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CHARACTERIZATION OF OPTIMA IN SMOOTH PARETO ECONOMIC SYSTEMS*

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Simple techniques of calculus and geometry are used to study and characterize the optima of pure exchange economies in which the utility functions are smooth but not necessarily convex. It is also shown how one can reduce the problem of optimizing p functions on the manifold of states to that of maximizing a single function on a submanifold of this space. Two models are described: one in which a person cannot trade to an optimum unless he starts at one; and one in which a person cannot even get near a local Pareto optimum along continuous 'trade curves' from most initial distributions. Finally, the set of optima is described for a generic set of utility mappings.

1. Introduction

In this paper, we use the tools of calculus and differential geometry to study various optima that arise naturally in a pure exchange economy. Our work has been motivated by the lectures and papers of Smale (1973a, b; 1974a, b, c, d) on Global Analysis and Economics. Not only do we attempt a very systematic treatment of the characterization of economic optima but we also supply complete proofs of some theorems which are only sketched in Smale's papers. We also restrict ourselves to simple tools of differential calculus although the sketches in Smale (1973a, b) rely on recent advances in the theory of singularities of mappings. More importantly, we employ a simpler model for trading and study different optima than Smale does. Our trade curves are simply piecewise smooth curves in the state space on which no person's utility function decreases and someone's utility function is increasing. Beside the classical Pareto optima (PO), we study local Pareto optima (LPO) and 'trade optima' (TO) (states from which there is no trade curve) – all of which, we feel, are more closely related to the classical optima of economics.

In section 2, we make precise our economic model and the axioms we will use. In section 3, we define our optima and our trade curves; in section 4, we give

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necessary conditions for a state to be an optimum. In section 5, we use second derivative tests to give sufficient conditions. In section 6, we derive other sufficient conditions using more geometric techniques. Here, we show how to reduce the problem of optimizing p functions on a state space to that of maximizing a single function on a submanifold of this state space (Theorems 7 and 10, and Corollary 2).

In section 7, we describe what the set of optima looks like for a generic utility mapping $(u_1, \ldots, u_p)$. Here, we use some elementary transversality and singularity theory for the first time. To emphasize that the techniques are indeed simple, we also provide short proofs of the basic theorems we use. In section 8, we discuss the problem of reaching a TO or LPO from a given initial distribution of commodities. We describe a model where no one can reach a TO without starting at one, and also a model where from most initial distributions one cannot even get near an LPO by continuous trade curves. These models, in which the utility functions are smooth and monotone but not convex, illustrate the importance of convexity assumptions in the theorems of Smale (1974d) and others which under various assumptions show the accessibility of an equilibrium from any initial distribution of wealth. These models also lead naturally to questions concerning the *generic* properties of utility mappings.

Besides the papers of Smale, we suggest Debreu (1959) and Intriligator (1958) as economic references and Abraham–Robbin (1967), Spivak (1965) and Golubitsky–Guillemin (1974) as mathematical references. It is a pleasure to acknowledge the assistance of Dennis Barden in setting up of models, of Jean Martinet in demystifying and applying singularity theory, and of Don Brown in renewing our interest in economics.

2. Pareto economic systems

We first describe the structure of certain relatively abstract Pareto systems and next, some smooth models of this system which we study in some detail in the following sections.

There are $p \geqq 2$ objects called *persons* and $c \geqq 2$ sets called *commodities*. Let $\mathbf{R}_+$ be the positive real numbers and $\overline{\mathbf{R}}_+$ the non-negative real numbers. On each commodity there is a measure which assigns to each measurable subset of the commodity a non-negative real number called an *amount* of the commodity. *A holding of the κth person* is a vector

$$x^\kappa = (x_1^\kappa, \ldots, x_c^\kappa) \in \overline{\mathbf{R}}_+^c,$$

where x_j^κ is the amount of the jth commodity held by the κth person. A *state* of the system is a vector

$$x = (x^1, \ldots, x^p) \in (\overline{\mathbf{R}}_+^c)^p,$$

which assigns a holding to each person. The *state space* is a collection of states.

To each (κth) person there is assigned an ordering on Ω, $x >^{\kappa} y$, which is transitive, reflexive and complete (complete means that, for every pair of states, x and y, either $x >^{\kappa} y$ or $y >^{\kappa} x$); this ordering is called *individual preference*. States x and y such that $x >^{\kappa} y$ and $y >^{\kappa} x$ are called *individually indifferent*, $x \sim^{\kappa} y$. Finally, a *Pareto system* has the following additional structure. A new ordering on Ω, $x > y$, called *Pareto preference* is defined by $x > y \Leftrightarrow x >^{\kappa} y$ for all κ and $x \nsim^{\kappa} y$ for some κ; in words, a state x is Pareto prefered to a state y if x is individually prefered to y by all and if some are not individually indifferent to x and y.

In order to use the techniques of calculus and geometry we will study only smooth mathematical models of Pareto systems; namely, we study Pareto systems which satisfy the following axioms [$A_n = (A1, A_n2, A_n3, A4)$].

(A1) The sum of the individual holdings of each commodity is a positive constant on Ω. Formally, this means that there exists a vector $a \in \mathbf{R}_c^+$ so that

$$\Omega = \{x = (x^1, \ldots, x^p) \in (\overline{\mathbf{R}}^c_+)^p \mid x^1 + \ldots + x^p = a\}.$$

The state space $\Omega \subset \mathbf{R}^{cp}$ is given the ordinary Euclidean topology, but the usual Euclidean metric on $\mathbf{R}^{cp}$ has no intrinsic significance.

(A_n2) For each (κth) person the individual preference on Ω is represented by a C^n-smooth, $n \geqq 1$, *utility function*, $u_\kappa : \Omega \to \mathbf{R}$, so that

$$x >^{\kappa} y \Leftrightarrow u_\kappa(x) \geqq u_\kappa(y).$$

(Clearly, many different sets of utility functions can represent the same set of individual preferences.)

(A_n3) The individual preference of each person depends only on his own holding. Formally, with the projections $\pi_\kappa : \Omega \to \overline{\mathbf{R}}^c_+$ defined by $x = (x^1, \ldots, x^p) \mapsto x^\kappa$, there exist C^n-smooth functions, $\bar{u}_\kappa : \pi_\kappa(\Omega) \subset \overline{\mathbf{R}}^c_+ \to \mathbf{R}$, so that

$$u_\kappa = \bar{u}_\kappa \circ \pi_\kappa.$$

The κth vertex of the system is the state x such that $\pi_\kappa(x) = a$; i.e., the state in which the κth person holds all of every commodity.

(A4) For each (κth) person the utility function u_κ has no critical point on Ω, except possibly at the κth vertex.

3. Trades and optima

The primary goal of the rest of this paper is to define and characterize various optimal states in systems which statisfy the Axioms A_n.

A state $x \in \Omega$ is a (Global) *Pareto Optimum* (PO) if there is no $y \in \Omega$ such that $y > x$; i.e., such that $u_\kappa(y) \geqq u_\kappa(x)$ for all κ and $u_\kappa(y) > u_\kappa(x)$ for some κ.

As usual in mathematics one treats a global optimization problem by first studying the local optima. Furthermore, if one assumes only that each person is 'aware' of his individual preference for holdings 'near' his present holding, then local optima become the relevant economic optima.

A state $x \in \Omega$ is a *Local Pareto Optimum*, LPO, if there is a neighborhood N of x, so that there is no $y \in N$ with $y > x$.

An allowable trade is a change from a state x to a state y, where $y > x$. In this way the definition of an LPO leads to the notion of sequences of 'small' allowable trades; such sequences can be continued unless the last state is an LPO. So it is natural to study the dynamics of the system by introducing a continuous analogue of allowable trade sequences.

In a system satisfying the Axioms A_n, a (allowable) *trade curve from the state* x is a continuous piecewise C^n-smooth mapping $\gamma: [0, \varepsilon) \to \Omega$, $\varepsilon > 0$, so that $\gamma(0) = x$ and $\gamma(s) < \gamma(t)$ for all $0 \leqq s < t < \varepsilon$.

Note that, although $\sum(u_\kappa \circ \gamma)$ is strictly a increasing function, its derivative may be zero on a totally disconnected set. On the other hand, the fact that $\sum(u_\kappa \circ \gamma)$ is strictly increasing implies that γ is one to one.

A state $x \in \Omega$ is a *trade optimum*, $x \in$ TO, if there is no trade curve from x.

It follows immediately from the definitions that PO $\subset$ LPO $\subset$ TO.

However, the converses are not true. To illustrate this, we construct Ω in the case $p = c = 2$. Here $\dim \Omega = c(p-1) = 2$ and Ω is the classical *Edgeworth Box*. For simplicity assume $\mathbf{a} = (1, 1)$. (x_1^1, x_2^1) is the first consumer's holding, while $(x_1^2, x_2^2) = (1-x_1^1, 1-x_2^1)$ is the second consumer's holding. So (x_1^1, x_2^1) give global coordinates on Ω. In these coordinates, $u_1 = \bar{u}_1$ and

$$u_2(x_1^1, x_2^1) = \bar{u}_2(1-x_1^1, 1-x_2^1) = \bar{u}_2(x_1^2, x_2^2).$$

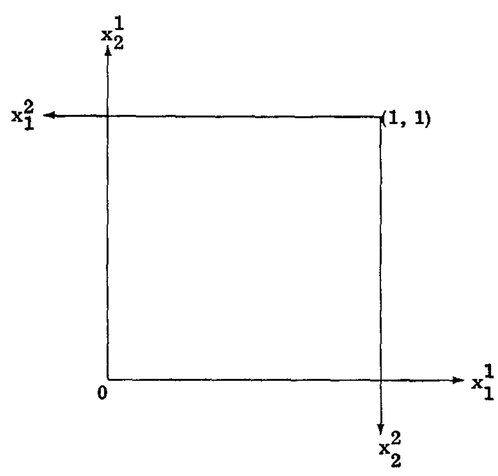

Fig. 1. Edgeworth box, Ω.

For an example of an LPO that is not a PO, let

$$\bar{u}_1(x_1^1, x_2^1) = -x_2^1.$$

The level curves of u_1 in Ω are horizontal. Let

$$u_2(x_1^1, x_2^1) = x_2^1 - 12(x_1^1 - \tfrac{1}{2})^3 + (x_1^1 - \tfrac{1}{2}).$$

It is simple to check that $Y_1 = (\frac{2}{3}, \frac{7}{18})$ is an LPO but $Y_2 = (\frac{1}{12}, \frac{1}{12})$ is a Pareto-preferred point $[u_i(Y_1) < u_i(Y_2)$ for $i = 1, 2]$. See fig. 2.

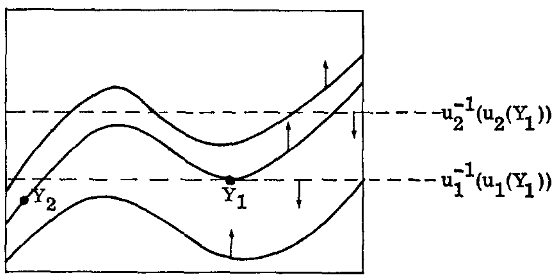

Fig. 2. Y_1 is an LPO, but not a PO.

For an example of a TO that is not an LPO consider the model

$$u_1(x_1^1, x_2^1) = -x_2^1,$$

$$u_2(x_1^1, x_2^1) = x_2^1 - f(x_1^1 - \tfrac{1}{2}),$$

where

$$f(s) = \begin{cases} e^{-1/s^2} \sin(1/s) & \text{for } s \neq 0, \\ 0 & \text{for } s = 0. \end{cases}$$

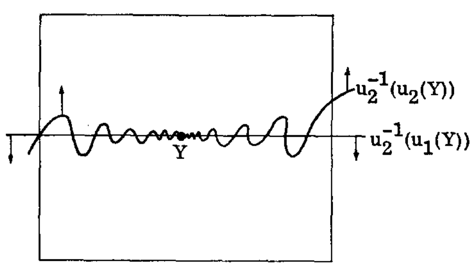

Fig. 3. Y is a TO, not an LPO.

Consider the point $Y = (\frac{1}{2}, \frac{1}{2})$. Since $u_2^{-1}(u_2(Y))$ crosses $u_1^{-1}(u_1(Y))$ infinitely often on both sides of Y, there is no trade curve from Y, i.e., Y is a TO. On the other hand, Y is not an LPO since arbitrarily close to Y, $(u_2^{-1}(u_2(Y))$ meets level lines of u_1 above $u_1^{-1}(u_1(Y))$.

4. First-order theory: Necessary conditions for optima

The tangent space to Ω at x is, from Axiom A1,

$$T\Omega_x = \{V = (V^1, \ldots, V^p) \in (\mathbf{R}^c)^p \mid V^1 + \ldots + V^p = 0\}.$$

Note that the dimension of $T\Omega_x$ is $c(p-1)$. For the derivative of kth utility function at x, we write $\mathrm{D}u_k(x) : T\Omega_x \to \mathbf{R}$. Call

$$u = (u_1, \ldots, u_p) : \Omega \to \mathbf{R}^p$$

the *utility mapping*. For its derivative we write $\mathrm{D}u(x) : T\Omega_x \to \mathbf{R}^p$. We denote the critical set of the utility mapping u by $S_1(u)$. So

$$S_1(u) = \{x \in \Omega \mid \text{Rank } \mathrm{D}u(x) < \text{Min.}\, [p, c(p-1)] = p\}.$$

Since Ω and $\bar{\mathbf{R}}^c_+$ both have non-empty boundaries, one must work carefully with the u_k, and $\bar{u}_k$ on these boundaries. For example, we require that each $\bar{u}_k$ be defined and C^n on an *open* neighborhood of $\Pi_k(\Omega)$ in $\mathbf{R}^c$. This will ensure that $u_k = \bar{u}_k \circ \Pi_k$ and u are defined and C^n on an open $c(p-1)$-dimensional manifold containing Ω. For $x \in \partial\Omega$, we will take $T\Omega_x$ to be the above $c(p-1)$-dimensional space instead of $T(\partial\Omega)_x$.

Theorem 1. Let $u : \Omega \to \mathbf{R}^p$ be a utility mapping of a Pareto system satisfying Axioms A_n. Then, the following statements are equivalent when x is not a vertex:

(1) $x \in S_1(u)$.
(2) Rank $\mathrm{D}u(x) = p-1$.
(3) *There exist $\lambda_1, \ldots, \lambda_p \in \mathbf{R}$ with no λ_i zero and*

$$\sum_{j=1}^{p} \lambda_j \, \mathrm{D}u_j(x) = 0.$$

(4) *There exist $\lambda_1, \ldots \lambda_p \in \mathbf{R}$ with no λ_i zero and*

$$\lambda_k \, \mathrm{D}\bar{u}_k(x^k) = \lambda_p \, \mathrm{D}\bar{u}_p(x^p), \quad k = 1, 2, \ldots, p-1.$$

Proof. We prove (1) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1). For (1) $\Rightarrow$ (4), since $x \in S_1(u)$, there exist $\lambda_1, \ldots, \lambda_p$ in $\mathbf{R}$, not all zero, such that

$$\sum \lambda_j \, \mathrm{D}u_j(x) V = 0,$$

for all $V \in T\Omega_x$. Since $u_j = \bar{u}_j \circ \pi_j$,

$$\sum_j \lambda_j \, D\bar{u}_j(x^j)V^j = 0,$$

for all $(V^1, \ldots, V^p) \in (\mathbf{R}^c)^p$ such that

$$\sum V^i = 0.$$

So,

$$\sum_{j=1}^{p-1} \lambda_j \, D\bar{u}_j(x^j)V^j + \lambda_p \, D\bar{u}_p(x^p)[-V^1 - \ldots - V^{p-1}] = 0,$$

for all $V^1, \ldots, V^{p-1}$ in $\mathbf{R}^c$. But this implies that

$$\lambda_j \, D\bar{u}_j(x^j) = \lambda_p \, D\bar{u}_p(x^p) \quad \text{for} \quad j = 1, \ldots, p.$$

Since no $D\bar{u}_k$ is zero and some λ_k is not zero, it follows that no λ_k is zero and (4) is proved. This argument also shows that (3) and (4) are equivalent since the λ_j in either statement serve as the λ_j in the other statement. But (4) implies that the ratios λ_j/λ_p are uniquely determined, which means that the corank of $Du(x)$ is equal to one; and so (2) follows. That (2) $\Rightarrow$ (1) is trivial and the proof is complete.

Remark. Theorem 1 tells us that axioms A_n3 and A4 imply the following condition which we will call *Axiom A3′*: the rank of $Du(x)$ is greater or equal to $p-1$ for all x in Ω. We will denote the (weaker) set of Axioms A1, A_n2, A3′, and A4 by A_n'.

Define $\theta = \theta(u) \subset S_1(u)$ to be the set of singular points x of u with the property that all of the λ_j in the linear combination $\sum \lambda_j \, Du_j(x) = 0$ have the same sign. The following characterizations of θ are essentially due to Smale (1973b, 1974b).

Theorem 2. Let $u : \Omega \to \mathbf{R}^p$ be a utility mapping for a Pareto system satisfying the Axioms A_n'. Let $x \in$ interior of Ω. Then, the following statements are equivalent:

(1) $x \in \theta$.
(2) *The image of* $Du(x)$ *does not meet* $\mathbf{R}^p_+$.
(3) *There is no trade curve γ from x such that*

$$(u_k \circ \gamma)'(0) = Du_k(x)\gamma'(0) > 0 \quad \textit{for all } k.$$

(4) *There is no $V \in T\Omega_x$ such that* $Du_k(x)V > 0$ *for all k.*

Proof. (1) implies (4) since for $x \in \theta$ there exist $\lambda_1, \ldots, \lambda_p$ all positive with $\sum \lambda_j \mathrm{D}u_j(x) = 0$. But if $\mathrm{D}u_j(x)V > 0$ for all j, then $\sum \lambda_j \mathrm{D}u_j(x)V$ would be positive and thus non-zero. (4) is equivalent to (3) by identifying V in (4) with $\gamma'(0)$ in (3). (2) is a restatement of (4). To see that (4) implies (1), note first that (4) implies x is in $S_1(u)$ since $\mathrm{D}u(x)$ is not onto. Suppose $\sum \lambda_j \mathrm{D}u_j(x) = 0$ and $\lambda_l > 0$. Choose $k \neq l$. Without loss of generality, take k to be p. Now there is a vector $w = (w_1, \ldots, w_{p-1})$ in $\mathbf{R}^{p-1}$ with $w_i > 0$ for all i and

$$\sum_{i<p} \lambda_i w_i > 0.$$

Since $\mathrm{D}(u_1, \ldots, u_{p-1})(x) : T\Omega_x \to \mathbf{R}^{p-1}$ is surjective (by Axiom A3′), there is a V in $T\Omega_x$ with $(\mathrm{D}u_1(x)V, \ldots, \mathrm{D}u_{p-1}(x)V) = w$. Now

$$\lambda_p \mathrm{D}u_p(x)V = -\sum_{j<p} \lambda_j \mathrm{D}u_j(x)V = -\sum \lambda_j w_j < 0.$$

Since each $w_j > 0$, (4) implies $\mathrm{D}u_p(x)V \leq 0$. Therefore, $\lambda_p > 0$.

Remark. We have preferred to use the differentials of utility functions instead of their gradients since the existence of the gradient depends on the choice of a Riemannian metric on Ω and therefore gradient vectors have no intrinsic significance in studying Pareto systems as we have defined them. The only intrinsically significant things that we defined by a utility function are its level (indifference) surfaces and which side of a given level surface is Pareto preferable to the surface. Nevertheless, the gradient vector grad $u_i(x)$ is sometimes helpful in interpreting definitions and theorems pictorially. For example, since

$$\langle \operatorname{grad} u_i(x), V \rangle = \mathrm{D}u_i(x)V,$$

where $\langle \cdot, \cdot \rangle$ is the inner product of the metric being used, statement (4) in Theorem 2 is equivalent to the statement that for any choice of Riemannian metric on Ω, the set of gradients $\{\operatorname{grad} u_k(x)\}_{k=1}^p$ do not lie in any open half space.

Remark. Smale (1973a, b) defines an *admissible curve* to be a trade curve γ such that $\mathrm{D}u_k(x)\gamma'(0) > 0$ for all k, as in (3) of Theorem 2. He uses only such curves to describe trades in his model.

The principal interest in θ comes from the fact that any optimum (PO, LPO, TO) in the interior of Ω must lie on θ. This follows from the contrapositive of (3) $\Rightarrow$ (1) in Theorem 2 and the fact that PO $\subset$ LPO $\subset$ TO.

Corollary 1. If x lies in $\dot{\Omega}$ (the interior of Ω) and x is a PO, LPO, or TO for a Pareto system satisfying Axioms A'_n ($n \geq 1$), then $x \in \theta$.

In what follows, we'll make frequent use of the following corollary of the implicit function theorem [e.g., Golubitsky–Guillemin (1974), Spivak (1965)]. Suppose $n \geq q$ and $g : M^n \to \mathbf{R}^q$ is a C^1 mapping with the rank of $\mathrm{D}g(x) : TM_x \to \mathbf{R}^q$ equal to q. Then, there is a neighborhood N of x in M^n such that $N \cap g^{-1}(g(x))$ is a smooth submanifold of M of codimension q (i.e., of dimension $n-q$).

The following definition isolates one of the principal ideas in all that follows. Let $U_x^{(k)}$ be the intersection of all but the kth level surface through x,

$$U_x^{(k)} = \bigcap_{\substack{j=1 \\ j \neq k}}^{p} u_j^{-1}(u_j(x)).$$

Theorem 3. Let $u = (u_1, \ldots, u_p)$ be the utility mapping for a Pareto system satisfying the Axioms A'_n. Then:

(1) *If $x \notin S_1(u)$,*

$$\bigcap_{j=1}^{p} u_j^{-1}(u_j(x))$$

is a smooth codimension p submanifold of Ω near x.

(2) *If x is in the interior of Ω, each $U_x^{(k)}$ is a codimension $p-1$ submanifold of Ω.*

(3) *$x \in S_1(u) \Leftrightarrow u_k \mid U_x^{(k)}$ has a critical point at x for all k; $\Leftrightarrow u_k \mid U_x^{(k)}$ has a critical point at x for some k.*

Proof. (1) and (2) follow immediately from the above corollary to the implicit function theorem. For (1), $\mathrm{D}(u_1, \ldots, u_p)(x) : T\Omega_x \to \mathbf{R}^p$ has rank p if $x \notin S_1(u)$. For (2), $\mathrm{D}(u_1, \ldots, u_{k-1}, u_{k+1}, \ldots, u_p)(x) : T\Omega_x \to \mathbf{R}^{p-1}$ has rank $p-1$ for all $x \in \dot{\Omega}$ by Axioms A3′ and A4. To see (3), suppose y is a critical point of $u_k \mid U_x^{(k)}$, i.e., $\mathrm{D}u_k(y) \mid (TU_x^{(k)})_y$ is 0. Since the other u_j's are constant on $U_x^{(k)}$, $\mathrm{D}u_j(y) \mid (TU_x^{(k)})_y$ is zero for all j. Since $u = (u_1, \ldots, u_p)$, the kernel of $\mathrm{D}u(y)$ contains a space of codimension $p-1$ and thus $\mathrm{D}u(y)$ is not of full rank at y. So, $y \in S_1(u)$. On the other hand, for $y \in S_1(u)$ there are $\lambda_1, \ldots, \lambda_p$ all non-zero with $\sum \lambda_j \mathrm{D}u_j(y) = 0$. For any k,

$$\mathrm{D}u_k(y) = \frac{1}{\lambda_k} \sum_{j \neq k} \lambda_j \mathrm{D}u_j(y).$$

As before, $\mathrm{D}u_j(y) \mid (TU_x^{(k)})_y$ is zero for all $j \neq k$. Therefore, $\mathrm{D}u_k(y) \mid (TU_x^{(k)})_y$ is 0 and y is a critical point of each $u_k \mid U_x^{(k)}$.

Remark. The above argument shows that for all k,

$$(TU_x^{(k)})_x = \ker \mathrm{D}u(x) \quad \text{for all} \quad x \in \operatorname{int} \Omega.$$

5. Characterization of LPO and TO in θ: Calculus techniques

In section 4, we saw that in order for a point Y in $\dot{\Omega}$ to be an optimum, Y must lie in θ. In this section, we discuss some sufficient conditions to determine which elements of θ are economic optima. As Smale (1973a, b; 1974b) does, we use the 'second intrinsic derivative' of our utility mapping for the optima which we are studying. However, we will use no singularity theory, only techniques of elementary calculus, and will write out the complete proofs.

Definition. Let u be a utility mapping satisfying the axioms A_n or A_n'. Let $x \in \theta(u)$. So, there are $\lambda_1, \ldots, \lambda_p$ all positive with

$$\sum_1^p \lambda_i \, \mathrm{D}u_i(x) = 0.$$

Without loss of generality, we will always assume $\lambda_1 = 1$ and so $\lambda_2, \ldots, \lambda_p$ are uniquely determined. Let K_x denote the kernel of $\mathrm{D}u(x) : T\Omega_x \to \mathbf{R}^p$. So, K_x lies in $T\Omega_x$ and is equal to

$$\bigcap_{i=1}^p \ker \{\mathrm{D}u_i(x) : T\Omega_x \to \mathbf{R}\}.$$

Let F_x^2 be the symmetric bilinear form

$$\sum_{i=1}^p \lambda_i \, \mathrm{D}^2 u_i(x) : K_x \times K_x \to \mathbf{R},$$

where $\mathrm{D}^2 u_i(x)$ is the second-order differential of u_i at x. F_x^2, with domain K_x, is related to what Porteous (1970) and Smale (1973a, b) call the second intrinsic derivative of u. Our first proposition states that F_x^2 is indeed intrinsic.

Proposition 1. For $x \in \theta(u)$, where u is a C^2 utility mapping $\Omega \to \mathbf{R}^p$ satisfying the Axioms A_2', F_x^2 is independent of the coordinate system used around x. More precisely, if $\gamma(t)$ is any smooth curve in Ω with $\gamma(0) = x$ and $\gamma'(0) = v \in K_x$, then

$$F_x^2(v, v) = \frac{\mathrm{d}^2}{\mathrm{d}t^2} \sum_{i=1}^p \lambda_i u_i(\gamma(t)) \Big|_{t=0} .$$

Proof.

$$\frac{\mathrm{d}}{\mathrm{d}t} u_i(\gamma(t)) = \mathrm{D}u_i(\gamma(t))\gamma'(t),$$

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2} u_i(\gamma(t))\bigg|_{t=0} &= \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{D}u_i(\gamma(t))\bigg|_{t=0} \gamma'(t) \\ &\quad + \mathrm{D}u_i(\gamma(t))\bigg|_{t=0} \frac{\mathrm{d}}{\mathrm{d}t}\gamma'(t)\bigg|_{t=0} \\ &= \mathrm{D}^2u_i(\gamma(0))(\gamma'(0), \gamma'(0)) + \mathrm{D}u_i(\gamma(0)(\gamma''(0)) \\ &= \mathrm{D}^2u_i(x)(v, v) + \mathrm{D}u_i(x)\gamma''(0),\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2} \sum_1^p \lambda_i u_i(\gamma(t))\bigg|_{t=0} &= \sum_i^p \lambda_i \frac{\mathrm{d}^2 u_i(\gamma(t))}{\mathrm{d}t^2}\bigg|_{t=0} \\ &= \sum_{i=1}^p \lambda_i \mathrm{D}^2u_i(x)(v, v) + \sum_1^p \lambda_i \mathrm{D}u_i(x)\gamma''(0).\end{aligned}$$

The first term is $F_x^2(v, v)$, and the second term is zero since $\sum \lambda_i \mathrm{D}u_i(x) = 0$.

To compute $F_x^2(v, w)$, use the polarization formula

$$F_x^2(v, w) = \tfrac{1}{2}[F_x^2(v+w, v+w) - F_x^2(v, v) - F_x^2(w, w)].$$

Definition. $x \in \mathrm{DGN}$ if and only if $x \in \theta(u)$ and $F_x^2 : K_x \times K_x \to \mathbf{R}$ is a *degenerate* bilinear form; that is, for some non-zero v in K_x, $F_x^2(v, w) = 0$ for all w in K_x. In the language of singularity theory [see Levine (1970)] such an x is in $S(S^1(u))$.

Before discussing sufficient conditions for optima, we first strengthen our necessary condition.

Theorem 4. Suppose u is a utility mapping satisfying the Axioms A'_2. Suppose x is in $\dot{\Omega}$ and in $\theta(u) \setminus \mathrm{DGN}$. If x is a TO, LPO, or PO, then F_x^2 is negative definite on K_x.

Proof. If F_x^2 is non-degenerate but not negative definite on K_x there is a non-zero v in K_x with $\sum \lambda_j \mathrm{D}^2u_j(x)(v, v) > 0$. Since the λ_j are all positive, there is some k with $\mathrm{D}^2u_k(x)(v, v) > 0$. Since $x \in \theta$,

$$\mathrm{D}u_k(x) = \sum_{i \neq k} \frac{\lambda_i}{\lambda_k} \mathrm{D}u_i(x).$$

So,

$$K_x = \bigcap_j \ker \mathrm{D}u_j(x) = \bigcap_{j \neq k} \mathrm{D}u_j(x) = (TU_x^{(k)})_x.$$

We have x a critical point of $u_k \mid U_x^{(k)} : U_x^{(k)} \to \mathbf{R}$ with $v \in (TU_x^{(k)})_x$, such that $\mathrm{D}^2(u_k \mid U_x^{(k)})(x)(v, v) \neq 0$. Therefore, there is a smooth curve $\gamma(t)$ on $U_x^{(k)}$ with $\gamma(0) = x$, $\gamma'(0) = v$, and $t \mapsto u_k(\gamma(t))$ strictly increasing for small t. So, γ is a trade curve and x is not a TO. ∎

Recall that $\gamma : [0, \varepsilon] \to \Omega$ is a trade curve if γ is a continuous, piecewise smooth curve with each $t \mapsto u_i(\gamma(t))$ non-decreasing and $t \to \sum_i u_i(\gamma(t))$ strictly increasing. If, in addition, γ is C^1 for $t = 0$ and $\gamma'(0) \neq 0$, we will call γ a *directed trade curve*. Such curves are more amenable to calculus techniques. If there is no directed trade curve from x, we say x is a TO$^+$. Clearly, LPO $\Rightarrow$ TO $\Rightarrow$ TO$^+$. In the last section, we'll give an example of a point that is a TO$^+$ but not a TO. Note that TO $\Leftrightarrow$ TO$^+$ for $c = p = 2$. Our use of directed trade curves is only temporary for in the next section we'll derive properties for LPO and TO's that are derived here for TO$^+$'s: The next theorem is the goal of this section.

Theorem 5. Suppose $u : \Omega \to \mathbf{R}^p$ is a utility mapping satisfying the Axioms A_2'. Suppose x is in $\mathring{\Omega}$ and in $\theta(u) \setminus DGN$. Then, x is a TO$^+$ if and only if $F_x^2 : K_x \times K_x \to \mathbf{R}$ is negative definite.

Proof. Suppose that F_x^2 is negative definite but x is not a TO$^+$. Let $\gamma(t)$ be a directed trade curve with $\gamma(0) = x$ and $\gamma'(0) = v$. So, $\mathrm{D}u_i(x)v \geq 0$ for all i. If $v \notin K_x$,

$$\sum \lambda_i \mathrm{D}u_i(x) > 0,$$

contradicting

$$\sum_i \lambda_i \mathrm{D}u_i(x) = 0.$$

So, $v \in K_x$. Expand $u_i(\gamma(t))$ in its Taylor series about $t = 0$,

$$\begin{aligned} u_i(\gamma(t)) - u_i(\gamma(0)) &= t \frac{\mathrm{d}}{\mathrm{d}t} u_i(\gamma(t))\bigg|_{t=0} + \frac{t^2}{2} \frac{\mathrm{d}^2}{\mathrm{d}t^2} u_i(\gamma(t))\bigg|_{t=0} + \dots \\ &= t\mathrm{D}u_i(x)v + \frac{t^2}{2}[\mathrm{D}u_i(x)w + \mathrm{D}^2u_i(x)(v, v)] + \dots, \end{aligned}$$

where $\gamma''(0) = w$. Since $\gamma(t)$ is a trade curve and $v \in K_x$,

$$0 \leq \frac{u_i(\gamma(t)) - u_i(\gamma(0))}{t} = \frac{t}{2}[\mathrm{D}u_i(x)w + \mathrm{D}^2u_i(x)(v, v)] + \dots.$$

Therefore, $Du_i(x)w + D^2u_i(x)(v, v) \geq 0$ for all i. Since λ_i's are positive and $x \in \theta$,

$$0 \leq \sum_i \lambda_i \, Du_i(x)w + \sum \lambda_i \, D^2u_i(x)(v, v)$$

$$= F_x^2(v, v).$$

But v non-zero contradicts the negative definiteness of F_x^2. So, x is a TO^+.

So except for states in DGN, we have an excellent analytical characterization of the economic optima in the interior of Ω. It now becomes important to study and characterize DGN, at least for generic utility mappings. This task will be carried out in section 7, where we indicate, among other things, that for an open and dense set of u's, $\theta(u)$ is a $(p-1)$-dimensional submanifold of Ω, and DGN sits in θ as a finite union of lower dimensional submanifolds. For $p = 2$, DGN is generically a finite set of states none of which are economic optima.

We conclude this section with a more geometric characterization of $\theta \setminus \mathrm{DGN}$. By Theorem 2, if $x \notin \theta$ and $x \in \dot{\Omega}$, there is a straight line (or 'first-order') trade curve from x, i.e., $\gamma(t) = x + tv$ for small t. Theorem 6 below states that if x is in θ but is not in DGN and is not a TO^+, then there is a 'second-order' trade curve from x, i.e., a curve of the form $\gamma(t) = x + tv + (t^2/2)w$. Presumably, if x is in DGN and DGN is a submanifold around x and x is not a critical point of $u \mid \mathrm{DGN}$, then if there is a trade curve from x, there is a third-order trade curve; and so on.

Theorem 6. Let u be a utility mapping satisfying the Axioms A_2. Suppose x is in $\dot{\Omega}$ but not in DGN. Then, there is a directed trade curve from x if and only if there is a curve of the form $\gamma(t) = x + tv + (t^2/2)w$, for small t and for some v and w in $T\Omega_x$.

Proof. If $x \notin \theta$, we can take $w = 0$ by Theorem 2. Suppose then that x is in $\theta(u)$. Let $\gamma(t)$ be a directed trade curve with $\gamma(0) = x$ and $\gamma'(0) = v = (v_1, \ldots, v_p) \in (\mathbf{R}^c)^p$ with $\sum v_i = 0$. The $u_i(\gamma(t))$ non-decreasing implies that $Du_i(x)v \geq 0$ for all i. If

$$v \notin K_x = \bigcap_{i=1}^{p} \ker Du_i(x),$$

some $Du_i(x)v > 0$. Since each $\ker Du_j(x)$ is codimension one in $T\Omega_x$, one can perturb v to v' in $T\Omega_x$ with $Du_j(x)v' > 0$ for all j. For small t, $\tilde{\gamma}(t) = x + tv'$ will be directed trade curve from x.

If $\gamma'(0) = v$ is in K, say $\gamma(t) = x + tv + (t^2/2)w + \ldots$. As in the proof of

Theorem 5, one finds

$$\mathrm{D}u_i(x)w+\mathrm{D}^2u_i(x)(v, v) \geq 0 \quad \text{for all } i,$$

and

$$F_x^2(v, v) = \sum \lambda_i \,\mathrm{D}^2u_i(x)(v, v) \geq 0.$$

Since $x \notin \mathrm{DGN}$, there is a v' in K_x arbitrarily close to v with $F_x^2(v', v') > 0$.

Now

$$\mathrm{D}u_i(x)w = \mathrm{D}\bar{u}_i(x^i)w_i,$$

where $u_i = \bar{u}_i \circ \pi_i$, $x = (x^1, \ldots, x^p)$, and $w = (w_1, \ldots, w_p)$. Choose $w' = (w'_1, \ldots, w'_p)$ with $\sum_j w'_j = 0$ and

$$\begin{aligned}\mathrm{D}u_i(x)w'+\mathrm{D}^2u_i(x)(v', v') &= \mathrm{D}\bar{u}_i(x^i)w'_i+\mathrm{D}^2u_i(x)(v', v')\\ &= 0 \quad \text{for} \quad i = 2, \ldots, p.\end{aligned}$$

Then,

$$\mathrm{D}u_1(x)w'+\mathrm{D}^2u_1(x)(v', v') = F_x^2(v', v') > 0.$$

Finally, choose $w'' = (w''_1, \ldots, w''_p)$ with $\sum_i w''_i = 0$,

$$\mathrm{D}u_i(x)w''+\mathrm{D}^2u_i(x)(v', v') > 0 \quad \text{for} \quad i = 2, \ldots, p,$$

and w'' so close to w' that $\mathrm{D}u_1(x)w''+\mathrm{D}^2u_1(x)(v', v')$ is still > 0. Now, let $\tilde{\gamma}(t) = x+tv'+(t^2/2)w''$. $\tilde{\gamma}(t)$ is a second-order directed trade-curve through x. ∎

6. Characterizations of LPO and TO in θ: Geometric techniques

In this section, we'll use some elementary geometry to improve the characterization of section 5. We'll rely on the properties described at the end of section 4, i.e., that the level surfaces of each u_i give a smooth codimension one partition or *foliation* of $\dot{\Omega}$, that on θ any $p-1$ of the level surfaces always meet transversally, while off θ all p of them meet transversally. All the results of this section are true under the weaker Axioms A'_n, i.e., no grad $u_i(x)$ is ever zero and the rank of $\mathrm{D}(u_1, \ldots, u_p)(x)$ is always greater than or equal to $p-1$.

The following lemma will be useful in this section. It enables us to replace the weak inequalities in the definition of PO and LPO by strong inequalities.

Lemma 1. $x \in \Omega$ *is a PO if and only if there is no* y *in* Ω *with* $u_i(y) > u_i(x)$ *for* all i; *x is an LPO if and only if for some neighborhood* W *of* x *there is no* y *in* W *with* $u_i(y) > u_i(x)$ *for all* i.

Proof. If x is not a PO there is a z in Ω with $u_i(z) \geq u_i(x)$ for all i and $u_k(z) > u_k(x)$ for some k. Since $u_1^{-1}(u_1(z))$ is a codimension one submanifold of Ω, we can find $z' \ni u_1(z') > u_1(z) \geq u_1(x)$ with z' still in the open set $u_k^{-1}(u_k(x), \infty)$ and $u_i(z') \geq u_i(x)$ for $i \neq 1, k$. Keep working one subindex at a time until a y is found with $u_i(y) > u_i(x)$ for all i. A similar argument works for LPO's with z', y, etc. in W. ▌

Since we'll be changing coordinate systems shortly, we will need to keep in mind the following lemma. We omit a formal proof since it follows directly from Theorems 1 and 2 and is a consequence of the fact that S_1 and θ can be defined solely in terms of the level surfaces of the u_i and their positive and negative sides.

Lemma 2. $S_1(u)$ and θ are independent of the choice of coordinate system or metric used on Ω.

Now, fix $x \in \theta$. Grad $u_1(x), \ldots,$ grad $u_{p-1}(x)$ are linearly independent vectors in $T\Omega_x$ throughout Ω. So, we can use $u_1, \ldots, u_{p-1}$ as part of a coordinate system about x. Choose coordinates $y_1, y_2, \ldots, y_{c(p-1)}$ in a nbhd. V about x, so that

$$
\begin{aligned}
y_i(x) &= 0 && \text{for all } i, \\
y_i &= u_i - u_i(x) && \text{for } i = 1, 2, \ldots, p-1.
\end{aligned}
$$

For $i < p$, $u_i(y_1, \ldots, y_{c(p-1)}) = y_i$ and $Du_i(y) =$ projection of the ith factor for all $y \in V$. Consequently, $D^2 u_i(y) = 0$ for all $i < p$ and for all $y \in W$. Note that $D^2 u_i(y)$ can be represented by the symmetric matrix

$$\left(\left(\frac{\partial^2 u_i}{\partial y_j \partial y_k}\right)\right).$$

So,

$$F_x^2 = \sum_{i=1}^{p} \mu_i \, D^2 u_i(x) = \mu_p \, D^2 u_p(x)$$

in these coordinates, where $\mu_1, \ldots, \mu_p$ are positive. These are bilinear maps on $K_x \times K_x$, where

$$K_x = \bigcap_{i=1}^{p-1} \ker Du_i(x)$$

can be viewed as the space $y_1 = 0, \ldots, y_{p-1} = 0$. The space K_x is also $T_x U_x^{(i)}$ for each i. Since we could have used any u_i for u_p in this paragraph, the following proposition follows:

Proposition 2. Let $x \in \theta \cap \dot{\Omega}$. So there exist $\lambda_1, \ldots, \lambda_p$ positive, such that

$$\sum_1^p \lambda_i \, Du_i(x) = 0.$$

(1) The index of the bilinear form

$$F_x^2 = \sum \lambda_i \, \mathrm{D}^2 u_i(x) : K_x \times K_x \to \mathbf{R}$$

is the same as the index of x as the critical point of *each* $u_k \mid U_x^{(k)}$.

(2) In particular, F_x^2 is negative definite on K_x if and only if x is a non-degenerate maximum for each of the functions $u_k \mid U_x^{(k)} : U_x^{(k)} \to \mathbf{R}$, $k = 1, 2, \ldots, p$.

To tie all this information in with our characterizations of TO's and LPO's, we will need the following lemma. Its proof follows very simply from standard techniques in the study of simplices and simplicial complexes and will be omitted.

Lemma 3. Let U be a neighborhood of 0 *in* $\mathbf{R}^n$ *and let* $f\colon U \to \mathbf{R}$ *be a smooth function with*

(a) 0 *a regular value*;
(b) $f^{-1}(0)$ *meeting each* $x_i = 0$ *transversally in* U;
(c) $\operatorname{grad} f(x) \cdot (\partial/\partial x_i) \leq \alpha < 0$ *for all* $x \in f^{-1}(0) \cap U$ *and for all i.*

[So the normal to $f^{-1}(0)$ always points into the negative orthant.] Then, (1) if $0 \in f^{-1}(0)$, near $0, f^{-1}(0)$ divides each orthant of $\mathbf{R}^n$ into 2 components except the positive and negative orthants, both of which it misses (fig. 4).

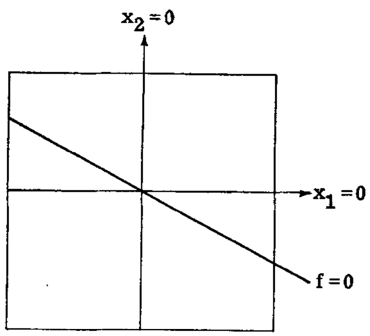

Fig. 4

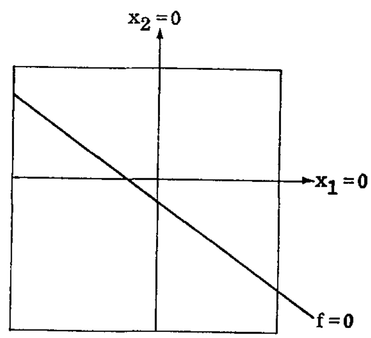

Fig. 5

(2) If $0 \notin f^{-1}(0)$, the hypersurfaces $f = 0$, $x_1 = 0, \ldots, x_n = 0$ divide some neighborhood V of 0 into $2^n - 1$ regions with all but one of the regions meeting the boundary of V and the remaining region, lying in the interior of V and in either the positive or negative orthant of $\mathbf{R}^n$, diffeomorphic to a standard simplex. The union of the intersections of any k of the sets $\{f = 0\}, \ldots, \{x_n = 0\}$ make up the $n-k$ skeleton (fig. 5).

We can now give a characterization of LPO's – including even those that lie in DGN.

Theorem 7. *For $x \in \dot{\Omega}$, the following are equivalent*:

(1) *x is an LPO.*
(2) *$x \in \theta$ and x is a weak local max. for every $u_k \mid U_x^{(k)}$.*
(3) *$x \in \theta$ and x is a weak local max. for some $u_k \mid U_x^{(k)}$.*

Proof. (1) $\Rightarrow$ (2) If x is an LPO, $x \in \theta$ by Corollary 1. If $\exists k \ni u_k \mid U_x^{(k)}$ does not have a weak local max. at x, then $\exists y$ arbitrarily close to x on $U_x^{(k)}$ with $u_k(y) > u_k(x)$. $y \in U_x^{(k)}$ implies $u_i(y) = u_i(x)$ for $i \neq k$. So, $x \notin$ LPO.

(2) $\Rightarrow$ (3) Trivially.

(3) $\Rightarrow$ (1) Without loss of generality, take $p = k$ in (3). $\nabla u_1, \ldots, \nabla u_{p-1}$ are independent throughout Ω. On a nbhd. N of x, choose coordinates $y_1, y_2, \ldots, y_{c(p-1)}$, so that $y_i = u_i - u_i(x)$ for $i = 1, 2, \ldots, p-1$, as we did in proving Proposition 2.

Choose a Riemannian metric m on N so that $\partial/\partial y_1, \ldots, \partial/\partial y_{c(p-1)}$ are orthonormal on N. So,

$$\operatorname{grad}_m u_1 = \partial/\partial y_1, \ldots, \operatorname{grad}_m u_{p-1} = \partial/\partial y_{p-1}$$

throughout N. Since θ is independent of choice of metric by Lemma 2 and since $x \in \theta$ there exist $r_1, \ldots, r_{p-1}$ all positive, so that

$$(+) \qquad \operatorname{grad}_m u_p(x) = - \sum_{i=1}^{p-1} r_i \operatorname{grad}_m u_i(x).$$

For $z \in N$, define P_z as the $(p-1)$ submanifold through z in N, satisfying

$$\begin{cases} y_p & = \text{constant}, \\ y_{p+1} & = \text{constant}, \\ \quad\vdots & \quad\vdots \\ y_{c(p-1)} & = \text{constant}. \end{cases}$$

Now, $\operatorname{grad}_m u_i(x)$ lies along P_x for $i = 1, \ldots, p$ with $\operatorname{grad}_m u_i(x)$ along the y_i-axis for $i < p$.

Using (+), we find that

$$\langle \operatorname{grad}_m u_p(x), \operatorname{grad}_m u_i(x) \rangle_m < 0 \quad \text{for all} \quad i < p,$$

i.e., $\operatorname{grad}_m u_p(x)$ points into the negative orthant of P_x.

Choose N so that all $z \in \bar{N}$:

1. the projection of $\mathrm{grad}_m\, u_p(z)$ on P_z points into the open negative orthant of P_z;
2. $u_p^{-1}(u_p(x))$ meets P_z and ∂P_z transversally with $u_p^{-1}(u_p(x)) \cap P_z$ a connected submanifold of codimension one in P_z separating P_z into exactly two components ($u_p > u_p(x)$ and $u_p < u_p(x)$) and meeting each $y_i = 0$ in P_z transversally for $i = 1, \ldots, p-1$;
3. if $z \in U_x^{(k)} \cap N$, $u_k(z) \leq u_k(x)$ for $k = 1, \ldots, p$.

For $j \leq p$, define $P_z^j = P_z \cap u_j^{-1}(u_j(x))$. So

$$P_z^j = \{y \mid y_j = 0\} \cap P_z \quad \text{for} \quad j \leq p-1.$$

The following two claims follow from Lemma 3 with $f = u_p$ and $x_i = y_i$.

Claim 1. If P_z^p contains the origin of P_z, then P_z^p divides every orthant of P_z but the positive and negative orthants. So, P_z is divided into 2^p-2 regions each of which meets ∂P_z (fig. 4 above).

Claim 2. If P_z^p does not contain the origin of P_z, then P_z is divided into 2^p-1 regions with exactly one region not meeting ∂P_z. This exceptional region is diffeomorphic to a standard $(p-1)$-simplex whose $(p-2)$ skeleton is composed of pieces of the P_z^j's and each of whose p vertices is an intersection of $(p-1)$ P_z^j's, i.e., an element of some $U_x^{(k)}$.

By Lemma 1, we need only show that

$$\bigcap_{i=1}^{p} u^{-1}(u_i(x), \infty) \cap N = \phi \quad \text{for } x \text{ to be an LPO.}$$

Suppose this intersection contains some z. Look at P_z.

Case 1. If P_z^p contains the origin of P_z, then

$$\bigcap_{i=1}^{p-1} u_i^{-1}(u_i(x), \infty) \cap P_z$$

is the open positive orthant of P_z while P_z^p does not meet this orthant and $\mathrm{grad}_m\, u_p$ points into the negative orthant at the origin. Thus,

$$\left[\bigcap_{i=1}^{p-1} u_i^{-1}(u_i(x), \infty) \cap P_z\right] \bigcap \left[u_p^{-1}(u_p(x), \infty) \cap P_z\right] = \phi.$$

Case 2. If P_z^p does not contain the origin of P_z, the only region that can correspond to

$$\bigcap_{i=1}^{p} u_i^{-1}(u_i(x), \infty)$$

in P_z is the $(p-1)$-simplex of Claim 2.

The origin w of P_z corresponds to $U_x^{(p)} \cap P_z$, as in fig. 6. Since $w \notin P_z^p$, $u_p(w) \neq u_p(x)$.

Since $u_p \mid U_x^{(p)}$ has a weak local max. at x, then $u_p(w) < u_p(x)$.

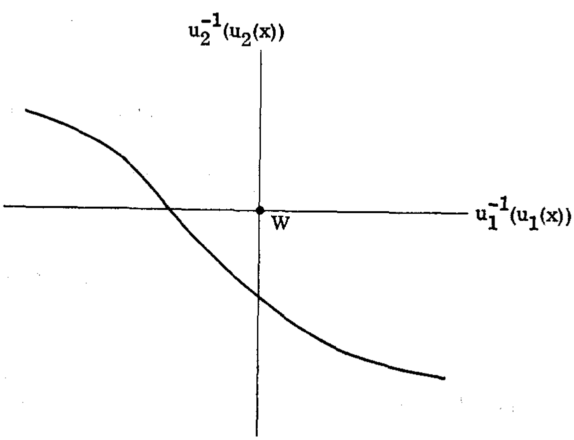

Fig. 6. P_z for $p = 3$.

This simplex lies on the negative side of $u_p^{-1}(u_p(x))$ so

$$\bigcap_{i=1}^{p} u_i^{-1}(u_i(x), \infty) \cap P_z = \phi,$$

and x is an LPO. Q.E.D.

We can drop the condition that $x \in \theta$.

Theorem 8. Let $x \in \dot{\Omega}$. Suppose that for each k, $u_k \mid U_x^{(k)}$ has a weak local max. at x; and for some k, $u_k \mid U_x^{(k)}$ is not locally constant at x. Then $x \in LPO$.

Proof. If $x \in \theta$, this follows from Theorem 7. The fact that x is a critical point for some $u_k \mid U_x^{(k)}$ insures that $x \in S_1(u)$. If $x \in S_1(u)\backslash\theta$, $\text{grad}_m\, u_p(x)$ would not point into the closed negative orthant in P_x in the above proof. Then, either $\mathrm{grad}_m\, u_p(x)$ points into the open positive orthant of P_x or P_x^p divides the positive orthant of P_x into 2 regions.

Choose p so that $u_p \mid U_x^{(p)}$ is not locally constant at x. So

$$\exists z \in U_x^{(p)} \quad \text{with} \quad u_p(z) < u_p(x).$$

P_z^p does not contain the origin z of P_z and a simplex appears in P_z as before (figs. 5 and 6). [Note that Lemma 3 holds as long as $\mathrm{grad}\, f(x)$ points into the same open orthant for all $x \in f^{-1}(0)$.]

If the simplex lies in the negative orthant, then $\mathrm{grad}_m\, u_p(x)$ points into the positive orthant and $u_p(z) > u_p(x)$, a contradiction.

If the simplex lies in some other orthant, there is a vertex w' not in the closure of the negative orthant; and for some k $w' \in U_x^{(k)}$ with $u_k(w') > u_k(x)$, a contradiction. $x \in \theta$ and the theorem follows. ▌

As a final geometric characterization of optima, we have the following:

Theorem 9. Suppose $x \in \dot{\Omega}$ and $x \notin S_1 \setminus \theta$. Then, if there is a trade curve from γ from x, there is a trade curve on every $U_x^{(k)}$.

Corollary 2. Suppose $x \in \dot{\Omega}$ and $x \notin S_1 \setminus \theta$. Then, x is a TO iff x is a TO for each $u_k \mid U_x^{(k)}$.

Proof of Theorem. If $x \notin S_1$, $U_x^{(k)}$ is transverse to $u_k^{-1}(u_k(x))\, \forall k$. Thus, for each k, there is a smooth curve on $U_x^{(k)}$ on which u_k is strictly increasing, i.e., a trade curve.

Suppose $x \in \theta$.

Construct coordinate system $y_1, \ldots, y_{c(p-1)}$, metric m, and submanifolds P_z as in the above theorems. Let $\gamma : [0, \varepsilon] \to \Omega$ be a trade curve with $\gamma(0) = x$. For each t, $\gamma(t)$ lies in the simplex

$$\bigcap_{i=1}^{p} u_i^{-1}[u_i(x), \infty)$$

in $P_{\gamma(t)}$ which lies in the positive orthant. For each k, one can take the new curve $\sigma_k : [0, \varepsilon] \to \Omega$, where

$$\sigma_k(t) \in U_x^{(k)} \cap P_{\gamma(t)}\ \forall t,$$

i.e.,

$$\sigma_k(t) = \bigcap_{j \neq k} P_{\gamma(t)}^j .$$

σ_k will be a trade curve from x. ▌

Remark. For $p = 2$, we could easily have

(1) $x \in S_1 \setminus \theta$,
(2) $u_1^{-1}(u_1(x)) \cap N = u_2^{-1}(u_2(x)) \cap N$ for some neighborhood N of x.

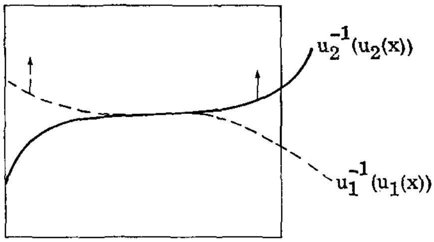

Fig. 7

Then, x is not a TO or LPO, but there is no trade curve on $u_1^{-1}(u_1(x))$ since u_1 and u_2 are constant there near x.

Finally, for x not in DGN the following proposition summarizes some of the results of the last two sections. For the optima that may lie in DGN, Theorems 7 and 8, and Corollary 2 apply.

Theorem 10. Suppose $x \in \dot{\Omega}$ and x is $\theta \setminus$ DGN. Suppose u satisfies the Axioms A_2 or A_2'. Then, the following are equivalent.

(1) $F_x^2 : K_x \times K_x \to \mathbf{R}$ *is negative definite.*
(2) *Each* $\mathrm{D}^2(u_k \mid U_x^{(k)}) : T_x U_x^{(k)} \times T_x U_x^{(k)} \to \mathbf{R}$ *is negative definite.*
(3) *x is a non-degenerate local maximum for each*

$$u_k \mid U_x^{(k)} : U_x^{(k)} \to \mathbf{R}, \qquad k = 1, \ldots, p.$$

(4) *x is an LPO.*
(5) *x is a TO.*
(6) *x is a TO$^+$.*

7. Generic properties of utility mappings

In this section, we will use some of the simpler tools of transversality theory and singularity theory to describe how $S^1(u)$, $\theta(u)$, and DGN sit in Ω for generic utility mappings $u: \Omega \to \mathbf{R}^p$ satisfying Axioms A_2 or A_2'. For some excellent survey articles on singularities and transversality, see Wall (1970) [especially, Levine (1970)] and Golubitsky–Guillemin (1974). Nearly all of this chapter is a result of conversations and correspondence with Jean Martinet, to whom we express our deep gratitude.

The main result of this section is that $\theta(u)$ is generically a $(p-1)$-dimensional submanifold of Ω and that DGN sits in $\theta(u)$ as a union of lower-dimensional algebraic sets. Smale (1974b) sketches a proof of the first statement and Wan (1973) handles DGN for $p = 2$. We have included this chapter not only for the sake of completeness but also to attempt to demystify the use of transversality techniques in economics. We include a simple, but complete, two paragraph proof of Thom's transversality theorem for finite-dimensional spaces of mappings and then we indicate how one can use finite-dimensional function spaces instead of infinite-dimensional ones to show generic properties of mappings and jets.

Let $f, g : \Omega \to \mathbf{R}^p$ be smooth (C^∞) maps with $x \in \Omega$ and $r \geq 0$ an integer. This means, of course, that f and g are defined and are C^∞ on some open neighborhood of Ω in

$$\mathbf{R}^{c(p-1)} = \{(x^1, \ldots, x^p) \in \mathbf{R}^{cp} \mid \sum x^i = a\}.$$

Write $f \sim_r g$ at x if $f(x) = g(x)$ and all partial derivatives of order $\leq r$ of f and of g agree at x. The partials are calculated with respect to some choice of local coordinates about x in Ω and about $f(x)$ in $\mathbf{R}^p$. So, $f \sim_r g$ at x if and only if the Taylor expansions of f and of g about x coincide up to and including the rth order terms. Such an equivalence class is called an r-jet at $(x, f(x))$ of maps $\Omega \to \mathbf{R}^p$.

If we use local coordinates at source and target, together with the partial derivatives listed above, to give charts, we can consider the space of all r-jets $\Omega \to \mathbf{R}^p$ as a smooth manifold and we denote it by $J^r(\Omega, \mathbf{R}^p)$. If we have a smooth map $f\colon \Omega \to \mathbf{R}^p$ and an x in Ω, the equivalence class of f at x in $J^r(\Omega, \mathbf{R}^p)$, written $j^r f(x)$, is an r-jet with source x. Letting x vary over Ω defines a smooth map (or 'section') $j^r f\colon \Omega \to J^r(\Omega, \mathbf{R}^p)$, called the *r-jet* of f.

We use r-jets to give a convenient topology to $C^r(\Omega, \mathbf{R}^p)$. For $r < \infty$, we give $C^r(\Omega, \mathbf{R}^p)$ the topology for which the following is a base of open subsets: $\{f\colon \Omega \to \mathbf{R}^p \mid j^r f(\Omega) \subset U\}$ for all open sets U in $J^r(\Omega, \mathbf{R}^p)$. Since Ω is compact, convergence in this topology is equivalent to uniform convergence of f with all its derivatives of order $\leq r$. The C^∞ topology on $C^\infty(\Omega, \mathbf{R}^p)$ is defined by taking the above for all r as a base of open subsets.

Suppose $g : \Omega \to N$ is a smooth map of smooth manifolds and Σ is a smooth submanifold of N. The map g is *transverse* to Σ if whenever $g(x) \in \Sigma$,

$$\mathrm{D}g(x)(T\Omega_x) + T\,\Sigma_{g(x)} = TN_{g(x)},$$

i.e., the image of Ω by g around $g(x)$ fills up a space complementary to Σ in N. It is a straightforward consequence of the implicit function theorem [e.g., Abraham (1967), Golubitsky–Guillemin (1974)] that when g is transverse to Σ, $g^{-1}(\Sigma)$ (if non-empty) is a submanifold of Ω whose dimension is equal to dim $\Omega +$ dim $\Sigma -$ dim N.

To illustrate the last paragraph, let us see what it states for linear maps. If E and F are finite-dimensional vector spaces, Σ is a subspace of F, and $A :\ E \to F$

is a linear map, then A is transverse to Σ if $A(E)+\Sigma = F$. Let G be a subspace of F with $\Sigma \oplus G = F$ and let π denote the projection of F onto G. 'A transverse to Σ' is equivalent to $\pi \circ A : E \to G$ being surjective. In this case, by a simple theorem of linear algebra, codimension of Σ in F = dim G = dim range $(\pi \circ A)$ = dim domain $(\pi \circ A)-$dim kernel $(\pi \circ A)$ = dim $E-$dim $A^{-1}(\Sigma)$ = codim $A^{-1}\Sigma$ in E. On manifolds, $g : \Omega \to N$ is transverse to Σ if whenever $g(x) \in \Sigma$, $\mathrm{D}g(x) : T\Omega_x \to TN_{g(x)}$ is transverse to $T\Sigma_{g(x)}$ as a linear map of linear spaces.

The following theorems of Thom are the central theorems of singularity and transversality theory. The proofs we use are basically Thom's original ones but seem much simpler than some more recent proofs [e.g., Abraham (1967)] because they use a finite-dimensional function space instead of the whole infinite-dimensional one.

Theorem 11 (Thom). Let M, P, and N be finite-dimensional manifolds with S a closed submanifold of N and M compact. Let $F: M \times P \to N$ be a C^∞ map with F transverse to S ($F \pitchfork S$). Then, for an open, dense of $p \in P$, the map $F_p : M \to N$ by $F_p(x) = F(x, p)$ is transverse to S.

[So, P is considered as a parameter space of maps from M to N.]

Proof. Openness follows easily from the fact that S is closed, M is compact, and transversal intersection is locally an open property.

To achieve the density, let $L = F^{-1}(S)$, a submanifold of $M \times P$ since $F \pitchfork S$. Let $\pi : M \times P \to P$ be the projection and let $\tilde{\pi} : L \to P$ be $\pi \mid L$. By Sard's theorem [Spivak (1965), Abraham (1967)], the regular values of $\tilde{\pi}$ are dense in P. So, we need only show that if p is a regular value of $\tilde{\pi}$, $F_p \pitchfork S$.

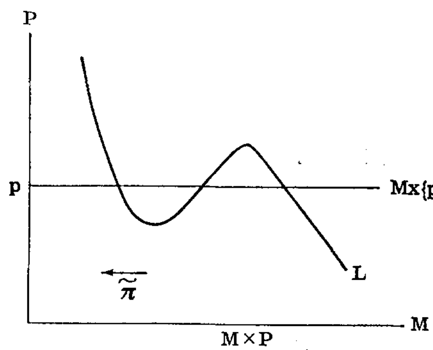

Fig. 8

Let $x \in M$ be such that $F_p(x) \in S$ and p is a regular value of $\tilde{\pi}$. By the latter condition, $\mathrm{D}\tilde{\pi}(x, p) = \tilde{\pi} : TL_{(x,p)} \to T_pP$ is surjective. Since

$$T(M \times P)_{(x,p)} \cong T(M \times \{p\})_{(x,p)} \oplus T(\{x\} \times P)_{(x,p)},$$

this means that

$$T(M\times\{p\})_{(x,p)}+TL_{(x,p)} = T(M\times P)_{(x,p)}.$$

Since $\mathrm{D}F(x,p)$ is linear and $F \pitchfork S$,

$$\mathrm{D}F(x,p)[T(M\times\{p\})_{(x,p)}]+\mathrm{D}F(x,p)(TL_{(x,p)})$$

contains a complement to $TS_{F_p(x)}$ in $TN_{F_p(x)}$. Since

$$\mathrm{D}F(x,p)(TL_{(x,p)}) \subseteq TS_{F_p(x)},$$

this means that

$$\mathrm{D}F(x,p)[T(M\times\{p\})_{(x,p)}] = \mathrm{D}F_p(x)[TM_x]$$

contains a complement to $TS_{F_p(x)}$ in $TN_{F_p(x)}$, i.e., that F_p is transverse to S. Q.E.D.

The Thom Jet Transversality Theorem follows easily from this simple transversality theorem.

Theorem 12 (Thom). Consider $C^\infty(M,\mathbf{R}^p)$ with the C^{r+1}-topology. Let Σ be any closed submanifold of $J^r(M,\mathbf{R}^p)$. The set of mappings f in $C^\infty(M,\mathbf{R}^p)$ such that $j^r(f): M\to J^r(M,\mathbf{R}^p)$ is transverse to Σ is open and dense in $C^\infty(M,\mathbf{R}^p)$.

Proof. Let $f\in C^\infty(M,\mathbf{R}^p)$. Embed M in $\mathbf{R}^m$. Let $\mathscr{P}_r$ be the finite-dimensional vector space of all polynomial mappings $\mathbf{R}^m\to\mathbf{R}^p$ of degree $\leqq r$. Define $F: M\times\mathscr{P}_r\to\mathbf{R}^p$ by

$$F(x,P) = f(x)+P(x).$$

Define $\hat{F} = j_1^r F: M\times\mathscr{P}_r\to J^r(M,\mathbf{R}^p)$ by $\hat{F}(x,P)$ as the Taylor series at x (up to and including terms of order r) of $f+P \mid M$. Since $TJ^r(M,\mathbf{R}^p)_{j^rf(x)}$ can be viewed as

$$\{j_rf(x)+P(x) \mid P\in\mathscr{P}_r\},$$

$\hat{F}$ is easily seen to be a submersion, i.e., its derivative is onto at each point. So $\hat{F}$ is transverse to any submanifold of $J^r(M,\mathbf{R}^p)$. Now, use the previous theorem to see that one can perturb f to g such that $j^r(g)\pitchfork\Sigma$ and that the set of all such g's is open in $C^\infty(M,\mathbf{R}^p)$. Q.E.D.

Now, let us try to state generic properties of $S^1(u)$ and $S(S^1(u))$ for utility mappings u by using Theorem 12. Because our utility mappings must satisfy Axioms A3 and A4 or Axioms A3′ and A4, we are not dealing with all of $C^\infty(\Omega, \mathbf{R}^p)$. We have

$$\Omega = \left\{(x^1, \ldots x^p) \in (\mathbf{R}^c)^p \,\middle|\, \sum_1 x^i = a\right\},$$

for some $a \in \mathbf{R}^C_+$. So $(x^1, \ldots, x^{p-1})$ give a good coordinate system for Ω. Let $C^\infty_S(\Omega, \mathbf{R}^p)$ be the subset of $C^\infty(\Omega, \mathbf{R}^p)$ consisting of maps $f = (f_1, \ldots, f_p) : \Omega \to \mathbf{R}^p$ such that

$$f_1(x^1, \ldots, x^{p-1}) = \hat{f}_1(x^1), \ldots, f_{p-1}(x^1, \ldots, x^{p-1}) = \hat{f}_{p-1}(x^{p-1}),$$

and

$$f_p(x^1, \ldots, x^{p-1}) = \hat{f}_p(a - x^1 - \ldots - x^{p-1}) = \hat{\hat{f}}_p(x^1 + \ldots + x^{p-1}),$$

for some C^∞ maps $\hat{f}_1, \ldots, \hat{f}_p, \hat{\hat{f}}_p : \mathbf{R}^c \to \mathbf{R}$. In fact, there is a natural bijection between $C^\infty_S(\Omega, \mathbf{R}^p)$ and $C^\infty(\mathbf{R}^c, \mathbf{R}) \times \ldots$ (p-times) $\ldots \times C^\infty(\mathbf{R}^c, \mathbf{R})$.

Since our utility mappings must satisfy Axiom A4, we will use $\mathscr{C}$ to denote the open subset of $[C^\infty(\mathbf{R}^c, \mathbf{R})]^p$ of $(\bar{u}_1, \ldots, \bar{u}_p)$ such that $\mathrm{D}\bar{u}_i$ is never zero. Since $\mathrm{D}\bar{u}_p$ is never zero and since transversality arguments are always local, we can assume without loss of generality that $\bar{u}_p(x_1, \ldots, x_c) = x_1$. By Theorem 1, $\mathbf{x} = (x^1, \ldots, x^{p-1})$ in Ω is in $S^1(u)$ if and only if there exist non-zero real numbers $\lambda_1, \ldots, \lambda_{p-1}$ such that

$$(*) \qquad \mathrm{D}\bar{u}_i(x^i) = \lambda_i\, \mathrm{D}\bar{u}_p(x^1 + \ldots + x^{p-1}).$$

This is equivalent to

$$(**) \qquad \frac{\partial \bar{u}_i}{\partial x^i_j}(x^i) = 0 \qquad \text{for } i = 1, \ldots, p-1 \quad \text{and} \quad j = 2, \ldots, c.$$

Theorem 13. Let $C^\infty_S(\Omega, \mathbf{R}^p)$ be the space of C^∞ utility mappings $\Omega \to \mathbf{R}^p$ that satisfy the Axioms A_2. For u in $C^\infty_S(\Omega, \mathbf{R}^p)$, let $S^1(u)$ and $\theta(u)$ denote the singularity sets defined in section 4. For an open, dense set of utility mappings u, $S^1(u)$ and $\theta(u)$ are $(p-1)$-dimensional submanifolds of $\dot{\Omega}$ (possibly empty).

Proof. We will sketch two (different) approaches to Theorem 13. First, since the $\bar{u}_1, \ldots, \bar{u}_p$ vary independently in $C^\infty(\mathbf{R}^c, \mathbf{R})$, the equations (**) are $(p-1)$ $(c-1)$ independent equations for generic $(\bar{u}_1, \ldots, \bar{u}_p)$ in $(C^\infty(\mathbf{R}^c, \mathbf{R}))^p$. Thus, for an open and dense set of $(\bar{u}_1, \ldots, \bar{u}_p)$, (**) define a submanifold $S^1(u)$ of Ω with codimension $(p-1)(c-1)$, i.e., dimension $c(p-1)-(p-1)(c-1) = p-1$.

More formally, the equations (**) define a closed submanifold $\mathscr{S}$ of $J^1(\mathbf{R}^c, \mathbf{R})\times\ldots(p-1 \text{ times})\ldots\times J^1(\mathbf{R}^c, \mathbf{R})$. For each $(\bar{u}_1, \ldots, \bar{u}_{p-1})$ in $[C^\infty(\mathbf{R}^c, \mathbf{R})]^{p-1}$, the mapping $j^1\bar{u} : \Omega \to [J^1(\mathbf{R}^c, \mathbf{R})]^{p-1}$ by

$$(x^1, \ldots, x^{p-1}) \to (j^1\bar{u}_1(x^1), \ldots, j^1\bar{u}_{p-1}(x^{p-1}))$$

is generically transverse to $\mathscr{S}$ by a proof analogous to that of Theorem 12. $S^1(u)$ is $(j^1\bar{u})^{-1}\mathscr{S}$.

Alternatively, one can work with $C_S^\infty(\Omega, \mathbf{R}^p)$ and with $J_S^1(\Omega, \mathbf{R}^p)$, the space of 1-jets of such mappings. For example, if $u = (u_1, \ldots, u_p)$ is in $C_S^\infty(\Omega, \mathbf{R}^p)$,

$$\mathrm{D}u(x) = \begin{bmatrix} \mathrm{D}\bar{u}_1(x^1) & 0 \;\ldots & 0 \\ 0 & \mathrm{D}\bar{u}_2(x^2)\ldots & 0 \\ \cdot & 0 & \mathrm{D}\bar{u}_{p-1}(x^{p-1}) \\ \cdot & \cdot & \\ -\mathrm{D}\bar{u}_p(x^p) & -\mathrm{D}\bar{u}_p(x^p) & -\mathrm{D}\bar{u}_p(x^p) \end{bmatrix}.$$

So, elements in $J_S^1(\Omega, \mathbf{R}^p)$ would be of the form

$$\{(x, y, A) \mid x \in \Omega, y \in \mathbf{R}^p), \quad A = \begin{bmatrix} A_1 & & 0 \\ & A_2 & \\ 0 & & \ddots A_{p-1} \\ A_p & A_p \ldots & A_p \end{bmatrix},$$

where the A_i are arbitrary $1\times c$ matrices$\}$. Let $\mathscr{A}$ be the set of all such matrices. Let Σ_S denote all 1-jets for which $\mathrm{D}u(\mathbf{x})$ has corank one. So Σ_S is the codimension $cp-p-c+1$ submanifold of $J_S^1(\Omega, \mathbf{R}^p)$ corresponding to those matrices A in $\mathscr{A}$, for which there exist non-zero constants $\lambda_1, \ldots, \lambda_{p-1}$ with $A_i = \lambda_i A_p$. Now, replace Σ, $\mathscr{P}_1$ and $J^1(M, \mathbf{R}^p)$ by Σ_S, $\mathscr{A}$, and $J_S^1(\Omega, \mathbf{R}^p)$ in the statement and proof of Theorem 12 to obtain Theorem 13.

Since $\theta(u)$ is open and closed in $S^1(u)$, it is generically a $(p-1)$-dimensional submanifold of Ω, also. Q.E.D.

Theorem 14. Let $C_S^\infty(\Omega, \mathbf{R}^p)$ be as above. For utility mapping u in $C_S^\infty(\Omega, \mathbf{R}^p)$, let DGN($u$) denote those x in $S^1(u)$ for which the second intrinsic derivative is degenerate. For generic u in $C_S^\infty(\Omega, \mathbf{R}^p)$, DGN($u$) is an 'algebraic-like' subset of Ω of dimension $p-2$, if non-empty. Thus, for $P = 2$, DGN(u) is generically a finite set of points, none of which are economic optima.

Remark. Recall that DGN(u) is the critical set of $u \mid S^1(u)$. Also, an algebraic set of codimension d of a manifold is a set defined as the zeroes of d independent real-valued polynomials. We call a set 'algebraic-like' if it is the inverse image of an algebraic set under a smooth map that is transverse to the algebraic set.

Proof. Again, let $u = (u_1, \ldots, u_p)$ be a utility mapping and let $\bar{u}_1, \ldots, \bar{u}_p$ denote the corresponding maps $\mathbf{R}^c \to \mathbf{R}$ with non-zero differential. Again, choose coordinates so that $\bar{u}_p(x_1, \ldots, x_c) = x_1$. Let $\alpha^i_j : \mathbf{R}^c \to \mathbf{R}$ be $\partial \bar{u}_i / \partial x^i_j$, for $i = 1, \ldots, p-1$ and $j = 2, \ldots c$. So $S^1(u)$ is defined by all $\alpha^i_j = 0$. To make our computations simpler and our notation less cumbersome, we will use the calculus k-forms and their wedge-product; see Spivak (1965). In particular, k covectors $\beta_1, \ldots, \beta_k$ are linearly independent if and only if the k-form $\beta_1 \wedge \ldots \wedge \beta_k$ is non-zero.

We will keep $\bar{u}_p$ fixed as above and work in $[C^\infty(\mathbf{R}^c, \mathbf{R})]^{p-1}$. Let $S^{1,1}(u)$ denote the set of points x in Ω such that

$$(+)\quad \begin{aligned} &\text{(a)}\quad \alpha^i_j(x^i) = 0 \qquad \text{for} \quad i = 1, \ldots, p-1 \quad \text{and} \quad j = 2, \ldots, c; \\ &\text{(b)}\quad \mathrm{d}\bar{u}_1(x^1) \wedge \ldots \wedge \mathrm{d}\bar{u}_{p-1}(x^{p-1}) \wedge \bigwedge_{i,j} \mathrm{d}\alpha^i_j(x^i) = 0. \end{aligned}$$

Since

$$\mathrm{d}\bar{u}_i(x^i_j) = \sum_{j=1}^{c} \frac{\partial u_i}{\partial x^i_j} \mathrm{d}x^i_j = \frac{\partial u_i}{\partial x^i_1} \mathrm{d}x^i_1 + \sum_{j=2}^{c} \alpha^i_j \, \mathrm{d}x^i_j,$$

$(+)$ is equivalent to

$$(++)\quad \begin{aligned} &\text{(a)}\quad \alpha^i_j(x^i) = 0 \qquad \text{for}\; i = 1, \ldots, p-1 \quad \text{and} \quad j = 2, \ldots, c; \\ &\text{(b}')\quad \mathrm{d}x^1_1 \wedge \ldots \wedge \mathrm{d}x^{p-1}_1 \wedge \bigwedge_{i,j} \mathrm{d}\alpha^i_j(x^i) = 0. \end{aligned}$$

Note that (b′) involves a form of degree $c(p-1)$ on a $c(p-1)$-dimensional space. Equations $(++)$ are $(p-1)(c-1)+1$ polynomial equations in the jet space $[J^2(\mathbf{R}^c, \mathbf{R})]^{p-1}$ and so they define a $(p-1)(c-1)+1$ codimensional algebraic subset $\Sigma^{1,1}$ of $[J^2(\mathbf{R}^c, \mathbf{R})]^{p-1}$. Now, define for $u \in C^\infty_S(\Omega, \mathbf{R}^p)$,

$$\tilde{j}^2 u : \Omega \to [J^2(\mathbf{R}^c, \mathbf{R})]^{p-1}$$

by

$$(x^1, \ldots, x^{p-1}) \mapsto (j^2\bar{u}_1(x^1), \ldots, j^2\bar{u}_{p-1}(x^{p-1})).$$

Again, using the techniques of Theorem 12, one sees that for an open and dense set of u in $C^\infty_S(\Omega, \mathbf{R}^p)$, $\tilde{j}^2(u)$ is transverse to $\Sigma^{1,1}$ and so, $S^{1,1}(u) = \tilde{j}^2(u)^{-1}(\Sigma^{1,1})$ is generically a codimension $(p-1)(c-1)+1$ dimensional algebraic-like subset of Ω.

To see that, in such cases, $S^{1,1}(u)$ is DGN(u), note that if $x \notin S^{1,1}(u)$, either (a) or (b′) of (+ +) does not hold. If (a) does not hold at x, then u has maximal rank at x, i.e., $x \notin S^1(u)$. If (b′) does not hold at x, but (a) does, then all

$$\alpha_j^i(x^i) = 0,$$

but

$$\bigwedge_{i,j} \mathrm{d}\alpha_j^i(x^i) \neq 0.$$

So, the equations $\alpha_j^i(x^i) = 0$ are independent and $S^1(u)$ is a manifold near x. In addition, the tangent space to $S^1(u)$, defined by

$$\mathrm{d}\alpha_j^i(x^i) = 0 \qquad \text{for} \quad i = 1, \ldots, p-1 \quad \text{and} \quad j = 2, \ldots, c,$$

and the kernel of Du, defined by

$$\mathrm{d}x_1^i = 0 \qquad \text{for} \quad i = 1, \ldots, p-1,$$

are transverse. So, $u \mid S^1(u)$ is of maximal rank at x and thus $x \notin$ DGN(u).

Remark 1. The simplication $\bar{u}_p(x) = x_1$ was actually only used to describe analytically the singular manifolds in our jet spaces. The proofs of the last two theorems can be written without this simplification, but the computations become unwieldly very quickly. Also, note that we exhibited generic properties for $[C^\infty(\mathbf{R}^c, \mathbf{R})]^{p-1}$ with u_p fixed. The required properties are generic in $[C^\infty(\mathbf{R}^c, \mathbf{R})]^p$, a fortiori.

Remark 2. To see that for $p = 2$, the elements in $S^{1,1}(u)$ are not TO's, one uses the Whitney normal form for generic elements in $S^{1,1}(u)$, when $u: \Omega \to \mathbf{R}^2$. If $\mathbf{0}$ is the generic element in $S^{1,1}(u)$, then there are coordinates $(y_1, \ldots, y_c)$, in which

$$u_1 = y_1,$$

$$u_2 = \sum_{i=3}^{c} \pm y_i^2 + y_2 y_1 + y_2^3/3.$$

Note that on the path $y_i = 0$, $i \neq 2$, u_1 is constant and u_2 is strictly increasing at $\mathbf{0}$. So the point $\mathbf{0} \in S^{1,1}(u)$ is not a TO. See Wan (1973).

Remark 3. If one uses the more general Axioms A_2' instead of the Axioms A_2, Theorems 13 and 14 are much easier to prove. Let $C_{S'}^\infty(\Omega, \mathbf{R}^p)$ be the space of utility mappings that satisfy Axioms A_2'. [So, D$u(x)$ always has rank $\geqq p-1$

and $Du_i(x)$ is never zero.] Then, $C^{\infty}_{S'}(\Omega, \mathbf{R}^p)$ is an *open subset* of $C^{\infty}(\Omega, \mathbf{R}^p)$. Since $S^1(u)$ is generically a submanifold of $\mathring{\Omega}$ of dimension $p-1$ and $S^{1,1}(u)$ is generically a submanifold of Ω of dimension $p-2$ in $C^{\infty}(\Omega, \mathbf{R}^p)$ [see Levine (1970) or Golubitsky–Guillemin (1974, section 4)], the same is true in $C^{\infty}_{S'}(\Omega, \mathbf{R}^p)$.

8. Accessibility of economic optima

After we have studied and characterized our economic optima, the following question arises naturally: given a Pareto economic system (i.e., a utility mapping $u:\Omega \to \mathbf{R}^p$ satisfying the Axioms A_2 or A_2') and an initial distribution of commodities, does there always exist a trade curve to an LPO or a TO? At first thought, the answer seems to be 'yes'; since if one is not at a TO, then one can trade some more and keep moving 'closer' to a TO. In fact, this is probably the case for all systems with two commodities and two agents. However, we will now describe a couple of interesting examples of Pareto systems with two persons and three commodities. In both models, the set of economic optima lie on a set of very small measure. In the first, no one can reach a TO from any initial state unless this initial state is already a TO, although one can approach arbitrarily close to a TO. In the second example, one cannot even get near a TO from most initial states.

Example 1. Since we are working with $c = 3$ and $p = 2$, Ω is the Edgeworth cube,

$$\{(x, y, z) \mid 0 \le x \le a_1, \quad 0 \le x \le a_2, \quad 0 \le x \le a_3\},$$

for some $(a_1, a_2, a_3) \in \mathbf{R}^3_+$.

To simplify our notations and constructions, we will take Ω to

$$\{(x, y, z) \mid \quad -2 \le x \le 2, \quad -2 \le y \le 2, \quad 0 \le z \le 4\}.$$

We will take $\bar{u}_1(x, y, z) = z$, i.e., $u_1(x, y, z) = 4-z$. So the first person is indifferent to the first two commodities and his indifference sets are planes parallel to the x, y-plane. Of course, u_2 is a bit more intricate.

We first describe the graphs of some auxiliary functions $\mathbf{R}^1 \to \mathbf{R}^1$. Let f_1: $[-2, 2] \to \mathbf{R}$ be a C^{∞} function with the following properties:

(1a) $\quad f_1(x) \in [0, 1]$ for all $x \in [-2, 2]$,

(1b) $\quad f_1(x) = 0$ if and only if $x \in [-\frac{1}{2}, \frac{1}{2}]$,

(1c) $\quad f_1'(x) < 0$ for $x \in (-2, -\frac{1}{2})$;
$\quad f_1'(x) > 0$ for $x \in (\frac{1}{2}, 2)$,

(1d) $\quad f_1(x) = f_1(-x)$.

Let $f_2 : [-2, 2] \to \mathbf{R}$ be a C^∞ function with the following properties:

(2a) $\quad f_2(x) \in [0, 1]$ for all $x \in [-2, 2]$,

(2b) $\quad f_2(x) \geq f_1(x) \geq 0$ for all x,

(2c) $\quad f_2' < f_1' < 0$ for $x \in (-2, -\frac{1}{2})$;
$\quad 0 < f_1' < f_2'$ for $x \in (\frac{1}{2}, 2)$,

(2d) $\quad f_2(x) = f_2(-x)$.

Let $f_3 : [-2, 2] \to \mathbf{R}$ be a C^∞ function with the following properties:

(3a) $\quad f_1(x) \leq f_3(x) \leq f_2(x)$ for all x,

(3b) $\quad f_3' < 0$ on $(-2, -\frac{1}{2})$;
$\quad f_3' > 0$ on $(\frac{1}{2}, 2)$,

(3c) $\quad$ the graph of f_3 oscillates infinitely often between the graphs of f_1 and f_2 both for $x < -\frac{1}{2}$ and $x > \frac{1}{2}$,

(3d) $\quad f_3$ is not symmetric; in fact, let $\{t_i\}$ and $\{s_i\}$ be sequences such that:
$-2 < t_1 < t_2 < t_3 < \ldots < -\frac{1}{2}$;
$2 > s_1 > s_2 > s_3 > \ldots > \frac{1}{2}$;
$t \in (-2, -\frac{1}{2})$ is a local max. of f_3 ($f_3(t) = f_2(t)$) if and only if $t = t_i$ for some i;
$s \in (\frac{1}{2}, 2)$ is a local max. of f_3 if and only if $s = s_i$ for some i;
$-t_1 > s_1 > -t_2 > s_2 > \ldots$, and therefore (by the symmetry of f_2) $f_3(t_1) > f_3(s_1) > f_3(t_2) > f_3(s_2) > \ldots$.

So, f_1, f_2 and f_3 are C^∞-flat at $x = \pm\frac{1}{2}$. The graphs of the functions are pictured in fig. 9.

Now disregard f_1 and f_2 and work with f_3. We use f_3 to construct the graph of a C^∞ function $f_4 : \mathbf{R}^2 \to \mathbf{R}^1$ as follows:

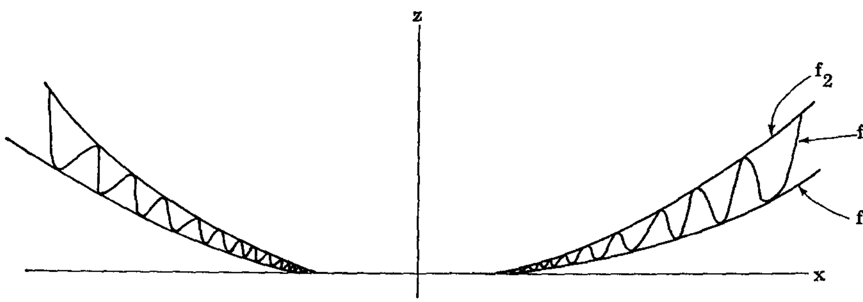

Fig. 9

Revolve the graph of f_3 around the z-axis 180°, at the same time bending it slightly so that after the half-turn, $(t_i, f_3(t_i))$ matches up with $(s_i, f_3(s_i))$,

$$\begin{aligned} &\operatorname{grad} f_4(x, z) \neq 0 \quad \text{for all} \quad |(x, z)| > \tfrac{1}{2}, \\ &\operatorname{grad} f_4(x, z) = 0 \quad \text{for all} \quad |(x, z)| \leq \tfrac{1}{2}, \\ &\operatorname{graph} f_4 \cap [x, z \text{ plane}] = \operatorname{graph} f_3. \end{aligned}$$

So, there is a spiraling smooth curve γ on the graph of f_4 with the properties:

(a) $(s_i, f_3(s_i)) \in \gamma$ and $(t_i, f_3(t_i)) \in \gamma$ for all i,

(b) if Q is any plane containing the z-axis and $p \in Q \cap \gamma$, then p is a local max. on the graph of $f_4 \cap Q$,

(c) γ is never horizontal,

(d) γ spirals in toward the circle $|(x, y)| = \frac{1}{2}, z = 0$.

A candidate for $f_4 : \mathbf{R}^2 \to \mathbf{R}$ would be the function

$$F(r, \theta) = \begin{cases} 0, & \text{if } r \leq \frac{1}{2} \\ e^{-1/(r-\frac{1}{2})^2} \left[\sin\left(\theta - \dfrac{1}{2(r-\frac{1}{2})^4} + 2 \right) \right], & \text{if } r > \frac{1}{2}. \end{cases}$$

We are now ready to construct our second utility function u_2. Recall that $\Omega = [-2, 2] \times [-2, 2] \times [0, 4]$ and $u_1(x, y, z) = 4 - z$. Let C be the solid cylinder $\{(x, y, z) \mid |(x, y)| \leq \frac{1}{2}\}$ in Ω. Construct u_2 so that $u_2^{-1}(2) \cap \Omega$ is the graph of $f_4 + 2$, $u_2^{-1}(a) \cap C$ lies in $z = a$, for all a, $u_2^{-1}(a)$ is basically the graph of $f_4 + a$, but squeezed into the 'box' $a \leq z \leq f_5(a)$, where $f_5 : [0, 4] \to \mathbf{R}$ is C^∞ strictly increasing with $f_5(2) = 3$ and $f_5(0) = 0$. So, $u_2^{-1}(0)$ lies in $z = 0$.

A candidate for $u_2 : \mathbf{R}^3 \to \mathbf{R}$ will be the function

$$U(r, \theta, z) = \frac{z}{1 + \frac{1}{2} F(r, \theta)},$$

where F is as above.

So, the level surfaces of u_2 intersected with the plane $y = 0$ in Ω are as in fig. 10.

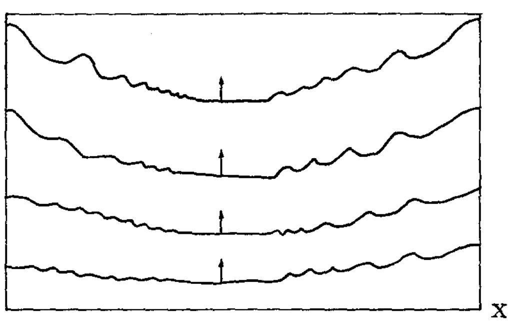

Fig. 10

Since grad $u_1(x, y, z) = (0, 0, -1)$ and since grad $u_2(x, y, z)$ is vertical only on C and on $z = 0$, the set of TO's, LPO's and PO's is exactly $C \cup \{z = 0\}$. [It is easy to see that there are no optima on $\Omega \cap \{z = 4\} \setminus C$; and that there are no optima on the four 'vertical' boundary planes of Ω since grad u_1 always points into Ω on these planes.]

Furthermore, because of the oscillatory behavior of the level surfaces of u_2, the only possible trade curves spiral in toward C. For, if one could trade toward C without spiralling around C, then one could trade toward C while remaining on some plane Q containing the z-axis. Because of u_1, the z-coordinate cannot increase along a trade curve. But because the level surfaces of u_2 on Q have infinitely many local minima with respect to z, once cannot reach C while staying on Q.

On the other hand, these spiralling trade curves approach C asymptotically but they never reach C and they do not approach any one optimal position on C since the limit set of any trade curve contains at least a circle on C. Thus, in this model, one cannot reach an economic optimum unless one starts at one.

Example 2. We will now modify Example 1 so that from most initial positions one cannot even get near to a TO. Let Ω and u_1 be as in Example 1. Let f_1, f_2, and f_3 be as above but modified only in $[-\frac{1}{2}, \frac{1}{2}]$ so that if $\bar{f}_3$ is the new f_3:

$$\bar{f}_3(x) = 0 \quad \text{if and only if} \quad x = -\tfrac{1}{2}, 0, \text{ or } \tfrac{1}{2},$$

and

$$\bar{f}_3''(x) > 0 \quad \text{for} \quad x \in (-\tfrac{1}{2}, \tfrac{1}{2}).$$

Again rotate $\bar{f}_3$ about the z-axis to obtain $\bar{f}_4 : \mathbf{R}^2 \to \mathbf{R}$. A candidate for $\bar{f}_4$ now is

$$\bar{F}(r, \theta) = \begin{cases} -\mathrm{e}^{-(r-\frac{1}{2})^{-2}} \ \mathrm{e}^{-(r+\frac{1}{2})^{-2}}, & \text{if } r < \frac{1}{2}, \\ \mathrm{e}^{-1/(r-\frac{1}{2})^2} \left[\sin\left(\theta - \dfrac{1}{2(r-\frac{1}{2})^4} \right) + 2 \right], & \text{if } r > \frac{1}{2}. \end{cases}$$

Construct the new u_2 from $\bar{F}$ as before and let $u = (u_1, u_2)$. $S^1(u) = \{z = 0\} \cup \partial C \cup z$-axis; and all the TO's now lie on $z = 0$ or on the z-axis. The TO's on the z-axis are accessible by trade curves from any position *in* C. However, if one starts in $\Omega \setminus C$, again the only possible trade curves are spirals that are asymptotic to ∂C, the boundary of C. However, ∂C acts as a barrier, preventing one from getting near the TO's on the z-axis.

Remark. One can easily obtain the phenomena of Examples 1 and 2 with $\bar{u}_1$ and $\bar{u}_2$ obeying the strong monotonicity conditions that are usually assumed in economic models.

Remark. We can use these examples to give an example of a state that is a TO but not a TO^+ as promised in section 5. Construct u_2 as in Example 1 but with the cylinder C replaced by the line $L: x = 0, y = 0$. Let $\hat{u}_2$ be the negative of this new u_2 and let $\hat{u}_1$ be the negative of the above u_1. So, if in this model the initial state lies on L, one can trade from L only by spiralling away from it, i.e., every smooth trade curve $\gamma: [0, \varepsilon) \to \Omega$ with $\gamma(0) \in L$ has $\gamma'(0) = 0$. Thus, the points of L are in TO^+ but not in TO.

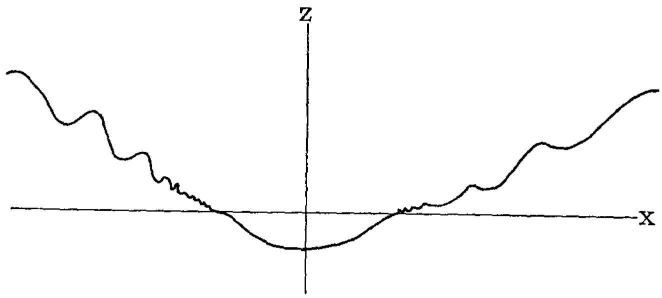

Fig. 11. Graph of $\bar{f}_3$.

We had to work a bit to construct u_2 in Examples 1 and 2. It was essential in both examples that u_2 was C^∞-flat on ∂C. Thus, the following conjectures seem natural:

Conjecture 1. For a generic set of utility mappings $u = (u_1, \ldots, u_p)$ in $C_S^\infty(\Omega, \mathbf{R}^p)$, one can reach a TO from any initial distribution of commodities by trading along trade curves.

Conjecture 2. If $\bar{u}_1, \ldots, \bar{u}_p : \Omega \to \mathbf{R}$ are all real-analytic functions, then one can reach a TO from any initial distribution of commodities by trading along trade curves.

Smale (1973a, 1974b) and Wan (1973) are concerned with this conjecture for the case where Ω is a compact manifold without boundary. In fact, Wan proves the conjecture in this case for $p = 2$. However, a real difficulty arises when trade curves are forced to the boundary of the usual commodity space Ω – a problem which has received very little attention in the literature and which we hope to deal with in a future paper. This problem cannot be avoided for it is simple to construct a model of a Pareto system in the Edgeworth Box ($p = c = 2$), where both u_1 and u_2 are strongly convex and monotone yet $S^1(u) = \phi$ and all the TO's lie on $\partial\Omega$.

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