UMM-82 = RL-2012

Studies in Radar Cross-Sections-II

The Zeros of the Associated Legendre Functions $P^m_n(\mu')$ of Non-Integral Degree

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Project MX-794
USAF Contract W33-038-ac-14222
April 1951
2nd Printing, November 1953

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University of Michigan
UMM-82.

TABLE OF CONTENTS

PREFACE

For several years the Willow Run Research Center has been interested in calculating the radar cross-sections of many shapes. Many approximate answers to composite bodies have been obtained in utilizing the methods of physical and geometric optics due to R. C. Spencer (Ref. 1, 2, 3). In the case of a prolate spheroid, answers have been obtained by four different methods: electromagnetic theory, scalar waves, physical optics and geometric optics (Ref. 4). The radar cross-section of an ogive is of particular interest because it is a typical missile shape.

Theoretically, it was expected that the radar cross-section of an ogive would approximate that of the ogive's tangent cone. Much analysis has been applied to this problem since 1946. On September 30, 1948, Hansen and Schiff presented an analytical theory for the scattering by a semi-infinite cone (Ref. 5). When put to use, this theory requires numerical solutions to certain functional equations.

Carrus and Treuenfels at Massachusetts Institute of Technology set out to compute solutions to these functional equations. Their methods of attack and recorded values appeared in an unpublished Cambridge Research Laboratory report entitled "Tables of Roots and Incomplete Integrals of Associated Legendre Functions of Fractional Orders." These values were then used by the authors and other investigators to obtain numerical solutions to this and other scattering problems. This Carrus and Treuenfels report was recently published (Ref. 6).

A review has been made of our scattering research program, and an attempt was made to find out what additional information was available in this field. This investigation showed that both theoretical and computational errors have been made in the published solutions to the cone problem. These computational errors originated in the Carrus and Treuenfels report and were copied by other investigators.

This report presents a new method of calculating some of the values of interest. In the process some limits of mathematical interest are obtained.

ABSTRACT

This paper derives (by a new method) an equation due to Macdonald for determining the zeros of the associated Legendre functions of order m and non-integral degree n when the argument is close to -1 (Ref. 7). A closed form solution is obtained for the values of $Q_n^m(\mu)$ and $Q_n^{-m}(\mu)$ for μ close to 1. Certain observations are made concerning errors in a recently published article (Ref. 6).

NOMENCLATURE

$P_n^{-m}(\mu)$	= the associated Legendre function of the first kind of degree n, order -m and argument μ.
$Q_n^{-m}(\mu)$	= the associated Legendre function of the second kind of degree n, order -m and argument μ.
θ	= usual polar coordinate angle
μ'	$\equiv \cos(\pi - \phi) \equiv \cos\theta$
μ	$\equiv \cos\phi$
ϕ	= an angle such that $0^\circ < \phi \leq 15^\circ$
$F(\alpha, \beta; \gamma; z)$	= Riemann hypergeometric function
II	= Gaussian operator
II (x)	= x! for positive integral x
II (x)	$= \Gamma(x+1)$ where $\Gamma(x+1)$ is the well known Gamma function
n	= a non-integral real number
m	= a real number
$\simeq$	= approximately equals
EC	= the essential contribution of the function as the variable μ approaches a value close to 1. μ differs from 1 by $(1 - \cos\phi)$. This latter value is at most 0.03407 for $\phi = 15^\circ$.

INTRODUCTION

In Part I are stated well-known equations involving the properties of spherical harmonics and the Riemann hypergeometric function.

In Part II we use these equations to derive the values of $Q_n^{m}(\mu)$ and $Q_n^{-m}(\mu)$ when μ is close to 1.

In Part III we derive, by a new method, a simple formula due to Macdonald for determining the zeros of either $P_n^{-m}(\mu')$ or $P_n^{m}(\mu')$ when ϕ is sufficiently small, namely:

$$n \simeq m + k + \frac{\Pi\,(2m + k)}{\Pi\,(m)\;\;\Pi\,(m - 1)\;\Pi\,(k)} \tan^{2m}(\phi/2)$$

In Part IV we analyze the values obtained in reference 6 and show that some of these values disagree significantly with the exact values.

PART I

FORMULAS

For m > 0:

$$P_n^{-m}(\mu) = \frac{1}{\Pi(m)} \left(\frac{1-\mu}{1+\mu}\right)^{m/2} F\left(-n, n+1; 1+m; \frac{1-\mu}{2}\right) \tag{1}$$

(Ref. 8, p. 404)

For $m \neq 1, 2, 3$, etc.:

$$P_n^{m}(\mu) = \frac{1}{\Pi(-m)} \left(\frac{1+\mu}{1-\mu}\right)^{m/2} F\left(-n, n+1; 1-m; \frac{1-\mu}{2}\right) \tag{2}$$

(Ref. 8, p. 386)

For m = 0, 1, 2, 3, etc.:

$$P_n^{m}(\mu) = \frac{1}{2^m \Pi(m)} \frac{\Pi(n+m)}{\Pi(n-m)} (1-\mu^2)^{m/2} F\left(m-n, n+m+1; 1+m; \frac{1-\mu}{2}\right) \tag{3}$$

(Ref. 8, p. 386)

For m an integer:

$$P_n^{m}(\mu) = \frac{\Pi(n+m)}{\Pi(n-m)} P_n^{-m}(\mu) \tag{4a}$$

(Ref. 8, p. 205)

and

$$Q_n^{m}(\mu) = \frac{\Pi(n+m)}{\Pi(n-m)} Q_n^{-m}(\mu) \tag{4b}$$

(Ref. 8, p. 196)

Combining (3) and (4a) we obtain:

$$P_n^{-m}(\mu) = \frac{1}{2^m \Pi(m)} (1-\mu^2)^{m/2} F\left(m-n, n+m+1; 1+m; \frac{1-\mu}{2}\right) \tag{5}$$

$$P_n^m(-\mu) = \cos(n+m)\pi \; P_n^m(\mu) - \frac{2\sin(n+m)\pi}{\pi} Q_n^m(\mu) \qquad (6)$$

(Ref. 8, p. 407)

$$Q_n^m(\mu) = \frac{\pi}{2\sin(m\pi)} \left\{ P_n^m(\mu)\cos m\pi - \frac{\Pi(n+m)}{\Pi(n-m)} P_n^{-m}(\mu) \right\} \qquad (6a)$$

(Ref. 8, p. 230)

$$F(a, b; c; z) = 1 + \frac{a\,b}{c}\frac{z}{1\,!} + \frac{a(a+1)\,b(b+1)}{c(c+1)}\frac{z^2}{2!} + \ldots \qquad (7)$$

(Ref. 9, p. 7)

$$\lim_{z \to 0} F(a, b; c; z) = 1 = E\,C\;\; F(a, b; c; 1-\mu) \qquad (8)$$

If $a + b - c < 0$

$$\lim_{\mu \to 1} F(a, b; c; \mu) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} = \frac{\Pi(c-1)\,\Pi(c-a-b-1)}{\Pi(c-a-1)\,\Pi(c-b-1)} \qquad (9)$$

(Ref. 9, p.8)

$$= E\,C\;\; F(a, b; c; \mu)$$

For $m > 0$, from (1) and (8):

$$EC\;\; P_n^{-m}(\mu) = \frac{1}{\Pi(m)} \left(\frac{1-\mu}{1+\mu}\right)^{m/2} \qquad (10)$$

For $m \neq 1, 2, 3$, etc., from (2) and (8)

$$EC\;\; P_n^{m}(\mu) = \frac{1}{\Pi(-m)} \left(\frac{1+\mu}{1-\mu}\right)^{m/2} \qquad (10a)$$

For $m = 0, 1, 2, 3$, etc., from (3) and (8)

$$EC\;\; P_n^{m}(\mu) = \frac{1}{2^m\,\Pi(m)} \frac{\Pi(n+m)}{\Pi(n-m)} (1-\mu^2)^{m/2} \qquad (11)$$

For m = 0, 1, 2, 3, etc., from (5) and (8):

$$\text{EC} \quad P_n^{-m}(\mu) = \frac{1}{2^m \, \Pi(m)} (1 - \mu^2)^{m/2} \tag{12}$$

$$\sin \pi z = \frac{\pi}{\Gamma(z) \; \Gamma(1 - z)} = \frac{\pi}{\Pi(z - 1) \; \Pi(- z)} \tag{13}$$

(Ref. 9, p. 1)

PART II

FORMULAS FOR $Q_n^{-m}(\mu)$ AND $Q_n^{m}(\mu)$ DERIVED WHEN m IS AN INTEGER, n IS NOT AN INTEGER AND μ IS CLOSE TO 1

In this section, m will be considered an integer.

Replacing m by -m in (6) we obtain:

$$P_n^{-m}(-\mu) = \cos(n-m)\pi \; P_n^{-m}(\mu) - \frac{2\sin(n-m)\pi}{\pi} Q_n^{-m}(\mu) \qquad (14)^*$$

Simplifying,

$$P_n^{-m}(-\mu) = (-1)^m \cos n\pi \; P_n^{-m}(\mu) - \frac{2(-1)^m \sin n\pi}{\pi} \; Q_n^{-m}(\mu) \qquad (15)$$

Replacing μ by $-\mu$ in equation (1) we obtain:

$$P_n^{-m}(-\mu) = \frac{1}{\Pi(m)}\left(\frac{1+\mu}{1-\mu}\right)^{m/2} F\left(-n, n+1; 1+m; \frac{1+\mu}{2}\right) \qquad (16)$$

Using equation (9)

$$\text{EC} \;\; P_n^{-m}(-\mu) = \left(\frac{1+\mu}{1-\mu}\right)^{m/2} \frac{\Pi(m-1)}{\Pi(m+n)\,\Pi(m-n-1)} \qquad (17)$$

Combining (15, (17) and (10) we obtain

$$\text{EC} \; Q_n^{-m}(\mu) = \frac{\dfrac{\Pi(m-1)\left(\dfrac{1+\mu}{1-\mu}\right)^{m/2}}{\Pi(m+n)\,\Pi(m-n-1)}}{\dfrac{2}{\pi}(-1)^{m+1}\sin n\pi} + \frac{\dfrac{\cos n\pi}{\Pi(m)}\left(\dfrac{1-\mu}{1+\mu}\right)^{m/2}}{\dfrac{2}{\pi}\sin n\pi} \qquad (18)$$

*This equation appears incorrectly in reference 7.

We observe that the second term is small compared to the first term.

Therefore, $$Q_n^{-m}(\mu) \simeq \frac{\pi\ \Pi(m-1)\ \cot^m(\phi/2)}{2\ \Pi(m+n)\ \Pi(m-n-1)\ (-1)^{m+1} \sin n\pi} \tag{19}$$

Using the equation (13), equation (19) becomes

$$Q_n^{-m}(\mu) \simeq \frac{(-1)^{m+1}\ \Pi(m-1)\ \Pi(n-1)\ \Pi(-n)}{2\ \Pi(m+n)\ \Pi(m-n-1)} \cot^m(\phi/2) \tag{20}$$

Now using (4b) we have

$$Q_n^{m}(\mu) \simeq \frac{(-1)^{m+1}\ \Pi(m-1)\ \Pi(n-1)\ \Pi(-n)}{2\ \Pi(n-m)\ \Pi(m-n-1)} \cot^m(\phi/2) \tag{21}$$

Noting from (13) that

$$\sin(m-n)\pi = \frac{\pi}{\Pi(m-n-1)\ \Pi(n-m)} = (-1)^{m+1} \sin(n\pi)$$

and $\sin n\pi = \dfrac{\pi}{\Pi(n-1)\ \Pi(-n)}$

then $$Q_n^{m}(\mu) \simeq \frac{(-1)^{m+1}\ \Pi(m-1)\ \sin(m-n)\pi}{2\sin(n\pi)} \cot^m(\phi/2) \tag{22}$$

and finally $$\boxed{Q_n^{m}(\mu) \simeq \frac{\Pi(m-1)}{2} \cot^m(\phi/2)} \tag{23}$$

By using equation (4b)

$$\boxed{Q_n^{-m}(\mu) \simeq \frac{\Pi(n-m)\ \Pi(m-1)}{2\ \Pi(n+m)} \cot^m(\phi/2)} \tag{24}$$

PART III

A. THE ZEROS OF $P_n^{-m}(\mu')$ WHEN μ' IS CLOSE TO - 1 AND m IS AN INTEGER

Rewriting equation (14) and replacing $-\mu$ by μ' we obtain

$$P_n^{-m}(-\mu) = P_n^{-m}(\mu') = \cos(n-m)\pi \, P_n^{-m}(\mu) - \frac{2}{\pi}\sin(n-m)\pi \, Q_n^{-m}(\mu) \tag{14}$$

We wish to find the values of n such that

$$P_n^{-m}(\mu') = 0$$

Upon division, equation (14) becomes

$$\tan(n-m)\pi = \frac{\pi}{2}\frac{P_n^{-m}(\mu)}{Q_n^{-m}(\mu)} \tag{25}$$

To find the value of the above expression when μ' is close to - 1, (i.e., μ close to 1) and when m is an integer, we may make use of equations (10) and (24), yielding:

$$\tan(n-m)\pi \simeq \frac{\pi \dfrac{1}{\Pi(m)} \left(\dfrac{1-\mu}{1+\mu}\right)^{m/2}}{\dfrac{\Pi(n-m)\,\Pi(m-1)\,\cot^m(\phi/2)}{\Pi(n+m)}} \tag{26}$$

and

$$n \simeq m + k + \frac{1}{\pi}\arctan\left\{\frac{\pi\,\Pi(n+m)\,\tan^{2m}(\phi/2)}{\Pi(n-m)\,\Pi(m)\,\Pi(m-1)}\right\} \tag{27}$$

where k = 0, 1, 2, 3, etc.

But since the argument of the arctangent is small, we may write

$$n \simeq m + k \tag{28a}$$

and:

$$\arctan (x) \simeq x \tag{28b}$$

Using (28a) and (28b), equation (27) now becomes:

$$n \simeq m + k + \frac{\Pi\,(2m + k)\ \tan^{2m}(\phi/2)}{\Pi\,(k)\ \Pi\,(m - 1)\ \Pi\,(m)} \tag{29}$$

Equation (29) is Macdonald's formula as derived here for **m** an integer.

We will now follow references 7 and 8 in deriving Macdonald's formula for non-integral m.

B. THE ZEROS OF $P_n^{-m}(\mu')$ WHEN μ' IS CLOSE TO-1 AND m IS NOT AN INTEGER

Using equations (6a) and (14) we obtain

$$\tan (n - m)\pi = \frac{\sin m\pi\ P_n^{-m}(\mu)}{\frac{\Pi\,(n - m)}{\Pi\,(n + m)}\, P_n^{m}(\mu) - \cos m\pi\ P_n^{-m}(\mu)} \tag{30*}$$

When μ is close to 1, by using equations (10) and (10a), equation (30) becomes:

$$\tan (n - m)\pi \simeq \frac{\frac{\sin m\pi}{\Pi\,(m)}\left(\frac{1-\mu}{1+\mu}\right)^{m/2}}{\frac{\Pi\,(n - m)}{\Pi\,(n + m)\ \Pi\,(-m)}\left(\frac{1+\mu}{1-\mu}\right)^{m/2} - \frac{\cos m\pi}{\Pi\,(m)}\left(\frac{1-\mu}{1+\mu}\right)^{m/2}} \tag{31}$$

*It should be pointed out that equation (30) appears incorrectly in reference 8. Our μ appears as $-\mu$ in this reference.

The second term in the denominator is small while the first term is large. Thus,

$$\tan (n - m)\pi \simeq \frac{\sin m\pi \; \Pi (n + m) \; \Pi (-m)}{\Pi (m) \; \Pi (n - m)} \tan^{2m} (\phi/2) \tag{32}$$

Using equation (13) then

$$\tan (n - m)\pi \simeq \frac{\pi \, \Pi (n + m) \; \tan^{2m}(\phi/2)}{\Pi (m - 1) \; \Pi (m) \; \Pi (n - m)} \tag{33}$$

and

$$n \simeq m + k + \frac{1}{\pi} \quad \arctan \left\{ \frac{\pi \, \Pi (m + n) \; \tan^{2m}(\phi/2)}{\Pi (m - 1) \; \Pi (m) \; \Pi (n - m)} \right\} \tag{34}$$

But the arctangent term is small when ϕ is small, therefore

$$n \simeq m + k \tag{35a}$$

$$\arctan (x) \simeq x \tag{35b}$$

and finally

$$n \simeq m + k + \frac{\Pi (2m + k) \; \tan^{2m}(\phi/2)}{\Pi (m - 1) \; \Pi (m) \; \Pi (k)} \tag{36}$$

Thus we have derived Macdonald's formula (without the use of Lagrange's theorem as suggested by references 7 and 8) for all m.

It should be pointed out that <u>for integral m</u>

$$P_n^m (\mu') = \frac{\Pi (m + n)}{\Pi (n - m)} P_n^{-m} (\mu') \tag{4a}$$

and that since $\frac{\Pi (m + n)}{\Pi (n - m)}$ has no zeros that <u>the zeros of $P_n^m (\mu')$ coincide with the zeros of $P_n^{-m} (\mu')$.</u>

PART IV

APPLICATION OF MACDONALD'S FORMULA

Macdonald's formula,

$$n \simeq m + k + \frac{\Pi\ (2m + k)}{\Pi\ (m)\ \Pi\ (m - 1)\ \Pi\ (k)} \tan^{2m} (\phi/2) \qquad (29)$$

as we have seen, is approximate.

It is of major interest in the back scattering from a cone to find the zeros of $P_n^1\ (\mu')$. We will use a cone angle of 30^o as the greatest angle to be treated by equation (29). If we analyze the case of axially symmetric back scattering, the angle ϕ will be one-half the cone angle.

Since the zeros of the associated Legendre functions for positive and negative integer m's coincide, we may use equation (29) to determine the zeros of either $P_n^{-m}\ (\mu')$ or $P_n^m\ (\mu')$.

The zeros of $P_n^1\ (\mu')$ for a 30^o cone are given by (29) as

$$n \simeq 1 + k + (k + 2)\ (k + 1)\ \tan^2 (7.5^o) \qquad (38)$$

where k = 0, 1, 2, 3, etc.

The first zero would occur when k = 0. Thus by equation (38) we have

$$n_1 \simeq 1.03466 \qquad (39)$$

Reference 6 states this value to be

$$n_1 \simeq 1.053 \qquad (40)$$

The exact value of n_1 lies between 1.0321 and 1.0316.

$$1.0321 > n_1 > 1.0316 \qquad (41)$$

One of these values was obtained by summing the first 50 terms of the hypergeometric series associated with $P^{-1}_{1.0321}$ (cos 165°) and comparing the remainder with two geometric progressions. One of these progressions was larger than the remainder, and one was less than the remainder. In this way we found for n = 1.0321

$$- 0.000043 > F\left(- 1.0321, 2.0321; 2; \frac{1 - \cos 165^\circ}{2}\right) > - 0.00027 \qquad (42)$$

We also computed the hypergeometric series for n = 1.0316 for 60 terms before summing and found that

$$+ 0.00000369 < F\left(- 1.0316, 2.0316; 2; \frac{1 - \cos 165^\circ}{2}\right) < 0.000125 \qquad (43)$$

Since we have established a change in sign, we have proven a root exists between n = 1.0321 and n = 1.0316. This is the way in which equation (41) was obtained (for details of this computation see Appendix).

Comparing equations (39), (40) and (41) we observe that Macdonald's formula yields a better result for the first zero than the Carrus and Treuenfels report.

It should be pointed out that Macdonald's formula cannot be used to find all the zeros, even for angles between $\frac{11}{12}\pi$ and π. This arises from the fact that in the derivation we replaced arctan x by x; this is only a good approximation when $x^2 \ll 1$.

That is,

$$\pi \frac{\Pi\,(2m + k) \tan^{2m} (\phi/2)}{\Pi\,(m)\ \Pi\,(m - 1)\ \Pi\,(k)} << 1$$

Working with m = 1 as previously, we find that

$$(k + 2)\ (k + 1) \ll \frac{1}{\pi \tan^2 (\phi/2)} \qquad (44)$$

For the case previously under consideration, i.e., $\phi = 15^\circ$,

$$(k + 2)\ (k + 1) \ll 18.37$$

Thus, in this case we are permitted to use Macdonald's formula for at most the first three zeros.

When $x^2 \gg 1$, the following series may be used for the arctangent:

$$\arctan (x) = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \dots \qquad (45)$$

It is also possible by improving upon many of the approximations in the derivation of the Macdonald formula to produce even better results.

OBSERVATIONS ON THE ARTICLE, "TABLES OF ROOTS AND INCOMPLETE INTEGRALS OF ASSOCIATED LEGENDRE FUNCTIONS OF FRACTIONAL ORDERS", BY P. A. CARRUS

C. G. TREUENFELS (Ref. 6)

Tables of Differences of Table 1, page 292, reference 6

Table 1

Zero No.	θ_o = 165	θ_o = 170	θ_o = 175
2 - 1	1.030	0.99	1.01
3 - 2	1.067	1.03	1.02
4 - 3	1.075	1.04	1.01
5 - 4	1.079	1.046	1.04
6 - 5	1.081	1.049	1.00
7 - 6	1.082	1.051	1.00
8 - 7	1.085	1.050	1.02
9 - 8	1.085	1.053	1.022
10 - 9	1.087	1.053	1.025
11 - 10	1.087	1.054	1.024
12 - 11	1.088	1.055	1.024
13 - 12	1.089	1.055	1.024
14 - 13	1.088	1.055	1.024
15 - 14	1.089	1.056	1.025
16 - 15	1.089	1.057	1.025
17 - 16	1.090	1.056	1.025
18 - 17	1.089	1.057	1.025
19 - 18	1.090	1.057	1.025
20 - 19	1.090	1.057	1.026
21 - 20	1.090	1.057	1.026
22 - 21	1.090	1.058	1.026
23 - 22	1.090	1.057	1.026
24 - 23	1.090	1.058	1.027
25 - 24	1.090	1.058	1.027
26 - 25	1.090	1.058	1.026

Table 1 (Continued)

Zero No.	$\theta_o = 165$	$\theta_o = 170$	$\theta_o = 175$
27 - 26	1.091	1.058	1.027
28 - 27	1.090	1.058	1.027
29 - 28	1.090	1.058	1.027
30 - 29	1.091	1.058	1.027
31 - 30	1.090	1.058	1.027
32 - 31	1.091	1.058	1.028
33 - 32	1.090	1.058	1.027
34 - 33	1.091	1.058	1.028
35 - 34	1.090	1.059	1.027
36 - 35	1.091	1.058	1.028
37 - 36	1.090	1.058	1.027
38 - 37	1.091	1.059	1.028
39 - 38	1.091	1.058	1.028
40 - 39	1.090	1.058	1.027
41 - 40	1.091	1.059	1.029
42 - 41	1.090	1.058	1.027
43 - 42	1.091	1.059	1.028
44 - 43	1.091	1.058	1.028
45 - 44	1.090	1.059	1.028
46 - 45	1.091	1.058	1.028
47 - 46	1.091	1.059	1.028
48 - 47	1.091	1.058	1.027
49 - 48	1.090	1.059	1.029
50 - 49	1.091	1.058	1.028

We had previously observed in equation (41) that the exact value of the first zero for $\theta = 165^{o}$ was lower than is indicated in reference 6.

$$1.0321 > n_1 > 1.0316 \qquad (41)$$

Now it should be pointed out that for $165^{o} \leq \theta \leq 180^{o}$ the value of n_1 should decrease with increasing θ (proven in reference 7). As a result, we observe that the first zero for $\theta = 170^{o}$ cannot be correct, for it is greater than 1.0321 (Ref. 6, Table 1). We also note that one difference is less than an integer; this is impossible. In one case, the successive difference for θ = constant decreases for increasing k by more than 1 in the last significant figure. This, too, is incorrect as it implies an error other than a mere rounding-off error.

References 7 and 8 show that when n is large, the zeros of $P_n^{-m}(\cos\theta)$ are given by:

$$(n_{k+1} + \frac{1}{2})\,\theta - \frac{\pi}{4} - \frac{m\pi}{2} \simeq (2k+1)\frac{\pi}{2} \qquad (46)$$

Successive differences in n are then given by:

$$n_{k+2} - n_{k+1} \simeq \frac{\pi}{\theta} \qquad (47)$$

Table 2

θ	$n_{k+2} - n_{k+1}$
165^{o}	1.0909
170^{o}	1.0588
175^{o}	1.0286

It should be pointed out that equation (46) should not be used when θ is very close to π. Although, if it is used, one obtains Table 2. This table predicts most of the differences, obtained from reference 6 and listed in Table 1 of this report, to two decimal places.

Thus, if one wants to extend Table 1 of reference 6, he has merely to add to each successive zero the values listed in Table 2 on the preceding page.

One may also note that the correct integer values for the associated functions may be obtained by observing the change in sign in the associated Legendre polynomials. An analysis of the associated Legendre polynomials, given in reference 10 for θ values from 0^{o} to 90^{o} in intervals of five degrees, makes it possible (after multiplying through by $(-1)^{k}$) to check* all the values listed in Table 1 (Ref. 6) from n = 1 to n = 24 (the extent of the table in reference 10).

Thus, any errors that exist in reference 6 should occur to the right of the decimal point, for one could know the exact integer values by examination of the signs of the polynomials.

*Example:

$$\frac{1}{\sin 165^{o}} P_{13}^{1} (\cos 165^{o}) \simeq 6.50140$$

$$\frac{1}{\sin 165^{o}} P_{14}^{1} (\cos 165^{o}) \simeq -.86327$$

$$\frac{1}{\sin 165^{o}} P_{15}^{1} (\cos 165^{o}) \simeq -5.23851$$

Thus, one would expect a zero of $P_n^1 (\cos 165^{o})$ to occur between n = 13 and n = 14, but none between n = 14 and n = 15 for $\theta = 165^{o}$.

APPENDIX

The hypergeometric series $F(-n, n+1; 2; x)$ may be written in the form $A_0 + A_1 x + A_2 x^2 + \dots$ where $A_0 = 1,\ A_1 = \frac{-n(n+1)}{2\cdot 1}, \dots$

$$A_k = \prod_{i=1}^{k} \frac{(i-n-1)(i+n)}{i(i+1)}$$

Then:

$$\frac{A_k}{A_{k-1}} = \frac{(k-n-1)(k+n)}{k(k+1)} = 1 - \frac{k + \frac{n(n+1)}{2}}{\frac{k(k+1)}{2}} = 1 - P_k$$

Let $S_k = \sum_{i=0}^{k} A_i x^i \qquad R_k = \sum_{i=k+1}^{\infty} A_i x^i$

Then:

$$R_k = A_{k+1} x^{k+1} + A_{k+2} x^{k+2} + \cdots = A_{k+1} x^{k+1}\left[1 + \frac{A_{k+2}}{A_{k+1}} x + \frac{A_{k+3}}{A_{k+1}} x^2 + \cdots\right]$$

$$= A_{k+1} x^{k+1}\left[1 + \frac{A_{k+2}}{A_{k+1}} x + \frac{A_{k+3}}{A_{k+2}} \frac{A_{k+2}}{A_{k+1}} x^2 + \cdots\right] = A_{k+1} x^{k+1} \lambda$$

Since $\lim_{k\to\infty} P_k = 0$ and $\lim_{k\to\infty} \frac{A_k}{A_{k-1}} = 1$ we have for $k > n$

$$0 < \frac{A_{k+2}}{A_{k+1}} < \frac{A_{k+3}}{A_{k+2}} < \frac{A_{k+4}}{A_{k+3}} \dots < 1$$

It follows that

$$\frac{1}{1 - \frac{A_{k+2}}{A_{k+1}} x} < \lambda < \frac{1}{1-x}$$

and hence R lies between two specific

values which are determined by the sums of the two infinite geometric series. Hence, we show that $S \equiv S_k + R_k$ satisfies the relation $\alpha > S > \beta$ where α and β are positive when n = 1.0316, and α and β are negative for n = 1.0321.

On the next page we show a few lines of the form used in the actual computation for the case n = 1.0316.

$$x = \frac{1 - \cos 165^{\circ}}{2} \simeq .98296291$$

① k	② $-n+k$	③ $n+1+k$	④ $(-n+k)\ (n+1+k)$ ②×③	⑤ $i(i+1)$ $(i=k+1)$	⑥ $B_i = \frac{④}{⑤}$	⑦ $A_i = \prod_1^i B_i$	⑧ $A_i\ x^i$	⑨ $\sum A_i\ x^i$
0	-1.0316	2.0316	-2.09579856	2	-1.04789928	-1.04789928	-1.03004613	
1	-0.0316	3.0316	-0.09579856	6	-0.01596643	0.01673121	0.01616596	
2	0.9684	4.0316	3.90420144	12	0.32535012	0.00544350	0.00516999	
3	1.9684	5.0316	9.90420144	20	0.49521007	0.00269568	0.00251661	
4	2.9684	6.0316	17.90420144	30	0.59680671	0.00160880	0.00147634	

Another procedure used on some of the computation is illustrated below:

$$\text{For } n = 1.0316, \quad \frac{n\ (n + 1)}{2} = 1.04789928$$

① k	② $\frac{k(k+1)}{2}$	③ $x \div ②$ $\frac{2x}{k(k+1)}$	④ $x(1-P_k)$	⑤ $A_k\ x^k$	⑥ $\sum A_k\ x^k$ (to be summed after 40 terms)
0			1.0000000000	1.0000000000	
1	1	.9829629131	-1.0300461289	-1.0300461289	
2	3	.3276543044	-.01569440537	0.01616596150	
3	6	.1638271522	.3198071017	0.005169989294	
4	10	.09829629131	.4867731350	0.002516611897	
5	15	.06553086087	.5866388668	0.001476342351	

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