A NONLINEAR MIXTURE THEORY REPRESENTATION
OF SATURATED SAND

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Abstract—Nonlinear constitutive equations for saturated sands are proposed. They exhibit the interaction of dilatation and shear stress observed in sands in its small strain range. The equations are based on a continuum theory derived from a Newtonian approach to a mixture of a nonlinear solid and compressible fluid. The determination of the constitutive parameters from common soil mechanics experiments is discussed and illustrated by qualitatively valid numerical results.

1. INTRODUCTION

THE BEHAVIOR of saturated sands when subjected to shear in the small strain range is a fundamental problem in analytical soil mechanics. The observed phenomenon is a change in fluid pore pressure and effective solid stresses when the confining overburden is kept constant[1–5]. The mechanism suggested to explain this phenomenon is the inherently nonlinear material behavior of dry sand. Even in the small strain range dry sand exhibits dilatational change when subjected to pure shear stress[1,5]. When the sand is saturated, the low compressibility of the water prevents the dilatation with consequent changes in the pore pressure and effective stress.

The purpose of this paper is to obtain constitutive equations for saturated sands which exhibit this observed behavior. Continuum mixture theory provides a theoretical framework for determining the form of these equations. In the next section a Newtonian approach is used to derive the field equations of a two constituent mixture in a form appropriate for the present application. The associated constitutive equations are derived in Section 3. For physical applications it is necessary, of course, to have numerical values for the constitutive parameters. For generalized continua their experimental determination is often a complex task. In the present application there are a number of standard tests that are commonly employed in experimental Soil Mechanics. In Section 4 these tests are interpreted in the context of the proposed theory providing a basis for the determination of the constitutive parameters. Finally, numerical results based on qualitative data in the literature are given in Section 5.

2. THE FIELD EQUATION OF THE MIXTURE

A formulation of mixture theory has been derived by Green and Naghdi[6, 7]. Various aspects and applications of the theory have been discussed by Green and Naghdi[8], Green and Steel[9] and Steel[10, 11]. A linear theory of fluid-solid mixtures has been discussed by Schneider[12]. Garg et al.[13, 14] have used various forms of mixture theory to examine wave propagation in fluid saturated porous media. Here we are concerned with the application of mixture theory to model the behavior of saturated sands. Anticipating future generalization to include soil plasticity, the field equations are established from a Newtonian approach for a mixture of a nonlinear solid and compressible fluid. For simplicity attention is restricted to chemically inert materials and to the case of zero heat flux through the boundaries. The kinematic formulation follows Green and Naghdi[7] and is briefly summarized here. Coordinates in the undeformed reference configuration of the mixture are X_i for the solid constituent and Y_i for the fluid constituent. Spatial coordinates in the deformed body are x_i and y_i for the solid and fluid constituents respectively. It is assumed that each point in the mixture is simultaneously occupied by both constituents. Thus in an Eulerian representation x_i and y_i are equivalent and may be used interchangeably as the independent spatial variable.

Denoting the solid and fluid particle velocities by u_i and v_i, respectively, the kinematic

†Deceased, 17 April 1978.

variables of interest in the sequel are the rate of deformation and rotation tensors for the solid and fluid constituents. They are respectively

$$d_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{1}$$

$$f_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial y_j} + \frac{\partial v_j}{\partial y_i}\right) \tag{2}$$

$$\Gamma_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \tag{3}$$

$$\Lambda_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial y_j} - \frac{\partial v_j}{\partial y_i}\right). \tag{4}$$

It is necessary to introduce stress quantities prior to obtaining the equations of motion. When regarding an element of the mixture, the common definition of stress is used leading to the well known symmetry of the total stress tensor. We view the surface tractions per unit area of a surface enclosing an arbitrary volume of the mixture as composed of tractions acting on the solid constituent t_i and tractions acting on the fluid constituent p_i. Considering the translational motion of an infinitesimal tetrahedron gives

$$t_i + p_i = s_{ij} n_j \tag{5}$$

where n_j is the j component of the normal to the surface on which t_i and p_i are defined, s_{ij} is the symmetric total stress tensor and the summation convention holds. Partial stress tensors are now introduced through the relations

$$t_j = (\sigma_{ij} + \alpha_{ij}) n_i \tag{6}$$

$$p_j = (\pi_{ij} + \beta_{ij}) n_i \tag{7}$$

where σ_{ij} is the symmetric part and α_{ij} the antisymmetric part of the solid partial stress tensor. Likewise π_{ij} and β_{ij} denote the symmetric and antisymmetric part of the fluid partial stress. Equations (5) and (6) imply

$$\alpha_{ij} + \beta_{ij} = 0 \tag{8}$$

$$t_j + p_j = (\sigma_{ij} + \pi_{ij}) n_i.$$

The motion of a material element of the total mixture is governed by Newton's second law of motion. It is

$$\int_{A_0} (t_j + p_j)\, \mathrm{d}A_0 + \int_{V_0} (\rho_1 F_j + \rho_2 G_j)\, \mathrm{d}V_0 = \frac{\mathrm{d}}{\mathrm{d}t} \int_{V_0} (\rho_1 u_j + \rho_2 v_j)\, \mathrm{d}V_0 \tag{9}$$

where F_j and G_j are body forces per unit mass of the solid and fluid, respectively, A_0 is the surface enclosing the arbitrary volume element of the mixture V_0, and ρ_1, ρ_2 are the mass densities of the solid and fluid constituents.

Through (8) and the divergence theorem, eqn (9) for small deformations and chemically inert materials reduces to

$$(\sigma_{ij} + \pi_{ij})_{,i} + \rho_1 F_j + \rho_2 G_j = \rho_1 \ddot{w}_j + \rho_2 \dot{v}_j \tag{10}$$

where a comma denotes a partial space derivative, and a dot denotes a partial time derivative. In (10) w_j denotes the displacement of the solid constituent, i.e.

$$w_j = x_j - X_j. \tag{11}$$

We next consider the motion of the mass of each constituent separately. The solid equation of motion is

$$\int_{A_0} t_j \, dA_0 + \int_{V_0} (\rho_1 F_j + \Phi_j^{\,1}) \, dV_0 = \frac{d}{dt} \int_{V_0} \rho_1 u_j \, dV_0 \tag{12}$$

where $\Phi_j^{\,1}$ is the additional body force exerted on the solid constituent due to the presence of the fluid. The fluid equation of motion is

$$\int_{A_0} p_j \, dA_0 + \int_{V_0} (\rho_2 G_j + \Phi_j^2) \, dV_0 = \frac{d}{dt} \int_{V_0} \rho_2 v_j \, dV_0 \tag{13}$$

where Φ_j^2 is the additional body force exerted on the fluid constituent due to the presence of the solid.

As in (9) above, eqns (12) and (13) reduce to

$$(\sigma_{ij} + \alpha_{ij})_{,i} + \rho_1 F_j + \Phi_j^{\,1} = \rho_1 \ddot{w}_j \tag{14}$$

$$(\pi_{ij} + \beta_{ij})_{,i} + \rho_2 G_j + \Phi_j^2 = \rho_2 \dot{v}_j. \tag{15}$$

For (10) to be compatible with (14) and (15) it follows that

$$\Phi_j^{\,1} + \Phi_j^2 = 0. \tag{16}$$

Thus the constituent interaction can be represented by a unique body force term

$$\Phi_j = -\Phi_j^{\,1} = \Phi_j^2. \tag{17}$$

With this the equations of motion are

$$(\sigma_{ij} + \alpha_{ij})_{,i} + \rho_1 F_j - \Phi_j = \rho_1 \ddot{w}_j \tag{18}$$

$$(\pi_{ij} + \beta_{ij})_{,i} + \rho_2 G_j + \Phi_j = \rho_2 \dot{v}_j. \tag{19}$$

This form is analogous to Green and Naghdi's[7] equations of motion, the main difference being the way apparent partial stresses and interaction forces are introduced. In [7] the nonsymmetric partial stresses are denoted by σ_{ij} and π_{ij}. The diffusive force term π_j introduced in [7] coincides with Φ_j for chemically inert materials.

To the equations of motion it is necessary to add the fluid continuity equation since the fluid's constitutive eqns (Section 2) are expressed as functions of the density change. According to Green and Naghdi, the continuity equation for small density changes reduces to

$$\dot{\eta} + \rho_2^0 v_{i,i} = 0 \tag{20}$$

where η is the change of the fluid's density from its initial value ρ_2^0 to its current value ρ_2.

3. CONSTITUTIVE RELATIONS

To complete the formulation constitutive relations are necessary for σ_{ij}, π_{ij}, α_{ij}, β_{ij} and Φ_j as functions of displacement, change of density and velocity fields. As discussed above, the two constituents considered are an elastic nonlinear solid and a compressible viscous fluid. Interactions are assumed to be of viscous and friction type, i.e. coupling terms in the constitutive equations depend (1) upon rates of deformation of the other constituent, (2) upon differences in the velocities, and (3) upon differences in the rotation rates. Thus, the assumed form of the constitutive equations is

$$\sigma_{ij} = A_{ij} + A_{ijrs} f_{rs} \tag{21}$$

$$\pi_{ij} = B_{ij} + B_{ijrs}f_{rs} + C_{ijrs}d_{rs} \tag{22}$$

$$\alpha_{ij} = -\beta_{ij} = K_1\epsilon_{ijk}(u_k - v_k) + L_1(\Gamma_{ij} - \Lambda_{ij}) + \epsilon_{ijk}b_k \tag{23}$$

$$\Phi_j = K_{ij}(u_i - v_i) + L_2\epsilon_{jki}(\Gamma_{ki} - \Lambda_{ki}) + a_{jki}f_{ki} + a_j \tag{24}$$

where d_{rs}, Γ_{ij}, f_{rs} and Λ_{ij} are defined in (1)–(4) and ϵ_{ijk} is the permutation tensor. All the coefficients are in general functions of the displacement gradient field $\partial x_i/\partial X_j$, the fluid density field ρ_2, and initial composition of the mixture.

The functional dependence of the constitutive coefficients on the displacement and density fields is obtained from thermodynamic considerations. According to Green and Naghdi[7], the entropy inequality when no heat sources are present reduces to

$$\frac{DU}{Dt} - \frac{DA}{Dt} - S\frac{DT}{Dt} \geq 0 \tag{25}$$

where U is the internal energy, A is the Helmholtz free energy, S is the entropy, all per unit mass of the mixture, and T is the absolute temperature field. The material derivative for the mixture is defined in [7] as

$$\rho\frac{D}{Dt} = \rho_1\frac{D^{(1)}}{Dt} + \rho_2\frac{D^{(2)}}{Dt} \tag{26}$$

where $D^{(1)}/Dt$ and $D^{(2)}/Dt$ are the material derivatives for the solid and fluid constituents respectively.

The functional form of U is obtained from the conservation of energy equation

$$\frac{d}{dt}\int_{V_0}\left(\rho U + \frac{1}{2}\rho_1 u_i u_i + \frac{1}{2}\rho_2 v_i v_i\right)dV_0 = \int_{V_0}(\rho_1 F_i u_i + \rho_2 G_i v_i)dV_0 + \int_{A_0}(t_i u_i + p_i v_i)dA_0 \tag{27}$$

Through the divergence theorem, eqns (14), (15) and (7) and the definitions of the rate of deformation and rotation tensors, it reduces to

$$\rho\frac{DU}{Dt} = \Phi_i(u_i - v_i) + \sigma_{ij}d_{ij} + \pi_{ij}f_{ij} + \alpha_{ij}(\Gamma_{ij} - \Lambda_{ij}). \tag{28}$$

The basic assumption† for a mixture of solid and fluid is

$$A = A(e_{ij}, \rho_2, T) \tag{29}$$

$$S = S(e_{ij}, \rho_2, T) \tag{30}$$

where

$$e_{ij} = \frac{1}{2}\left(\frac{\partial x_k}{\partial x_i}\frac{\partial x_k}{\partial x_j} - \delta_{ij}\right) \tag{31}$$

is the strain tensor and δ_{ij} is the Kroneker Delta.

Substituting (28)–(30) and the constitutive eqns (21)–(24) into (25) yields

$$-\rho\left(S + \frac{\partial A}{\partial T}\right)\frac{DT}{Dt} + \left(A_{pq} - \rho\frac{\partial A}{\partial e_{ij}}\frac{\partial x_p}{\partial X_i}\frac{\partial x_q}{\partial X_j}\right)d_{pq} + \left(\rho\rho_2\frac{\partial A}{\partial \rho_2}\delta_{ij} + B_{ij}\right)f_{ij} + B_{ijrs}f_{rs}f_{ij} \tag{32}$$

$$+ \left(a_i - \rho_1\frac{\partial A}{\partial \rho_2}\frac{\partial \rho_2}{\partial x_i} + \rho_2\frac{\partial A}{\partial e_{pq}}\frac{\partial e_{pq}}{\partial x_i}\right)(u_i - v_i) + K_{ij}(u_i - v_i)(u_j - v_j)$$

$$+ \epsilon_{ijk}b_k(\Gamma_{ij} - \Lambda_{ij}) + L_1(\Gamma_{ij} - \Lambda_{ij})(\Gamma_{ij} - \Lambda_{ij})$$

$$+ (A_{ijrs} + C_{rsij})d_{ij}f_{rs} + (L_2 + K_1)\epsilon_{jki}(u_j - v_j)(\Gamma_{ki} - \Lambda_{ki}) + a_{jki}f_{ki}(u_j - v_j) \geq 0.$$

†See, e.g. [9].

At a given state of deformation (e_{ij} and ρ_2 specified), the system may have arbitrary rate of deformation velocities and values of DT/Dt. The inequality (32) has to hold for all possible choices of these fields. It follows that the coefficients of the linear terms must vanish. Thus,

$$S + \frac{\partial A}{\partial T} = 0 \tag{33}$$

$$A_{ij} = \rho \frac{\partial x_i}{\partial X_p} \frac{\partial x_j}{\partial X_q} \frac{\partial A}{\partial e_{pq}} \tag{34}$$

$$B_{ij} = -\rho \rho_2 \frac{\partial A}{\partial \rho_2} \delta_{ij} \tag{35}$$

$$a_i = \rho_1 \frac{\partial A}{\partial \rho_2} \rho_{2,i} - \rho_2 \frac{\partial A}{\partial e_{pq}} e_{pq,i} \tag{36}$$

$$b_i = 0. \tag{37}$$

With this the left hand side of (32) reduces to a quadratic form in the elements of the kinematic variables. The elements of d_{ij} appear only in the term $d_{ij}f_{rs}$. Consequently, the coefficient of this term must vanish giving

$$A_{ijrs} = -C_{rsij}. \tag{38}$$

Some additional restrictions on the constitutive parameters could be obtained by further analysis of the quadratic form. Here, however, we restrict ourselves to an isotropic mixture. Requiring that the form of (21)–(24) be invariant under arbitrary orthogonal transformations leads to

$$B_{ijrs} = \lambda \delta_{ij} \delta_{rs} + \mu(\delta_{ir}\delta_{js} + \delta_{is}\delta_{jr}) \tag{39}$$

$$A_{ijrs} = -C_{rsij} = \gamma_3 \delta_{ij}\delta_{rs} + \gamma_4(\delta_{ir}\delta_{js} + \delta_{is}\delta_{jr}) \tag{40}$$

$$L_2 = K_1 = 0 \tag{41}$$

$$K_{ij} = K_2 \delta_{ij} \tag{42}$$

$$a_{ijk} = 0 \tag{43}$$

where λ, μ, γ_3, γ_4 and K_2 are constants. With this it is readily verified that the inequality (32) is satisfied.

The above results imply that for an isotropic mixture the antisymmetric part of the partial stress tensors depend only on the relative rotation rate $(\Gamma_{ij} - \Lambda_{ij})$ whereas Φ_j is independent of this quantity. For simplicity in the present application we suppress dependence on the rotation rate by choosing $L_1 = 0$ with the consequence that the antisymmetric partial stresses vanish.

The Helmholtz free energy function A is equivalent to the strain energy function [9]. The choice of its functional form determines uniquely the elastic properties of the mixture. The properties of saturated sand are inherently nonlinear. The characteristic feature is dilatation under shear. Including cubic terms involving the solid strains e_{ij} and fluid density change η in the free energy A is sufficient to model this nonlinear behavior. Thus we choose

$$A = \alpha_3 e_{mm} + \alpha_4 e_{mm} e_{nn} + \alpha_5 \eta^2 + \alpha_6 e_{mm}\eta + \alpha_7 e_{nm} e_{ml} e_{ln} + \alpha_8 \eta^3 + \alpha_9 \eta^2 e_{nn}$$

$$+ \alpha_{10}\eta e_{mm} e_{nn} + \alpha_{11}\eta e_{mn} e_{nm} + \alpha_{12} e_{mm} e_{nn} e_{ll} + \alpha_{13} e_{mm} e_{nl} e_{ln}$$

where $\alpha_3, \alpha_4, \ldots \alpha_{13}$ are material constants still to be determined.

Introducing the above results into the constitutive eqns (21)–(24) yields the general form of the nonlinear stress-strain relations. They are

$$\sigma_{ij} = (2\alpha_4 e_{mm} + \alpha_6\eta + \alpha_9\eta^2 + 2\alpha_{10}\eta e_{mm} + 3\alpha_{12}e_{mm}e_{nn} + \alpha_{13}e_{nl}e_{ln})\delta_{ij}$$
$$+ 2\alpha_3 e_{ij} + (4\alpha_4 + 2\alpha_{13})e_{ij}e_{mm} + (2\alpha_6 + 2\alpha_{11})e_{ij}\eta$$
$$+ (4\alpha_3 + 3\alpha_7)e_{is}e_{sj} + 0(e^3) + \gamma_3 f_{mm}\delta_{ij} + 2\gamma_4 f_{ij}, \tag{45}$$

$$\pi_{ij} = -\rho_2^0\{2\alpha_5\eta + \alpha_6 e_{mm} + 3\alpha_8\eta^2 + 2\alpha_9\eta e_{mm} + \alpha_{10}e_{mm}e_{nn}$$
$$+ \alpha_{11}e_{mn}e_{nm} - \frac{1}{\rho^0}(\alpha_3 e_{mn}e_{nm} + \alpha_4 e_{mm}e_{nn} + \alpha_5\eta^2 + \alpha_6 e_{mm}\eta$$
$$+ 0(e^3))\}\delta_{ij} + \lambda f_{mm}\delta_{ij} + 2\mu f_{ij} - \gamma_3 d_{mm}\delta_{ij} - 2\gamma_4 d_{ij}, \tag{46}$$

$$\alpha_{ij} = -\beta_{ij} = 0 \tag{47}$$

$$\Phi_i = K_2(u_i - v_i) + \frac{\rho_1^0}{\rho^0}\left(2\frac{\rho_2^0}{\rho^0}\alpha_5\eta + \alpha_6 e_{mm}\right)\eta_{,i}$$
$$- 2\frac{\rho_2^0}{\rho^0}\alpha_3 e_{kj}e_{kj,i} - \frac{\rho_2^0}{\rho^0}(2\alpha_4 e_{mm} + \alpha_6\eta)e_{nn,i} + 0(e^3) \tag{48}$$

where $0(e^3)$ stands for terms of third order or higher in e_{ij} and η and ρ^0, ρ_1^0 and ρ_2^0 are the initial densities.

For application to soils we must determine numerical values of the constitutive parameters. This is a formidable task for the general form of eqns (45), (46) and (48). Here we reduce the number of parameters by requiring the coefficients of assumed negligible terms to vanish. For example assuming that the pure fluid is a linear compressible material leads to the choice

$$\alpha_8 - \frac{1}{\rho^0}\alpha_5 = 0. \tag{49}$$

In a similar manner the number of parameters necessary to account for solid-fluid interaction is reduced to one static and one dynamic coefficient. In the fluid partial stresses we neglect nonlinear terms in the solid dilatation and fluid density change ($e_{mm}e_{nn}$ and $e_{mm}\eta$). The latter term is also neglected in the solid partial stresses as is the term $e_{ij}e_{mm}$. The latter assumption confines the coupling of longitudinal and shear strain with shear stress in the solid to one term governed by the parameter $(4\alpha_3 + 3\alpha_7)$. Consequently, we choose

$$\alpha_{10} - \frac{\alpha_4}{\rho^0} = 0, \quad 2\alpha_9 - \frac{\alpha_6}{\rho^0} = 0, \quad \alpha_6 + \alpha_{11} = 0, \quad 4\alpha_4 + 2\alpha_{13} = 0. \tag{50}$$

Finally, for application to soils the fluid considered is water. Moreover, interest is primarily-focused on the effect of shear deformation and the fluid partial normal stresses. Thus, for the present study we neglect viscosity, i.e.

$$\lambda = \mu = \gamma_3 = \gamma_4 = 0. \tag{51}$$

With this the final form of the constitutive equations proposed for a saturated sand is

$$\sigma_{ij} = \left(2\alpha_4 e_{mm} + \alpha_6\eta + \frac{\alpha_6}{2\rho^0}\eta^2 + 2\frac{\alpha_4}{\rho^0}\eta e_{mm} + 3\alpha_{12}e_{mm}e_{nn} - \alpha_4 e_{mn}e_{nm}\right)\delta_{ij}$$
$$+ (4\alpha_3 + 3\alpha_7)e_{im}e_{mj} + 2\alpha_3 e_{ij} \tag{52}$$

$$\pi_{ij} = \left[-2\rho_2^0\alpha_5\eta - \rho_2^0\alpha_6 e_{mm} + \rho_2^0\left(\frac{\alpha_3}{\rho^0} + \alpha_6\right)e_{mn}e_{nm}\right]\delta_{ij} \tag{53}$$

$$\Phi_i = K_2(u_i - v_i) + \frac{\rho_1^\circ}{\rho^\circ}\left(2\frac{\rho_2^\circ}{\rho^\circ}\alpha_5\eta + \alpha_6 e_{mm}\right)\eta_{,i}$$

$$-2\frac{\rho_2^\circ}{\rho^\circ}\alpha_3 e_{kj}e_{kj,i} - \frac{\rho_2^\circ}{\rho^\circ}(2\alpha_4 e_{mm} + \alpha_6\eta)e_{mm,i}. \tag{54}$$

Characteristic features of these equations are illustrated by several simple deformation cases. An imposed normal strain produces normal solid stresses and hydrostatic fluid stress but no shear stress. Likewise, only normal stresses are produced by compressing the fluid constituent. Pure shear deformation, however, generates normal stresses in both constituents in addition to shear stresses in the solid. The normal stresses depend upon the square of the shear strain, thus being unaffected by the shear direction.

Finally, the field equations in terms of solid displacements and fluid velocities and density change are obtained by substituting (52)–(54) into (18)–(20). They are

$$2\alpha_4 w_{m,mi} + \alpha_3 w_{k,ik} + \alpha_3 w_{i,kk} - K_2\dot{w}_i - \rho_1^\circ\ddot{w}_i + K_2 v_i + \alpha_6\eta_{,i}$$

$$+\left(\frac{\alpha_6}{\rho^\circ} - 2\frac{\rho_1^\circ}{\rho^\circ}\frac{\rho_2^\circ}{\rho^\circ}\alpha_5\right)\eta\eta_{,i} + \left(2\frac{\alpha_4}{\rho^\circ} - \frac{\rho_1^\circ}{\rho^\circ}\alpha_6\right)w_{m,m}\eta_{,i} + \left(2\frac{\alpha_4}{\rho^\circ} + \frac{\rho_2^\circ}{\rho^\circ}\alpha_6\right)\eta w_{m,mi}$$

$$+\left(6\alpha_{12} + 2\frac{\rho_2^\circ}{\rho^\circ}\alpha_4\right)w_{m,m}w_{n,ni} + \frac{1}{2}\left(2\frac{\rho_2^\circ}{\rho^\circ}\alpha_3 - 2\alpha_4\right)w_{m,n}w_{n,mi}$$

$$+\frac{1}{2}\left(2\frac{\rho_2^\circ}{\rho^\circ}\alpha_3 - 2\alpha_4\right)w_{m,n}w_{m,ni} + \frac{1}{4}(4\alpha_3 + 3\alpha_7)(w_{n,m}w_{m,in} + w_{m,n}w_{m,in}$$

$$+ w_{n,m}w_{i,mn} + w_{m,n}w_{i,mn} + w_{n,mn}w_{m,i} + w_{m,nn}w_{m,i} + w_{n,mn}w_{i,m} + w_{m,nn}w_{i,m}) = 0. \tag{55}$$

$$-2\rho_2^\circ\alpha_5\eta_{,i} - \rho_2^\circ\alpha_6 w_{m,mi} + K_2\dot{w}_i - K_2 v_i - \rho_2^\circ\dot{v}_i$$

$$+2\frac{\rho_1^\circ}{\rho^\circ}\frac{\rho_2^\circ}{\rho^\circ}\alpha_5\eta\eta_{,i} + \frac{\rho_1^\circ}{\rho_2^\circ}\alpha_6 w_{m,m}\eta_{,i} - \frac{\rho_2^\circ}{\rho^\circ}\alpha_6\eta w_{m,mi}$$

$$-2\frac{\rho_2^\circ}{\rho^\circ}\alpha_4 w_{m,m}w_{n,ni} + \rho_2^\circ\alpha_6 w_{n,m}w_{n,mi} + \rho_2^\circ\alpha_6 w_{n,m}w_{m,ni} = 0. \tag{56}$$

$$\dot{\eta} + \rho_2^\circ v_{i,i} = 0. \tag{57}$$

4. ESTABLISHING THE CONSTITUTIVE COEFFICIENTS FROM COMMON SOIL MECHANICS EXPERIMENTS

In the constitutive relations developed for a saturated sand mixture seven constitutive parameters are involved. Six of them $(\alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_{12})$ are associated with static deformations; the seventh (K_2) is associated with the relative velocities. Of the first group five parameters $(\alpha_3, \alpha_4, \alpha_5, \alpha_7, \alpha_{12})$ relate to the behavior of the single materials in the absence of the other; the other parameter (α_6) relates to the static coupling between the two constituents.

In this section we briefly describe a number of common soil mechanics experiments which permit determination of the parameters for the proposed mixture theory. The quantities α_3, α_4, α_7 and α_{12} are obtained from dry sand test results, and α_5 is obtained from the compressibility of the fluid. The parameter α_6 is derived by considering the influence of one constituent's static deformations on the partial stresses of both, i.e. drained test results, and K_2 is derived from the permeability test. The mathematical representation implied by the proposed theory is given below for a number of standard tests:

(a) *Hydrostatic compression test of dry sand* $(\rho_2^\circ = 0)$
Under ideal test conditions

$$e_{11} = e_{22} = e_{33} = -\frac{1}{3}e_0 \tag{58}$$

where e_0 is the compressional dilatation. The stresses are

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = -p_0 \tag{59}$$

where p_0 is the hydrostatic pressure applied to the specimen.

Using (52), (58) and (59) it follows that

$$p_0 = \left(2\alpha_4 + \frac{2}{3}\alpha_3\right)e_0 - \left[3\alpha_{12} + \frac{1}{9}(4\alpha_3 + 3\alpha_7 - 3\alpha_4)\right]e_0^2 \tag{60}$$

(b) *Confined compression test of dry sand* $(\rho_2^0 = 0)$

The specimen is restrained from lateral movement. The strain components are given by

$$e_{11} = -e_1$$
$$e_{ij} = 0 \qquad \text{for } i \neq 1 \text{ or } j \neq 1. \tag{61}$$

The applied compressive stress q is related to the stress components by

$$\sigma_{11} = -q. \tag{62}$$

By (52) the relation of q to e_1 is

$$q = 2(\alpha_3 + \alpha_4)e_1 - (3\alpha_{12} + 4\alpha_3 + 3\alpha_7 - \alpha_4)e_1^2 \tag{63}$$

(c) *Undrained confined compression test of saturated sand*

Under quasistatic load conditions the continuity eqn (57) yields by integration

$$\eta = -\rho_2^0 e_1. \tag{64}$$

The external pressure q is applied to both constituents simultaneously, i.e.

$$\sigma_{11} + \pi_{11} = -q. \tag{65}$$

From (58) and (59) this yields

$$q = (2\alpha_3 + 2\alpha_4 - 2\rho_2^0\alpha_6 + 2\rho_2^{0^2}\alpha_5)e_1 - \left[3\alpha_{12} + \left(4 + \frac{\rho_2^0}{\rho^0}\right)\alpha_3 \right.$$
$$\left. - \left(1 + 2\frac{\rho_2^0}{\rho^0}\right)\alpha_4 + 3\alpha_7 + \rho_2^0\left(1 + \frac{\rho_2^0}{2\rho^0}\right)\alpha_6\right]e_1^2. \tag{66}$$

(d) *Drained confined compression test of saturated sand*

Drained conditions are simulated by the requirement that no partial stresses are generated in the fluid, i.e.

$$\pi_{ij} = 0 \tag{67}$$

$$\sigma_{ij} = q.$$

From (58) and (60) it follows that

$$q = \left(2\alpha_3 + 2\alpha_4 - \frac{\alpha_6^2}{2\alpha_5}\right)e_1 - \left[3\alpha_{12} + 4\alpha_3 + 3\alpha_7 - \alpha_4 \right.$$
$$\left. + \frac{\alpha_6}{2\alpha_5}\left(\frac{\alpha_3}{\rho^0} + \alpha_6 + \frac{\alpha_6^2}{4\rho^0\alpha_5} - \frac{2\alpha_4}{\rho^0}\right)\right]e_1^2 + 0(e_1^3) \tag{68}$$

(e) *Triaxial shear test of dry sand* ($\rho_2^\circ = 0$)

This is a pure shear test in the sense that the first stress invariant σ_{mm} is kept constant, whereas the octahedral shear stress is varied.

For an initial hydrostatic compression state with initial dilatation of $-e_0$ and pressure p_0 the additional strains are Δe_{ij}. From (52) and (60) it follows that during loading

$$\left[6\alpha_4 + 2\alpha_3 - \frac{2}{3}e_0(4\alpha_3 + 3\alpha_7 - 3\alpha_4 + 27\alpha_{12})\right]\Delta e_{mm} + 9\alpha_{12}\Delta e_{mm}^2$$
$$+ (4\alpha_3 + 3\alpha_7 - 3\alpha_4)(\Delta e_{11}^2 + \Delta e_{22}^2 + \Delta e_{33}^2) = 0. \qquad (69)$$

(g) *Triaxial compression test of dry sand* ($\rho_2^\circ = 0$)

This test starts from a hydrostatic compression state with e_0 and p_0 as above. Loading is then performed along one axis so that

$$\Delta\sigma_{22} = \Delta\sigma_{33} = 0 \qquad (70)$$

$$\Delta\sigma_{11} = \Delta\sigma$$

where $\Delta\sigma_{ij}$ are the additional stresses and $\Delta\sigma$ is the loading stress along the axis.

Using (52), (60) and (70) yields

$$\Delta\sigma = \left[6\alpha_4 + 2\alpha_3 - \frac{2}{3}e_0(4\alpha_3 + 3\alpha_7 - 3\alpha_4 + 27\alpha_{12})\right]\Delta e_{mm}$$
$$+ 9\alpha_{12}\Delta e_{mm}^2 + (4\alpha_3 + 3\alpha_7 - 3\alpha_4)(\Delta e_{11}^2 + 2\Delta e_{22}^2), \qquad (71)$$

$$\Delta\sigma = \left[2\alpha_3 - \frac{2}{3}e_0(4\alpha_3 + 3\alpha_7)\right](\Delta e_{11} - \Delta e_{22}) + (4\alpha_3 + 3\alpha_7)(\Delta e_{11}^2 - \Delta e_{22}^2). \qquad (72)$$

(h) *Hydrostatic compression of a pure fluid* ($\rho_1^\circ = 0$)

The hydrostatic stress p causes a fluid dilatation of ϵ. The bulk modulus K_w of the pure fluid is defined by

$$p = K_w\epsilon = K_w\frac{\Delta V_w}{V_w} \qquad (73)$$

where ΔV_w is the change from the initial fluid volume V_w. Here p is defined per unit area of the fluid cross-section, whereas the definition of partial stresses used in the present formulation is per total area of the mixture. This yields the relation

$$\pi_{mm} = 3\beta p \qquad (74)$$

where β is the porosity factor defined as the ratio of fluid area to total area in a cross-section of the mixture.

Integration of the continuity eqn (57) yields

$$\eta = -\rho_2^\circ\frac{\Delta V_w}{V_0} = -\rho_w V_w \frac{\Delta V_w}{V_w}\frac{V_w}{V_0} \qquad (75)$$

where ρ_w is the specific mass of the fluid and V_0 is the initial volume of the mixture. Introducing the approximation $\beta = V_w/V_0$ for an isotropic mixture, it follows from (73) to (74) that

$$2\rho_2^\circ\alpha_5 = \frac{K_w}{\rho_w}. \qquad (76)$$

This implies that the coefficient of η in eqn (53) is a constant for all solid fluid mixtures with the same fluid constituent.

(i) *Permeability test of a saturated sand*

Flow of the fluid constituent is generated in one direction relative to the solid. The velocity of the fluid v_1 is related to the head gradient I_H by Darcy's Law. It is

$$v_1 = k_D I_H \tag{77}$$

where k_D is Darcy's coefficient.

Assuming that at the steady state stage the solid strain field is uniform, eqns (19) and (54) yield

$$\pi_{11,1} - K_2 v_1 = 0. \tag{78}$$

The relation between the partial fluid stress gradient and the total stress in the fluid just outside of the mixture is

$$\pi_{11,1} = \frac{P_0 - P_L}{L} \cdot \beta \tag{79}$$

where $P_0 - P_L$ is the pressure loss along L. The head gradient for water is

$$I_H = \frac{P_0 - P_L}{L \rho_w g} \tag{80}$$

where g is the gravity acceleration. From (77) and (80) it follows that

$$K_2 = \frac{\rho_w g \beta}{k_D}. \tag{81}$$

5. NUMERICAL APPLICATION

The coefficients α_5, K_2 have been directly related to commonly known fluid properties. The remaining parameters must be determined from the standard soil mechanics experiments described above.

The coefficients obtained from dry sand tests are established in the following manner:

1. From hydrostatic compression test results $(6\alpha_4 + 2\alpha_3)$ and $(27\alpha_{12} + 4\alpha_3 + 3\alpha_7 - 3\alpha_1)$ are obtained by a parabolic least square fit.

2. With this the parameter α_{12} is found from a linear least square fit of (69) for the triaxial shear test.

3. Finally, triaxial compression test results may be used to establish the individual values of α_3 and α_7. If test results are available for both axial and lateral strains, this may be done directly by fitting (72) to the data. Equation (71) then becomes a check on the validity of the established coefficients. If lateral strain data is not available, eqn (71) is used to numerically eliminate Δe_{22}.

The value of α_6 is now established by fitting (68) to quasistatic drained confined compression test results. We note, however, that the load deformation relations of the dry and drained tests differ only in terms that have α_6 as a factor. According to Lambe and Whitman[5], drained tests provide the same final load deformation relations as dry tests. It is thus concluded that for a saturated sand

$$\alpha_6 = 0. \tag{82}$$

To carry out the above procedure requires that each of the postulated experiments be conducted on a particular sand at a given void ratio. At the present time such a consistent set of data is not readily available for any one sand. For the purpose of illustration here we use the above procedure to obtain the constitutive parameters for a hypothetical sand at three different void ratios using a qualitatively correct data set based on actual and extrapolated soil data reported in Refs.[1, 5]. The results are given in Table 1. The resulting load deformation

Table 1. Numerical data

Sand type	Loose	Medium	Dense	Units
Void Ratio (V.R.)†	0.60	0.55	0.485	
Porosity (β)	0.375	0.35	0.33	
Solid Density (ρ_1^s)	1.56×10^{-4}	1.62×10^{-4}	1.67×10^{-4}	psi sec²/in.²
Fluid Density (ρ_2^s)	0.35×10^{-4}	0.33×10^{-4}	0.31×10^{-4}	psi sec²/in.²
K_2	7.5	7.0	6.5	psi sec/in.²
$2\rho_2^s \alpha_5$	3.3×10^9	3.3×10^9	3.3×10^9	in.²/sec²
$2\alpha_4$	0.9×10^4	1.2×10^4	1.5×10^4	psi
α_3	2.2×10^3	2.6×10^3	3.2×10^3	psi
α_{12}	-1.5×10^6	-1.8×10^6	-2.0×10^6	psi
α_7	2.0×10^5	1.5×10^5	3.0×10^3	psi

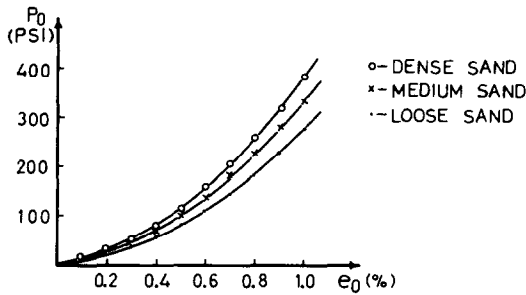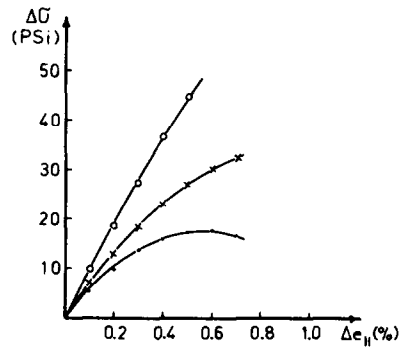†The void ratios refer to the values used by Ko[1].

Fig. 1.

Fig. 2.

Fig. 1. Generated data for hydrostatic test of dry sands according to parameters represented in Table 1 (stress vs strain).

Fig. 2. Generated data for triaxial compression test of dry sands (deviator stress vs axial strain).
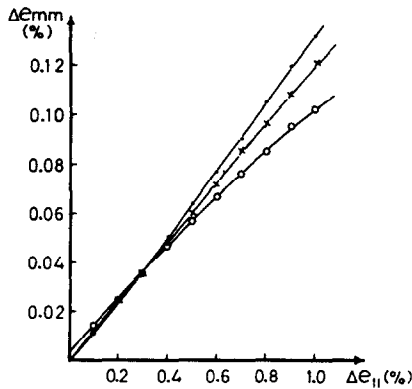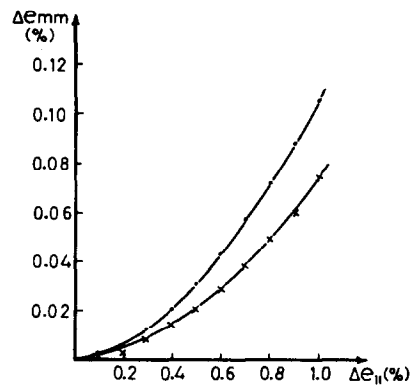

Fig. 3.

Fig. 4.

Fig. 3. Generated data for triaxial compression test of dry sands (dilatation vs axial strain).

Fig. 4. Generated data for triaxial shear test of dry sands (dilatation vs strain).

behavior for the various test conditions is shown in Figs. 1–4 demonstrating the relative effect of void ratio on stress-strain behavior. The characteristic behavior illustrated is in accordance with typical experimental results[1, 5].

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