W_n.'$ This finishes Case 2.

This concludes the proof that $W_n' = W_n$. $\square$

4.2 The Multiplicities of V_n and W_n

In this section, we compute the multiplicities of V_n and W_n. Because V_n is standard-graded and has dimension 3, its Hilbert function eventually agrees with a polynomial of degree 2 called the Hilbert polynomial. The Hilbert polynomial can be used to compute the multiplicity of V_n localized at its homogeneous maximal ideal. Put another way, we can extend the notion of multiplicity from the local setting to the standard-graded setting in a compatible manner. See Chapter 1 for more details.

On the other hand, the ring W_n is not standard-graded and in particular, the Hilbert function of W_n is not the same as the Hilbert function of the associated graded ring of W_n localized at its maximal ideal. In fact, the Hilbert function of W_n does not eventually agree with a polynomial. By "the multiplicity of W_n," we mean the multiplicity of W_n localized at its homogeneous maximal ideal.

Throughout this section, it will be convenient to switch between the local setting and the graded setting. For simplicity of notation, we will write V_n and W_n for both the graded rings and their localizations at their respective homogeneous maximal ideals when it is clear from context which setting we are working in.

Proposition 4.2.1. *Let $n \geq 2$. Let W_n be the ring*

$$k[x, y, x^n s, y^n s, xys, xys^2, \ldots, xys^{n-1}]$$

localized at the maximal ideal $\mathfrak{m}_{W_n} = (x, y, x^n s, y^n s, xys, xys^2, \ldots, xys^{n-1})$. Then

$$I = (x - y^n s, y - x^n s, xys^{n-1})W_n$$

is a minimal reduction for $\mathfrak{m}_{W_n}$. The multiplicity of W_n is $n+1$ and the type of W_n is n.

Proof. For simplicity of notation, we will write $\mathrm{gr}(W)$ for the associated graded ring of W_n with respect to its maximal ideal $\mathfrak{m}_W$. To show that $I = (x - y^n s, y - x^n s, xys^{n-1})$ is a minimal reduction of $\mathfrak{m}_{W_n}$, it is enough to show that the images of $x - y^n s, y - x^n s$, and xys^{n-1} in $\mathrm{gr}(W)$ form a linear system of parameters for $\mathrm{gr}(W)$.

Now the associated graded ring $\mathrm{gr}(W)$ has the form $k[x, y, u, v, w_1, \ldots w_{n-1}]/J$ where J is the ideal generated by the leading forms (i.e., lowest degree terms) in the kernel of the surjection

$$\varphi : k[x, y, u, v, w_1, \ldots, w_{n-1}] \twoheadrightarrow k[x, y, x^n s, y^n s, xys, \ldots, xys^{n-1}]$$

where $x \mapsto x$, $y \mapsto y$, $u \mapsto x^n s$, $v \mapsto y^n s$, and $w_i \mapsto xys^i$ for $1 \leq i \leq n-1$.

We have the following relations in $\ker(\varphi)$:

- $w_i^{n-1} - (xy)^{n-1-i} w_{n-1}^i = 0$ for $1 \leq i < n-1$.
- $xv - y^{n-1} w_1 = 0$.
- $yu - x^{n-1} w_1 = 0$.

Then xv, yu, and w_i where $1 \leq i < n-1$ are nilpotent in the associated graded ring $\mathrm{gr}(W)$. Killing w_{n-1}, $x - v$, and $y - u$ make x, y, u, v and w_{n-1} nilpotent in

$$\mathrm{gr}(W)/(w_{n-1}, x - v, y - u)\mathrm{gr}(W).$$

So w_{n-1}, $x - v$, and $y - u$ are a linear system of parameters for gr(W) and

$$I = (x - y^n s, y - x^n s, xys^{n-1})$$

is a minimal reduction of $\mathfrak{m}_{W_n}$.

The multiplicity of W_n is the length of $\overline{W}_n = W_n/(x - y^n s, y - x^n s, xys^{n-1})W_n$, which is the same as the k-vector space dimension of $\overline{W}_n$.

Claim. *The element xy is 0 in $\overline{W}_n$.*

Proof. In $\overline{W}_n$, we have $x = y^n s$, $y = x^n s$, and $xys^{n-1} = 0$. Then

$$\begin{aligned} xy = (y^n s)(x^n s) &= (s^2 xy)(xy)^{n-1} \\ &= (s^2 xy)((y^n s)(x^n s))^{n-1} \\ &= (s^2 xy)(s^2 xy)^{n-1}(xy)^{(n-1)(n-1)} \\ &= (s^2 xy)(s^{n-1} xy)^2 (xy)^{n-3}(xy)^{(n-1)(n-1)} \\ &= 0. \end{aligned}$$

□

Claim. *Let $\overline{\mathfrak{m}}_{W_n}$ be the maximal ideal of $\overline{W}_n$. Then $(\overline{\mathfrak{m}}_{W_n})^2 = 0$.*

Proof. We already showed that $xy = 0$. Now consider the products $w_i w_j = (xys^i)(xys^j)$ where $1 \leq i, j \leq n-2$. If $i + j \geq n-1$, then

$$w_i w_j = (xys^{n-1})(xys^{(i+j)-(n-1)}) = 0.$$

If $i + j < n - 1$, then

$$w_i w_j = xys^{i+j}(xy) = 0.$$

Next, notice that

$$x^2 = x(y^n s) = (sxy)(y^{n-1}) = 0.$$

By symmetry $y^2 = 0$. Finally, we have

$$\begin{aligned} xw_i = x(xys^i) &= (y^n s)(xys^i) \\ &= (y^n)(xys^{i+1}) = (y^{n-1})(x^n s)(xys^{i+1}) \\ &= (xy)^{n-2} x(xys)(xys^{i+1}) = 0. \end{aligned}$$

By symmetry $yw_i = 0$. □

Now there are no linear relations between x, y, and the w_i for $1 \le i \le n-2$. So then we must have

$$\overline{W}_n \cong \frac{k[x, y, w_i \mid 1 \le i \le n-2]}{(x, y, w_i \mid 1 \le i \le n-2)^2}.$$

Thus the multiplicity of W_n is $n+1$ and the type is n. □

Proposition 4.2.2. *Let $n \ge 2$. Let V_n be the standard-graded ring*

$$V_n = k[\mathcal{L}_{n+1}, \mathcal{L}_n s, xy\mathcal{L}_{n-3}s^2, xy\mathcal{L}_{n-4}s^3, \ldots, xy\mathcal{L}_1 s^{n-2}, xy\mathcal{L}_0 s^{n-1}].$$

Then the multiplicity of V_n is n^2+1

Proof. We will compute the multiplicity of V_n by using the Hilbert function of V_n which is defined as

$$\mathrm{Hilb}_{V_n}(m) = \ell([V_n]_m) = \dim_k([V_n]_m).$$

where the m-th graded piece $[V_n]_m$ which consists of forms of degree $m(n+1)$. We will consider the case where $n = 2$ and $n \ge 3$ separately.

Let $n = 2$. Then recall from the previous section that a monomial $x^a y^b s^j \in V_2$ if and only if the following conditions hold:

(a) $a + b + j = 3m$.

(b) $j \le m$.

Since $a, b \ge 0$, we can rewrite these conditions as

(a′) $a + b \le 3m$ and

(b′) $a + b \geq 2m$.

The area defined by these inequalities is a trapezoid whose vertices are integer coefficients. We want to find the number of lattice points in this region. Pick's theorem says that the area of a polygon whose vertices have integer coordinates is determined by the number of lattice points in the following way: Let $i(m)$ be the number of interior lattice points and $b(m)$ be the number of boundary lattice points. Then the area of the region is

$$A(m) = i(m) + \frac{b(m)}{2} - 1.$$

Notice that $i(m)$ is quadratic in m and $b(m)$ is linear in m. Since

$$i(m) < A(m) < i(m) + b(m),$$

we have $A(m)$ is asymptotic to $(i(m) + b(m))$. That is, $\mathrm{Hilb}_{V_2}(m)$ is asymptotic to $A(m)$. We have

$$A(m) = \frac{1}{2}(9m^2 - 4m^2) = \frac{1}{2}(5m^2).$$

Thus, the multiplicity of V_2 is $5 = n^2 + 1$

Now let $n \geq 3$. Recall from the previous section that a monomial $x^a y^b s^j \in V_n$ if and only if the following conditions hold:

(a) $a + b + j = m(n + 1)$.

(b) $a, b \geq (j - m)/(n - 2)$.

(c) $j \leq m(n - 1)$.

Since $a, b \geq 0$, we can rewrite these conditions as (a′) $a + b \le m(n+1)$,

(b′) $(n-1)a + b \ge mn$,

(c′) $a + b(n-1) \ge mn$, and

(d′) $a + b \ge 2m$.

The last condition (d′) is redundant and can be obtained from (b′) and (c′). We want to determine the number of lattice points in the region determined by these inequalities. On (a, b)-plane, the intercepts of the above conditions are

(a′) $(m(n+1), 0), (0, m(n+1))$

(b′) $(mn/(n-1), 0), (0, mn)$, and

(c′) $(mn, 0), (0, mn/(n-1))$.

This is a pentagon with vertices that have integer coordinates.

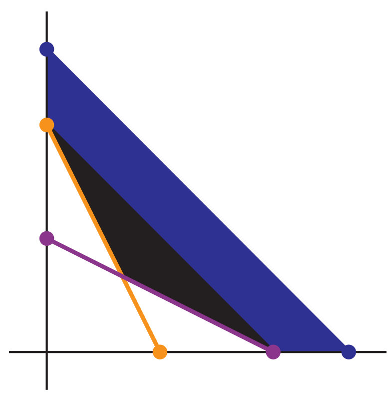

Figure 4.1: Area $A(m)$ split into a triangle and a trapezoid

We want to find the area $A(m)$ of the pentagon. We can split up $A(m)$ as a triangle with vertices $(0, mn), (m, m)$, and $(mn, 0)$ and a trapezoid. See Figure 4.1. The area of the trapezoid is

$$\frac{1}{2}m^2(n+1)^2 - \frac{1}{2}(mn)^2 = \frac{1}{2}(2n+1)m^2.$$

The area of the triangle is

$$\frac{1}{2}(n^2-2n)m^2.$$

Then

$$A(m) = \frac{1}{2}(n^2+1)m^2.$$

Thus, the multiplicity of V_n is n^2+1. $\square$

Proposition 4.2.3. *Let $n \geq 2$. Let V_n be the localization of*

$$k[\mathcal{L}_{n+1}, \mathcal{L}_n s, xy\mathcal{L}_{n-3}s^2, xy\mathcal{L}_{n-4}s^3, \ldots, xy\mathcal{L}_1 s^{n-2}, xy\mathcal{L}_0 s^{n-1}].$$

at the homogeneous maximal ideal

$$\mathfrak{m}_{V_n} = (\mathcal{L}_{n+1}, \mathcal{L}_n s, xy\mathcal{L}_{n-3}s^2, xy\mathcal{L}_{n-4}s^3, \ldots, xy\mathcal{L}_1 s^{n-2}, xy\mathcal{L}_0 s^{n-1}).$$

Then $I = (x^{n+1} - y^n s, y^{n+1} - x^n s, xys^{n-1})V_n$ is a minimal reduction of m_{V_n}.

Proof. For large n, it can be cumbersome to work directly with V_n because of the number of monomial generators. We can work with W_n instead. The multiplicity of W_n as a V_n-module is $e_{V_n}(W_n)\text{rank}_{V_n}(W_n) \cdot e_{V_n}(V_n)$. The fraction field of W_n is

$$k(x,y,s) = k(x, y/x, s/x)$$

and the fraction field of V_n is

$$k(x^{n+1}, y/x, s/x).$$

So $\operatorname{rank}_{V_n}(W_n) = n+1$ and $e_{V_n}(W_n) = (n+1)(n^2+1)$. In order to prove that $I = (x^{n+1} - y^n s, y^{n+1} - x^n s, xys^{n-1})V_n$ is a minimal reduction for $\mathfrak{m}_{V_n}$, by Rees's theorem, it suffices to show that $e_I(W_n)$ is $(n+1)(n^2+1)$. Note that $e_I(W_n) \geq (n+1)(n^2+1)$ and $e_I(W_n)$ is the length of $Q_n \coloneqq W_n/(x^{n+1} - y^n s, y^{n+1} - x^n s, xys^{n-1})W_n$. So if we find a spanning set for Q_n as a k-vector space with $(n+1)(n^2+1)$ elements, we are done.

Claim. *The following elements span Q_n as a k-vector space:*

- $x^i y^j$ *where* $0 \leq i, j, \leq n+1$ *but both* i, j *cannot be equal to* $n+1$*.*
- $x^i y^j w_m$ *where* $w_m = xys^m$, $1 \leq m \leq n-2$, *and* $0 \leq i, j \leq n$*.*

Proof. Note that in Q_n, we have $x^{n+1} = y^n s$, $y^{n+1} = x^n s$, and $xys^{n-1} = 0$. We want to show that all monomials in Q_n can be written as a sum of the basis elements. We first consider the case $x^i y^j w_m$ where either $i > n$ or $j > n$. Choose $t = \max\{i, j\}$. Because the relations we are quotienting out are symmetric in x and y, we may assume $i > j$ without loss of generality. Then

$$x^i y^j w_m = x^{i-(n+1)}(y^n s) y^j w_m = x^{i-(n+1)}(y^{j+n}) w_{m+1}.$$

Notice that the total degree of x and y in the monomial has decreased by 1 while the subscript on the w term has increased by 1. Let $i' = (i - (n+1))$ and $j' = j + n$. Repeat the process until either the exponents on x and y are at most n or until the subscript on the w reaches $n-1$. In the latter case, the monomial is 0 because $w_{n-1} = xys^{n-1} = 0$. The algorithm must terminate because at each step, the total degree of x and y decreases by 1 and the subscript on w increases by 1.

Next, we consider the case $x^i y^j$ where either $i > n+1$ or $j > n+1$. Without loss of generality, suppose $i > n+1$. Then

$$x^i y^j = (y^n s) x^{i-(n+1)} y^j = x^{i-(n+1)} y^j w_1.$$

So we have reduced to the previous case.

Finally, in the case where $i = j = n+1$, we have

$$x^{n+1} y^{n+1} = (x^n s)(y^n s) = (s^2 xy) x^{n-1} y^{n-1} = x^{n-1} y^{n-1} w_2. \qquad \square$$

The total number of elements in the spanning set above is

$$((n+2)^2 - 1) + (n+1)^2 (n-2) = n^3 + n^2 + n + 1.$$

But this is the same as $(n+1)(n^2+1)$. $\square$

Proposition 4.2.4. *Let $n \geq 2$. Let U_n be the localization of*

$$k[x^n s, x^{n+1}, x^n y, y^n s, y^{n+1}, xy^n, xys^{n-1}]$$

at the homogeneous maximal ideal $\mathfrak{m}_{U_n} = (x^n s, x^{n+1}, x^n y, y^n s, y^{n+1}, xy^n, xys^{n-1})$. Then

$$I = (x^{n+1} - y^n s, y^{n+1} - x^n s, xys^{n-1}) U_n$$

is a minimal reduction for $\mathfrak{m}_{U_n}$, and the multiplicity of U_n is $n^2 + 1$.

Proof. Since $x^{n+1} - y^n s, y^{n+1} - x^n s$, and xys^{n-1} generate minimal reduction for the maximal ideal of V_n which contains $\mathfrak{m}_{U_n}$ and V_n is integral over U_n, it follows that $\mathfrak{m}_{U_n}$ is integral over $(x^{n+1} - y^n s, y^{n+1} - x^n s, xys^{n-1}) U_n$.

To get the multiplicity of U_n, it suffices to compute the length of

$$V_n/(x^{n+1} - y^n s, y^{n+1} - x^n s, xys^{n-1})V_n$$

because V_n is a rank 1 Cohen–Macaulay module over U_n. But IV_n is a minimal reduction for the maximal ideal of V_n. So this length is precisely the multiplicity of V_n which is $n^2 + 1$. □

4.3 Relationships between U_n, V_n, and W_n

Theorem 4.3.1. *Let $(R, \mathfrak{m}_R, k) \to (S, \mathfrak{m}_S, l)$ be a module-finite extension such that $\mathfrak{m}_S$ is integral over $\mathfrak{m}_R$. It suffices for the generators of $\mathfrak{m}_S$ to be integral over $\mathfrak{m}_R$. Then the parameters that give a minimal reduction of $\mathfrak{m}_R$ give a minimal reduction of $\mathfrak{m}_S$. Moreover, if M is an Ulrich module of S, then M is an Ulrich module of R.*

Proof. Let $d = \dim(R) = \dim(S)$. Let $x_1, \ldots, x_d$ be a system of parameters of R such that the ideal they generate $I = (x_1, \ldots, x_d)R$ is a minimal reduction for $\mathfrak{m}_R$. Now $\mathfrak{m}_S$ is integral over $\mathfrak{m}_R$, and $\mathfrak{m}_R$ is integral over I. So $\mathfrak{m}_S$ is integral over IS. But IS has $d =$dim(S) generators. So IS is a minimal reduction for $\mathfrak{m}_S$.

Let M be an Ulrich module of S. Clearly M is still an MCM module over R. Now

$$IM \subseteq \mathfrak{m}_R M \subseteq \mathfrak{m}_S M.$$

Then we have $IM = \mathfrak{m}_S M$ because M is an Ulrich module of S. So $IM = \mathfrak{m}_R M$. But this means M is an Ulrich module of R.

□

Corollary 4.3.2. *Let $(S, \mathfrak{n}, k)$ be a complete local ring of dimension d containing k. Let $(x_1, \ldots, x_d)$ be a minimal reduction of $\mathfrak{n}$. If S has an Ulrich module, then any ring R such that $k[\![x_1, \ldots, x_d]\!] \subseteq R \subseteq S$ has an Ulrich module.*

Applying Theorem 4.3.1, we see that any Ulrich module of W_n is an Ulrich module over U_n, and any Ulrich module over V_n is an Ulrich module over U_n. Then to show that V_n or W_n has no Ulrich modules (if this is indeed the case), it is enough to show that U_n has no Ulrich modules.

Theorem 4.3.3. *The ring $U_2 = V_2$ has an Ulrich module.*

Proof. We show that U_2 has a rank one Ulrich module. If M is a rank one Ulrich, then M must be an ideal $I \subseteq U_2$. Now an ideal I is MCM over U_2 if and only if the depth of I is 3. Because U_2 is a domain, I is MCM over U_2 if and only if the quotient ring U_2/IU_2 is a Cohen–Macaulay ring of dimension 2 and I has pure height one (i.e., all the primes in its primary decomposition have height one). The multiplicity of U_2 is $n^2 + 1 = 5$. So then I is a rank one Ulrich module over U_2 if and only if I is an ideal of pure height one such that the quotient ring U_2/IU_2 is a Cohen–Macaulay ring of dimension 2. We show that such an ideal I exists.

Using Macaulay2 [M2], we compute a presentation of U_2. In particular, we have the following isomorphism:

$$\varphi : k[u_1, u_2, u_3, v_1, v_2, v_3, w]/J \xrightarrow{\sim} k[x^3, x^2y, xy^2, y^3, x^2s, y^2s, xys] = U_2$$

where J is generated by the relations

1. $v_2v_3 - w^2$

2. $u_3v_3 - v_1w$

3. $u_2v_3 - u_3w$

4. $u_1v_3 - u_2w$

5. $v_1v_2 - u_3w$

6. $u_3v_2 - u_2w$

7. $u_2v_2 - u_1w$

8. $u_3^2 - u_2v_1$

9. $u_2u_3 - u_1v_1$

10. $u_2^2 - u_1u_3$

which were computed using Macaulay2 [M2].

Claim 4.3.3.1. *The ideal* $I = (u_2, u_3, v_1, v_3, w)$ *is an Ulrich module.*

Proof. We want to compute the quotient U_2/IU_2. In this quotient, the relations listed above become trivial. Then $U_2/IU_2 \cong k[u_1, v_2]$, which is a two-dimensional Cohen–Macaulay ring. So I is a rank one MCM module of U_2. Since I has five generators, it is an Ulrich module. □

□

Remark 4.3.4. The same approach does not appear to work for U_n where $n \geq 3$. This is because the multiplicity of U_n is $n^2 + 1$, but the number of monomial generators of the maximal ideal remains the same. So finding an "obvious" Ulrich module becomes more difficult. One could possibly try to work with V_n since the monomial generators increases quadratically, but the process quickly becomes very computationally laborious.

Lemma 4.3.5. *Let $n \geq 2$. Let W_n and V_n be the monomial algebras defined in the previous sections. Then considering W_n as a V_n-module, we have*

$$W_n = \oplus_{\rho=0}^{n} W_{n,\rho} = V_n \oplus (\oplus_{\rho=1}^{n} W_{n,\rho})$$

where $W_{n,\rho}$ is spanned by all homogeneous elements of degree ρ modulo $n+1$. Each $W_{n,\rho}$ has rank 1 as a V_n-module (also as a U_n-module by restriction of scalars) and is Cohen–Macaulay over V_n (resp. over U_n).

Proof. It is clear that $W_n = V_n \oplus (\oplus_{\rho=1}^{n} W_{n,\rho})$, and that because W_n is a Cohen–Macaulay V_n-module, each $W_{n,\rho}$ is a Cohen–Macaulay V_n-module. To see that $W_{n,\rho}$ has rank 1 as a V_n-module, it is enough to see that each $W_{n,\rho} \neq 0$. This is because $\operatorname{rank}_{V_n}(W_n) = n+1$, and there are $n+1$ summands. But $x^{\rho} \cdot V_n \subset W_{n,\rho}$. □

Theorem 4.3.6. *Let $n \geq 3$ and $0 \leq \rho \leq n$. The rank one Cohen-Macaulay summands $W_{n,\rho}$ are not Ulrich modules of U_n or V_n.*

Proof. It is enough to show that the $W_{n,\rho}$ are not Ulrich modules of U_n. The multiplicity of each $W_{n,\rho}$ is n^2+1. So it suffices to show that the minimal number of generators of $W_{n,\rho}$ is less than n^2+1.

Let $\mathfrak{m}_{U_n}$ be the maximal homogeneous ideal of U_n. From the previous section, we know that a minimal reduction of $\mathfrak{m}_{U_n}$ is $I = (x^{n+1} - y^n s, y^{n+1} - x^n s, xys^{n-1})$. A basis for $Q_n = W_n/IW_n$ is

1. $x^a y^b$ where $0 \leq a, b, \leq n+1$ but both a, b cannot be equal to $n+1$, and
2. $x^a y^b w_i$ where $w_i = xys^i$, $1 \leq i \leq n-2$, and $0 \leq a, b \leq n$.

Then $Q_n/(x^{n+1}, y^{n+1}, xy^n, x^n y)Q_n = W_n/\mathfrak{m}_{U_n} W_n$. So, we want to determine what basis elements of Q_n are multiples of x^{n+1}, y^{n+1}, xy^n, and $x^n y$. Removing the obvious multiples, we have the following spanning set for $W_n/\mathfrak{m}_{U_n} W_n$:

1. x^n, y^n, $x^a y^b$ where $0 \leq a, b, < n$, and
2. $x^a y^b w_i$ where $w_i = xys^i$, $1 \leq i \leq n-2$, and $0 \leq a, b \leq n-1$.

Note that for $i \geq 2$, we have

$$x^n w_i = x^n(xys^i) = (x^n s)w_{i-1} = y^{n+1} w_{i-1} = 0.$$

If $i = 1$, then

$$x^n w_1 = x^n(xys) = (x^n s)xy = y^{n+1}xy = 0.$$

Similarly, $y^n w_i = 0$ for $0 \leq i \leq n-2$.

The elements of this spanning set that have degree ρ modulo $n+1$ form a spanning set of $W_{n.\rho}/\mathfrak{m}_{U_n} W_{n,\rho}$. Note that degree of the elements of the first type (i.e., monomials in x and y only) ranges from 0 to $2n-2$. The degree of the elements of the second type ranges from 3 to $3n-2$

Claim 4.3.6.1. *The number of elements of type one (i.e., monomials of the form $x^a y^b$) in the spanning set that have degree ρ modulo $n+1$ is max$\{n-1, \rho+1\}$.*

Proof. The number of elements of degree ρ is $\rho + 1$. Consider the elements that have degree $\rho + (n+1)$. If $\rho \geq n-2$, then there are no basis elements of this degree. If $\rho \leq n-3$, then we have $a, b > \rho + 1$ because $a, b < n$. So there are $n - \rho - 2$ elements. Finally, there are no elements of the form $x^a y^b$ that have degree $\rho + 2(n+1)$. Then the number of elements of degree ρ modulo $n+1$ is

$$\rho + 1 + \max\{n - \rho - 2, 0\} = \max\{n-1, \rho+1\}$$

□

It remains to count the number of elements of type 2 that have degree ρ modulo $n+1$. These are elements of the form $x^a y^b w_i$ where $0 \leq a, b \leq n-1$. Now $w_i = xys^i$ and so $x^a y^b w_i = x^a y^b s^i xy$ has degree $a+b+i+2$. Then counting the number of elements $x^a y^b w_i$ that have degree ρ modulo $n+1$ is the same as counting the number of monomials $x^a y^b s^i$ such that

- $a + b + i \equiv \rho - 2 \pmod{n+1}$,
- $0 \leq a, b \leq n-1$, and
- $1 \leq i \leq n-2$.

We can get an upper bound on the number of monomials $x^a y^b s^i$ satisfying the above conditions by counting the monomials $x^a y^b s^i$ satisfying

- $a + b + i \equiv \rho - 2 \pmod{n+1}$,
- $0 \leq a, b \leq n-1$, and
- $0 \leq i \leq n-2$.

This is the same as the sum of the Hilbert function of $T := k[x,y,s]/(x^n, y^n, s^{n-1})$ evaluated at $\rho - 2$, $\rho - 2 + (n+1)$, and $\rho - 2 + 2(n+1)$. Let $\mathrm{Hilb}_T(d)$ be the Hilbert function of T. We can compute a closed form of $\mathrm{Hilb}_T(d)$ using the Koszul complex $K_\bullet(x^n, y^n, s^{n-1}; k[x,y,s])$, which is a free resolution of T as a module over $k[x,y,s]$. Note that we will want to keep track of the grading. Let $R = k[x,y,s]$ and $d = a+b+i$. Then we have the following Koszul complex for T:

$$\begin{aligned} 0 \to R(-3n+1) \to R(-2n) \oplus R(-2n+1) \oplus R(-2n+1) \\ \to R(-n) \oplus R(-n) \oplus R(-n+1) \to R\ (\to T) \to 0. \end{aligned}$$

Let $H_R(d)$ be the Hilbert function of $k[x,y,s]$. So we have

$$H_R(d) = \binom{d+2}{2}.$$

Then the Hilbert function of $T = k[x,y,s]/(x^n, y^n, s^{n-1})$ has the form

$$H_R(d) - 2H_R(d-n) - H_R(d-n+1) + 2H_R(d-2n+1) + H_R(d-2n) - H_R(d-3n+1)$$

where $H_R(m) = 0$ if $m < 0$. Note that $H_R(m) = 1$ if $m = 0$. Thus, in order to get more explicit formulas, we need to consider the following cases:

- $d < n-1$,
- $d = n-1$,
- $d = n$,
- $n < d < 2n-1$,
- $d = 2n-1$,
- $d = 2n$, and
- $2n < d < 3n-1$.

We compute $\mathrm{Hilb}_T(d)$:

- $\underline{d < n-1}$

$$\mathrm{Hilb}_T(d) = H_R(d) = \frac{1}{2}(d^2 + 3d + 2)$$

- $\underline{d = n-1}$

$$\mathrm{Hilb}_T(d) = H_R(d) - 1 = \frac{1}{2}(d^2 + 3d + 2) - 1$$

- $\underline{d = n}$

$$\mathrm{Hilb}_T(d) = H_R(d) - 5 = \frac{1}{2}(d^2 + 3d + 2) - 5$$

- $\underline{n < d < 2n - 1}$

$$\begin{aligned}\mathrm{Hilb}_T(d) &= H_R(d) - 2H_R(d-n) - H_R(d-n+1)\\ &= \frac{1}{2}(-2d^2 + 6dn - 8d - 3n^2 + 11n - 8)\end{aligned}$$

- $\underline{d = 2n - 1}$

$$\begin{aligned}\mathrm{Hilb}_T(d) &= H_R(d) - 2H_R(d-n) - H_R(d-n+1) + 2\\ &= \frac{1}{2}(-2d^2 + 6dn - 8d - 3n^2 + 11n - 8) + 2\end{aligned}$$

- $\underline{d = 2n}$

$$\begin{aligned}\mathrm{Hilb}_T(d) &= H_R(d) - 2H_R(d-n) - H_R(d-n+1) + 7\\ &= \frac{1}{2}(-2d^2 + 6dn - 8d - 3n^2 + 11n - 8) + 7\end{aligned}$$

- $\underline{2n < d < 3n - 1}$

$$\begin{aligned}\mathrm{Hilb}_T(d) &= H_R(d) - 2H_R(d-n) - H_R(d-n+1) + 2H_R(d-2n+1)\\ &\quad + H_R(d-2n) - H_R(d-3n+1)\\ &= \frac{1}{2}(d^2 - 6dn + 5d + 9n^2 - 15n + 6)\end{aligned}$$

Recall that: $0 \leq \rho \leq n$. We want to sum the cases where the values of d are $d = \rho - 2$,

$d = \rho + n - 1$, and $d = \rho + 2n$. This will give us an upper bound for the number of basis elements of the form $x^a y^b w_i$.

- Case 1: $\rho = 0$
  - When $d = \rho - 2 < 0$, we have 0.
  - When $d = \rho + n - 1 = n - 1$, we have

$$\frac{1}{2}(d^2 + 3d + 2) - 1 = \frac{1}{2}(n^2 + n) - 1.$$

  - When $d = \rho + 2n = 2n$, we have

$$\frac{1}{2}(n^2 - 5n - 8) + 7$$

  Summing the three terms, we have $\mathbf{n^2 - 2n + 2}$.

- Case 2: $\rho = 1$
  - When $d = \rho - 2 < 0$, we have 0.
  - When $d = \rho + n - 1 = n$, we have

$$\frac{1}{2}(n^2 + 3n + 2) - 5.$$

  - When $d = \rho + 2n = 2n + 1$, we have

$$\frac{1}{2}(n^2 - 7n + 12).$$

  Summing the three terms, we have $\mathbf{n^2 - 2n + 2}$.

- Case 3: $2 \leq \rho < n - 1$
  - When $d = \rho - 2$, we have
$$\frac{1}{2}(\rho(\rho - 1)).$$
  - When $d = \rho + n - 1$, we have
$$\frac{1}{2}(n^2 - 2n\rho + n - 2\rho^2 - 4\rho - 2).$$
  - When $d = \rho + 2n$, we have
$$\frac{1}{2}(n^2 - 2n\rho - 5n + \rho^2 - 5\rho + 6)$$

  Summing the three terms, we have $\mathbf{n^2 - 2n + 2}$.

- Case 4: $\rho = n - 1$
  - When $d = \rho - 2 = n - 3$, we have
$$\frac{1}{2}(\rho(\rho - 1)) = \frac{1}{2}(n^2 - 3n + 2).$$
  - When $d = \rho + n - 1 = 2n - 2$, we have
$$\frac{1}{2}(n^2 - 2n\rho + n - 2\rho^2 - 4\rho - 2) = \frac{1}{2}(n^2 - n)$$
  - When $d = \rho + 2n = 3n - 1$, we have 0.

  Summing the three terms, we have $\mathbf{n^2 - 2n + 1}$.

- Case 5: $\rho = n$

– When $d = \rho - 2 = n - 2$, we have

$$\frac{1}{2}(\rho(\rho - 1)) = \frac{1}{2}(n^2 - n).$$

– When $d = \rho + n - 1 = 2n - 1$, we have

$$\frac{1}{2}(n^2 - 3n - 2) + 2$$

– When $d = \rho + 2n = 3n$, we have 0.

Summing the three terms, we have $\mathbf{n^2 - 2n + 1}$.

Recall that the upper bound for the elements of type 1 (monomials in x and y) with degree ρ modulo $n+1$ is $\max\{n-1, \rho+1\}$. Adding these to the upper bounds for the elements of type 2 (i.e., monomials of the form $x^a y^b w_i$) with degree ρ modulo $n+1$, we have the following upper bounds for the minimal number of generators of $W_{n,\rho}/\mathfrak{m}_{U_n} W_{n,\rho}$

- Case 1: $\rho = 0$

  $$\max\{n-1, \rho+1\} + n^2 - 2n + 2 = n - 1 + n^2 - 2n + 2 = \mathbf{n^2 - n + 1}.$$

- Case 2: $\rho = 1$

  $$\max\{n-1, \rho+1\} + n^2 - 2n + 2 = n - 1 + n^2 - 2n + 2 = \mathbf{n^2 - n + 1}$$

- Case 3: $2 \le \rho < n - 1$

  $$\max\{n-1, \rho+1\} + n^2 - 2n + 2 = n - 1 + n^2 - 2n + 2 = \mathbf{n^2 - n + 1}$$

- Case 4: $\rho = n - 1$

  $$\max\{n-1, \rho+1\} + n^2 - 2n + 1 = n + n^2 - 2n + 1 = \mathbf{n^2 - n + 1}$$

- Case 5: $\rho = n$

  $$\max\{n-1, \rho+1\} + n^2 - 2n + 1 = n + 1 + n^2 - 2n + 1 = \mathbf{n^2 - n + 2}.$$

But $n \geq 2$, and so $n^2 - n + 2 < n^2 + 1$ and $n^2 - n + 1 < n^2 + 1$. Thus, the $W_{n,\rho}$ are not Ulrich modules for U_n and consequently, they are not Ulrich modules for V_n.

□

Chapter 5

Lim Ulrich Sequences and Weakly Lim Ulrich Sequences for Domains of Dimension 2

5.1 (Weakly) lim Cohen–Macaulay and (weakly) lim Ulrich sequences over domains of dimension 2

Definition 5.1.1. Let $(R, \mathfrak{m}, k)$ be a local ring. Let $\mathcal{M} = \{M_n\}$ be a sequence of nonzero finitely generated R-modules. Let $\nu_R(M_n)$ be the minimal number of generators of M_n. Let $\{a_n\}$ and $\{b_n\}$ be a sequence of positive integers. We define $\sim_{\mathcal{M}}$ to be the equivalence relation $\sim_{\mathcal{M}}$ where $\{a_n\} \sim_{\mathcal{M}} \{b_n\}$ if

$$\lim_{n\to\infty} \frac{a_n - b_n}{\nu_R(M_n)} = 0.$$

For the sake of simplicity, we write $a_n \sim_{\mathcal{M}} b_n$ instead of $\{a_n\} \sim_{\mathcal{M}} \{b_n\}$.

Lemma 5.1.2. *Let $(R, \mathfrak{m}, k)$ be a local ring. Let $\mathcal{M} = \{M_n\}$ and $\mathcal{N} = \{N_n\}$ be two sequences of nonzero finitely generated R-modules. Let $\{a_n\}$ and $\{b_n\}$ be a sequence of non-negative integers. Suppose $\nu_R(N_n) \sim_{\mathcal{M}} \nu_R(M_n)$. If $a_n \sim_{\mathcal{M}} b_n$, then $a_n \sim_{\mathcal{N}} b_n$. In particular, $\nu_R(M_n) \sim_{\mathcal{N}} \nu_R(N_n)$.*

Theorem 5.1.3. *Let $(R, \mathfrak{m}, k)$ be a local domain of dimension 2. Let $\{M_n\}$ be a weakly lim Cohen–Macaulay (resp. weakly lim Ulrich) sequence over R. Let $C_n \subseteq M_n$ be a torsion submodule such that the quotient $\overline{M}_n \coloneqq M_n/C_n$ has no finite length submodules. Then the sequence $\{\overline{M}_n\}$ is a lim Cohen–Macaulay (resp. lim Ulrich) sequence over R.*

Proof. Let $\mathcal{M} \coloneqq \{M_n\}$ be a weakly lim Cohen–Macaulay sequence over R. Let $I = (\underline{x})$ be a system of parameters of the maximal ideal $\mathfrak{m}$ and let $\nu_R(M_n)$ be the minimal number of generators of M_n. First, we check that

$$\nu_R(M_n) \sim_{\mathcal{M}} \nu_R(\overline{M}_n).$$

Consider the short exact sequence

$$0 \to C_n \to M_n \to \overline{M}_n \to 0.$$

We know that $\nu_R(\overline{M}_n) \leq \nu_R(M_n) \leq \nu_R(\overline{M}_n) + \nu_R(C_n)$. So it suffices to show that

$$\nu_R(C_n) \sim_{\mathcal{M}} 0.$$

From the short exact sequence, we get the long exact sequence of Koszul homology

$$
\begin{aligned}
&0 \to H_2(\underline{x}; C_n) \to H_2(\underline{x}; M_n) \to H_2(\underline{x}; \overline{M}_n) \\
&\quad \to H_1(\underline{x}; C_n) \to H_1(\underline{x}; M_n) \to H_1(\underline{x}; \overline{M}_n) \to H_0(\underline{x}; C_n) \to H_0(\underline{x}; M_n) \to H_0(\underline{x}; \overline{M}_n) \to 0.
\end{aligned}
$$

Now $\overline{M}_n$ has no finite length torsion submodules, so $H_2(\underline{x}; \overline{M}_n) = 0$. We observe the following:

(a) $H_2(\underline{x}; C_n) \cong H_2(\underline{x}; M_n)$

(b) $h_1(\underline{x}; C_n) \leq h_1(\underline{x}; M_n)$

(c) $\chi_1(\underline{x}; M) \geq 0$ for any finitely generated R-module M [S]

(d) $\chi(\underline{x}; C_n) = 0$

(e) $0 \leq h_0(\underline{x}; M_n) - h_0(\underline{x}; \overline{M}_n) \leq h_0(\underline{x}; C_n)$

From (a), (b), and (c), it follows that

$$0 \leq \chi_1(\underline{x}; C_n) \leq \chi_1(\underline{x}; M_n).$$

and because $\mathcal{M}$ is weakly lim Cohen–Macaulay, we have

$$\chi_1(\underline{x}; C_n) \sim_{\mathcal{M}} 0. \tag{5.1}$$

But $\chi(\underline{x}; C_n) = 0$ and so, $\chi_1(\underline{x}; C_n) = h_0(\underline{x}; C_n) = \ell(C_n/(\underline{x})C_n)$. Then the inequality

$$\nu_R(C_n) = \ell(C_n/\mathfrak{m}C_n) \leq \ell(C_n/(\underline{x})C_n)$$

yields

$$\nu_R(C_n) \sim_{\mathcal{M}} 0.$$

Next, we show that $\{\overline{M}_n\}$ is a lim Cohen-Macaulay sequence over R. We already know that $h_2(\underline{x};\overline{M}_n) = 0$. It remains to show

$$\lim_{n\to\infty} \frac{h_1(\underline{x};\overline{M}_n)}{\nu_R(\overline{M}_n)} = 0.$$

By Lemma 5.1.2, it is enough to show that

$$\lim_{n\to\infty} \frac{h_1(\underline{x};\overline{M}_n)}{\nu_R(M_n)} = 0.$$

because $\nu_R(\overline{M}_n) \sim_{\mathcal{M}} \nu_R(M_n)$. Take the alternating sum of the lengths of the Koszul homology in the exact sequence

$$0 \to H_1(\underline{x};C_n) \to H_1(\underline{x};M_n) \to H_1(\underline{x};\overline{M}_n) \to H_0(\underline{x};C_n) \to H_0(\underline{x};M_n) \to H_0(\underline{x};\overline{M}_n) \to 0.$$

This is the sum

$$h_1(\underline{x};C_n) - h_1(\underline{x};M_n) + h_1(\underline{x};\overline{M}_n) - h_0(\underline{x};C_n) + h_0(\underline{x};M_n) - h_0(\underline{x};\overline{M}_n) = 0.$$

Then

$$\begin{aligned} h_1(\underline{x}; \overline{M}_n) &= -h_1(\underline{x}; C_n) + h_0(\underline{x}; C_n) + h_1(\underline{x}; M_n) - h_0(\underline{x}; M_n) + h_0(\underline{x}; \overline{M}_n) \\ &= -h_2(\underline{x}; C_n) + h_1(\underline{x}; M_n) - h_0(\underline{x}; M_n) + h_0(\underline{x}; \overline{M}_n) \\ &= -h_2(\underline{x}; M_n) + h_1(\underline{x}; M_n) - h_0(\underline{x}; M_n) + h_0(\underline{x}; \overline{M}_n) \\ &= \chi_1(\underline{x}; M_n) - (h_0(\underline{x}; M_n) - h_0(\underline{x}; \overline{M}_n)). \end{aligned}$$

Now we know that

$$\chi_1(\underline{x}; M_n) \sim_{\mathcal{M}} 0$$

and by (e) and 5.1 above, we have

$$0 \leq h_0(\underline{x}; M_n) - h_0(\underline{x}; \overline{M}_n) \leq h_0(\underline{x}; C_n) = \chi_1(\underline{x}; C_n) \sim_{\mathcal{M}} 0.$$

Thus

$$\lim_{n\to\infty} \frac{h_1(\underline{x}; \overline{M}_n)}{\nu_R(M_n)} = 0$$

and the sequence $\{\overline{M}_n\}$ is lim Cohen–Macaulay. It remains to check that

$$\lim_{n\to\infty} \frac{e_R(\overline{M}_n)}{\nu_R(\overline{M}_n)} = 1.$$

But $e_R(\overline{M}_n) = e_R(M_n)$ and $\nu_R(\overline{M}_n) \sim_{\mathcal{M}} \nu_R(M_n)$, so the condition immediately follows and thus $\{\overline{M}_n\}$ is a lim Ulrich sequence of R. □

Definition 5.1.4. Let $(R, \mathfrak{m}, k)$ be a local domain and let M be finitely generated torsion-free R-module. Let $(S, \mathfrak{n}, \ell)$ be a local module-finite extension domain of R. Suppose $\mathcal{K} = frac(R) = frac(S)$. Then we define MS to be the S-module generated by

M in $M \otimes_R \mathcal{K}$.

Remark 5.1.5. In the case where R is a local domain with a S_2-ification S that is local, if M is an MCM module of R, then $MS = M$ by Lemma 3.1.1.

Lemma 5.1.6. *Let $(R, \mathfrak{m}, k)$ be a local domain of dimension 2 and let M be a finitely generated torsion-free R-module. Let $(S, \mathfrak{n}, \ell)$ be a local module-finite extension domain of R. Suppose $S \subseteq$ frac(R) and S/R has finite length. Choose a fixed constant t such that $\mathfrak{m}^t S \subseteq R$. Let x, y be a system of parameters for R. Then*

(a) $MS \subseteq M :_{\mathcal{K} \otimes_R M} (x^t, y^t)R$,

(b) $(M :_{M \otimes_R \mathcal{K}} (x^t, y^t))/M \cong H_1(x^t, y^t; M)$,

(c) $\ell(MS/M) \leq h_1(x^t, y^t; M)$.

Proof. Part (a) is clear by the choice of t. Part(c) follows immediately from parts (a) and (b). It remains to prove part (b). Define

$$\varphi : H_1(x^t, y^t; M) \to (M :_{M \otimes_R \mathcal{K}} (x^t, y^t))/M$$

to be the map

$$[(u, v)] \mapsto \left[\frac{u}{y^t}\right] = \left[\frac{-v}{x^t}\right]$$

where the equality follows from the relation $ux^t + vy^t = 0$. This map is well-defined. If $[(u, v)]$ is trivial, then there exists $w \in M$ such that $[(u, v)] = [y^t w, -x^t w]$. But $[y^t w, -x^t w]$ is mapped to $[(y^t w)/y^t] = [w/1] = 0$.

For the map going the other direction, define

$$\psi : (M :_{M \otimes_R \mathcal{K}} (x^t, y^t))/M \to H_1(x^t, y^t; M)$$

to be the map

$$[f] \mapsto [(y^t f, -x^t f)].$$

This is clearly well-defined. The maps φ and ψ are inverses, so we are done.

□

Theorem 5.1.7. *Let $(R, \mathfrak{m}, k)$ be a local domain of dimension 2. Let $(S, \mathfrak{n}, \ell)$ be a local module–finite extension domain of R such that $S \subseteq frac(R)$ and S/R has finite length. Let $\mathcal{M} = \{M_n\}$ be a lim Cohen–Macaulay (resp. lim Ulrich) sequence of torsion-free R-modules. Then the sequence $\mathcal{N} = \{M_n S\}$ is a lim Cohen–Macaulay (resp. lim Ulrich) sequence of R-modules and also a lim Cohen–Macaulay sequence of S-modules.*

Proof. We first prove that

$$\nu_R(M_n) \sim_{\mathcal{M}} \nu_R(M_n S).$$

Let $Q_n = M_n S/M_n$. Note that Q_n has finite length because S/R has finite length. The short exact sequence

$$0 \to M_n \to M_n S \to Q_n \to 0$$

yields the long exact sequence

$$\ldots \to \operatorname{Tor}_1^R(Q_n, k) \to M_n \otimes_R k \to M_n S \otimes_R k \to Q_n \otimes_R k \to 0.$$

Then

$$\nu_R(M_n) \le \nu_R(M_n S) + \ell(\operatorname{Tor}_1^R(Q_n, k)) \le \nu_R(M_n) + \nu_R(Q_n) + \ell(\operatorname{Tor}_1^R(Q_n, k)).$$

and so it suffices to show that

$$\ell(\mathrm{Tor}_1^R(Q_n, k)) \sim_{\mathcal{M}} 0 \quad \text{and} \quad \nu_R(Q_n) \sim_{\mathcal{M}} 0.$$

Let $\underline{x} = x_1, x_2$ be a system of parameters for R. By Lemma 5.1.6, we know that $\ell(Q_n) \leq h_1(x_1^t, x_2^t; M_n)$ for some fixed t. But $\mathcal{M}$ is a lim Cohen-Macaulay sequence, so $h_1(x_1^t, x_2^t; M_n) \sim_{\mathcal{M}} 0$ and

$$\ell(Q_n) \sim_{\mathcal{M}} 0.$$

Then

$$\nu_R(Q_n) = \ell(Q_n/\mathfrak{m}Q_n) \sim_{\mathcal{M}} 0.$$

Next, by taking a prime cyclic filtration of Q_n, one can observe that

$$\ell(\mathrm{Tor}_1^R(Q_n, k)) \leq \ell(Q)\ell(\mathrm{Tor}_1^R(k, k)).$$

Then it immediately follows that

$$\ell(\mathrm{Tor}_1^R(Q_n, k)) \sim_{\mathcal{M}} 0.$$

We now show that $\mathcal{N} = \{M_n S\}$ is a lim Cohen–Macaulay sequence of R-modules. It is enough to show that

$$h_1(\underline{x}; M_n S) \sim_{\mathcal{M}} 0.$$

Because M_n and M_nS are torsion-free over R, we have the long exact sequence

$$0 \to H_2(\underline{x}; Q_n) \to H_1(\underline{x}; M_n) \to H_1(\underline{x}; M_nS) \to H_1(\underline{x}; Q_n)$$
$$\to H_0(\underline{x}; M_n) \to H_0(\underline{x}; M_nS) \to H_0(\underline{x}; Q_n) \to 0.$$

Observe that for all $i \geq 0$

$$h_i(\underline{x}; Q_n) \sim_{\mathcal{M}} 0.$$

We see that

$$h_2(\underline{x}; Q_n) \sim_{\mathcal{M}} 0$$

because $H_2(\underline{x}; Q_n)$ injects into $H_1(\underline{x}; M_n)$. We already proved that $\ell(Q_n) \sim_{\mathcal{M}} 0$. It immediately follows that

$$h_0(\underline{x}; Q_n) = \ell(Q_n/\underline{x}Q_n) \sim_{\mathcal{M}} 0.$$

Then, it follows from $\chi(\underline{x}; Q_n) = 0$ that

$$h_1(\underline{x}; Q_n) \sim_{\mathcal{M}} 0.$$

From the long exact sequence on Koszul homology, we have

$$h_1(\underline{x}; M_nS) \leq h_1(\underline{x}; M_n) + h_1(\underline{x}; Q_n).$$

But $h_1(\underline{x}; M_n) \sim_{\mathcal{M}} 0$ and $h_1(\underline{x}; Q_n) \sim_{\mathcal{M}} 0$. Therefore, $\mathcal{N} = \{M_nS\}$ is a lim Cohen–Macaulay sequence over R.

If $\mathcal{M}$ is lim Ulrich, it immediately follows that $\mathcal{N}$ is lim Ulrich because

$e_R(M_n) = e_R(M_nS)$ and $\nu_R(M_n) \sim_{\mathcal{M}} \nu_R(M_nS)$. It remains to check that $\mathcal{N} = \{M_nS\}$ is a lim Cohen–Macaulay sequence for S.

For any i, the Koszul homology $H_i(\underline{x}; M_nS)$ does not change for whether we think of M_nS as an R-module or an S-module. We also have

$$\nu_R(M_nS) \leq \nu_R(S)\nu_S(M_nS),$$

which yields

$$\frac{\nu_R(M_nS)}{\nu_R(S)} \leq \nu_S(M_nS).$$

Then

$$\lim_{n\to\infty} \frac{h_1^S(\underline{x}; M_nS)}{\nu_S(M_nS)} \leq \lim_{n\to\infty} \frac{\nu_R(S)h_1^R(\underline{x}; M_nS)}{\nu_R(M_nS)} = \nu_R(S) \lim_{n\to\infty} \frac{h_1^R(\underline{x}; M_nS)}{\nu_R(M_nS)} = 0.$$

Thus $\mathcal{N} = \{M_nS\}$ is a lim Cohen-Macaulay sequence over S.

□

Theorem 5.1.8. *A sequence of finitely generated nonzero torsion–free modules* $\{N_n\}$ *over a regular local ring* S *of dimension* 2 *is lim Cohen–Macaulay if and only if for the minimal free resolution*

$$0 \to S^{b_n} \to S^{a_n} \to N_n \to 0$$

we have $\lim_{n\to\infty} b_n/a_n = 0$. *Such a sequence is always lim Ulrich.*

Proof. Let x, y be a regular system of parameters for S. We have

$$a_n = \nu_S(N_n) = h_0(x, y; N_n)$$

and

$$b_n = h_1(x, y; N_n).$$

Then $\{N_n\}$ is lim Cohen–Macaulay over S if and only if

$$\lim_{n\to\infty} \frac{h_1(x, y; N_n)}{\nu_S(N_n)} = \lim_{n\to\infty} \frac{b_n}{a_n} = 0.$$

Moreover, we have

$$\lim_{n\to\infty} \frac{e_S(N_n)}{\nu_S(N_n)} = \lim_{n\to\infty} \frac{a_n - b_n}{a_n} = 1.$$

Thus $\{N_n\}$ is a lim Ulrich sequence for S. □

5.2 Weakly lim Ulrich sequences do not always exist for local domains

Theorem 5.2.1. *Let $(R, \mathfrak{m}, k)$ be a local domain of dimension 2. Suppose R has an S_2-ification S that is a regular local ring. The following are equivalent:*

(a) R has a weakly lim Ulrich sequence.

(b) R has an Ulrich module.

(c) S is an Ulrich module of R.

(d) For any minimal reduction I of $\mathfrak{m}$, we have $IS = \mathfrak{m}S$.

Proof. First, (c) and (d) are equivalent by definition. Next (b) and (c) are equivalent by Theorem 3.1.2. It is clear that (b) implies (a). It remains to show that (a) implies (b).

Suppose R has a weakly lim Ulrich sequence. Then by Theorems 5.1.3 and 5.1.7, there exists a lim Ulrich sequence $\mathcal{M} = \{M_n\}$ of torsion-free R modules that are also S modules. Consider the minimal free resolution

$$0 \to S^{b_n} \to S^{a_n} \to M_n \to 0$$

where $a_n = \nu_S(M_n)$. Now

$$\nu_R(S^{b_n}) = b_n \nu_R(S)$$

and

$$e_R(S^{b_n}) = b_n e_R(S).$$

Then Theorem 5.1.8 yields

$$\lim_{n\to\infty} \frac{\nu_R(S^{b_n})}{\nu_R(M_n)} = \lim_{n\to\infty} \frac{\nu_R(S)b_n}{\nu_R(M_n)} \leq \lim_{n\to\infty} \frac{\nu_R(S)b_n}{\nu_S(M_n)} = \lim_{n\to\infty} \frac{\nu_R(S)b_n}{a_n} = 0,$$

and

$$\lim_{n\to\infty} \frac{e_R(S^{b_n})}{\nu_R(M_n)} = \lim_{n\to\infty} \frac{e_R(S)b_n}{\nu_R(M_n)} \leq \lim_{n\to\infty} \frac{e_R(S)b_n}{\nu_S(M_n)} = \lim_{n\to\infty} \frac{e_R(S)b_n}{a_n} = 0.$$

Consequently, by the minimal free resolution above, we have

$$\nu_R(M_n) \sim_{\mathcal{M}} \nu_R(S^{a_n}) = \nu_R(S)a_n, \tag{5.2}$$

and

$$e_R(M_n) \sim_{\mathcal{M}} e_R(S^{a_n}) = e_R(S)a_n. \tag{5.3}$$

Combining equivalences 5.2 and 5.3, we have

$$\lim_{n\to\infty}\frac{e_R(S)}{v_R(S)} = \lim_{n\to\infty}\frac{e_R(M_n)}{v_R(M_n)} = \frac{e_R(S)}{v_R(S)} = 1.$$

Thus S is an Ulrich module of R. □

Theorem 5.2.2. *Weakly lim Ulrich sequences do not always exist for (complete) local domains.*

Proof. This is immediate by Theorem 3.1.3 and Theorem 5.2.1. □

Corollary 5.2.3 (Localization)**.** *Weakly lim Ulrich sequences do not always localize for local domains. More precisely, there exist local domains $(R, \mathfrak{m}, k)$ that have a weakly lim Ulrich sequence $\{M_n\}$ and a prime ideal $\mathfrak{p}$ such that $\{(M_n)_\mathfrak{p}\}$ is not a weakly lim Ulrich sequence for $R_\mathfrak{p}$. Moreover, there exist local domains that have weakly lim Ulrich sequences and a prime ideal $\mathfrak{p}$ such that $R_\mathfrak{p}$ has no weakly lim Ulrich sequences.*

Proof. This is immediate by taking k to be perfect and $\text{char}(k) > 0$ in Counterexample 3.1.4 and applying Theorem 5.2.1. □

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