and the difference would increase as Re_{x_1} increases. A compromise for a solution at the vicinity of the slot may be obtained by letting $Re_{x_1} = 3 \times 10^6$ and replacing the first term in the denominator of equation (8) by $1{\cdot}9\,Pr^{2/3}$. This replacement forces η to be unity at the edge of the slot, hence satisfying the initial temperature requirement. The adjusted solution is shown in Fig. 2 as a comparison with the results of [1]. The value of $\sin\alpha$ is taken to be unity, corresponding to normal injection. The agreement is quite satisfactory. The only discrepancy is at small values of ξ. Also, Re_{x_1} has been selected as a fixed quantity because the velocity profile in the boundary layer would approach a power-law profile depending on the value of Re_{x_1} rather than ξ. The value of n at $Re_{x_1} = 3 \times 10^6$ is approximately 5·5 [3] in the absence of secondary flow.

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SOLUTION OF THE NATURAL CONVECTION PROBLEM BY PARAMETER DIFFERENTIATION

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INTRODUCTION

THE PROBLEM of free convection over a semi-infinite isothermal flat plate has been analysed in the literature [1]. Such analysis involved a numerical solution of a boundary value problem which required iteration in order to satisfy the conditions stated at the boundaries for every value of Prandtl number, Pr. In the present note we present a non-iterative method, known as the method of parameter differentiation, to solve the same problem for various values of Pr. The method requires no iteration once a solution or the initial conditions of such a solution are known for one value of Pr. In the event a starting solution or conditions are not available, iteration is then required only for one value of Pr. Results for other values of Pr are then obtained by integrating the rate of change of the solution with respect to the parameter Pr. Each step in the calculation involves only a small perturbation in the parameter. By this approach, the equations solved are linear differential equations which can be solved noniteratively. Even though this method has been applied to the solution of simpler equations [2, 3], its application to the simultaneous nonlinear ordinary differential equations for the purpose of eliminating iteration is not evident in the literature. We like to note that Narayana and Ramamoorthy [4], in their analysis of the compressible boundary layer equations, attempted to eliminate iteration using the method of parameter differentiation. However, their attempt was not successful. This is due to the fact that they chose a two-parameter two-term superposition technique for their solution instead of a two-parameter three-term superposition like the one given by equations (7) and (8) in the present note. The choice of the number of terms in the solution, as it will become evident later on in the note, depends upon the number of the missing initial conditions in the solution.

ANALYSIS

The nonlinear ordinary differential equations governing the natural convection boundary layer flow over a semi-infinite isothermal flat plate in terms of similarity variables can be written in the usual notations as

$$F_{\eta\eta\eta} + 3FF_{\eta\eta} - 2F_\eta^2 + \theta = 0 \tag{1}$$

$$\theta_{\eta\eta} + 3PrF\theta_\eta = 0 \tag{2}$$

subject to the boundary conditions

$$\eta = 0: F(0) = F_\eta(0) = 0, \quad \theta(0) = 1$$
$$\eta = \infty: F_\eta(\infty) = 0, \quad \theta(\infty) = 0.$$

The subscript η implies differentiation with respect to η. In this note solutions are sought for different values of the parameter Pr. Differentiating equations (1) and (2) with respect to Pr we get

$$g_{\eta\eta\eta} + 3F_{\eta\eta}g + 3Fg_{\eta\eta} - 4F_\eta g_\eta + T = 0 \tag{3}$$

$$T_{\eta\eta} + 3F\theta_\eta + 3Prg\theta_\eta + 3PrFT_\eta = 0 \tag{4}$$

where

$$g = \frac{\partial F}{\partial Pr} \quad \text{and} \quad T = \frac{\partial \theta}{\partial Pr} \tag{5}$$

with the boundary conditions

$$\begin{aligned} &\eta = 0: g(0) = g_\eta(0) = 0, \quad T(0) = 0 \\ &\eta = \infty: g_\eta(\infty) = 0, \quad T(\infty) = 0. \end{aligned} \tag{6}$$

Equations (3) and (4) are now linear, and their solutions can be obtained by separating the dependent variables as

$$g = g_1 + \lambda g_2 + \mu g_3 \tag{7}$$

$$T = T_1 + \lambda T_2 + \mu T_3. \tag{8}$$

In equations (7) and (8) g and T are expressed in terms of two parameters (λ and μ). This is necessary as there are two missing initial conditions in the present problem. Substituting these expressions for g and T in equations (3) and (4) and separating the resulting equations result in three sets of initial value problems. These sets are

$$g_{1\eta\eta\eta}+3F_{\eta\eta}g_1+3Fg_{1\eta\eta}-4F_\eta g_{1\eta}+T_1 = 0 \quad (9)$$

$$T_{1\eta\eta}+3F\theta_\eta+3Prg_1\theta_\eta+3PrFT_{1\eta} = 0 \quad (10)$$

$$g_1(0) = g_{1\eta}(0) = g_{1\eta\eta}(0) = 0$$

$$T_1(0) = T_{1\eta}(0) = 0$$

$$g_{2\eta\eta\eta}+3F_{\eta\eta}g_2+3Fg_{2\eta\eta}-4F_\eta g_{2\eta}+T_2 = 0 \quad (11)$$

$$T_{2\eta\eta}+3Prg_2\theta_\eta-3PrFT_{2\eta} = 0 \quad (12)$$

$$g_2(0) = g_{2\eta}(0) = g_{2\eta\eta}(0) = 0$$

$$T_2(0) = 0, \quad T_{2\eta}(0) = 1$$

$$g_{3\eta\eta\eta}+3F_{\eta\eta}g_3+3Fg_{3\eta\eta}-4F_\eta g_{3\eta}+T_3 = 0 \quad (13)$$

$$T_{3\eta\eta}+3Prg_3\theta_\eta+3PrFT_{3\eta} = 0 \quad (14)$$

$$g_3(0) = g_{3\eta}(0) = 0, \quad g_{3\eta\eta}(0) = 1$$

$$T_3(0) = 0, \quad T_{3\eta}(0) = 0.$$

The initial conditions in the three sets of initial value problems were based on the boundary conditions specified in equation (6) and were separated such that

$$\mu = g_{\eta\eta}(0) \quad \text{and} \quad \lambda = T_\eta(0). \quad (15)$$

In addition the boundary condition at infinity in equation (6) yields

$$\lambda g_{2\eta}(\infty)+\mu g_{3\eta}(\infty) = -g_{1\eta}(\infty) \quad (16)$$

$$\lambda T_2(\infty)+\mu T_3(\infty) = -T_1(\infty) \quad (17)$$

from which

$$\lambda = \frac{T_1(\infty)g_{3\eta}(\infty)-T_3(\infty)g_{1\eta}(\infty)}{T_3(\infty)g_{2\eta}(\infty)-T_2(\infty)g_{3\eta}(\infty)} \quad (18)$$

$$\mu = \frac{g_{1\eta}(\infty)T_2(\infty)-g_{2\eta}(\infty)T_1(\infty)}{g_{2\eta}(\infty)T_3(\infty)-g_{3\eta}(\infty)T_2(\infty)}. \quad (19)$$

The solution procedure, similar to those discussed in detail in [2] and [3], can be summarized as follows: Consider the solutions of equations (1) and (2) for $Pr > 1$. The solutions of these equations for $Pr = 1$ are known from the literature [1]. The initial conditions of such a solution are

$$Pr = 1{\cdot}00: F_{\eta\eta}(0) = 0{\cdot}6421, \quad \theta_\eta(0) = -0{\cdot}5671.$$

With these conditions and the functions F, F_η, $F_{\eta\eta}$, θ and θ_η in equations (9)–(14) known, equations (9)–(14) can now be integrated for $Pr = 1+\Delta Pr$ to give g_1, g_2, g_3, T_1, T_2 and T_3 along with their derivatives. These results are used in turn to evaluate λ and μ from equations (18) and (19). It was found that λ and μ both approached constant values for η larger than approximately 6·00. The solutions of g and T can then be found from equations (7) and (8) respectively. Finally the integration of equation (5) yields the solutions of equations (1) and (2) for $Pr = 1+\Delta Pr$ as follows

$$F(\eta)\Big|_{Pr=1+\Delta Pr} = F(\eta)\Big|_{Pr=1} + g(\eta)\Big|_{\text{from equation (7)}} . \Delta Pr$$

$$\theta(\eta)\Big|_{Pr=1+\Delta Pr} = \theta(\eta)\Big|_{Pr=1} + T(\eta)\Big|_{\text{from equation (8)}} . \Delta Pr$$

This procedure can be repeated to calculate the solutions of equations (1) and (2) for $Pr = 1+2\Delta Pr$, $1+3\Delta Pr$, ... etc.

RESULTS

Using the present method, selected results were obtained and are shown in Table 1. For each set of the solutions, the initial slopes, $F_{\eta\eta}(0)$ and $\theta_\eta(0)$, are known from the starting solution for a designated Pr. The other solutions are then obtained by parameter differentiation. The results agree very closely with those of Ostrach [1].

Table 1. Selected solutions based on parameter differentiation

	Present method		Ostrach [1]	
Pr	$F_{\eta\eta}(0)$	$\theta_\eta(0)$	$F_{\eta\eta}(0)$	$\theta_\eta(0)$
0·72*	0·6760†	−0·5046†	0·6760	−0·5046
0·60	0·6946	−0·4725		
0·50	0·7131	−0·4420		
0·40	0·7354	−0·4066		
0·30	0·7633	−0·3641		
0·20	0·8009	−0·3101		
0·10	0·8590	−0·2326		
0·06	0·8961	−0·1864		
0·04	0·9221	−0·1556		
0·01	0·9887	−0·0817	0·9862	−0·0812
1·00‡	0·6421†	−0·5671†	0·6421	−0·5671
1·10	0·6323	−0·5860		
1·20	0·6234	−0·6036		
1·30	0·6152	−0·6202		
1·40	0·6076	−0·6358		
1·50	0·6006	−0·6506		
1·60	0·5940	−0·6646		
1·70	0·5879	−0·6780		
1·80	0·5821	−0·6908		
1·90	0·5767	−0·7031		
2·00	0·5715	−0·7149	0·5713	−0·7165
2·00§	0·5713†	−0·7165†	0·5713	−0·7165
3·00	0·5312	−0·8145		
4·00	0·5036	−0·8898		
5·00	0·4827	−0·9517		
6·00	0·4660	−1·0047		
7.00	0·4522	−1·0512		
8·00	0·4405	−1·0930		
9.00	0·4304	−1·1309		
10·00	0·4215	−1·1658	0·4192	−1·1694

*$\Delta Pr = -0{\cdot}005$ in this set of solutions.
†Given starting solutions from Ostrach [1]. Other solutions were obtained by parameter differentiation.
‡$\Delta Pr = 0{\cdot}01$ in this set of solutions.
§$\Delta Pr = 0{\cdot}05$ in this set of solutions.

Note: $\Delta\eta = 0{\cdot}01$ throughout the above sets of solutions. Based on our experience the choice of ΔPr in each set of solutions should not be larger than 1% of the range of Pr investigated in the set, with an upper bound being 0·05.

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GAS ABSORPTION INTO TURBULENT LIQUIDS AT INTERMEDIATE CONTACT TIMES

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NOMENCLATURE

a, eddy diffusivity parameter [s^{-1}];
c, concentration of diffusing gas [g moles/cm^3];
c_0, initial concentration [g moles/cm^3];
c_s, interfacial concentration [g moles/cm^3];
d, film thickness [cm];
D, molecular diffusivity [cm^2/s];
k_c, local mass transfer coefficient

$$= \frac{-D}{(c_s - c_0)} \frac{\partial c}{\partial y}\bigg|_{y=0} \text{ [cm/s]};$$

$\bar{k}_c$, average mass transfer coefficient

$$= 1/x \int_0^x k_c \, dx \text{ [cm/s]};$$

L_e, mass transfer entrance length [cm];
$\bar{L}_e$, dimensionless mass transfer entrance length = $(L_e D)/(ud^2)$;
q, volumetric flow rate per unit perimeter [cm^2/s];
Re, Reynolds number = $4q/\nu$;
Sh, local Sherwood number = $k_c d/D$;
Sh_m, mean Sherwood number = $\bar{k}_c d/D$;
u, velocity [cm/s];
$\overline{v'c'}$, time averaged value of the fluctuating components of the v-velocity and concentration [g moles/cm^3];
x, distance in direction of flow [cm];
$\bar{x}$, dimensionless axial distance = $(xD)/(ud^2)$;
x_1, limit of applicability of equation (9) [cm];
y, distance normal to the interface [cm];
$\bar{y}$, dimensionless normal distance = y/d.

Greek letters

β, dimensionless diffusion parameter = $(ad^2)/D$;
ε, eddy diffusivity = $\dfrac{\overline{v'c'}}{dc/dy}$ [cm^2/s];
η, dimensionless $y = y/2(Dx/u)^{1/2}$;
θ, dimensionless concentration, $(c_s - c)/(c_s - c_0)$;
θ_0, concentration function defined by equation (7)

$$= \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-\eta^2} \, d\eta;$$

θ_1, concentration function defined by equation (7);
ν, kinematic viscosity [cm^2/s];
π, = 3·1416;
ρ, density [g/cm^3].

UNDER usual operating conditions for gas absorption the controlling mass transfer resistance resides in the liquid phase. The work reported here is concerned with describing liquid phase mass transfer in terms of an eddy diffusivity for gas absorption in turbulent film flow ($4q/\nu > 1200$) for the case where the concentration profiles are not fully developed. The motivation for this work is that for many applications, such as liquid flow in packed columns, the film lengths or contact times between complete mixing may not be sufficiently long to achieve fully developed conditions in the liquid. In this case information regarding the entrance region mass transfer coefficient would be of interest.

Liquid phase mass transfer across a free surface has been treated in terms of an eddy diffusivity by Levich [1], Davies [2] and King [3]. Lamourelle and Sandall [4] have experimentally determined the behavior of the eddy diffusivity near a free surface by absorbing four different gases into turbulent water films in a long wetted-wall column. These authors found that the eddy diffusivity varies as the square of the distance from the free surface.

$$\varepsilon = ay^2. \quad (1)$$

For water at 25°C, a was found to be

$$a = 7{\cdot}90 \times 10^{-5} \, Re^{1{\cdot}678}. \quad (2)$$

Equation (1) is valid for the region adjacent to the free surface. The major resistance to mass transfer occurs close to the surface because of the large Schmidt numbers usually encountered in gas absorption, and thus it is important to know the eddy diffusivity accurately only for this region.

The differential equation describing diffusion in two-dimensional, fully developed flow may be written in terms of an eddy diffusivity as

$$u(y)\frac{\partial \theta}{\partial x} = \frac{\partial}{\partial y}\left\{(D + \varepsilon(y))\frac{\partial \theta}{\partial y}\right\}. \quad (3)$$