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Critical Multitype Branching Processes with Infinite Variance

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The exponential limit law for the critical multitype Bienaymé–Galton–Watson process is extended to a class of offspring distributions some or all of whose second moments are infinite. Several asymptotic consequences pertaining to transition probabilities and invariant measures are derived.

1. INTRODUCTION

In this paper we extend to the multitype case some results of Slack [9] concerning critical Bienaymé–Galton–Watson processes without variance. Let $\mathbf{Z}_n \equiv (Z_n^{(1)},..., Z_n^{(d)})$ denote a critical, d-type, nonsingular and positively regular BGW process. By $\mathbf{F}(\mathbf{s}) \equiv (F^{(1)}(\mathbf{s}),..., F^{(d)}(\mathbf{s}))$ we denote the offspring p.g.f., M is the expectation matrix, $\mathbf{v}$ and $\mathbf{u}$ its left and right eigenvectors, respectively, corresponding to the maximal eigenvalue 1, and normalized so that $\mathbf{v} \cdot \mathbf{u} = 1$, $\mathbf{1} \cdot \mathbf{u} = 1$, with $\mathbf{1}$ the vector $(1,..., 1)$. $\mathbf{e}_i$ $(1 \leqslant i \leqslant d)$ is a unit vector consisting of zeros except for a 1 in position i, and $\mathbf{s} \equiv (s_1,..., s_d)$ is a generic point in $\mathbb{R}^d$. The state space for the process is $\mathscr{K}$, the collection of all d-tuples, $\mathbf{k} \equiv (k_1,..., k_d)$ of non-negative integers. We shall employ the convenient notation $\mathbf{s} \leqslant \mathbf{t}$ whenever $s_i \leqslant t_i$ for all $1 \leqslant i \leqslant d$, $\exp \mathbf{s} = (e^{s_1},..., e^{s_d})$, and $\mathbf{s}^{\mathbf{k}} = s_1^{k_1} \cdots s_d^{k_d} \cdot$ $\| \mathbf{s} \|_2$ denotes the L_2 norm of $\mathbf{s}$, and, finally, $\mathbf{F}_n$ is the nth functional iterate of $\mathbf{F}$.

If

$$E[Z_1^{(i)} Z_1^{(j)} \mid \mathbf{Z}_0 = \mathbf{e}_k] < \infty \qquad \text{for all } 1 \leqslant i, j, k \leqslant d \tag{1.1}$$

then $n^{-1}\mathbf{Z}_n \mid \mathbf{Z}_n \neq \mathbf{0}$ converges in distribution (see [6, 8]) to an exponential

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random variable whose mass is concentrated on the ray $c\mathbf{v}$. In one dimension and for a suitable class of p.g.f. Slack [9] has demonstrated that normalization by $1 - F_n(0)$ in place of n^{-1} produces a nondegenerate limit, which is not exponential in general, if the variance is infinite.

The approach taken in this paper is as follows. First we show that there are suitable constants a_n such that $a_n \mathbf{Z}_n \cdot \mathbf{u} \mid \mathbf{Z}_n \neq \mathbf{0}$ converges in distribution. Then we show that $\mathbf{Z}_n/(\mathbf{Z}_n \cdot \mathbf{u}) \mid \mathbf{Z}_n \neq \mathbf{0}$ converges in probability to a fixed direction $\mathbf{v}$. These two results are then combined to give convergence in distribution of the vector $a_n \mathbf{Z}_n \mid \mathbf{Z}_n \neq \mathbf{0}$.

It is fitting to note in passing that our considerations include the classical situation when (1.1) holds, in which case the proof given is different from the usual one (which requires a certain uniformity lemma [2, Sect. V.5]) and adds some geometric insight. In fact the lemma of Section 3 is valid without any assumptions on the offspring distribution other than those stated in the first paragraph above.

The final section of this paper contains several asymptotic results on the n-step transition probabilities and invariant measures which sharpen some earlier work [5].

2. Scalar Convergence

The foundation block of this section is the following expansion of Joffe and Spitzer [6],

$$\mathbf{1} - \mathbf{F}(\mathbf{s}) = (M - E(\mathbf{s}))\,(\mathbf{1} - \mathbf{s}), \tag{2.1}$$

where the matrix $E(\mathbf{s})$ is nonincreasing in $\mathbf{s}$, with respect to the partial order induced by $\leqslant$, and tends to zero as $\mathbf{s} \to \mathbf{1}$. Let $\Delta(\mathbf{s}) = (\mathbf{v}E(\mathbf{1} - \mathbf{s})\,\mathbf{s})/(\mathbf{v} \cdot \mathbf{s})$ and then for all sufficiently small and positive scalars x define $\Lambda(x) \equiv \Delta(x\mathbf{u}) = \mathbf{v}E(\mathbf{1} - x\mathbf{u})\,\mathbf{u}$.

Assumption:

$$\Lambda(x) = x^{\alpha} L(x), \tag{2.2}$$

for some $0 < \alpha \leqslant 1$ and a function L slowly varying at 0. The motivation for (2.2) is apparent when we reduce it to an equivalent condition when $d = 1$. In this case (2.1) becomes $F(s) = s + E(s)\,(1 - s)$ and (2.2) reduces to $E(s) = (1 - s)^{\alpha} L(1 - s)$ giving $F(s) = s + (1 - s)^{1+\alpha} L(1 - s)$, precisely the form of p.g.f. singled out by (1.1) of [9].

Lemma 1. $\lim_{x\to 0}(x\Lambda'(x)/\Lambda(x)) = \alpha$.

Proof. From (2.1) $\mathbf{v} \cdot (\mathbf{F}(\mathbf{1} - x\mathbf{u}) - \mathbf{1} + x\mathbf{u}) = x\Lambda(x)$. Differentiation shows that the left side has a monotone increasing derivative in x and then we may apply the one-dimensional argument to deduce the lemma. ∎

For the sequel we let $a_n = \mathbf{v} \cdot (\mathbf{1} - \mathbf{F}_n(\mathbf{0}))$ and $\Delta_n = \Delta(\mathbf{1} - \mathbf{F}_n(\mathbf{0}))$.

LEMMA 2. $\lim_{n\to\infty}(\Delta_n/\Lambda(a_n)) = 1$.

Proof. Given $\epsilon > 0$, by (3.3) of [6] for all n sufficiently large, $(1-\epsilon)\, a_n\mathbf{u} \leqslant \mathbf{1} - \mathbf{F}_n(\mathbf{0}) \leqslant (1+\epsilon)\, a_n\mathbf{u}$ and invoking the monotonicity of E, $(1-\epsilon)\,\Lambda((1-\epsilon)\, a_n) \leqslant \Delta_n \leqslant (1+\epsilon)\,\Lambda((1+\epsilon)\, a_n)$. Dividing these inequalities by $\Lambda(a_n)$, letting $n \to \infty$ and using the definition of a regularly varying function we have

$$(1-\epsilon)^{1+\alpha} \leqslant \liminf_{n\to\infty} \frac{\Delta_n}{\Lambda(a_n)} \leqslant \limsup_{n\to\infty} \frac{\Delta_n}{\Lambda(a_n)} \leqslant (1+\epsilon)^{1+\alpha}.$$

Finally let $\epsilon \downarrow 0$. ∎

LEMMA 3. $\lim_{n\to\infty} n\Lambda(a_n) = 1/\alpha$.

Proof. We shall mimic Lemma 2 of [9]. To this end substitute $\mathbf{s} = \mathbf{F}_n(\mathbf{0})$ into (2.1) and take the scalar product with $\mathbf{v}$ to yield $a_{n+1} = a_n - a_n\Delta_n$. Next apply the mean value theorem to the function Λ to obtain $\Lambda(a_n) - \Lambda(a_{n+1}) = a_n\Delta_n\Lambda'(a_n - \theta_n a_n\Delta_n)$ for some $0 < \theta_n < 1$. Make the decomposition

$$\begin{aligned}\frac{1}{\Lambda(a_{n+1})} - \frac{1}{\Lambda(a_n)} &= \frac{\Lambda(a_n) - \Lambda(a_{n+1})}{\Lambda(a_n)\,\Lambda(a_{n+1})}\\ &= \frac{a_n\Delta_n\Lambda'(a_n - \theta_n a_n\Delta_n)}{\Lambda(a_n)\,\Lambda(a_n - a_n\Delta_n)}\\ &= A_nB_nC_nD_n\end{aligned}$$

where

$$A_n = \frac{a_n}{a_n - \theta_n a_n\Delta_n},$$

$$B_n = \frac{(a_n - \theta_n a_n\Delta_n)\,\Lambda'(a_n - \theta_n a_n\Delta_n)}{\Lambda(a_n - \theta_n a_n\Delta_n)},$$

$$C_n = \frac{\Lambda(a_n - \theta_n a_n\Delta_n)}{\Lambda(a_n - a_n\Delta_n)},$$

and

$$D_n = \frac{\Delta_n}{\Lambda(a_n)}.$$

$\Delta_n \to 0$, which implies that $A_n \to 1$. By Lemma 1, $B_n \to \alpha$. The uniform convergence property of slowly varying functions [7] shows that $C_n \to 1$, and we have just seen by Lemma 2 that $D_n \to 1$. Summarizing, we have shown that

$\Lambda(a_{n+1})^{-1} - \Lambda(a_n)^{-1} \to \alpha$. The Cesaro sums of these differences also converge to α, and so the lemma follows. ∎

COROLLARY.

$$\lim_{n\to\infty} nL(a_n) \Pr[\mathbf{Z}_n \neq \mathbf{0} \mid \mathbf{Z}_0 = \mathbf{i}]^\alpha = (\mathbf{i}\cdot\mathbf{u})^\alpha/\alpha, \qquad \mathbf{i} \in \mathscr{K} \mid \{\mathbf{0}\}.$$

THEOREM 1. *Let $G(x)$ be the distribution function with Laplace–Stieltjes transform*

$$\phi(t) = 1 - t(1 + t^\alpha)^{-1/\alpha}.$$

Then for any $x \geqslant 0$, $\mathbf{i} \in \mathscr{K} \mid \{\mathbf{0}\}$,

$$\lim_{n\to\infty} \Pr[a_n\mathbf{Z}_n \cdot \mathbf{u} \leqslant x \mid \mathbf{Z}_n \neq \mathbf{0}, \mathbf{Z}_0 = \mathbf{i}] = G(x).$$

Proof. Let $\phi_n(t)$ be the LS transform of the conditional distribution $a_n\mathbf{Z}_n \cdot \mathbf{u} \mid \mathbf{Z}_n \neq \mathbf{0}$, $\mathbf{Z}_0 = \mathbf{i}$. For fixed $0 < t < \infty$ put $\mathbf{y}_n = \exp(-ta_n\mathbf{u})$. Then $\phi_n(t) = 1 - [(1 - \mathbf{F}_n{}^{\mathbf{i}}(\mathbf{y}_n))/(1 - \mathbf{F}_n{}^{\mathbf{i}}(\mathbf{0}))]$. We want to show that $\phi_n(t) \to \phi(t)$ as $n \to \infty$. By (3.3) of [6] and (3.6) of [5] it suffices to show that $\psi_n(t) \equiv \mathbf{v} \cdot (\mathbf{1} - \mathbf{F}_n(\mathbf{y}_n))/\mathbf{v} \cdot (\mathbf{1} - \mathbf{F}_n(\mathbf{0}))$ converges to $1 - \phi(t)$. To achieve this we first select ϵ_1 and ϵ_2 arbitrarily small and positive. If n is sufficiently large then $ta_n(1 + \epsilon_1)/(1 - \epsilon_2) < 1$ and we may therefore find an integer $k \equiv k(n)$ such that

$$a_{k+1} \leqslant ta_n(1 + \epsilon_1)/(1 - \epsilon_2) < a_k\,. \tag{2.3}$$

By (3.3) of [6] if k is sufficiently large

$$(1 - \epsilon_2)\, a_k\mathbf{u} \leqslant \mathbf{1} - \mathbf{F}_k(\mathbf{0}) \leqslant (1 + \epsilon_2)\, a_k\mathbf{u}. \tag{2.4}$$

It is also true that for n sufficiently large

$$1 - t(1 + \epsilon_1)\, a_n\mathbf{u} \leqslant \mathbf{y}_n \leqslant 1 - t(1 - \epsilon_1)\, a_n\mathbf{u}. \tag{2.5}$$

Since $k \to \infty$ as $n \to \infty$ it is clear that we may find an integer N such that if $n \geqslant N$ then (2.3), (2.4), and (2.5) hold simultaneously. Multiply the right hand inequality of (2.3) by the vector $(1 - \epsilon_2)\,\mathbf{u}$, obtaining $t(1 + \epsilon_1)\, a_n\mathbf{u} < (1 - \epsilon_2)\, a_k\mathbf{u}$, which is $\leqslant \mathbf{1} - \mathbf{F}_k(\mathbf{0})$ by (2.4). Now invoke (2.5) to deduce the implication $\mathbf{F}_k(\mathbf{0}) \leqslant \mathbf{y}_n$.

We may also define an integer $l \equiv l(n)$ by

$$a_l \leqslant ta_n(1 - \epsilon_1)/(1 + \epsilon_2) < a_{l-1} \tag{2.3'}$$

so that for n sufficiently large,

$$(1 - \epsilon_2)\, a_l\mathbf{u} \leqslant \mathbf{1} - \mathbf{F}_l(\mathbf{0}) \leqslant (1 + \epsilon_2)\, a_l\mathbf{u}. \tag{2.4'}$$

Then play the same game as in the previous paragraph, replacing (2.3) and (2.4) with (2.3′) and (2.4′), respectively, to get $\mathbf{y}_n \leqslant \mathbf{F}_l(\mathbf{0})$. We have thereby succeeded in sandwiching $\mathbf{y}_n$ between two iterates of $\mathbf{F}$, namely,

$$\mathbf{F}_k(\mathbf{0}) \leqslant \mathbf{y}_n \leqslant \mathbf{F}_l(\mathbf{0}). \tag{2.6}$$

Furthermore, the asymptotic behavior of k and l as $n \to \infty$ may be readily ascertained. In particular, divide (2.3) by a_k and since $a_{k+1}/a_k \to 1$ as $k \to \infty$ [5], conclude that

$$\lim_{n\to\infty} \frac{a_k}{a_n} = t\,\frac{1+\epsilon_1}{1-\epsilon_2}.$$

The uniform convergence property of slowly varying functions in conjunction with Lemma 3 produces

$$\lim_{n\to\infty} \frac{n}{k} = \lim_{n\to\infty} \left(\frac{a_k}{a_n}\right)^{\alpha} = t^{\alpha}\,\frac{(1+\epsilon_1)^{\alpha}}{(1-\epsilon_2)^{\alpha}}. \tag{2.7}$$

Similarly we may show that

$$\lim_{n\to\infty} \frac{a_l}{a_n} = t\,\frac{1-\epsilon_1}{1+\epsilon_2}$$

and then

$$\lim_{n\to\infty} \frac{n}{l} = t^{\alpha}\,\frac{(1-\epsilon_1)^{\alpha}}{(1+\epsilon_2)^{\alpha}}.$$

By Lemma 3

$$\frac{a_{n+k}}{a_n} \sim \left(\frac{n+k}{n}\right)^{-1/\alpha} \left[\frac{L(a_{n+k})}{L(a_n)}\right]^{-1/\alpha}, \qquad n \to \infty. \tag{2.8}$$

Because of (2.7) for n sufficiently large, $n + k \leqslant \lambda n$ for some integer λ. Hence

$$1 \geqslant \frac{a_{n+k}}{a_n} \geqslant \prod_{j=n}^{\lambda n-1} \frac{a_{j+1}}{a_j} = \prod_{j=n}^{\lambda n-1} (1 - \Delta_j). \tag{2.9}$$

From Lemmas 2 and 3, $j\Delta_j \to \alpha^{-1}$ as $j \to \infty$ so if j is sufficiently large $\Delta_j \leqslant 2(j\alpha)^{-1}$. Thus the right hand side of (2.9) exceeds $\prod_{j=n}^{\lambda n-1} (1 - 2(j\alpha)^{-1})$ for n sufficiently large and this is greater than the expression $(1 - 2(n\alpha)^{-1})^{(\lambda-1)n}$, which remains bounded away from zero as $n \to \infty$. It follows from the uniform convergence of slowly varying functions that $L(a_{n+k})/L(a_n) \to 1$ as $n \to \infty$. Equations (2.7) and (2.8) then give

$$\lim_{n\to\infty} \frac{a_{n+k}}{a_n} = [1 + t^{-\alpha}(1+\epsilon_1)^{-\alpha}(1-\epsilon_2)^{\alpha}]^{-1/\alpha}, \tag{2.10}$$

and we also have

$$\lim_{n\to\infty} \frac{a_{n+l}}{a_n} = [1 + t^{-\alpha}(1 - \epsilon_1)^{-\alpha}(1 + \epsilon_2)^{\alpha}]^{-1/\alpha}. \tag{2.10'}$$

Finally, from (2.6), $1 - \mathbf{F}_{n+l}(\mathbf{0}) \leqslant 1 - \mathbf{F}_n(\mathbf{y}_n) \leqslant 1 - \mathbf{F}_{k+n}(\mathbf{0})$, which forces

$$\frac{a_{n+l}}{a_n} \leqslant \psi_n(t) \leqslant \frac{a_{n+k}}{a_n}.$$

Let $n \to \infty$, invoke (2.10) and (2.10′), and then let $\epsilon \downarrow 0$, to conclude

$$\lim_{n\to\infty} \psi_n(t) = (1 + t^{-\alpha})^{-1/\alpha} = t(1 + t^{\alpha})^{-1/\alpha} = 1 - \phi(t).$$

Although not explicitly stated in [9], the distribution function $G(x)$ can be shown to be continuous for all $x \geqslant 0$ by Theorem 6.2.5 and its corollary of [3]. The continuity theorem then finishes up the proof. ∎

3. Vector Convergence

Lemma 4. *Given any* $\delta > 0$

$$\lim_{n\to\infty} \Pr\left[\left\|\frac{\mathbf{Z}_n}{\mathbf{Z}_n \cdot \mathbf{u}} - \mathbf{v}\right\|_2 \geqslant \delta \mid \mathbf{Z}_n \neq \mathbf{0}\right] = 0.$$

Proof. The details are similar to those of Athreya's work [1] in the supercritical case except that here we must deal with the conditioning $\mathbf{Z}_n \neq \mathbf{0}$. The following variant of the weak law of large numbers is required:

Let $\{X_1, X_2, ...\}$ be a sequence of independent random variables whose distribution functions are taken from a finite set each member of which has mean zero. Then for any $\gamma > 0$

$$\lim_{n\to\infty} \Pr\left[\left|\frac{X_1 + \cdots + X_n}{n}\right| \geqslant \gamma\right] = 0. \tag{3.1}$$

Returning to the proof of this lemma, let n and m be positive integers (to be specified later) and decompose $\mathbf{Z}_{n+m}$ into the sum of the progeny produced at time $n + m$ by the nth generation,

$$\mathbf{Z}_{n+m} = \sum_{i=1}^{d} \sum_{j=1}^{Z_n^{(i)}} \mathbf{Z}_m(i, j; n),$$

where $\mathbf{Z}_m(i, j; n)$ is the offspring vector at $n + m$ descended from the jth particle of type i at the nth generation. Since $\mathbf{v} \cdot (\mathbf{1} - \mathbf{F}(\mathbf{s}))/\mathbf{v} \cdot (\mathbf{1} - \mathbf{F}_n(\mathbf{0})) \to 1$ as $n \to \infty$ for all $\mathbf{0} \leqslant \mathbf{s} < \mathbf{1}$ it follows that

$$\lim_{n\to\infty} \Pr[\mathbf{Z}_n \cdot \mathbf{1} \leqslant \lambda \mid \mathbf{Z}_n \neq \mathbf{0}] = 0 \tag{3.2}$$

for each fixed integer λ.

Select $\delta > 0$ and arbitrary. Then choose $\epsilon > 0$ and $0 < \eta < 1$ to satisfy the inequality $\delta > (1 - \eta)^{-1} (\eta + \eta \| \mathbf{v} \|_2 + \epsilon)$. By [2, Lemma V.6.1] let $m \geqslant m_0$ be so large that $\| \mathbf{s}M^m - \mathbf{v} \|_2 < \epsilon$ for all $\mathbf{0} \leqslant \mathbf{s}$ satisfying $\mathbf{s} \cdot \mathbf{u} = 1$. Therefore

$$\Pr \left[\left\| \frac{\mathbf{Z}_n}{\mathbf{Z}_n \cdot \mathbf{u}} M^m - \mathbf{v} \right\|_2 \geqslant \epsilon \mid \mathbf{Z}_n \neq \mathbf{0} \right] = 0 \qquad \text{for all } n. \tag{3.3}$$

Next define $\mathbf{s}_{nm} = (\mathbf{Z}_n \cdot \mathbf{u})^{-1} \sum_{i=1}^{d} \sum_{j=1}^{Z_n^{(i)}} (\mathbf{Z}_m(i, j; n) - \mathbf{e}_i M^m)$, and $r_{nm} = \mathbf{s}_{nm} \cdot \mathbf{u}$. The difference $\mathbf{Z}_{n+m}/(\mathbf{Z}_{n+m} \cdot \mathbf{u}) - \mathbf{v}$ can then be expanded as

$$(1 + r_{nm})^{-1} \left(\mathbf{s}_{nm} - \mathbf{v} r_{nm} + \frac{\mathbf{Z}_n}{\mathbf{Z}_n \cdot \mathbf{u}} M^m - \mathbf{v} \right).$$

Consider the conditional probability $\Pr[| r_{nm} | > \eta \mid \mathbf{Z}_n \neq \mathbf{0}]$. This may be written as

$$\sum_{\mathbf{k}\neq\mathbf{0}} \Pr[| r_{nm} | > \eta \mid \mathbf{Z}_n = \mathbf{k}] \frac{\Pr[\mathbf{Z}_n = \mathbf{k}]}{\Pr[\mathbf{Z}_n \neq \mathbf{0}]}. \tag{3.4}$$

Since $u_i > 0$ for all i, $\mathbf{Z}_n \cdot \mathbf{u} \geqslant u_0 \mathbf{Z}_n \cdot \mathbf{1}$ where u_0 is the smallest component of the vector $\mathbf{u}$ and is positive. Hence

$$\begin{aligned} &\Pr[| r_{nm} | > \eta \mid \mathbf{Z}_n = \mathbf{k}] \\ &\quad \leqslant \Pr \left[\left| (\mathbf{Z}_n \cdot \mathbf{1})^{-1} \sum_{i=1}^{d} \sum_{j=1}^{Z_n^{(i)}} (\mathbf{Z}_m(i, j; n) \cdot \mathbf{u} - u_i) \right| > u_0 \eta \mid \mathbf{Z}_n = \mathbf{k} \right]. \end{aligned} \tag{3.5}$$

Let $\theta > 0$ be chosen arbitrarily. Then by (3.1) applied to the right hand side of (3.5) with γ replaced by $u_0 \eta$ there exists an integer k such that if $\mathbf{1} \cdot \mathbf{1} \geqslant k$, then (3.5) $\leqslant \theta$ for all n. We may thus bound (3.4) from above by

$$\sum_{\mathbf{k}:\mathbf{k}\cdot\mathbf{1}<k} \Pr[\mathbf{Z}_n = \mathbf{k} \mid \mathbf{Z}_n \neq \mathbf{0}] + \theta.$$

Letting $n \to \infty$, the sum goes to zero by (3.2) and then we let $\theta \downarrow 0$ to obtain

$$\lim_{n\to\infty} \Pr[| r_{nm} | > \eta \mid \mathbf{Z}_n \neq \mathbf{0}] = 0. \tag{3.6}$$

A totally analogous argument produces

$$\lim_{n\to\infty} \Pr[\| \mathbf{s}_{nm} \|_2 > \eta \mid \mathbf{Z}_n \neq \mathbf{0}] = 0. \tag{3.7}$$

Thus

$$\Pr\left[\left\|\frac{\mathbf{Z}_{n+m}}{\mathbf{Z}_{n+m}\cdot\mathbf{u}}-\mathbf{v}\right\|_2 \geqslant \delta \mid \mathbf{Z}_n \neq \mathbf{0}\right]$$

$$\leqslant \Pr\left[|1+r_{nm}|^{-1}\left[\|\mathbf{s}_{nm}\|_2+|r_{nm}|\,\|\mathbf{v}\|_2+\left\|\frac{\mathbf{Z}_n}{\mathbf{Z}_n\cdot\mathbf{u}}M^m-\mathbf{v}\right\|_2\right] \geqslant \delta \mid \mathbf{Z}_n \neq \mathbf{0}\right]$$

$$\leqslant \Pr\left[|1+r_{nm}|^{-1}\left[\|\mathbf{s}_{nm}\|_2+|r_{nm}|\,\|\mathbf{v}\|_2+\left\|\frac{\mathbf{Z}_n}{\mathbf{Z}_n\cdot\mathbf{u}}M^m-\mathbf{v}\right\|_2\right] \geqslant \delta,\right.$$

$$\left.\|\mathbf{s}_{nm}\|_2 \leqslant \eta,\ |r_{nm}| \leqslant \eta,\ \left\|\frac{\mathbf{Z}_n}{\mathbf{Z}_n\cdot\mathbf{u}}M^m-\mathbf{v}\right\|_2 < \epsilon \mid \mathbf{Z}_n \neq \mathbf{0}\right]$$

$$+\Pr[\|\mathbf{s}_{nm}\|_2 > \eta \mid \mathbf{Z}_n \neq \mathbf{0}] + \Pr[|r_{nm}| > \eta \mid \mathbf{Z}_n \neq \mathbf{0}]$$

$$+\Pr\left[\left\|\frac{\mathbf{Z}_n}{\mathbf{Z}_n\cdot\mathbf{u}}M^m-\mathbf{v}\right\|_2 \geqslant \epsilon \mid \mathbf{Z}_n \neq \mathbf{0}\right]$$

$$\leqslant \Pr[(1-\eta)^{-1}(\eta+\eta\|\mathbf{v}\|_2+\epsilon) \geqslant \delta \mid \mathbf{Z}_n \neq \mathbf{0}]$$

$$+\Pr[\|\mathbf{s}_{nm}\|_2 > \eta \mid \mathbf{Z}_n \neq \mathbf{0}] + \Pr[|r_{nm}| > \eta \mid \mathbf{Z}_n \neq \mathbf{0}]$$

$$+\Pr\left[\left\|\frac{\mathbf{Z}_n}{\mathbf{Z}_n\cdot\mathbf{u}}M^m-\mathbf{v}\right\|_2 \geqslant \epsilon \mid \mathbf{Z}_n \neq \mathbf{0}\right].$$

The first probability equals zero by choice of ϵ and η, and as $n \to \infty$ the remaining three probabilities tend to zero by virtue of (3.3), (3.6) and (3.7). We have therefore shown that

$$\lim_{n\to\infty}\Pr\left[\left\|\frac{\mathbf{Z}_{n+m}}{\mathbf{Z}_{n+m}\cdot\mathbf{u}}M^m-\mathbf{v}\right\|_2 \geqslant \delta \mid \mathbf{Z}_n \neq \mathbf{0}\right] = 0.$$

Lemma 4 then follows because

$$\Pr\left[\left\|\frac{\mathbf{Z}_{n+m}}{\mathbf{Z}_{n+m}\cdot\mathbf{u}}M^m-\mathbf{v}\right\|_2 \geqslant \delta \mid \mathbf{Z}_{n+m} \neq \mathbf{0}\right]$$

$$\leqslant \Pr\left[\left\|\frac{\mathbf{Z}_{n+m}}{\mathbf{Z}_{n+m}\cdot\mathbf{u}}M^m-\mathbf{v}\right\|_2 \geqslant \delta \mid \mathbf{Z}_n \neq \mathbf{0}\right]\frac{\Pr[\mathbf{Z}_n \neq \mathbf{0}]}{\Pr[\mathbf{Z}_{n+m} \neq \mathbf{0}]}$$

and the ratio of probabilities on the right tends to 1 as $n \to \infty$ for each fixed m. ∎

THEOREM 2. *$\lim_{n\to\infty} Pr[a_n\mathbf{Z}_n \leqslant \mathbf{s} \mid \mathbf{Z}_n \neq \mathbf{0}] = H(\mathbf{s})$ where $H(\mathbf{s}) = G(\hat{s})$ and $\hat{s} = \min_{1\leqslant i\leqslant d}(s_i/v_i)$. The limit distribution is concentrated on the ray $c\mathbf{v}$, $c > 0$.*

Proof. Fix $\mathbf{0} \leqslant \mathbf{s}$. First we need some extra notation. $C(\delta)$ will be the positive cone with apex at the origin consisting of all $\mathbf{0} \leqslant \boldsymbol{\xi}$ such that $\| \boldsymbol{\xi}/(\boldsymbol{\xi} \cdot \mathbf{u}) - \mathbf{v} \|_2 \leqslant \delta$. $C_x(\delta) = \{\boldsymbol{\xi}: \boldsymbol{\xi} \cdot \mathbf{u} \leqslant x \text{ and } \boldsymbol{\xi} \in C(\delta)\}$. $K(\delta) = \{\boldsymbol{\xi}: \mathbf{0} \leqslant \boldsymbol{\xi} \leqslant \mathbf{s} \text{ and } \boldsymbol{\xi} \in C(\delta)\}$. We claim that given $\delta > 0$ sufficiently small, $\exists x_1 \equiv x_1(\delta)$ and $x_2 \equiv x_2(\delta)$ such that

$$C_{x_1}(\delta) \subseteq K(\delta) \subseteq C_{x_2}(\delta), \tag{3.8}$$

and furthermore

$$x_1 \uparrow \hat{s} \downarrow x_2 \qquad \text{as} \qquad \delta \downarrow 0. \tag{3.9}$$

To prove the claim, select $\delta > 0$ so small that $C(\delta)$ lies strictly within the positive d-dimensional orthant (excepting the apex $\mathbf{0}$). By $\partial(\delta)$ we denote the intersection of $C(\delta)$ with those parts of the hyperplanes $\xi_i = s_i$, $1 \leqslant i \leqslant d$ which form the boundary of the cube $\mathbf{0} \leqslant \boldsymbol{\xi} \leqslant \mathbf{s}$. By convexity follows that $\boldsymbol{\xi} \in K(\delta)$ iff $\boldsymbol{\xi} = r\boldsymbol{\gamma}$ for some $\boldsymbol{\gamma} \in \partial$ and $0 \leqslant r \leqslant 1$. Let $f(\boldsymbol{\xi})$ be the linear functional defined by $f(\boldsymbol{\xi}) = \boldsymbol{\xi} \cdot \mathbf{u}$. Let $x_1 = \inf\{f(\boldsymbol{\xi}): \boldsymbol{\xi} \in \partial\}$ and $x_2 = \sup\{f(\boldsymbol{\xi}): \boldsymbol{\xi} \in \partial\}$. To show that (3.8) holds, suppose that $\boldsymbol{\xi} \in C_{x_1}(\delta)$. Then $\boldsymbol{\xi} \cdot \mathbf{u} \leqslant x_1$. It is also clear from the definition of the cone $C(\delta)$ that $\boldsymbol{\xi} = c\boldsymbol{\gamma}$ for some $c \geqslant 0$ and $\boldsymbol{\gamma} \in \partial$. Thus $c\boldsymbol{\gamma} \cdot \mathbf{u} \leqslant x_1$ and also $x_1 \leqslant \boldsymbol{\gamma} \cdot \mathbf{u}$. This implies that $0 \leqslant c \leqslant 1$ and therefore $\boldsymbol{\xi} \in K(\delta)$. Next suppose that $\boldsymbol{\xi} \in K(\delta)$. Then $\boldsymbol{\xi} = c\boldsymbol{\gamma}$, $0 \leqslant c \leqslant 1$, $\boldsymbol{\gamma} \in \partial$. Hence $\boldsymbol{\xi} \cdot \mathbf{u} = c\boldsymbol{\gamma} \cdot \mathbf{u} \leqslant cx_2 \leqslant x_2$. This proves that $\boldsymbol{\xi} \in C_{x_2}(\delta)$, and the inclusions of (3.8) are valid.

Moving on to the proof of (3.9), the sets $\partial(\delta)$ are nested decreasing as $\delta \downarrow 0$ and their intersection is the point where the ray $c\mathbf{v}$ first hits the union of the hyperplanes $\xi_i = s_i$, $1 \leqslant i \leqslant d$. The value of c is obviously $\min s_i/v_i$. By continuity of f, $x_1 \uparrow c\mathbf{v} \cdot \mathbf{u} \equiv c \downarrow x_2$ as $\delta \downarrow 0$.

Next, by (3.8) we obtain

$$\begin{aligned} \Pr[a_n \mathbf{Z}_n \in C_{x_1}(\delta) \mid \mathbf{Z}_n \neq \mathbf{0}] &\leqslant \Pr[a_n \mathbf{Z}_n \in K(\delta) \mid \mathbf{Z}_n \neq \mathbf{0}] \\ &\leqslant \Pr[a_n \mathbf{Z}_n \in C_{x_2}(\delta) \mid \mathbf{Z}_n \neq \mathbf{0}]. \end{aligned} \tag{3.10}$$

Now

$$\begin{aligned} \Pr[a_n \mathbf{Z}_n \leqslant \mathbf{s} \mid \mathbf{Z}_n \neq \mathbf{0}] = \Pr &\left[a_n \mathbf{Z}_n \leqslant \mathbf{s}, \left\| \frac{\mathbf{Z}_n}{\mathbf{Z}_n \cdot \mathbf{u}} - \mathbf{v} \right\|_2 \leqslant \delta \mid \mathbf{Z}_n \neq \mathbf{0}\right] \\ &+ \Pr\left[a_n \mathbf{Z}_n \leqslant s, \left\| \frac{\mathbf{Z}_n}{\mathbf{Z}_n \cdot \mathbf{u}} - \mathbf{v} \right\|_2 > \delta \mid \mathbf{Z}_n \neq \mathbf{0}\right]. \end{aligned}$$

Therefore by Lemma 4,

$$\liminf_{n\to\infty} \Pr[a_n \mathbf{Z}_n \leqslant \mathbf{s} \mid \mathbf{Z}_n \neq \mathbf{0}] = \liminf_{n\to\infty} \Pr[a_n \mathbf{Z}_n \in K(\delta) \mid \mathbf{Z}_n \neq \mathbf{0}]$$

and

$$\limsup_{n\to\infty} \Pr[a_n\mathbf{Z}_n \leqslant \mathbf{s} \mid \mathbf{Z}_n \neq \mathbf{0}] = \limsup_{n\to\infty} \Pr[a_n\mathbf{Z}_n \in K(\delta) \mid \mathbf{Z}_n \neq \mathbf{0}].$$

Similarly

$$\begin{aligned}\liminf_{n\to\infty} \Pr[a_n\mathbf{Z}_n \in C_{x_1}(\delta) \mid \mathbf{Z}_n \neq \mathbf{0}] &= \liminf_{n\to\infty} \Pr[a_n\mathbf{Z}_n \cdot \mathbf{u} \leqslant x_1 \mid \mathbf{Z}_n \neq \mathbf{0}] \\ &= G(x_1), \qquad \text{by Theorem 1,}\end{aligned}$$

and

$$\begin{aligned}\limsup_{n\to\infty} \Pr[a_n\mathbf{Z}_n \in C_{x_2}(\delta) \mid \mathbf{Z}_n \neq \mathbf{0}] &= \limsup_{n\to\infty} \Pr[a_n\mathbf{Z}_n \cdot \mathbf{u} \leqslant x_2 \mid \mathbf{Z}_n \neq \mathbf{0}] \\ &= G(x_2).\end{aligned}$$

Applying these results to (3.10)

$$\begin{aligned}G(x_1) &\leqslant \liminf_{n\to\infty} \Pr[a_n\mathbf{Z}_n \leqslant \mathbf{s} \mid \mathbf{Z}_n \neq \mathbf{0}] \leqslant \limsup_{n\to\infty} \Pr[a_n\mathbf{Z}_n \leqslant \mathbf{s} \mid \mathbf{Z}_n \neq \mathbf{0}] \\ &\leqslant G(x_2).\end{aligned}$$

Let $\delta \downarrow 0$ and use the continuity of G together with (3.9) to complete the proof of the theorem. ∎

4. Asymptotic Results

Let $\{\pi(\mathbf{i})\}$ be the unique invariant measure (up to the normalization $P(\mathbf{F}(\mathbf{0})) = 1$ where $P(\mathbf{s})$ is the generating function of the measure, [5]). According to [5, Theorem 2.2],

$$P_n(\mathbf{i}, \mathbf{j}) \sim \pi(\mathbf{j})\, (\mathbf{i} \cdot \mathbf{u})\, \mathbf{v} \cdot (\mathbf{F}_{n+1}(\mathbf{0}) - \mathbf{F}_n(\mathbf{0})), \qquad n \to \infty$$

where $\{P_n(\mathbf{i}, \mathbf{j})\}$ are the n-step transition probabilities. From (2.1)

$$\begin{aligned}\mathbf{v} \cdot (\mathbf{F}_{n+1}(\mathbf{0}) - \mathbf{F}_n(\mathbf{0})) &= a_n\Delta_n \\ &\sim a_n\Delta(a_n), &&\text{by Lemma 2,} \\ &\sim \left(\frac{1}{\alpha n}\right)^{1+1/\alpha} L(a_n)^{-1/\alpha}, &&\text{by Lemma 3.}\end{aligned}$$

Therefore

$$(\alpha n)^{1+1/\alpha} L(a_n)^{1/\alpha}\, P_n(\mathbf{i}, \mathbf{j}) \to \pi(\mathbf{j})\, (\mathbf{i} \cdot \mathbf{u}) \qquad \text{as} \qquad n \to \infty. \tag{4.1}$$

Let $U(x)$ be the measure assigned by $\pi(\mathbf{i})$ to the region $\mathbf{i} \cdot \mathbf{u} \leqslant x$, that is $U(x) = \sum_{\mathbf{i} \cdot \mathbf{u} \leqslant x} \pi(\mathbf{i})$. It follows that $P(e^{-t\mathbf{u}})$ is the LS transform of $U(x)$,

$$P(e^{-t\mathbf{u}}) = \int_0^\infty e^{-ty} U(dy).$$

Our interest is in the asymptotic behavior of $P(e^{-t\mathbf{u}})$ as $t \to 0$ since we plan to apply a Tauberian theorem. We shall proceed as in Theorem 1. In fact we assert that given ϵ_1 and ϵ_2 small and positive then for all t sufficiently small there exist integers $k = k(t)$ and $l = l(t)$ such that

$$\mathbf{F}_k(\mathbf{0}) \leqslant e^{-t\mathbf{u}} \leqslant \mathbf{F}_l(\mathbf{0}), \tag{4.2}$$

and

$$a_k \sim t \frac{1 + \epsilon_1}{1 - \epsilon_2}, \qquad a_l \sim t \frac{1 - \epsilon_1}{1 + \epsilon_2} \qquad \text{as} \qquad t \to 0. \tag{4.3}$$

Since P satisfies Abel's equation

$$P(\mathbf{F}(\mathbf{s})) = P(\mathbf{s}) + 1, \qquad P(\mathbf{0}) = 0$$

we get from (4.2)

$$k \leqslant P(e^{-t\mathbf{u}}) \leqslant l. \tag{4.4}$$

Since by Lemma 3, $k a_k^\alpha L(a_k) \to \alpha^{-1}$ we use (4.3) and (4.4) to obtain

$$\liminf_{t \to \infty} P(e^{-t\mathbf{u}})\, t^\alpha L(t) \geqslant \alpha^{-1}(1 - \epsilon_2)^\alpha (1 + \epsilon_1)^{-\alpha},$$

and similarly

$$\limsup_{t \to \infty} P(e^{-t\mathbf{u}})\, t^\alpha L(t) \leqslant \alpha^{-1}(1 + \epsilon_2)^\alpha (1 - \epsilon_1)^{-\alpha}.$$

Let $\epsilon \downarrow 0$ to conclude that

$$P(e^{-t\mathbf{u}}) \sim t^{-\alpha} \frac{1}{\alpha L(t)}, \qquad t \to 0.$$

Karamata's Tauberian Theorem [4, Theorem XIII.5.2] then yields

$$U(x) \sim \frac{x^\alpha}{\alpha \Gamma(\alpha + 1) L(x^{-1})}, \qquad x \to \infty. \tag{4.5}$$

Relations (4.1) and (4.5) are the d-dimensional analogs of Theorem 2 of [9].

5. Final Remark

The converse to the main result of this paper, as considered in [10], is presently under investigation, and will be forthcoming in a future publication.

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