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THERMAL INSTABILITY IN SPHERICAL LIQUID SHELLS INDUCED BY SURFACE TENSION*

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ABSTRACT

A linear perturbation method is employed to determine the condition for neutral stability in spherical liquid shells induced by surface tension mechanism. Three possible boundary conditions are considered: at least one boundary free or both. The critical Marangoni numbers for the onset of cellular convections are found for two types of steady radial temperature distributions in the spherical shells. Results are compared with those induced by buoyancy mechanism. It is concluded that surface tension forces are much more effective than buoyancy forces in producing thermal instability and a parabolic steady temperature distribution is more susceptible than a linear one to thermal disturbances due to surface tension forces. Heat transfer between a free surface and the ambient promotes thermal stability in liquid shells.

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Introduction

Surface tension variations due to temperature gradients generally induce fluid motion which would not otherwise occur. Such phenomena are often called the Marangoni effect. Effects due to temperature gradients are also referred to as thermocapillarity. Kenning [1] reviewed the processes by which surface-tension variations influence two-phase flow. Surface tension effects may also be the origin of the cellular convection in thin liquid layers. Block [2] suggested surface tension as the cause of Benard cells and liquid deformation in a liquid film. Using small perturbation analysis, Pearson [3] demonstrated that instability due to surface tension would occur at a critical Marangoni number. Scriven and Sternling [4] extended Pearson's study to include the effect of surface deformation. Analyzing the coupling between surface tension and buoyancy, Nield [5] disclosed that the critical Marangoni number decreased with an increase in the Rayleigh number. Berg, Boudart, and Acrivos [6] observed three basic structural forms of flows during the evaporation of liquid less than 1cm deep. They found a simple criterion for distinguishing visual flow patterns as being induced by surface tension, buoyancy or surface contamination. Scanlon and Segal [7] analyzed finite amplitude cellular canvection induced by surface tension. All these studies concern with thin liquid films on a flat surface.

The present work deals with thermal instability in spherical liquid shells with at least one free surface. Initially, a steady temperature of certain profile prevails in the shell. Conditions for the onset of stationary instability caused by surface tension effects are determined using a linear perturbation method. Results are compared with those caused by the buoyancy mechanism in reference 8.

Analysis

Thermal conditions which lead to the onset of cellular convection in spherical liquid shells due to the action of surface tension are to be determined. Let R_1 and R_2 be respectively the inner and outer radii of a liquid shell. Depending on the nature of the boundary surfaces, free or rigid, three combinations are considered: a free surface at R_1 and a rigid surface at R_2, a free surface at R_2 and a rigid surface at R_1, and free surfaces at R_1 and R_2. Steady radial temperature distribution in a liquid shell may take two forms: parabolic and linear. A parabolic distribution in temperature is the base state when the fluid has a constant heat source per unit volume. The linear dependence of temperature on r is only valid in the small gap limit.

(i) Parabolic distribution of steady temperature in spherical shells

A. Rigid inner surface and free outer surface

Consider a quiescent liquid shell whose inner surface at $r = R_1$ lies against a solid sphere, whose outer surface at $r = R_2$ is in contact with an inviscid fluid. The origin of (r,θ,ϕ) co-ordinates is fixed at the center of the shell. At undisturbed, steady state, the temperature gradient in the liquid shell is a linear function of the r co-ordinate alone, that is

$$\frac{dT_o}{dr} = -2\beta r$$

where T_o denotes the unperturbed temperature in the liquid shell and β is a constant.

Next, one superimposes an infinitesimal disturbance and linearizes the equations of motion and heat transport. Let v represent the velocity in the r direction; and T' be the perturbation temperature. The equations of motion and heat transport become

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\nabla^2(rv) = 0 \tag{2}$$

$$\left(\frac{\partial}{\partial t} - \alpha\nabla^2\right) T' = 2\beta rv \tag{3}$$

Here t denotes the time; ν , kinematic viscosity; and

α, thermal diffusivity. One writes the surface tension of the liquid S to vary with the perturbed liquid surface temperature T_s' as

$$S = S_o - \sigma T_s' \tag{4}$$

where $-\sigma = (\partial S/\partial T)$ evaluated at steady surface temperature T_{so}, and S_o is the surface tension at T_{so}. The liquid temperature T is equal to $T' + T_o$.

The boundary conditions at rigid surface $r = R_1$ are:

$$v = \frac{dv}{dr} = 0, \; T' = 0 \tag{5}$$

at free surface $r = R_2$:

$$v = 0, \; \frac{\rho\nu}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) = -\sigma\nabla_1{}^2 T', \; T' = -\frac{k}{h}\frac{\partial T'}{\partial r} \tag{6}$$

Here

$$\nabla_1^2 = \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] \tag{7}$$

and its eigen value λ is

$$\lambda^2 = n(n+1) \tag{8}$$

where n is an integer. The boundary conditions at $r = R_1$ are obvious. At the boundary $r = R_2$, the first expression states the condition of zero liquid velocity normal to the interface; the second relation states that the change in surface tension along the boundary must be balanced by shear force. The last one states that the continuity of heat exists at the interface.

Suppose that the perturbations v and T' have the forms

$$rv = \nu\, Y_n^m(\theta,\phi) f(\eta) e^{p\tau} \tag{9-a}$$

$$T' = \beta R_2^2 \, Y_n^m \, (\theta,\phi) g(\eta) e^{p\tau} \tag{9-b}$$

in which

$$\tau = \frac{\nu t}{R_2^2}, \quad \eta = \frac{r}{R_2}, \quad p = \frac{\partial}{\partial\tau} \tag{10}$$

and the spherical harmonics $Y_n^m \, (\theta,\phi)$ are the eigen function. h is the liquid-fluid heat transfer coefficient and k denotes the thermal conductivity of the liquid.

Equations (2) and (3) now become

$$(D_1 - p) \, D_1 f = 0 \tag{11}$$

$$(p \, Pr - D_1) \, g = f \tag{12}$$

and

$$\frac{1}{\eta^2} \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \, \frac{\partial Y_n^m}{\partial\theta}\right) + \frac{1}{\sin^2\theta} \, \frac{\partial^2 Y_n^m}{\partial\phi^2}\right] + \lambda^2 Y_n^m = 0 \tag{13}$$

Here, the operator D_1 is defined as

$$D_1 = \frac{1}{\eta^2} \left[\frac{d}{d\eta} \left(\eta^2 \frac{d}{d\eta}\right) - \lambda^2\right] \tag{14}$$

and $Pr = \nu/\alpha$ is the Prandtl number. Equations (11) and (12) are subject to the boundary conditions

$$f(d) = f'(d) = 0; \quad g(d) = 0 \tag{15}$$

$$f(1) = 0; \quad f''(1) = -2\lambda^2 \, Ma \; g(1); \quad g'(1) = -Bi \; g(1) \tag{16}$$

where $Ma = \frac{\sigma\beta R_2^3}{\rho\nu\alpha}$ is the Marangoni number and $Bi = \frac{hR_2}{k}$ is the Biot number. d is defined as R_1/R_2.

By setting $p = 0$ for marginal stability, equations (11) and (12) reduce to

$$D_1^2 f = 0 \tag{17}$$

$$D_1 g = -f \tag{18}$$

Solutions to these eqautions must be found subject to the boundary conditions (15) and (16). For a fixed value of Bi, Ma is minimized as a function of the wave number λ to obtain the critical Marangoni number for the onset of cellular convection.

The solutions for f and g are found to be

$$f = C_1(\eta^n + A\eta^{n+2} + B\eta^{-n-1} + C\eta^{-n+1}) \tag{19}$$

and $$g = -C_1\left[\frac{\eta^{n+2}}{2(2n+3)} + \frac{A\eta^{n+4}}{4(2n+5)} + \frac{B\eta^{-n+1}}{2(-2n+1)} + \frac{C\eta^{-n+3}}{4(-2n+3)} - \frac{D\eta^{-n-1}}{2n+1} + E\eta^n\right]$$

Here, the coefficients are defined as

$$A = [-2d^{-2n-1} + (2n+1)d^{-2} - (2n-1)]/F$$

$$B = [2d^{2n+1} - (2n+1)d^2 + (2n-1)]/F$$

$$C = [-2d^{2n+1} - (2n+1)d^{-2} + (2n+3)]/F$$

$$F = 2d^{-2n-1} + (2n+1)d^2 - (2n+3)$$

$$P = \frac{d^{n+2}}{2(2n+3)} + \frac{Ad^{n+4}}{4(2n+5)} + \frac{Bd^{-n+1}}{2(-2n+1)} + \frac{Cd^{-n+3}}{4(-2n+3)}$$

$$H = [(Bi-n-1)d^n - (Bi+n)d^{-n-1}]/(2n+1)$$

$$D = [d^n Q - (Bi+n)P]/H$$

$$E = [d^{-n-1} Q - (Bi-n-1)P]/[H(2n+1)]$$

$$Q = \frac{Bi+n+2}{2(2n+3)} + \frac{A(Bi+n+4)}{4(2n+5)} + \frac{B(Bi-n+1)}{2(-2n+1)} + \frac{C(Bi-n+3)}{4(-2n+3)}$$

The subsitutions of the solutions (19) and (20) into the second expression of equation (16) yields a relation between Ma, Bi, and n as

$$Ma = \frac{n(n-1) + A(n+1)(n+2) + B(n+1)(n+2) + Cn(n-1)}{2n(n+1)\left[\frac{1}{2(2n+3)}+\frac{A}{4(2n+5)}+\frac{B}{2(-2n+1)}+\frac{C}{4(-2n+3)}-\frac{D}{2n+1}+E\right]} \quad (21)$$

B. Free inner surface and rigid outer surface

The outer surface of a quiescent liquid shell at $r = R_2$ lies against a spherical rigid surface, while the inner surface of the shell is in contact with an inviscid fduid. The equations governing marginal stability (17) and (18) are subject to the boundary conditions:

$$f(1) = f'(1) = 0; \quad g(1) = 0 \quad (22)$$

$$f(d) = 0; \quad f''(d) = -2\lambda^2 Mag(d); \quad g'(d) = -Big(d) \quad (23)$$

The solutions for f and g take the same form as equations (19) and (20), respectively, where

$$A = [(-2n+1)d^{-n-1} - 2d^n + (2n+1)d^{-n+1}]/F$$

$$B = [-(2n+1)d^n + (2n-1)d^{n+1} + 2d^{-n+1}]/F$$

$$C = [-2d^{-n-1} - (2n+1)d^{n+2} + (2n+3)d^n]/F$$

$$F = (2n+1)d^{-n-1} + 2d^{n+2} - (2n+3)d^{-n+1}$$

$$P = \frac{1}{2(2n+3)} + \frac{A}{4(2n+5)} + \frac{B}{2(-2n+1)} + \frac{C}{4(-2n+3)}$$

$$Q = \frac{(n+2)d^{n+1}+Bid^{n+2}}{2(2n+3)} + \frac{A[(n+4)d^{n+3}+Bid^{n+4}]}{4(2n+5)} + \frac{B[(-n+1)d^{-n}+Bid^{-n+1}]}{2(-2n+1)} + \frac{C[(-n+3)d^{-n+2}+Bid^{-n+3}]}{4(-2n+3)}$$

$$D = \frac{(2n+1)[Q-(nd^{n-1}+Bid^{n})P]}{nd^{n-1}+Bid^{n}+(n+1)d^{-n-2}-Bid^{-n-1}}$$

$$E = \frac{Q+[(n+1)d^{-n-2}-Bid^{-n-1}]P}{nd^{n-1}+Bid^{n}+(n+1)d^{-n-2}-Bid^{-n-1}}$$

The critical Marangoni number is then

$$Ma = \frac{n(n-1)d^{n-2}+A(n+1)(n+2)d^{n}+B(n+1)(n+2)d^{-n-3}+Cn(n-1)d^{-n-1}}{2(n+1)n\left[\frac{d^{n+2}}{2(2n+3)}+\frac{Ad^{n+4}}{4(2n+5)}+\frac{Bd^{-n+1}}{2(-2n+1)}+\frac{Cd^{-n+3}}{4(-2n+3)}-\frac{Dd^{-n-1}}{2n+1}+Ed^{n}\right]} \tag{24}$$

C. Free inner and outer surface

Equations (17) and (18) are solved subject to the boundary conditions

$$f(1) = 0;\ f''(1) = -2\lambda^2 Ma_1 g(1);\ g'(1) = -Bi_1 g(1) \tag{25}$$

$$f(d) = 0;\ f'' = -2\lambda^2 Ma_2 g(d);\ g'(d) = -Bi_2 g(d) \tag{26}$$

where $Ma = \frac{\sigma_1 \beta R_2^3}{\rho \nu \alpha}$, $Ma_2 = \frac{\sigma_2 \beta R_2^3}{\rho \nu \alpha}$, $Bi_1 = \frac{h_1 R_2}{k_1}$ and $Bi_2 = \frac{h_2 R_2}{k_2}$.

The subscript 1 refers to the fluid enclosed within the shell, while the subscript 2 indicates the fluid enclosing the shell.

The solutions are found to be

$$f = C_1(\eta^n + A_1\eta^{-n-1} + B_1\eta^{-n+1}) + C_2(\eta^{n+2} + A_2\eta^{-n-1} + B_2\eta^{-n+1}) \quad (27)$$

$$g = -C_1\left[\frac{\eta^{n+2}}{2(2n+3)} + \frac{A_1\eta^{-n+1}}{2(-2n+1)} + \frac{B_1\eta^{-n+3}}{4(-2n+3)} - \frac{D_1\eta^{-n-1}}{2n+1} + E_1\eta^n\right]$$

$$-C_2\left[\frac{\eta^{n+4}}{4(2n+5)} + \frac{A_2\eta^{-n+1}}{2(-2n+1)} + \frac{B_2\eta^{-n+3}}{4(-2n+3)} - \frac{D_2\eta^{-n-1}}{2n+1} + E_2\eta^n\right] \quad (28)$$

Here, $j = 1,2$

$$A_1 = \frac{d^2-d^{2n+1}}{1-d^2};\quad A_2 \frac{d^2-d^{2n+3}}{1-d^2};\quad B_1 = \frac{d^{2n+1}-1}{1-d^2};\quad B_2 = \frac{d^{2n+3}-1}{1-d^2}$$

$$D_j = [P_j(nd^{n-1}+Bi_2d^n) - Q_j(n+Bi_1)]/F$$

$$E_j = P_j\left[\frac{-(n+1)d^{-n-2}+Bi_2d^{-n-1}}{2n+1}\right] - Q_j\left[\frac{-n-1+Bi_1}{2n+1}\right] /F$$

$$F = (-n-1+Bi_1)(nd^{n-1}+Bi_2d^n)-(n+Bi_1)[(-n-1)d^{-n-2}+Bi_2d^{-n-1}] /(2n+1)$$

$$P_j = \frac{n+2j+Bi_1}{2j(2n+2j+1)} + \frac{A_j(-n+1+Bi_1)}{2(-2n+1)} + \frac{B_j(-n+3+Bi_1)}{4(-2n+3)}$$

$$Q_j = \frac{(n+2j)d^{n+2j-1}+Bi_2d^{n+2j}}{2j(2n+2j+1)} + \frac{A_j[(-n+1)d^{-n}+Bi_2d^{-n+1}}{2(-2n+1)}$$

$$+ \frac{B_j[(-n+3)d^{-n+2}+Bi_2d^{-n+3}}{4(-2n+3)}$$

Through the substitution of equations (27) and (28) into the second expressions of equations (25) and (26), one gets the equation relating the critical Marangoni numbers and the Biot numbers

$$\frac{L_1(d)-2\lambda^2 Ma_2J_1(d)}{L_1(1)-2\lambda^2 Ma_1J_1(1)} = \frac{L_2(d)-2\lambda^2 Ma_2J_2(d)}{L_2(1)-2\lambda^2 Ma_1J_2(1)} \quad (29)$$

wherein both L_j and J_j are a function of d as defined by

$$L_j(d) = (n+2j-2)(n+2j-3)d^{n+2j-4}+A_j(n+1)(n+2)d^{-n-3}+B_j(n-1)nd^{-n-1}$$

$$J_j(d) = \frac{d^{n+2j}}{2j(2n+2j+1)} + \frac{A_j d^{-n+1}}{2(-2n+1)} + \frac{B_j d^{-n+3}}{4(-2n+3)} - \frac{D_j d^{-n-1}}{2n+1} + E_j d^n$$

for $j = 1,2$.

(ii) Linear distribution of steady temperature in spherical shells

Another interesting case is that the temperature distribution in the liquid shell at steady state takes a linear form, instead of equation (1). That is

$$\frac{dT_o}{dr} = -\beta^* \tag{30}$$

where β^* is a constant. Equation (2) remains unchanged, while $2\beta r$ on the RHS of equation (3) should be replaced by β^*. Both equations are subject to the same boundary conditions (5) and (6). In the solutions, r on the LHS of equation (9-a) must be replaced by R, and βR^2 on the RHS of equation (9-b) becomes $\beta^* R$. 2Ma in the second expression of equation (16) reduces to $Ma^* = \frac{\sigma\beta^* R^2}{\rho\ \alpha}$. The critical Marangoni number for the onset of cellular convection then reads

$$Ma^* = 2Ma \tag{31}$$

for all three boundary conditions. This indicates that under the same values of n, d, and Bi, a linear steady temperature profile is twice more stable than a parabolic one.

In a special case for $d = 0$ which corresponds to a liquid sphere, one obtains

$$Ma^* = \frac{(2n+1)(2n+3)(2n+5)(n+Bi)}{n(n+1)}$$

Results and Discussion

The Biot number Bi is a measure of relative importance between surface conductance and internal conductance of a thermal system in contact with its ambient. It has two limiting values: zero and infinity. Bi = 0, referred to as "insulating", signifies an adiabatic free surface. The opposite situation $Bi = \infty$ is called "conducting", meaning no thermal resistance between a free surface and its ambient.

For convenience, case C with $Ma_1 = Ma_2 = Ma$ and $Bi_1 = Bi_2 = Bi$ is employed for comparison. Equations (21), (24), and (29) for Bi = 0 are graphically illustrated in figures 1-a, 1-b, and 1-c, respectively, to exhibit the dependence of Ma on n and on the boundary conditions. d is R_1/R_2, while (1-d) signifies a dimensionless shell thickness $(R_2-R_1)/R_2$. d of unity corresponds to zero shell thickness and zero value of d refers to a liquid sphere. In the figures, higher values of Ma mean more stability, requiring larger surface tension forces to induce cellular convection. At a given d, case B is the most stable, case A comes next, and case C is the most susceptible to thermal instability among the three possible boundary conditions. Each curve has a minimum value of Ma, called the critical Marangoni number for the onset of instability, Ma_c. From figures 1-a to 1-c, it is apparent that as the thickness of the shell decreases, the pattern of the convection which manifests itself at marginal stability shifts progressively to harmonics of the higher orders.

Figure 2 is a plot of Ma_c against d for Bi = 0. It is seen that as shell thickness increases, the value of Ma_c decreases monotonically in case A, while case B has a minimum Ma_c of about 0.55 at d. The Ma_c-d relationship in case C is quite complex.

It is interesting to compare the role of buoyancy and surface tension forces on the onset of cellular motion. As Ma is relevant for the surface tension mechanism, the Rayleigh number

$$Ra = \frac{2\beta\gamma g R_2^5}{\nu\alpha}$$

is relevant for the density-dependent mechanism, where g is the gravitational acceleration and γ denotes the coefficient of thermal expansion of the liquid. The dependence of Ra on n [8] is superimposed in figures 1-a through 1-c for each corresponding case. Obviously, surface tension forces are more effective than buoyancy forces in producing thermal instability (for the same value of d) in all three cases. The critical Rayleigh number for the onset of marginal stability Ra_c is plotted against d in figure 2. A comparison of Ma_c and Ra_c yields the conclusions that (i) The degree of stability follows the order of cases B,A, and C in both mechanisms; (ii) The onset of cellular motion could be attributed to surface tension rather than buoyancy. These conclusions may be extended to non-zero values of Bi.

Next is a quantitative comparison of the two mechanisms. From the definition of Ma and Ri, one gets a critical radius of the outer spherical boundary

$$R_S = \left(\frac{Ma\,\rho\,\nu\alpha}{\sigma\beta}\right)^{\frac{1}{3}}$$

for surface tension mechanism and a radius

$$R_B = \left(\frac{Ra\ \nu\alpha}{2g\ \beta\gamma}\right)^{\frac{1}{5}}$$

for buoyancy effect. They will be equal for a value R_{SB} given by

$$R_{SB}^2 = \frac{\sigma\ Ra}{2g\rho\gamma Ma} \qquad (33)$$

Equation (33) is plotted in figure 3 for case A with zero Bi using the physical properties of water-air system. When a radius of the outer shell surface is less than R_{SB}, corresponding to the region below the curves, surface tension forces would be more effective than buoyancy forces in producing instability. On the other hand, the region above the curves signifies buoyancy mechanism controlling the onset of cellular convection. It is observed in the figure that the value of R_{SB} reduces with an increase in Bi, indicating buoyancy forces become more important in causing thermal instability as heat transfer between the free surface and the ambient increases. At Bi = 5, the curves for different modes converge at large values of d.

Finally, the effect of Bi on Ma is illustrated in figure 4 for case A. It is seen that heat transfer between the free surface and the ambient results in an upward shift of marginal stability curves, indicating more stability to thermal disturbances.

Conclusions

The criteria for marginal stability in spherical liquid shells induced by surface tension mechanism are determined for three possible boundary conditions: case A for a free outer surface and a rigid inner surface, case B for a rigid outer surface and a free inner surface, and case C for free inner and outer surfaces . Parabolic and linear temperatures at steady state are considered. The Marangoni number is found to be functions of the wave number n, the ratio of inner and outer radii d, and the Biot number Bi. The effects of n,d, and Bi on the neutral stability are determined. It is concluded that a linear steady temperature profile in a liquid shell is twice more stable than a parabolic one. Marginal stability decreases in the order of cases B, A, and C. The onset of cellular convection in spherical shells could be attributed to surface tension forces rather than buoyancy forces. An increase in Bi results in higher Ma, promoting thermal stability.

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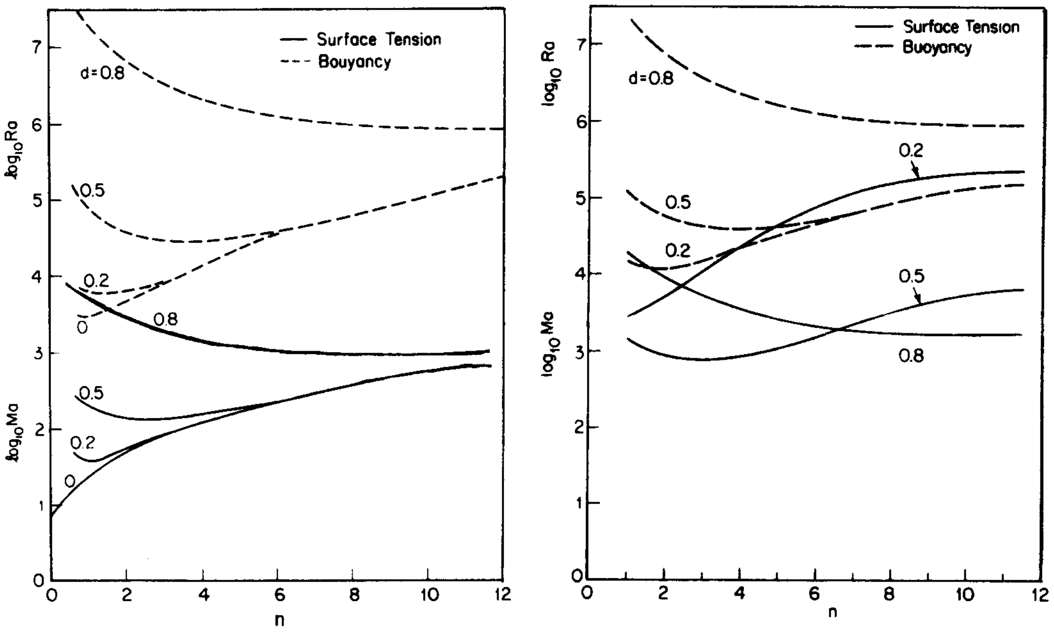


FIG. 1-a and 1-b

The Marangoni and Rayleigh numbers for the onset of convection in case A: rigid inner surface and free outer surface and for case B: free inner surface and rigid outer surface

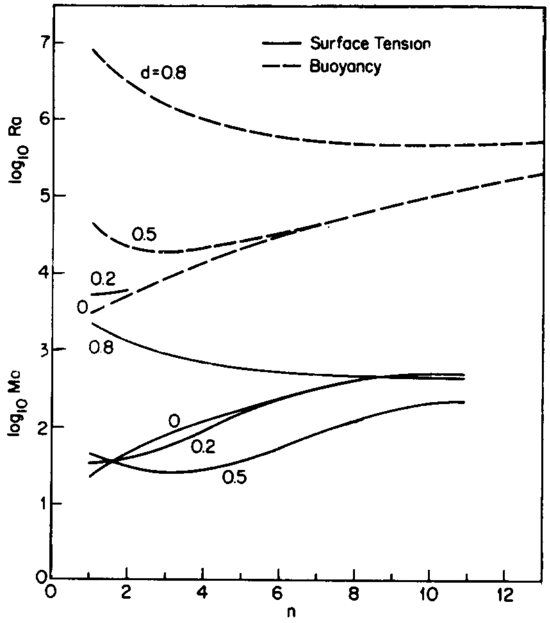
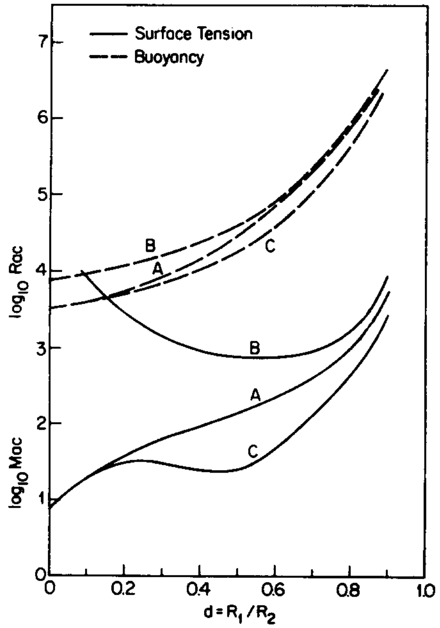


FIG. 1-c

The Marangoni and Rayleigh numbers for the onset of convection in case C: free inner and outer surfaces

FIG. 2

The critical Marangoni and Rayleigh numbers versis d



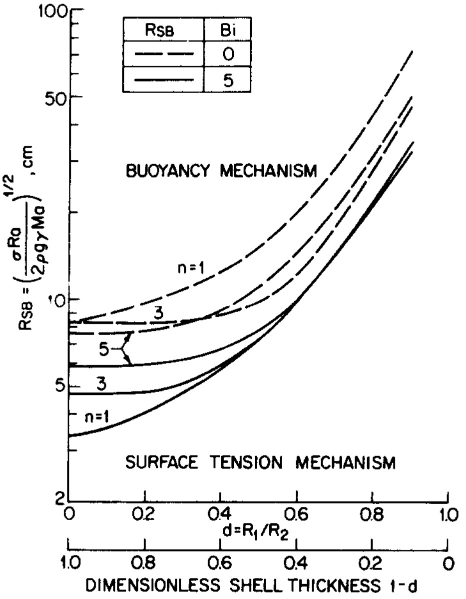


FIG. 3

Comparison of surface tension and buoyancy mechanisms for case A

FIG. 4

Effect of Biot number on thermal instability in spherical liquid shells for the case A