LETTERS IN HEAT AND MASS TRANSFER Volume 4, Pages 75-84, Pergamon Press, 1977. Printed in Gt. Britain.

TRANSIENT HEAT TRANSFER IN A THICK THERMAL BOUNDARY LAYER WITH SPHERICAL SYMMETRY

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(Communicated by J.P. Hartnett and W.J. Minkowycz)

ABSTRACT

Unsteady heat diffusion in a thick thermal boundary layer over a spherical surface in radial motion is studied. The boundary layer is divided into two parts, one towards the outer edge of the layer, and the other adjacent to the surface. A method of successive approximations is employed to obtain the solutions appropriate to these regions. An explicit expression for the temperature distribution is presented in the zero order when the temperature at infinity and the temperature gradient at the spherical surface are specified. The convergence of the approximation procedure and the joining of the inner and outer solutions are discussed. Results may be applied to the problems of bubble growth or collapse in a liquid and droplet evaporation and particle sublimation in a hot environment.

Introduction

Unsteady transport phenomena in a spherically symmetrical system with a boundary in radial motion are commonly observed in nature and in industry. Typical examples of such a physical event include the growth and collapse of a vapor bubble, evaporation of a droplet, and sublimation of a solid particle.

Plesset and Zwick [1] have obtained a solution in successive approximations for the heat diffusion associated with the dynamics of a vapor bubble in a liquid. A crucial assumption states that the thermal boundary layer adjacent to the spherical boundary is very thin or equivalently the Fourier number is small. Their approximate solution has been applied to the growth

[2,3] and collapse [4] of a spherical vapor bubble in a liquid. In these bubble dynamics problems, the assumption of the thin thermal boundary layer is quite reasonable since not only the heat capacity is much greater in the liquid state than in the vapor state but the thermal diffusivity is also much smaller. Theory, therefore, agrees very well with experiments.

Due to the restriction of small Fourier numbers, the thin thermal boundary layer solution is equivalent to the small-time solution which is valid over short time intervals immediately following the introduction of a vapor bubble into a liquid. In the asymptotic phase after certain time has elapsed, however, the asymptotic solution to the temperature field can be obtained by means of a similarity transformation. A thermodynamic equilibrium under the saturation state is assumed to prevail at the spherical boundary R(t) which obeys the square law, namely $R^2(t)$ varies linearly with time. Because of simplicity in mathematical treatment, numerous articles dealing with the asymptotic behavior of the temperature (or concentration in mass diffusion) field have been published, for example references 5-7.

In the case of droplet evaporation or sublimation of solid particles in a hot environment, the heat capacity is much smaller in the gas phase than in the liquid or solid phase while the thermal diffusivity is also much larger. Consequently, the thermal boundary layer adjacent to the moving spherical boundary is thicker than the droplet or particle radius. The thin thermal boundary layer solution is not valid any more.

The present study deals with the unsteady heat diffusion in a thick thermal boundary layer. By means of a method of successive approximations, the approximate solutions are obtained based on a division of the boundary layer into two parts.

Analysis

Consider the physical situation in which a spherical particle, either a liquid droplet or a solid particulate, is suddenly placed in a hot environment of an infinite extent at temperature T_{∞}. As the radius of the particle changes due to phase transformation, heat flows toward the moving spherical interface. It is postulated that the gas is inviscid and incompressible, and that thermal properties remain constant with temperature. Using spherical coordinates with the origin fixed at the center of the particle, the gas temperature T satisfies the equation

$$\frac{\partial T}{\partial t} = \frac{\alpha}{r^2}\frac{\partial}{\partial r}(r^2 \frac{\partial T}{\partial r}) \tag{1}$$

with the initial and boundary conditions

$$T\ (0,r) = T_\infty \tag{2}$$

$$T\ (t,\infty) = T_\infty \quad ; \quad \frac{\partial T(t,R)}{\partial r} = \rho_\ell \lambda \dot{R}/k \tag{3}$$

Here, t denotes the time; α, thermal diffusivity; r, radial distance from the origin; k, thermal conductivity; ρ, particle density; λ, latent heat of evaporation or sublimation; and $\dot{R}$, time derivative of R(t). The temperature gradient at the spherical boundary in Eq. (3) is specified.

In dealing with a diffusion problem with moving boundaries, it is advantageous to transform Eqs. (1) through (3) from Eulerian to Lagrangian coordinates. With Lagrange coordinates of

$$x = [r^3 - R^3(t)]/3 \tag{4-a}$$

$$t = t \tag{4-b}$$

Eq. (1) reduces to

$$\frac{\partial T}{\partial t} = \alpha\frac{\partial}{\partial x}(r^4\frac{\partial T}{\partial x}) \tag{5}$$

One sets

$$\frac{\partial u}{\partial x} = T - T_\infty \tag{6}$$

so that

$$u = \int_0^x (T - T_\infty)\ dx + a(t) \tag{7}$$

where a(t) is the integration constant. Equation (5) can be rewritten as

$$\frac{\partial}{\partial x}(\frac{\partial u}{\partial t} - \alpha r^4 \frac{\partial^2 u}{\partial x^2}) = 0$$

so that upon an integration with respect to x, one gets

$$\frac{\partial u}{\partial t} - \alpha r^4 \frac{\partial^2 u}{\partial x^2} = b(t) \tag{8}$$

in which b(t) is an arbitrary function of time. With the incorporation of Eq. (2), we can select the function a(t) so that

$$u(0,x) = 0 \tag{9}$$

and b(t)=0. Equation (8) now becomes

$$\frac{\partial u}{\partial t} - \alpha r^4 \frac{\partial^2 u}{\partial x^2} = 0 \tag{10}$$

The boundary conditions (3) can be rewritten as

$$\frac{\partial u(t,\infty)}{\partial x} = 0 \tag{11}$$

$$\frac{\partial^2 u}{\partial x^2} = f(t) \tag{12}$$

where

$$f(t) = \frac{\rho_\ell \lambda \dot{R}}{k\, R^2} = \frac{1}{R^2}\,\frac{\partial T(t,R)}{\partial r} \tag{13}$$

The boundary layer is divided into two parts: one adjacent to the spherical boundary and the other towards the outer edge of the layer. Then, the inner and outer solutions appropriate to these regions are sought.

For the outer region $x \gg R^3(t)$, one can write Eq. (4-a) in the form of

$$r^4 = (3x)^{4/3} \left[1 + \frac{R^3}{3x}\right]^{4/3}$$

which may be expanded into a series as

$$r^4 = (3x)^{4/3} \left[1 + \frac{4}{3}\left(\frac{R^3}{3x}\right) + \ldots\ldots \right] \tag{14-a}$$

The series expansion of Eq. (4-a) appropriate to the inner region $x \ll R^3(t)$ is

$$r^4 = R^4 \left[1 + \frac{4}{3}\left(\frac{3x}{R^3}\right) + \ldots\ldots \right] \tag{14-b}$$

As a routine procedure in successive approximation, let

$$u = u_0 + \epsilon u_1 + \ldots\ldots, \quad T = T_0 + \epsilon T_1 + \ldots\ldots\ldots \tag{15}$$

Here, the subscript indicates the order of the approximation in power of the perturbation parameter ϵ which has the same order of magnitude as R^3/x in the outer region or x/R^3 in the inner region.

A. Inner Solution

It is convenient to introduce in Eq. (10) a new time variable τ defined as

$$\tau = \int_0^t R^4(t)\, dt \tag{16}$$

With the substitution of Eqs. (14-b), (15) and (16), one finds that the zero-order approximation, u_0, is determined by

$$\frac{\partial u_o}{\partial \tau} - \alpha\frac{\partial^2 u_o}{\partial x^2} = 0 \tag{17}$$

subject to the conditions (9) and (12). By taking the Laplace transform on the variable τ, Eqs. (17) and (12) become

$$p\bar{U}_o - \alpha\frac{d^2\bar{U}_o}{dx^2} = 0 \tag{18}$$

and

$$\frac{d^2\bar{U}_o(p,0)}{dx^2} = \bar{F}(p) \tag{19}$$

respectively, where p represents the Laplace variable of τ and $\bar{U}_o$ and $\bar{F}$ are the Laplace transformed functions of u_o and f respectively. The general solution of Eq. (18) reads

$$\bar{U}_o(p,x) = C_1(p)\exp(\sqrt{p/\alpha}\,x)+C_2(p)\exp(-\sqrt{p/\alpha}\,x) \tag{20}$$

Since a small value of τ corresponds to a large value of P, the positive exponential term must vanish for the temperature to be finite at small times: $C_1=0$. The combination of Eqs. (18), (19) and (20) at x=0 produces

$$C_2(p) = (\alpha/p)\bar{F}(p)$$

so that Eq. (20) becomes

$$\bar{U}_o(p,x) = (\alpha/p)\exp(-\sqrt{p/\alpha}\,x)\,\bar{F}(p) \tag{21}$$

An inverse Laplace transform on the first derivative of Eq. (21), $d\bar{U}_o/dx$, leads to

$$T_o(t,x) - T_\infty = -\sqrt{\frac{\alpha}{\pi}}\int_0^t \frac{g(t')\exp(-\frac{x^2}{4\alpha Y})}{Y^{1/2}}dt' \tag{22}$$

wherein

$$g(t') = R^2(t')(\partial T/\partial r)_{r=R(t')} \tag{23}$$

$$Y = \int_{t'}^{t} R^4(y)\,dy \tag{24}$$

One finds from Eq. (22) the boundary temperature

$$T_o(t,0) - T_\infty = -\sqrt{\frac{\alpha}{\pi}}\int_0^t \frac{g(t')}{Y^{1/2}}dt' \tag{25}$$

and the temperature gradient

$$\frac{\partial T_o(t,x)}{\partial r} = \frac{R^2(t)x}{2(\pi\alpha)^{1/2}}\int_0^t \frac{g(t')\exp(-\frac{x^2}{4\alpha Y})}{Y^{3/2}}dt' \tag{26}$$

Upon the transformation of

$$\xi = \frac{x}{2(\alpha Y)^{1/2}} \tag{27}$$

Eqs. (25) and (26) can be reduced to

$$T_o(t,x)-T_\infty = -\frac{x}{\pi^{1/2}}\int_X^\infty \frac{(\partial T/\partial r)_{r=R(t')}}{R^2(t')}\xi^{-2}\exp(-\xi^2)d\xi \tag{28}$$

and

$$\frac{\partial T_o(t,x)}{\partial r} = \frac{2}{\sqrt{\pi}}R^2(t)\int_X^\infty \frac{(\partial T/\partial r)_{r=R(t')}}{R^2(t')}\exp(-\xi^2)d\xi \tag{29}$$

respectively, where

$$X = \frac{x}{2\left[\alpha\int_0^t R^4(y)dy\right]^{1/2}} = 1/F_o^{\ 1/2} \tag{30}$$

The Fourier number F_o in the inner region is defined as

$$F_o = 4\alpha\int_0^t R^4(y)dy\ /x^2 \tag{31}$$

It is important to note that the inner solution (22) is identical to the thin thermal boundary layer solution of reference 1 in which C_1 is made zero through imposing the boundary condition at infinity (11). In the present study, the inner solution is valid only at the spherical boundary and its immediate vicinity.

B. Outer Solution

With the aid of Eqs. (14-a) and (15), the zero-order approximation is governed by

$$\frac{\partial u_o}{\partial t} - \alpha(3x)^{4/3}\frac{\partial^2 u_o}{\partial x^2} = 0 \tag{32}$$

with the boundary conditions (9) and (11). The Laplace transform on t yields

$$\frac{d^2U_o}{dx^2} - \frac{s}{\alpha}(3x)^{-4/3}U_o = 0 \tag{33}$$

and

$$\frac{dU_o(s,\infty)}{dx} = 0 \tag{34}$$

Here, s signifies the Laplace variable of t and $U_o(s,x)$ corresponds to the Laplace transformed function of $u_o(t,x)$. Equation (32) has the general solution [8,9]

$$U_o(s,x)=C_3(s)x^{1/2}I_{3/2}\left[(3x)^{1/3}(\tfrac{s}{\alpha})^{1/2}\right]+C_4(s)x^{1/2}K_{3/2}\left[(3x)^{1/3}(\tfrac{s}{\alpha})^{1/2}\right] \tag{35}$$

wherein $I_{3/2}$ is known as the modified Bessel function of the first kind, of order 3/2 and $K_{3/2}$ of the second kind. The boundary condition (34) requires that $C_3(s)$ must vanish. One then has

$$U_o(s,x) = C_4(s)x^{1/2}K_{3/2}\left[(3x)^{1/3}(s/\alpha)^{1/2}\right] \tag{36}$$

Now, C_4 is to be determined by the joining of the inner and outer solutions at the junctions of these regions: x = +0. There, the inner solution (21) can be rewritten in the Laplace domain of s as

$$U_o(s,+0) = \alpha Y(s)/s \tag{37}$$

where

$$Y(s) = L\left[f(t)R^4(t)\right]$$

On equating (37) to the outer solution (36) at the junction, one gets [10]

$$C_4(s) = (6/\pi)^{1/2}(\alpha/s)^{1/4}Y(s)$$

Equation (36) can now be written as

$$U_o(s,x)=(6/\pi)^{1/2}x^{1/2}(\alpha/s)^{1/4}Y(s)K_{3/2}\left[(3x)^{1/3}(s/\alpha)^{1/2}\right] \tag{38}$$

so that

$$\frac{dU_o}{dx} = -\frac{Y(s)}{(3x)^{1/3}}\exp\left[-(3x)^{1/3}(s/\alpha)^{1/2}\right] \tag{39}$$

An inverse Laplace transform on Eq. (39) followed by the substitution of Eq. (13) gives

$$T_o(t,x)-T_\infty=-\frac{1}{2\sqrt{\pi\alpha}}\int_0^t \frac{g(t')}{(t-t')^{3/2}}\exp\left[-\frac{(3x)^{2/3}}{4\alpha(t-t')}\right]dt' \tag{40}$$

Upon performing the transformation of

$$\zeta = \frac{3x)^{1/3}}{2\left[\alpha(t-t')\right]^{1/2}} \tag{41}$$

Eq. (40) reduces to

$$T_o(t,x)=-\frac{2}{\sqrt{\pi}}\frac{1}{(3x)^{1/3}}\int_Z^\infty \exp(-\zeta^2)g\left[t-\frac{(3x)^{2/3}}{4\alpha\zeta^2}\right]d\zeta +T_\infty \tag{42}$$

in which

$$Z = \frac{(3x)^{1/3}}{2(\alpha t)^{1/2}} = 1/F_o^{1/2} \tag{43}$$

The Fourier number in the outer region is defined as

$$F_o = 4\alpha t/(3x)^{2/3} \tag{44}$$

The temperature gradient can be found from Eq. (42) to be

$$\frac{\partial T_o(t,x)}{\partial r} = \frac{4}{\sqrt{\pi}}\frac{1}{(3x)^{2/3}}\int_Z^\infty \zeta^2\exp(-\zeta^2)g\left[t-\frac{(3x)^{2/3}}{4\alpha\zeta^2}\right]d\zeta \tag{45}$$

At large values of F_o, Z approaches zero and the variable of the g function approach t. Equations (42) and (45) can then be simplified as

$$T_o(t,x) - T_\infty = -\frac{g(t)}{(3x)^{1/3}} \tag{46}$$

and

$$\frac{\partial T_o(t,x)}{\partial r} = \frac{g(t)}{(3x)^{2/3}} \tag{47}$$

respectively.

If the curvature of the boundary r=R(t) is neglected, the use of Lagrange coordinates

$$x = r - R(t), \qquad t = t \tag{48}$$

leads to the plane approximation

$$T_o(t,x)-T_\infty=-\left(\frac{\alpha}{\pi}\right)^{1/2}\int_0^t \frac{R(t')(\partial T/\partial r)_{r=R(t')}}{(t-t')^{1/2}[x+R(t')]}\exp\left[-\frac{x^2}{4\alpha(t-t')}\right]dt' \tag{49}$$

The solution is valid for the entire boundary layer.

Results and Discussion

For estimating the behavior of the convergence of the approximation theory, it is necessary to examine the first-order correction to the solution. The first-order correction u_1 is determined by

$$\frac{1}{\alpha}\frac{\partial u_1}{\partial t} - \frac{\partial^2 u_1}{\partial x^2} = \frac{4x}{R^3}\frac{\partial u_o}{\partial x^2} \tag{50-a}$$

in the inner region and

$$\frac{1}{\alpha(3x)^{4/3}}\frac{\partial u_1}{\partial t} - \frac{\partial^2 u_1}{\partial x^2} = \frac{4R^3}{9x}\frac{\partial^2 u_o}{\partial x^2} \tag{50-b}$$

in the outer region. The initial and boundary conditions are

$$u_1(0,x) = \frac{\partial u_1(t,\infty)}{\partial x} = \frac{\partial^2 u_1(t,0)}{\partial x^2} = 0 \tag{51}$$

Following the procedure taken in the zero-order approximation, the first-order temperature correction in the inner region can be determined by solving Eq. (50-a) and the first and third expressions of Eq. (51) together with the requirement for finite u_1 at small times. The solution is identical with that obtained by the thin thermal boundary layer approximation [1]. It gives the first-order temperature correction at the boundary r=R(t) bound by

$$\frac{\alpha}{R^3}\int_0^\tau f(\tau')d\tau' \leq T_1(\tau,0) \leq \frac{4\alpha}{3R_o^3}\int_0^\tau f(\tau')d\tau' \tag{52}$$

where R_o=R(0). The requirements for rapid convergence of the approximation solution as indicated by the bounds (52) are satisfied for the particular problem of the growth or collapse of a spherical vapor bubble in a liquid [1-4].

Since the lifetime of an evaporating droplet [11] or a sublimating particle in a hot environment is about the same order of magnitude as that of a vapor bubble collapsing in a subcooled liquid [2,4], the zero-order approximation as given by Eqs. (22) and (25) are sufficient. This conclusion is believed to hold also for the outer solution (40) or (42) as it has been derived through matching with the inner solution.

One example is given below to demonstrate the joining of the inner and outer solutions: The boundary R and its radial velocity $\dot{R}$ must be known so that the boundary condition (3) is specified. So, let the time history of the boundary be

$$R = at^n \quad \text{so that} \quad \dot{R} = ant^{n-1} \tag{53}$$

where both a and n are constant. a is dependent upon physical properties. Equation (25) and (46) for high F_o are then reduced to

$$\frac{T_o(t^*,0)-T_\infty}{T_s - T_\infty} = -\frac{J}{\sqrt{\pi}}\,\frac{a^*(t^*)^n}{2}\int_0^{\pi/2} (\sin x)^{1/2}\,dx \tag{54}$$

and

$$\frac{T_o(t^*,x^*)-T_\infty}{T_s - T_\infty} = -\frac{nJ(a^*)^3(t^*)^{3n-1}}{x^*} \tag{55}$$

respectively, where T_s denotes the saturation temperature and

$$J = \frac{\rho_\ell \lambda}{\rho c_p \Delta T}\,;\ \Delta T = T_s - T_\infty\ ;\ a^* = \frac{aR_o^{2n-1}}{n}\ ;\ t^* = \frac{\alpha t}{R_o^2} \tag{56}$$

An examination of Eqs. (54) and (55) has disclosed that the value of n is not arbitrary but subject to some physical constraints. In case of bubble growth for example, a faster temperature change in the inner region than in the outer region requires that the exponent of t* in Eq. (54) be greater or equal to that of t* in Eq. (55). The upper bound of n is thus found to be $n \leq 1/2$. On the other hand, the exponent of t* in Eq. (55) must be greater than zero for $T_o(t,x)$ in the outer region to be a monotonic increasing function of time. It yields the lower bound as $n > 1/3$. Thus, one gets

$$1/2 \geq n > 1/3 \tag{57}$$

The n range is in good agreement with the square law (the square of bubble, droplet or particle diameter varies linearly with time) observed in the asymptotic stage of bubble growth [2,3,5-7] or droplet evaporation [11, with negative a] which gives n = 1/2.

The ratio of the inner to outer solutions is obtained from Eqs. (54) and (55) as

$$\frac{T_o(t^*,0)-T_\infty}{T_o(t^*,x^*)-T_\infty} = \frac{2n\pi(a^*)^2(t^*)^{1-2n}x^*}{\int_0^{\pi/2} (\sin x)^{1/2}dx} \tag{58}$$

A smooth joining of the outer solution to the inner solution represented by the surface temperature is warranted.

Conclusions

A method of successive approximations is employed to solve the problem of unsteady heat diffusion in a thick thermal boundary layer over a spherical boundary in radial motion. The approximate solutions are obtained based on a division of the boundary layer into the inner and outer regions. The zero-order approximation is sufficient to represent the temperature field. A smooth joining of the inner and outer solutions is warranted. The Fourier numbers appropriate to the regions are defined. The approximate solutions are valid in all ranges of the Fourier numbers. The boundary temperature is identical with that obtained by the thin thermal boundary layer approximation.

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