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The Theory of d-Sequences and Powers of Ideals

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INTRODUCTION

In this paper we study the question of finding depth R/I^n given an ideal I in a commutative ring.

This problem is difficult in general; even for simple examples depth R/I^2 can be difficult to compute. Brodmann has shown that for any commutative Noetherian ring depth R/I^n become stable for large n; finding this value, however, is quite hard.

This problem originally arose through consideration of the following example: suppose k is a field and $X = (x_{ij})$ is an $n \times (n+1)$ matrix of indeterminates. Let $R = k[x_{ij}]$ and let I be the ideal generated by the maximal minors of X. Then, question: compute the projective dimensions of R/I^n for large n, or equivalently find the depth R/I^n.

In this case the maximal minors are an example of a d-sequence. A d-sequence $x_1,..., x_n$ is any system of elements in a commutative ring R which satisfy two conditions:

(i) x_i is not in the ideal generated by the rest of the x_j,

(ii) For all $k \geqslant i+1$ and all $i \geqslant 0$,

$$((x_1,..., x_i): x_{i+1}x_k) = ((x_1,..., x_i): x_k)$$

(See page 9 for more on this.) Thus d-sequences may be regarded as "weak" R-sequences. Indeed they turn out to be examples of weak R-sequences in the sense of [28] and of relative regular sequences as in [9].

The general problem of computing depth R/I^n for a given ideal I can often be simplified by writing $I = J + K$ and comparing depth (R/I^n) to depth $R/(J, K^n)$. In some cases this has been done; indeed considerably more has been done in the case K is generated by an R-sequence modulo J. These theorems were studied originally by Hironaka and later by Herrmann, Schmidt, Robbiano and Valla. In [26], the following is shown:

THEOREM [26]. *Let I be an ideal in A, a commutative local ring and*

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suppose $J=(x_1,\dots,x_t,I)$ *where the* $x_1,\dots,x_t$ *form an* A/I*-sequence. Then the following are equivalent*:

(1) $\mathrm{gr}_I(A)$ *is a free* A/I*-module*, $\mathrm{gr}_I(A)=A/I\oplus I/I^2\oplus\cdots$

(2) $\mathrm{gr}_J(A)$ *is a free* A/J*-module and* $\mathrm{gr}_{I_p}(A_p)$ *is a free* $(A/I)_p$ *module for every prime* p *in* $\mathrm{Ass}(A/J)$.

This is called the "transitivity of normal flatness." If assumption (1) occurs then depth $A/I^n=$ depth A/I for every positive n. If we set $K=(x_1,\dots,x_t)$ then as was well known depth $A/(I,K^n)=$ depth $A/(I,K)$. The conclusion then implies depth $A/(I,K)^n=$ depth $A/J^n=$ depth $A/(I,K^n)=$ depth $A/(I,K)$ for every positive n.

We weaken both the hypothesis and conclusion of this to obtain a "transitivity of depth." What is needed to pass from depth $R/(I+J)^n$ to depth $R/(I,J^n)$ is a condition controlling the powers of each ideal, namely that $J^mI^{n-1}\cap I^n\subseteq J^{m-1}I^n$ for every $m\geqslant 2$ and $n\geqslant 1$. We show that under suitable conditions if J is generated by elements which form a d-sequence in R/I, then this condition is satisfied.

The main result which allows us to obtain the transitivity of depth as well as compute specific depths in section four is the following proposition.

PROPOSITION 3.1. *Let* R *be a commutative ring,* I *and* J *two ideals which satisfy*

$$I^{k+1}\cap I^kJ^m\subseteq I^{k+1}J^{m-1}.$$

Let $Q=I+J$. *Then* R/Q^n *has a filtration* $M_0=R/Q^n,\dots,M_n=0$ *of* R*-modules such that* M_k/M_{k+1} *is isomorphic to* $I^k/I^k(I,J^{n-k})$.

The depth R/I^n for large n is connected with the analytic properties of the ideal I. If $l(I)$ is used to represent the analytic spread of the ideal I [24], then L. Burch [6] showed if R is local then (also see Brodmann [3]).

$$l(I)\leqslant \dim R-\inf_n \operatorname{depth}(R/I^n).$$

Using this, Cowsik and Nori have shown that ideals I with R/I^n Cohen–Macaulay are close to complete intersections. Specifically, they proved the result below [7].

THEOREM. *If* I *is a self-radical ideal in a Cohen–Macaulay local ring* R *such that*

(i) R_p *is regular for each minimal prime* p *containing* I *and*

(ii) R/I^n *is Cohen–Macaulay for every* n, *then* I *is a complete intersection.*

Recently, Brodmann [3] has given an improvement of this.

We show every d-sequence in a local ring is analytically independent and give a criterion for when one obtains equality in the equation of Burch.

Finally we note d-sequences have proved effective in the study of numerous problems, ranging from properties of the symmetric algebra [18] to conditions for $P^{(n)} = P^n$ for a prime P generated by a d-sequence. [19].

We now describe the contents of this paper more precisely.

Section 1 introduces the notion of a d-sequence and proves several elementary remarks concerning their behavior. The main body of this section deals with giving examples of d-sequences. These examples include the maximal minors of an $n \times n+1$ generic matrix, the images of these minors in the symmetric algebra of the ideal they generate, the generic Pfaffians of a 5×5 matrix, any two elements of an integrally closed ring, and most almost complete intersections, in particular ideals algebraically linked to Gorenstein primes. We show in "good" two dimensional local rings any system of parameters will have a sufficiently high power which forms a d-sequence. Local rings where every system of parameters form a d-sequence are exactly Buchsbaum rings. ([28], [29], or [30].)

Section 2 develops the basic properties of d-sequences most notable of which is the following.

THEOREM 2.1. *Let R be a commutative ring.*

If I is an ideal of R and $x_1,\dots,x_n$ *are elements which form a d-sequence modulo I, then*

$$(x_1,\dots,x_n)^m \cap I \subseteq (x_1,\dots,x_n)^{m-1} I.$$

We also show d-sequences in local rings are analytically independent. Most of this section is devoted to the spade-work necessary to apply the results of Section 3.

Section 3 is a technical section whose purpose is to prove the transitivity of depth under suitable conditions. The main theorem states:

THEOREM 3.1. (Transitivity of Depth). *Let I and J be ideals in a commutative Noetherian ring R and let* $p \supset I + J$. *Suppose* I^n/I^{n+1} *are free R/I-modules for* $n = 1,\dots$. *If J is an ideal such that* $I \cap J^m \subseteq IJ^{m-1}$ *then for* $n = 1, 2\dots$

$$\operatorname{depth}_p R/(I+J)^n \geqslant \min_{0 \leqslant k \leqslant n} \operatorname{depth}_p R/(I, J^k).$$

The final section applies the above result to various of the d-sequences described in Section 1. In particular we compute depth R/I^n, where $R = k[x_{ij}]$ and I is the ideal generated by the maximal minors of the

$n \times (n+1)$ matrix $X = (x_{ij})$. Hochster [14] has shown that $gr_I(R)$ is Cohen–Macaulay in this case and this yields the asymptotic value of depth R/I^n. Robbiano [27] has also studied these depths in certain cases. These computations are carried out for several of the other examples of Section 1. Finally the equality in Burch's equation is shown to hold under fairly general assumptions if the ideal in question is generated by a d-sequence.

All rings will be commutative Noetherian with one unless stated otherwise. The basic definitions and notations used throughout may be found in [22].

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1. d-SEQUENCES

In this section we give the definition of a d-sequence and give several classes of examples, to which we will later apply the results on the transitivity of depth.

DEFINITION 1.1. Let R be a commutative ring. A sequence of elements $x_1,..., x_n$ belonging to an ideal I is said to be a d-sequence in I (or simply a d-sequence) if

(1) $x_i \notin Rx_1 + \cdots + Rx_{i-1} + Rx_{i+1} + \cdots + R_{xn}$ for $i = 1,..., n$

(2) For all $k \geqslant i+1$ and all $i \geqslant 0$, $(x_0 = 0)$

$$((x_0,..., x_i): x_{i+1}x_k) = ((x_0,..., x_i): x_k)$$

Here, if I is an ideal and $x \in R$, $(I: x) = \{r \in R \mid rx \in I\}$. If $x_1,..., x_n$ is a d-sequence in any order, we will say $x_1,..., x_n$ are an *unconditioned* d-sequence.

Remarks. (1) Condition (2) is equivalent to saying x_{i+1} is not a zero divisor module the ideal $((x_1,..., x_i): x_k)$, and hence a d-sequence can be thought of as a weak R-sequence.

(2) As $((x_1,..., x_i): x_{i+1}) \subseteq ((x_1,..., x_i): x_{i+1}x_k)$, condition 2) shows $((x_1,..., x_i): x_{i+1}) \subseteq ((x_1,..., x_i): x_k)$ for $k > i$.

(3) If $x_1,..., x_n$ form a d-sequence then the images of $x_i,..., x_n$ in the ring $R/(x_1,..., x_{i-1})$ form a d-sequence.

(4) The single element x is a d-sequence if and only if $(0: x) = (0: x^2)$.

(5) Any R-sequence is trivially a d-sequence.

(6) Let $R \to S$ be a faithfully flat extension of rings and suppose

$x_1,..., x_n$ are in R. Then $x_1,..., x_n$ is a d-sequence in R if and only if $x_1,...,x_n$ is a d-sequence in S.

Proof. $x_1,..., x_n$ is a d-sequence in S. As $x_i \notin (x_1,..., \hat{x}_i,..., x_n) S$, x_i is certainly not in $(x_1,..., \hat{x}_i,..., x_n) R$. Now as S is faithfully flat, if $w \in R$ and I is an ideal in R,

$$(I\colon wR) S = (IS\colon wS).$$

Hence

$$((x_1,...,x_i) R\colon x_k R) S = ((x_1,..., x_i) S\colon x_k S)$$

and by assumption x_{i+1} is not a zero-divisor modulo this ideal. Set $J = ((x_1,..., x_i)\colon x_k R))$. We have shown x_{i+1} is not a zero divisor modulo the ideal JS. Hence there is an exact sequence $0 \to S/JS \to^{x_{i+1}} S/JS$; set $N =$ kernel of the map $R/JR \to^{x_{i+1}} R/JR$. From $0 \to N \to R/JR \to^{x_{i+1}} R/JR$ we obtain by $\otimes_R S$ the sequence

$$0 \to N \otimes_R S \to S/JS \xrightarrow{x_{i+1}} S/JS.$$

Hence $N \otimes_R S = 0$ and as S in faithfully flat over R, $N = 0$ and this shows $x_1,..., x_n$ is a d-sequence.

Now suppose $x_1,..., x_n$ is a d-sequence in R. Since S is faithfully flat over R,

$$(x_1,..., \hat{x}_i,..., x_n) S \cap R = (x_1,..., \hat{x}_i,..., x_n) R$$

and this shows $x_i \notin (x_1,..., \hat{x}_i,..., x_n) S$.

We must show x_{i+1} is not a zero divisor modulo $((x_1,..., x_i) S\colon x_k S)$. Again, this ideal is equal to $((x_1,..., x_i) R\colon x_k R) S$.

Let J be as above; from the exact sequence $0 \to R/J \to^{x_{i+1}} R/J$ we obtain

$$0 \to S/SJ \xrightarrow{x_{i+1}} S/JS,$$

which establishes our claim.

(7) Two maximal d-sequences in an ideal I need not have the same length. For example, consider the ideal (X) in the ring of polynomials $k[X, Y, Z]$. X itself is certainly a maximal d-sequence in (X). However, XY, XZ also form a d-sequence in the ideal (X), since $(XY\colon XZ) = (Y)$ and XZ is not a zero divisor modulo (Y).

We now begin our list of examples.

EXAMPLE 1.1. Let $X = (x_{ij})$ be an n by $n+1$ matrix of indeterminates over $k(k = Z$ or a field). Let μ_i be the determinant of the matrix formed by deleting the $n+2-i$th column of X. Set $R = k[x_{ij}]$.

PROPOSITION 1.1. *$\mu_1, \ldots, \mu_{n+1}$ form an unconditioned d-sequence.*

Proof. As rearrangement of the columns of X affects nothing, it is enough to show that $\mu_i, \ldots, \mu_{n+1}$ are not zero divisors modulo the ideal $((\mu_1, \ldots, \mu_{i-1}): \mu_i)$.

Let Y be the n by $(n+2-i)$ matrix obtained by deleting the last $i-1$ columns of X. (Here we may assume $i \geqslant 2$ as any two of the maximal minors form an R-sequence.)

Let λ be any maximal minor of Y; we claim $\lambda \in ((\mu_1, \ldots, \mu_{i-1}): \mu_i))$. λ is fixed by choosing $n+2-i$ rows of X; let σ be such a choice. Expand λ along the $(n+1-i)$th column. We obtain

$$\lambda = \sum_{j \in \sigma} x_{jn+1-i} \lambda_j,$$

where the λ_j are minors of order $n+1-i$. If $m \neq n+1, n, \ldots, n+2-i$, then by elementary linear algebra,

$$\sum_{j \in \sigma} x_{jm} \lambda_j = 0. \tag{1}$$

We also know

$$\sum_{j=1}^{n+1} x_{rj} \mu_{n+2-j} = 0. \tag{2}$$

Multiplying Eq. (2) by λ_s when $r = s$ and summing for $s \in \sigma$, we obtain

$$\sum_{s \in \sigma} \lambda_s \sum_{j=1}^{n+1} x_{sj} \mu_{i+2-j} = 0,$$

so that

$$\sum_{j=1}^{n+1} \mu_{n+1-j} \left(\sum_{s \in \sigma} \lambda_s x_{sj} \right) = 0. \tag{3}$$

By (1), the inner sum is zero when $j \neq n+1, n, \ldots, n+2-i$. When $j = n+2-i$, $\sum_{s \in \sigma} \lambda_s x_{sj} = \lambda$. Thus Eq. (3) becomes

$$\mu_i \lambda + \mu_{i-1} \left(\sum_{s \in \sigma} \lambda_s x_{sn+3-i} \right) + \cdots + \mu_1 \left(\sum_{s \in \sigma} \lambda_s x_{sn+1} \right) = 0.$$

Hence $\lambda \in (\mu_1, \ldots, \mu_{i-1} : \mu_i)$ as claimed.

Let J be the ideal generated by the maximal minors of Y. Note $J \supseteq (\mu_1, \ldots, \mu_{i-1})$. We have shown $J \subseteq (\mu_1, \ldots, \mu_{i-1} : \mu_i)$, and it is clear $\mu_i \notin J$. By the work of Hochster and Eagon [15], J is a prime ideal. But then

$$\mu_i(\mu_1, \ldots, \mu_{i-1} : \mu_i) \subseteq (\mu_1, \ldots, \mu_{i-1}) \subseteq J$$

implies

$$(\mu_1, \ldots, \mu_{i-1} : \mu_i) \subseteq J$$

and hence is equal to J. As J is prime and $\mu_k \notin J$ for $k = i, i+1, \ldots, n+1$ we see that $\mu_1, \ldots, \mu_{n+1}$ form a d-sequence.

EXAMPLE 1.2. Let the notation be as in Example 1. Set $I = (\mu_1, \ldots, \mu_{n+1})$ and let $S = S(I)$ be the symmetric algebra of I. (See [17].) In general, if I is an ideal generated by $t_1, \ldots, t_n$ the symmetric algebra of I is equal to $R[Y_1, \ldots, Y_n]/q$, where q is the ideal of $R[Y_1, \ldots, Y_n]$ generated by the linear forms $b_1 Y_1 + \cdots + b_n Y_n$ such that $b_1 t_1 + \cdots b_n t_n = 0$.

As is well known [3] the linear relations on $\mu_1, \ldots, \mu_{n+1}$ are generated by the rows of X. Hence for I as above,

$$S(I) = k[x_{ij}, Y_1, \ldots, Y_{n+1}] \Big/ \left(\sum_{j=1}^{n+1} x_{ij} Y_j \right).$$

Write "$^-$" for the image of an element in $R = k[x_{ij}, Y_1, \ldots, Y_{n+1}]$ under the map $R \to S(I)$.

As in Example 1, it is enough to show the ideal $J = (\bar{\mu}_1, \ldots, \bar{\mu}_{q-1} : \bar{\mu}_q)$ is a prime ideal which does not contain $\bar{\mu}_q, \ldots, \bar{\mu}_{n+1}$. As above, if we let $\bar{Z}$ be the $n \times (n+2-\bar{q})$ matrix obtained by deleting the last $q-1$ columns of $\bar{X}$, any maximal minor of $\bar{Z}$ is in J. We may assume $i \geqslant 1$ as $S(I)$ is known to be a domain. ([9] or [18].)

I claim $\bar{\mu}_{n+2-i} \bar{Y}_j \pm \bar{\mu}_{n+2-j} \bar{Y}_i = 0$ in S.

Delete the ith column of $\bar{X}$ and expand $\bar{\mu}_{n+2-i}$ along the jth column;

$$\bar{\mu}_{n+2-i} = \sum_{k=1}^{n} \bar{x}_{kj} \bar{\Delta}_k,$$

where $\bar{\Delta}_k$ are minors of order $n-1$ in ($\bar{X}$ – the ith column.)

$$\sum_{k=1}^{n+1} \bar{x}_{mk} \bar{y}_k = 0$$

and hence

$$\sum_{m=1}^{n} \bar{\Delta}_m \left(\sum_{k=1}^{n+1} \bar{x}_{mk} \bar{y}_k \right) = 0.$$

Consequently, rearranging the two sums gives

$$\sum_{k=1}^{n+1} \bar{y}_k \left(\sum_{m=1}^{n} \bar{\Delta}_m \bar{x}_{mk} \right) = 0. \tag{4}$$

Now if $m=j$, the inner sum is $\bar{\mu}_{n+2-i}$. If $m=i$, we are simply expanding $\pm\bar{\mu}_{n+2-j}$ along the ith column of $\bar{X}$ – the jth column. If $m\neq i,j$ by elementary linear algebra,

$$\sum_{m=1}^{n} \bar{\Delta}_m \bar{x}_{mk} = 0.$$

Hence (4) yields

$$\bar{\mu}_{n+2-i}\,\bar{y}_j \pm \bar{\mu}_{n+2-j}\,\bar{y}_i = 0$$

If $i=n+2-q$ and $n+3-q \leqslant j \leqslant n+1$, then this shows

$$\bar{\mu}_q\,\bar{y}_j \in (\bar{\mu}_1,..., \bar{\mu}_{q-1}).$$

Set $J=$ the ideal generated by the maximal minors of $\bar{Z}$ together with $\bar{y}_{n+3-q},..., \bar{y}_{n+1}$. J is of a type of ideal shown to be prime in [15]. Also $\bar{\mu}_q,..., \bar{\mu}_{n+1} \notin J$. As $(\bar{\mu}_1,..., \bar{\mu}_{q-1}) \subseteq J \subseteq (\bar{\mu}_1,..., \bar{\mu}_{q-1} : \bar{\mu}_q)$ we see $\bar{\mu}_q(\bar{\mu}_1,..., \bar{\mu}_{q-1} : \bar{\mu}_q) \subseteq (\bar{\mu}_1,..., \bar{\mu}_{q-1}) \subseteq J$ implies $J=(\bar{\mu}_1,..., \bar{\mu}_{q-1} : \bar{\mu}_q)$.

By the remarks above, this shows the claim.

EXAMPLE 1.3. Let $X=(x_{ij})$ be an $r\times s$ matrix of indeterminates with $r\leqslant s$, and set I equal to the ideal generated by all $m\times m$ minors of X for $0\leqslant m\leqslant r$. Set $S=R/I$. Then the image of the first row of X form an unconditioned d-sequence in S.

Proof. As rearrangement will affect nothing, consider the ideal $(\bar{x}_{11},..., \bar{x}_{1j} : \bar{x}_{1j+1})$ in S. It is clear that this ideal contains $\bar{x}_{11},..., \bar{x}_{1j}$ and all the $(m-1)\times(m-1)$ minors of the first j columns of $\bar{X}$, which can be seen by considering the $m\times m$ minor determined from the given $(m-1)\times(m-1)$ minor by adding the $(j+1)^{st}$ column and the first row. The ideal determined as above is prime by the work of Hochster–Eagon [15] and as $\bar{x}_{1j+1}, \bar{x}_{1j+1},..., \bar{x}_{1s}$ are not in it, this shows as above that the $\bar{x}_{1i}$ do indeed form a d-sequence.

EXAMPLE 1.4. Let R be a commutative Noetherian ring containing an infinite field k which satisfies Serre's condition S_n:

$$\operatorname{depth} R_p \geqslant \min[htp, n]$$

for all primes p in R. Suppose $a_1,..., a_n \in R$ such that

(1) height $(a_1,..., a_{n-1}) = n-1$,

(2) height $((a_1,..., a_{n-1}) : a_n), a_n) = n$.

Then the ideal $I=(a_1,..., a_n)$ can be generated by a d-sequence of length n, obtained from $a_1,..., a_n$ by elementary transformations.

Proof. Since R satisfies S_n and height $(a_1, \ldots, a_{n-1}) = n-1$, it is easy to see using the following lemma that $a_1, \ldots, a_{n-1}$ may be changed by elementary transformations to $a'_1, \ldots, a'_{n-1}$ forming an R-sequence with $(a_1, \ldots, a_{n-1}) = (a'_1, \ldots, a'_{n-1})$.

LEMMA [20, Theorem 124]. *Let $P_1, \ldots, P_n$ be prime ideals in a commutative ring R, and let I be an ideal in R, x an element of R such that $(x, I) \not\subseteq P_1 \cup \cdots \cup P_n$. Then there exists an element i in I such that $x+i$ is not in the union of the P_i.*

By changing notation we may assume $a_1, \ldots, a_{n-1}$ form an R-sequence. Then $(a_1, \ldots, a_{n-1})$ is unmixed since R satisfies S_n. Hence every associated prime of $(a_1, \ldots, a_{n-1})$ (and thus of $((a_1, \ldots, a_{n-1}): a_n))$ has height $n-1$. In particular, since $(((a_1, \ldots, a_{n-1}): a_n), a_n)$ has height n, a_n annot be in any of these and so a_n is not a zero divisor modulo $((a_1, \ldots, a_{n-1}): a_n)$. This shows $a_1, \ldots, a_n$ form a d-sequence.

Recall a local ring R is said to be Gorenstein if it has finite injective dimension over itself. A well-known result is the following [21].

PROPOSITION. *Let (R, m) be a Gorenstein local ring, I an ideal of height zero such that R/I is Gorenstein. Then*

(1) $(0: I) = (s)$ *is principal,*

(2) $(0: s) = I$.

If R is a regular local ring, an ideal I is said to be Gorenstein if R/I is a Gorenstein ring.

COROLLARY 1.4. *Let R be a regular local ring and p be a Gorenstein prime. Choose an R-sequence $x_1, \ldots, x_m$ in p such that $(x_1, \ldots, x_m) R_p = pR_p$. Then $((x_1, \ldots, x_m): p)$ can be generated by a d-sequence.*

Proof. By the quoted Proposition, there is an s such that $((x_1, \ldots, x_m): p) = (x_1, \ldots, x_m, s)$. $s \notin p$ as $x_1, \ldots, x_m$ generate p in R_p and so $((x_1, \ldots, x_m): p) \not\subseteq p$. We also have $((x_1, \ldots, x_m): s) = p$ and so as p is prime s is not a zero divisor modulo $((x_1, \ldots, x_m): s)$. The above Proposition now applies to conclude $x_1, \ldots, x_m, s$ is a d-sequence.

Peskine and Szpiro [25] said two ideals I and J were algebraically linked if there was an R-sequence $(x_1, \ldots, x_m)$ in I such that $((x_1, \ldots, x_m): I) = J$ and $((x_1, \ldots, x_m): J) = I$. In this terminology the Corollary reads any ideal in a regular local ring R algebraically linked to a Gorenstein prime is generated by a d-sequence.[1]

[1] This is true only if the $x_1, \ldots, x_m$ generate p locally as in Corollary 1.4.

It is known [12] that the ideal I in $k[[X_1, X_2, X_3]]$ defining the curve $k[[t^{n_1}, t^{n_2}, t^{n_3}]]$ is an almost complete intersection whose defining equations are given by

$$F_1 = X_2^{r_{12}} X_3^{r_{13}} - X_1^{c_1}, \qquad F_2 = X_1^{r_{21}} X_3^{r_{23}} - X_2^{c_2},$$
$$F_3 = X_1^{r_{31}} X_2^{r_{32}} - X_3^{c_3},$$

where the c_i are the smallest positive integers such that there exist integers $r_{ij} \geqslant 0$ with

$$c_i n_i = \sum_{j \neq i} \sum_{j=1}^{3} r_{ij} n_j.$$

Set $v_i = (-c_1, r_{12}, r_{13})$, $v_2 = (r_{21}, -c_2, r_{23})$ and $v_3 = (r_{32}, r_{32}, -c_3)$. Then in the case above it was shown [12] that

(i) $v_1 + v_2 + v_3 = 0$,

(ii) $v_1 \times v_2 = v_2 \times v_3 = v_1 \times v_3 = (n_1, n_2, n_3)$.

Using (i) and (ii) one may easily check that

$$(X_2^{r_{12}}, X_1^{r_{21}}) \subseteq ((F_1, F_2): F_3).$$

But F_3 is not a zero dvisor modulo $(X_2^{r_{12}}, X_1^{r_{21}})$ and so the two ideals are qual, and F_3 is not a zero divisor modulo $((F_1, F_2): F_3)$. Hence I is generated by a d-sequence.

EXAMPLE 1.5.

PROPOSITION 1.5. *Let A be an integrally closed Noetherian domain. If $J = (a, b)$ is an ideal of A minimally generated by two elements, then J may be generated by two elements which form a d-sequence.*

Proof. If we can find an a' such that $((a': b), b)$ has height > 1 and $(a', b) = (a, b)$ then we may apply the proposition of Example 4 since A satisfies S_2 [22]. Let $r \in A$. If $(a - rb: b) = (a - rb: b^2)$ we are done. Hence for every r in A we may assume b is a zero divisor modulo $(a - rb: b)$. Let $Q_1, \ldots, Q_k$ be the associated primes of b.

Let v_i be the valuation associated with Q_i. Suppose $v_i(a) \geqslant v_i(b)$ for $i = 1, \ldots, k$ while $v_i(a) < v_i(b)$ for $i = k + 1, \ldots, m$. Choose r in A such that $v_i(r) \geqslant 1$ for $i = 1, \ldots, k$ and $v_i(r) = 0$ for $i = k + 1, \ldots, m$. Thus for a suitably large integer, s, $v_i(a) < v_i(r^s b)$ for all $i = 1, \ldots, m$. This implies that $b/(a - r^s b)$ is in A_{Q_i} for $i = 1, \ldots, m$.

The associated primes of $(a - r^s b: b)$ by construction are distinct from $Q_1, \ldots, Q_m$. Hence b is not in any associated prime of $(a - r^s b: b)$ and this shows $(a - r^s b: b) = (a - r^s b: b^2)$ as is required.

EXAMPLE 1.6.

PROPOSITION 1.6. *Let (R, m) be a two dimensional local domain. Then there exists an N such that for every system of parameters x, y and $n \geqslant N$, x^n, y^n is a d-sequence if and only if $R^{(1)}$ is a finitely generated R-module. (Here $R^{(1)} = \cap R_p$ taken over all height one primes.)*

Proof. If such an n exists, then $(x^n : y^n) = (x^n : y^{2n})$ for all $n \geqslant N$. In the language of [23] we have $S(x, y, n) \leqslant n$ for every $n \geqslant N$. By [23], $R^{(1)}$ is then a finitely generated R-module. Conversely if $R^{(1)}$ is a finitely generated R-module, let $J = \{r \in R \mid rR^{(1)} \subseteq R\}$. Then by [23] there is a k such that $m^k \subseteq J$ and $C(b, c, -) \leqslant k + 1$. By [23], $S(c, b, -) \leqslant k$. Finally for all $n \geqslant k$, $S(x, y, n) \leqslant k \leqslant n$ which shows $(x^n : y^n) = (x^n : y^{2n})$ for x, y any system of parameters and any $n \geqslant k$. Take $N = k$.

Now suppose R is as above and $\operatorname{char} R = p > 0$. Let $F: R \to R$ be the Frobenius map. Thus $F(r) = r^p$. $F^e(R)$ is the subring of R consisting of all (p^e)th powers of elements of R. The map $R \to^{F^e} R$ is said to be pure if for any R-module M the map $M \to M \otimes_R f_R$ via $m \to m \otimes 1$ is injective where f_R is R regarded as an R module via F^e. We see that $F^e(R) \subseteq R$ (i.e., R is reduced) and if I is an ideal in $F^e(R)$ then $IR \cap F^e(R) = I$. See [16] for details on F-purity.

COROLLARY 1.6. *Let R be as in Proposition 1.6 and suppose $\operatorname{char} R = p > 0$ and $R \to^{Fe} R$ is pure for some e. Then every system of parameters form a d-sequence.*

Proof. The assumption shows that for every sufficiently large f, if $x, y, w \in R$ and $x^{p^f} w = y^{p^f}$, then $w = u^{p^f}$ for some $u \in R$. (That is, $x^{p^f} R \cap F^f(R) = x^{p^f} F^f(R)$). Let N be as in Proposition 1.6 and choose f such that $p^f \geqslant N$. By Proposition 1.6, if x and y are a system of parameters for R, then x^{p^f}, y^{p^f} form a d-sequence. We wish to demonstrate that $\{x, y\}$ is a d-sequence.

Let $s \in (x : y^2)$ so that $y^2 s = xr$. Then $y^{2p^f} s^{p^f} = x^{p^f} r^{p^f}$. As x^{p^f}, y^{p^f} is a d-sequence (in R) we must have $y^{p^f} s^{p^f} \in x^{p^f} R$ so that $y^{p^f} s^{p^f} = x^{p^f} t$ for some t. Then $y^{p^f} t = r^{p^f}$. By purity $t = u^{p^f}$ for some $u \in R$. But as R is reduced (again by purity) the equation $(yu - r)^p = 0$ forces $yu = r$. Then $y^2 s = xyu$ shows $ys = xu$ and $s \in (x : y)$. We may interchange x and y without loss of generality and conclude that x, y form a d-sequence. This establishes Corollary 1.6.

Rings in which every system of parameters form a d-sequence do occur "in nature". In fact the next example shows these rings are precisely Buchsbaum rings.

EXAMPLE 1.7. At the end of Example 1.6 we showed how to find a local ring in which every system of parameters is a d-sequence. In fact such rings turn out to be precisely Buchsbaum rings, which have been studied extensively by Vogel and others. (See [28–30].)

DEFINITION [24]. A sequence of elements $\{a_1,..., a_r\}$, $r \leqslant d = \dim R$, R a local ring, is said to be a weak R-sequence if and only if for each $i = 1,..., r$

$$m[((a_1,..., a_{i-1}): a_i)/(a_1,..., a_{i-1})] = 0.$$

If every system of parameters for R is a weak R-sequence we say R is a Buchsbaum ring.

LEMMA 1.7. *(R, m) local. If every system of parameters forms a d-sequence, then R_p is Cohen–Macaulay for every $p \neq m$.*

Proof. Let p be a minimal prime in R with R_p not Cohen–Macaulay. If height $p = n$, choose $a_1,..., a_n$ in p such that height $(a_1,..., a_i) = i$. Complete $a_1,..., a_n$ to a system of parameters $a_1,..., a_n, a_{n+1},..., a_d$ of R. Since p is the minimal prime which is not Cohen–Macaulay, we may assume p is associated to $(a_1,..., a_i)$ with $i < n$. We will show (Proposition 2.1) that

$$(a_1,..., a_i) = ((a_1,..., a_i): a_{i+1}) \cap (a_1,..., a_d).$$

Now since $(a_1,..., a_d)$ is primary to m, this decomposition shows that p is associated to $((a_1,..., a_i): a_{i+1})$. However $a_{i+1} \in p$ and a_{i+1} is not a zero divisor modulo $((a_1,..., a_i): a_{i+1})$. This contradiction proves the lemma.

PROPOSITION 1.7. *Let R, m be local. Then R, m is a Buchsbaum ring if and only if every system of parameters form a d-sequence.*

Proof. The condition of being a Buchsbaum ring is the following: for every system of parameters $a_1,..., a_d$ of R,

$$((a_1,..., a_{i-1}): a_i) \subseteq ((a_1,..., a_{i-1}): m).$$

As $((a_1,..., a_{i-1}): m) \subseteq ((a_1,..., a_{i-1}): a_i)$ we see equality must hold. Let $a_1,..., a_d$ be any system of parameters. Then $a_1,..., a_i, a_{i+1}a_k$ and $a_1,..., a_i, a_k$ are also a part of a system of parameters. By the above comment we then see $((a_1,..., a_i): a_{i+1}a_k) = ((a_1,..., a_i): m) = ((a_1,..., a_i): a_k)$ and so $a_1,..., a_d$ for a d-sequence.

Now suppose every system of parameters is a d-sequence. Let $a_1,..., a_d$ be such a system of parameters. We must show that $((a_1,..., a_{i-1}): a_i) \subseteq ((a_1,..., a_{i-1}): m)$ or equivalently if $q_1,..., q_n$ are the isolated primary ideals associated to $(a_1,..., a_{i-1})$, we must show $((a_1,..., a_{i-1}): m) = q_1 \cap \cdots \cap q_n$. But we know x is not a zero divisor modulo $((a_1,..., a_{i-1}): x)$ for every

$x \in m \backslash p_1 \cup \cdots \cup p_m$, by the assumption. This implies $((a_1, \ldots, a_{i-1}) : x) = q_1 \cap \cdots \cap q_n$.

For Lemma 1.7 shows the associated primes of $(a_1, \ldots, a_{i-1})$ are either $\{p_1, \ldots, p_m\}$ or $\{p_1, \ldots, p_m, m\}$. Hence the same holds for $((a_1, \ldots, a_{i-1}) : x)$ and our first statement shows that m is not one of the associated primes. As x is not in the union of the p_i, $((a_1, \ldots, a_{i-1}) : x) = q_1 \cap \cdots \cap q_m$.

Now let $t \in q_1 \cap \cdots \cap q_n$. We have shown if $x \in m \backslash p_1 \cup \cdots \cup p_n$ then $tx \in (a_1, \ldots, a_{i-1})$. Hence

$$m \subseteq p_1 \cup \cdots \cup p_n \cup ((a_1, \ldots, a_{i-1}) : q_1 \cap \cdots \cap q_n)$$

and now the prime avoidance Lemma [10] shows

$$m \subseteq ((a_1, \ldots, a_{i-1}) : q_1 \cap \cdots \cap q_n)$$

and hence

$$((a_1, \ldots, a_{i-1}) : m) = q_1 \cap \cdots \cap q_n$$

as the reverse containment always holds.

Hence local catenary Buchsbaum rings give us examples of rings abundant with d-sequences. For example of such rings, see [28] where Buchsbaum rings of dimension d and depth r are given. We give one example here found in [28].

Let $X \subset \mathbb{P}^3_C$ be the nonsingular curve of Macaulay given parametrically by $\{t_1^4, t_1^3 t_2, t_1 t_2^3, t_2^4\}$.

Let A be the local ring of the vertex of the cone over X. A is not Cohen–Macaulay but it was shown in [28] that A is a Buchsbaum ring. The cone is embedded in $\mathbb{A}^4_C$ by the defining ideal

$$(wx - yz, z^2 x - y^2 w, y^3 - x^2 z, z^3 - w^2 y).$$

Suppose R, m is local and char $R = p > 0$. Let $F: R \to R$ be the Frobenius. Thus $F(r) = r^p$.

COROLLARY 1.7. *Suppose R is a reduced local catenary ring of characteristic $p > 0$. If $F^e(R)$ is Buchsbaum for some e, then R is Buchsbaum.*

Proof. By Proposition 1.7, it is enough to show every system of parameters form a d-sequence. Accordingly, let $x_1, \ldots, x_d$ be a system of parameters. Then we must show $((x_1, \ldots, x_k) : x_{k+1} x_n) = ((x_1, \ldots, x_k) : x_n)$. However, $x_1^{p^e}, \ldots, x_d^{p^e}$ form a system of parameters $F^e(R)$, and since $F^e(R)$ is Buchsbaum, these elements form a d-sequence in $F^e(R)$.

Now suppose $r x_{k+1} x_n = \sum_{i=1}^k s_i x_i$ and raise this equation to the (p^e)th power; we obtain

$$r^{p^e} x_{k+1}^{p^e} x_n^{p^e} = \sum_{i=1}^{k} s_i^{p^e} x_i^{p^e}.$$

In $F^e(R)$, $((x_1^{p^e},..., x_k^{p^e}): x_{k+1}^{p^e} x_n^{p^e}) = ((x_1^{p^e},..., x_k^{p^e}): x_n^{p^e})$. This shows $r^{p^e} x_n^{p^e} = \sum_{i=1}^{k} t_i^{p^e} x^{p^e}$. As R is reduced, there are unique (p^e)th roots and thus

$$rx_m = \sum_{i=1}^{k} t_i x_i.$$

This shows $x_1,..., x_d$ is a d-sequence and establishes our claim.

EXAMPLE 1.8. We return here to generic determinantal phenomena. The maximal minors of an n by $n+1$ matrix arise by the Hilbert–Burch structure theorem for perfect ideals if codimension 2, which states:

THEOREM (Hilbert). *Let R be a commutative Noetherian ring. If*

$$0 \to R^n \xrightarrow{f_2} R^{n+1} \xrightarrow{f_1} R \to R/I \to 0$$

is a free resolution of a cyclic module R/I, where I is an ideal of R, then I is a multiple of the ideal generated by the $n \times n$ minors of the matrix f_2.

See [4] for a discussion of this and related topics. Recently Buchsbaum and Eisenbud [5] have proved a similar theorem for Gorenstein ideals of codimensions three. We describe their main result.

If R is a commutative ring and F a finitely generated free R-module, a map $f: F^* \to F$ is said to be alternating if with respect to some (and therefore every) basis and dual basis of F and F^* the matrix of f is skew symmetric and all the diagonal entries are zero. If rank F is even and $f: F^* \to F$ is alternating then $\det(f)$ is a square of a polynomial function of the entries of the matrix for f, called the pfaffian of F. In general if F has odd rank n, then the determinant of the matrix resulting from f by deleting the ith column and ith row is a square of a polynomial function of the corresponding entries and the ideal generated by these pfaffians will be denoted $Pf_{n-1}(f)$.

The theorem of [5] gives the structure of Gorenstein ideals of grade 3.

THEOREM [5]. *Let R be a Noetherian local ring with maximal ideal J.*

(1) *Let $n \geqslant 3$ be an odd integer and let F be a free R-module of rank n. Let $f: F^* \to F$ be an alternating map whose image is contained in JF. Suppose $Pf_{n-1}(f)$ has grade 3. Then $Pf_{n-1}(f)$ is a Gorenstein ideal, minimally generated by n elements.*

(2) *Every Gorenstein ideal of grade 3 arises as in (1).*

Now let

$$X = \begin{pmatrix} 0 & x_{12} & x_{13} & x_{14} & x_{15} \\ -x_{12} & 0 & x_{23} & x_{24} & x_5 \\ -x_{13} & -x_{23} & 0 & x_{34} & x_{35} \\ -x_{14} & -x_{24} & -x_{34} & 0 & x_{45} \\ -x_{15} & -x_{25} & -x_{35} & -x_{45} & 0 \end{pmatrix}$$

be a generic 5×5 alternating matrix. Denote the pfaffian determined by omitting the ith row and ith column by p_i, and let

$$P = \text{the column vector} \begin{pmatrix} p_1 \\ \vdots \\ p_5 \end{pmatrix}.$$

Then $XP = 0$, and from this it follows that

$$(p_1,..., p_4 : p_5) \supseteq (x_{15},..., x_{45})$$

As $(p_1,..., p_4) \subseteq (x_{15},..., x_{45})$ as p_5 is not in the later ideal which is prime, it follows that $(p_1,..., p_4 : p_5) = (x_{15},..., x_{45})$ and p_5 is not a zero divisor modulo this.

Let us calculate $((p_1, p_2, p_3): p_4)$. This will clearly contain x_{45}. But (p_1, p_2, p_3, x_{45}) = the ideal generated by x_{45} and all 2×2 determinants of

$$\begin{pmatrix} x_{14} & x_{24} & x_{34} \\ x_{15} & x_{25} & x_{35} \end{pmatrix}$$

and this is a prime ideal not containing p_4 or p_5.

It follows that $((p_1, p_2, p_3): p_4)$ is equal to this ideal and neither p_4 nor p_5 is a zero divisor modulo this ideal. Hence $p_1,..., p_5$ is indeed a d-sequence.

(David Eisenbud has communicated a proof that $k[x_{ij}]$ form an algebra with straightening law on the poset of Pfaffians and this should imply that the generic Pfaffians always form a d-sequence).

2. Properties of d-Sequences

In this section we develop the basic properties of d-sequences which will allow us to apply the results of Section 3.

First we show d-sequences are relative regular sequences in the sense of [9].

Definition [9]. A sequence of elements $\{a_1,..., a_n\}$ is a commutative ring R is said to be a *relative regular sequence* if $((a_1,\dots, a_i)I: a_{i+1}) \cap (a_1,\dots, a_n) = (a_1,\dots, a_i)$. Such a sequence is said to be *unconditioned* if any permutation of the sequence is relative regular. $(I = (a_1,\dots, a_n))$.

In [9], Fiorenteni proved that the maximal minors of a generic $n \times (n+1)$ matrix formed a relative regular sequence. This follows immediately from Example 1 of Section 1 and the proposition below.

PROPOSITION 2.1. *Any d-sequence $x_1,\dots, x_n$ is relative regular sequence.*

Proof. By moding out the ideal $(x_1,\dots, x_i)$ it is clearly enough to show if $y_1,\dots, y_d$ is a d-sequence then

$$(0: y_1) \cap (y_1,\dots, y_d) = (0).$$

Show this by induction on d. If $d = 1$, $(0: y_1) = (0: y_1^2)$ shows $(0: y_1) \cap (y_1) = (0)$.

Suppose $d > 1$. Let $\sum_{i=1}^{d} r_i y_i \in (0: y_1)$. As $(0: y_1) \subseteq (0: y_d y_1) = (0: y_d)$ we see $r_d y_d^2 \in (y_1,\dots, y_{d-1})$ and hence as $y_1,\dots, y_d$ form a d-sequence, $r_d y_d \in (y_1,\dots, y_{d-1})$. But then $\sum_{i=1}^{d} r_i y_i \in (y_1,\dots, y_{d-1}) \cap (0: y_1) = (0)$ by induction.

The next proposition and theorem are keys to the usefulness (and understanding) of d-sequences.

PROPOSITION 2.2. *Suppose $x_1,\dots, x_d$ is a d-sequence in R. Then the images of $x_1,\dots, x_d$ form a d-sequence in $R/(0: x_1)$.*

Proof. It is enough to show that x_{j+1} is not zero divisor modulo the ideal $I = (((0: x_1), x_1,\dots, x_j): x_k)$ for $k \geqslant j+1$. This will follow immediately if we can show

$$I = (((0: x_1), x_1,\dots, x_j): x_k) = ((x_1,\dots, x_j): x_k).$$

Let $c \in I$ so that there is an equation

$$cx_k = \sum_{k=1}^{j} r_i x_i + w, \qquad \text{where} \quad wx_1 = 0.$$

But then,

$$cx_k - \sum_{k=1}^{j} r_i x_i \in (x_1,\dots, x_d) \cap (0: x_1) = (0)$$

by Proposition 2.1, and this shows $c \in ((x_1,\dots, x_j): x_k)$.

THEOREM 2.1. *Let R be a commutative ring and $x_1,\dots,x_n$ a d-sequence modulo an ideal I of R. Let $X=(x_1,\dots,x_n)$. Then*

$$X^m \cap I \subseteq X^{m-1}I.$$

for all $m \geqslant 1$.

Further suppose either R is local or positively graded with R_0 a field. Suppose $I=(a_1,\dots,a_d)$ and $x_1,\dots,x_n$ are elements such that $a_1,\dots,a_d$, $x_1,\dots,x_n$ form a d-sequence. In the graded case suppose $a_1,\dots,a_d$, $x_1,\dots,x_n$ are forms. Set M equal to the maximal ideal of R in the local case and the irrelevant ideal of R in the graded case. Then if $X=(x_1,\dots,x_n)$

$$X^m \cap I \subseteq MX^{m-1}I.$$

for all $m \geqslant 1$.

Proof. We prove these by induction on n. First suppose $n=1$. If $x_1^m r \in (x_1^m) \cap I$ then as $(I: x_1) = (I: x_1^2)$ we obtain $x_1 r \in I$ and so $x_1^m r = x_1^{m-1}(x_1 r) \in x_1^{m-1}I$. This handles the case $n=1$ for (1). Suppose $I=(a_1,\dots,a_d)$ as in (2). We claim $x_1 r \in I$ implies $x_1 r \in MI$. For we have $x_1 r = \sum_{i=1}^d s_i a_i$. If some $s_j \notin M$ then we may assume s_j is a unit and so $a_j \in (a_1,\dots,a_{j-1}, a_{j+1},\dots,a_d, x_1)$ which contradicts the definition of a d-sequence.

Now assume (1) holds for all $k<n$. Let $J=(x_2,\dots,x_n)$ and let $x=x_1$. The induction applied to (I,x) shows $J^m \cap (I,x) \subseteq J^{m-1}(I,x)$ for all $m \geqslant 1$. In addition, since $X^m \cap (I: x) \subseteq X \cap (I: x) \subseteq I$ by Proposition 2.1, we see that

$$X^m \cap (I: x) = X^m \cap I.$$

Now induct on m.

Now suppose $a \in X^m \cap I$. As $X^m = J^m + X^{m-1}x$

$$a = b + cx, \qquad \text{where} \quad c \in X^{m-1},$$

and $b \in J^m$. Then

$$b \in J^m \cap (I,x)$$

and so by the above comments,

$$b \in J^{m-1}(I,x).$$

Write $b = u + xv$, where $u \in J^{m-1}I$ and $v \in J^{m-1}$. Then

$$a = u + x(v+c)$$

and so

$$v + c \in (I: x) \cap X^{m-1} = X^{m-1} \cap I$$

by above. The induction on m shows

$$X^{m-1} \cap I \subseteq X^{m-2}I,$$

and so $a = u + x(v + c)$ is in

$$J^{m-1}I + xX^{m-2}I \subseteq X^{m-1}I.$$

This shows (1). The proof for (2) is exactly the same, putting in M in each of the induction steps.

As stated in the Introduction, the behavior of the ideals I^n have much to do with the analytic properties of the ideal I. In particular as the powers of an ideal generated by a d-sequence will be shown to be well behaved, one would expect d-sequences to have nice analytic properties. In fact we show d-sequences in local rings are analytically independent.

DEFINITION [31]. Let R be a local ring with maximal ideal m. Elements $z_1,..., z_n$ are said to be analytically independent if for every homogeneous polynomial $F(X_1,..., X_n) \in R[X_1,..., X_n]$, $F(z_1,..., z_n) = 0$ implies all the coefficients of $F(X_1,..., X_n)$ lie in m.

It is well known any system of parameters is analytically independent. This shows analytic independence is not a sufficient condition for a set of elements to be a d-sequence, else by Example 1.7 every local ring would be Buchsbaum.

THEOREM 2.2. *Let R be local with maximal ideal m. Suppose $x_1,..., x_n$ are a d-sequence. Then $x_1,..., x_n$ are analytically independent.*

Proof. Induct on n. If $n = 1$ the result is clear as from $(0: x_1) = (0: x_1^2)$ it follows that x_1 is not nilpotent and hence is analytically independent.

Now assume the result for all local rings and all d-sequences of length $<n$. Assume $x_1,..., x_n$ are not analytically independent. Then there is a homogeneous polynomial $F(T_1,..., T_n)$ in n-variables with a unit coefficient in one of the monomials and such that $F(x_1,..., x_n) = 0$. Induct on the degree of F for all d-sequences of length n in any local ring; we may suppose F is a relation of minimal degree. Write

$$F(T_1,..., T_n) = T_1 G(T_1,..., T_n) + H(T_2,..., T_n),$$

where H is homogeneous of degree d in $T_2,..., T_n$. By induction $H(T_2,..., T_n)$ cannot have a monomial with a unit coefficient as $H(\bar{x}_2,..., \bar{x}_n) = 0$ in R/Rx_1 would show $\bar{x}_2,..., \bar{x}_n$ are analytically dependent in R/Rx_1.

Hence $H(T_2,\ldots, T_n) \in m[T_1,\ldots, T_n]$ and $G(T_1,\ldots, T_n)$ must have a unit coefficient. Now $H(x_2,\ldots, x_n) + x_1 G(x_1,\ldots, x_n) = 0$ shows

$$w = H(x_2,\ldots, x_n) \in J^d \cap (x_1),$$

where $J = (x_2,\ldots, x_n)$. By Theorem 2.1, $J^d \cap (x_1) \subseteq mJ^{d-1}x_1$ as J is generated by a d-sequence modulo (x_1). Hence there is a homogeneous polynomial $K(T_2,\ldots, T_n) \in R[T_2,\ldots, T_n]$ of degree $d-1$ with coefficients in m such that $w = x_1 K(x_2,\ldots, x_n)$. Then $x_1,\ldots, x_n$ is a solution to the equation

$$T_1 G(T_1,\ldots, T_n) + T_1 K(T_2,\ldots, T_n) = 0$$

but

$$x_1(G(x_1,\ldots, x_n) + K(x_2,\ldots, x_n)) = 0$$

implies

$$[G(x_1,\ldots, x_n) + K(x_2,\ldots, x_n)] \in (0: x_1).$$

If $d > 1$, then $G(x_1,\ldots, x_n) + K(x_2,\ldots, x_n) \in (x_1,\ldots, x_n)$ and so Proposition 2.1 implies $G(x_1,\ldots, x_n) + K(x_2,\ldots, x_n) = 0$. Now $G(T_1,\ldots, T_n) + K(T_2,\ldots, T_n)$ is a homogeneous polynomial of degree $d-1$ and has a *unit coefficient* in some monomial as G did and $K(T_2,\ldots, T_n)$ has coefficients in m. The induction gives the required contradiction.

3. Transitivity of Depth

We list the basic definitions and conventions. For further information see [22].

Definition. Let R be a commutative ring and M an R-module. A sequence $x_1,\ldots, x_n$ is called an M-sequence if x_i is not a zero-divisor on the module $M/(x_1,\ldots, x_{i-1})M$ for $i = 1,\ldots, n$ and $(x_1,\ldots, x_n)M \neq M$.

The following characterization of depth is well known [22].

Theorem. *Let R be a commutative Noetherian ring and M a finitely generated R-module. Let I be an ideal such that $IM \neq M$. Then the following are equivalent*:

(i) *There exists an M-sequence of length n in the ideal I,*

(ii) $\operatorname{Ext}_R^i(R/I, M) = 0$ *for* $i < n$.

Further there exist finite maximal M-sequences in I. If $x_1, \ldots, x_n$ is maximal M-sequence in I then $\mathrm{Ext}^n_R(R/I, M) \neq 0$.

This shows the length of any two maximal M-sequences in I is the same, that number being the least n such that $\mathrm{Ext}^n_R(R/I, M) \neq 0$. This number is denoted $\mathrm{depth}_I(M)$. $\mathrm{depth}_I(R)$ is also called the *grade* of I. If $I = m$ is the maximal ideal of a local ring R, $\mathrm{depth}_m(M)$ is simply called depth M.

We now prove the main technical result.

PROPOSITION 3.1. *Let R be a commutative ring, I and J two ideals which satisfy*

$$I^{k+1} \cap I^k J^m \subseteq I^{k+1} J^{m-1}.$$

Let $Q = I + J$. Then R/Q^n has a filtration $M_0 = R/Q^n, \ldots, M_n = 0$ such that M_k/M_{k+1} is isomorphic to $I^k/I^k(I, J^{n-k})$.

Proof. Set $M_k = I^k/I^kQ^{n-k}$ where by convention we set $I^0 = R$. Then $M_0 = R/Q^n$, $M_{n-1} = I^{n-1}/I^{n-1}Q = I^{n-1}/(I^n + I^{n-1}J)$, while $M_n = 0$.

There is a surjective map from M_k onto $I^k/I^{k+1} + I^kJ^{n-k}$; it is enough to show the kernel of this map is isomorphic to M_{k+1}.

The kernel is $(I^{k+1} + I^kJ^{n-k})/(I^kQ^{n-k}) = (I^{k+1} + I^kJ^{n-k})/(I^{k+1}Q^{n-k-1} + I^kJ^{n-k})$, which is isomorphic to

$$I^{k+1}/(I^{k+1}Q^{n-k-1} + I^{k+1} \cap I^kJ^{n-k}).$$

However, $I^{k+1} \cap I^kJ^{n-k} \subseteq I^{k+1}J^{n-k-1}$ by assumption and as $I^{k+1}J^{n-k-1} \subseteq I^{k+1}Q^{n-k-1}$ we find the kernel is just

$$I^{k+1}/I^{k+1}Q^{n-k-1},$$

which is by definition M_{k+1} as required.

This gives us knowledge of the depth R/Q^n as the following lemmas show.

LEMMA 3.1. (See [22].) *Suppose R is Noetherian ring and I an ideal. Suppose*

$$0 \to K \to N \to L \to 0$$

is an exact sequence of finitely generated R-modules. Then either

(a) $\mathrm{depth}_I K \geqslant \mathrm{depth}_I N = \mathrm{depth}_I L$

(b) $\mathrm{depth}_I N \geqslant \mathrm{depth}_I K = \mathrm{depth}_I L + 1$

(c) $\mathrm{depth}_I L > \mathrm{depth}_I K = \mathrm{depth}_I N$.

In particular if $\operatorname{depth}_I K = \operatorname{depth}_I L$, *then*

$$\operatorname{depth}_I K = \operatorname{depth}_I L = \operatorname{depth}_I N.$$

Also if $\operatorname{depth}_I K \geqslant \operatorname{depth}_I L$ *then* $\operatorname{depth}_I N \geqslant \operatorname{depth}_I L$.

Proof. Well known and easy from the homological characterizations of depth.

LEMMA 3.2. *Suppose R is a Noetherian ring, I is an ideal, and M is a finitely generated modules with a filtration* $\{M_k\}_{k=0}^{n}$. *If*

$$t = \min_{0 \leqslant k \leqslant n-1} \operatorname{depth}_I (M_k/M_{k+1})$$

then $\operatorname{depth}_I M \geqslant t$.

Proof. Induct on the length of the filtration. We may assume depth $M_1 \geqslant t$. We have an exact sequence,

$$0 \to M_1 \to M \to M/M_1 \to 0$$

and depth $M_1 \geqslant t$, depth $M/M_1 \geqslant t$. By Lemma 2.3, depth $M \geqslant t$.

To obtain the transitivity of depth it remains to calculate the depth$I^k/I^{k+1} + I^k J^{n-k}$ in terms of depth $R/(I, J^n)$.

LEMMA 3.3. *Suppose I is an ideal in a commutative Noetherian ring and* I^n/I^{n+1} *are free R/I modules. Suppose J is any ideal and p is an ideal containing* $I + J$. *Then*

$$\min_{0 \leqslant k \leqslant n} \operatorname{depth}_p R/(I, J^k) \leqslant \operatorname{depth}_p(R/I^m + I^{m-1}J^n)$$

for all $m \geqslant 1$ *and all* $n \geqslant 0$.

Set $t = \min_{0 \leqslant k \leqslant n} \operatorname{depth}_p R/(I, J^k)$

Proof. Induct on t. If $t = 0$ there is nothing to show so suppose $t > 0$ and choose $z \in p$, z not a zero divisor on $(R/I, J^k)$ for $0 \leqslant k \leqslant n$. Set $\bar{R} = R/(z)$. We claim $(I^k/I^{k+1}) \otimes_R \bar{R}$ is isomorphic to $(I^k, z)/(I^{k+1}, z)$. For

$$\frac{(I^k, z)}{(I^{k+1}, z)} \simeq \frac{I^k}{I^{k+1} + (I^k \cap (z))}.$$

By an easy induction using the fact that z is not a zero divisor modulo I and the I^k/I^{k+1} are free R/I-modules, one sees z is not a zero divisor modulo I^k for every k. Hence $I^k \cap (z) = I^k z$ and

$$\frac{(I^k, z)}{(I^{k+1}, z)} \simeq \frac{I^k}{I^{k+1} + I^k z} \simeq (I^k/I^{k+1}) \otimes_R R/(z).$$

As I^k/I^{k+1} is a free R/I module, $I^k/I^{k+1} \otimes_R \bar{R}$ is a free $\bar{R}/\bar{I}$ module where $\bar{I} = (I, z)/(z)$. Hence $(I^k, z)/(I^{k+1}, z) \simeq \bar{I}^k/\bar{I}^{k+1}$ is a free module for every k. But $\text{depth}_p \bar{R}/(\bar{I}, \bar{J}^k) = t - 1$ and so the induction shows

$$t - 1 \leqslant \text{depth}_{\bar{p}} \left(\frac{\bar{R}}{\bar{I}^m + \bar{I}^{m-1}\bar{J}^n}\right),$$

and it is enough to show z is not a zero divisor modulo $I^m + I^{m-1}J^n$. If

$$zc \in I^m + I^{m-1}J^n \qquad \text{then} \quad c \in I^{m-1}$$

since z is not a zero divisor modulo I^{m-1}. Choose a free R/I basis $\bar{y}_1, \ldots, \bar{y}_q$ for I^{m-1}/I^m. Reading mod I_m, $zc = \sum_{i=1}^q r_i y_i$, where each r_i can be assumed to be in J^n. But $c \in I^{m-1} \Rightarrow$ modulo I^m that $c \equiv \sum c_i y_i$. Hence,

$$(zc_i - r_i) y_i \equiv 0 \bmod I^m$$

and as $\bar{y}_1, \ldots, \bar{y}_p$ are a R/I-basis we see $zc_i - r_i \in I$. Hence $zc_i \in (I, J^n)$. But z is not a zero-divisor modulo this ideal and so $c_i \in (I, J^n)$. Then $c \in I^{m-1}(I, J^n) = I^m + I^{m-1}J^n$ and this finishes the claim and the lemma.

LEMMA 3.4. *Suppose I is an ideal with I^n/I^{n+1} free R/I modules and J another ideal such that $I \cap J^m \subseteq IJ^{m-1}$. Then I and J satisfy $I^n \cap I^{n-1}J^m \subseteq I^nJ^{m-1}$.*

Proof. We must show $J^mI^{n-1} \cap I^n \subseteq J^{m-1}I^n$. Let $y_1, \ldots, y_p$ be elements of I^{n-1} which are a free R/I basis of I^{n-1}/I^n. If $\sum_{i=1}^p z_i y_i \in I^n$ where the $z_i \in J^m$, then by the freeness of the basis $y_1, \ldots, y_p$, $z_i \in I$ and hence $z_i \in I \cap J^m \subseteq J^{m-1}I$. Then $\sum_{i=1}^p z_i y_i \in J^{m-1}I^n$ as required.

THEOREM 3.1. (Transitivity of depth.) *Let I and J be ideals in a commutative Noetherian ring R and suppose p is an ideal containing $I + J$. Further assume I^n/I^{n+1} are free R/I modules for every $n \geqslant 0$, and $I \cap J^m \subseteq IJ^{m-1}$ for all $m \geqslant 1$. Then for all $n \geqslant 1$,* $\text{depth}_p R/(I + J)^n \geqslant \min_{0 \leqslant k \leqslant n} \text{depth}_p(R/(I, J^k))$ *where $R/(I, J^0) = R/I$.*

Proof. By Lemma 3.4, $I^n \cap I^{n-1}J^m \subseteq I^nJ^{m-1}$ for every $n \geqslant 1$ and $m \geqslant 1$ and so Proposition 3.1 combined with Lemma 3.2 shows that

$$\text{depth}_p R/(I + J)^n \geqslant \min_{0 \leqslant k \leqslant n} \text{depth}_p I^k/(I^{k+1} + I^kJ^{n-k}).$$

Set $t = \min_{0 \leqslant k \leqslant n} \text{depth}_p R/(I, J^k)$ and consider the exact sequences

$$0 \to I^k/(I^{k+1} + I^kJ^{n-k}) \to R/(I^{k+1} + I^kJ^{n-k}) \to R/I^k \to 0.$$

Since $t \leqslant \operatorname{depth}_p R/I$ and the I^k/I^{k+1} are free, it is easy to see $t \leqslant \operatorname{depth}_p R/I^k$ for every $k \geqslant 1$. Lemma 3.3 shows $t \leqslant \operatorname{depth}_p(R/(I^{k+1}+I^kJ^{n-k}))$. Now using Lemma 3.2 we see

$$t \leqslant \operatorname{depth}_p I^k/(I^{k+1}+I^kJ^{n-k})$$

for $0 \leqslant k \leqslant n$. This proves Theorem 3.1.

4. Calculation of Depth

In this section we apply the results of the previous sections to explicitly calculate the depth of R/I^n for ideals I generated by d-sequences. Theorem 4.1 below shows that these depths depend only on the linear relations of the d-sequence.

LEMMA 4.1. *Suppose $x_1,..., x_n$ is a d-sequence in a commutative ring R. Set $x = x_1$, and $J = (x_2,..., x_n)$. Then $(x^{k+1}) \cap x^kJ^m \subseteq x^{k+1}J^{m-1}$ for $k \geqslant 0$.*

Proof. Suppose $x^{k+1}a = x^kb$, where b is in J^m. Then $x^k(xa-b)=0$ and so $x(xa-b)=0$. $xa-b \in (0\colon x) \cap (J,x) = (0)$ by Proposition 2.1 and so $xa=b$ and $b \in J^m \cap (x)$. By Theorem 2.1, $b \in J^{m-1}x$ and so $x^kb \in x^{k+1}J^{m-1}$ as is needed.

DEFINITION 4.1. An ideal I will be said to be *related* to the d-sequence $x_1,..., x_n$ if I is of the form

$$(((x_1,..., x_i)\colon x_{i+1}), x_{i+1},..., x_n)$$

for $0 \leqslant i \leqslant n-1$.

THEOREM 4.1. *Let R be a commutative ring and $x_1,..., x_n$ a d-sequence in R. Let $I = (x_1,..., x_n)$. Then R/I^n has a filtration $\{M_j\}_{j=0}^d$ such that for all k, $0 \leqslant k \leqslant d-1$, M_k/M_{k+1} is isomorphic to R/K, where K is a related ideal.*

Proof. Induct on n. If $n=1$, then since $(0\colon x_1^2) = (0\colon x_1)$,

$$(x_1)/(x_1^2) \simeq (x_1^2)/(x_1^3) \simeq \cdots \simeq (x_1^n)/(x_1^{n+1}) \simeq \cdots.$$

As $(x_1)/(x_1^2) \simeq R/((0\colon x_1), x_1)$ we see $R/(x_1^2)$ has the required filtration from the exact sequence

$$0 \to (x_1)/(x_1^2) \to R/(x_1^2) \to R/(x_1) \to 0.$$

From the exact sequences

$$0 \to (x_1^n)/(x_1^{n+1}) \to R/(x_1^{n+1}) \to R/(x_1^n) \to 0$$

and the isomorphisms $(x_1^n)/(x_1^{n+1}) \simeq (x_1)/(x_1^2) \simeq R/((0: x_1), x_1)$ we obtain the result by induction.

Suppose then $n > 1$. Set $x = x_1$ and $J = (x_2,..., x_n)$. By Lemma 4.1 and Proposition 3.1 we may conclude R/I^n has a filtration whose factors are isomorpic to $(x^k)/(Rx^{k+1} + x^k J^{n-k})$. Map R onto this by multiplication by x^k; since $(0: x) = (0: x^2)$ we see

$$(x^k)/(Rx^{k+1} + x^k J^{n-k}) \simeq R/(x, (0: x), J^{n-k}).$$

Now set $S = R/(x, (0: x))$. By Proposition 2.2 the images of $x_2,..., x_n$ form a d-sequence in S. By induction, the modules $S/\bar{J}^k$ have filtrations whose factors are of the form $S/\bar{K}$, where $\bar{K}$ is a related ideal to $\bar{x}_2,..., \bar{x}_n$. Reading this back in R, we see $R/(x, (0: x), J^{n-k})$ have filtrations whose factors are isomorphic to

$$R/((x, (0: x), x_2,..., x_i): x_{i+1}), x_{i+1},..., x_n).$$

However, we claim

$$((0: x), x, x_2,..., x_i): x_{i+1}) = ((x, x_2,..., x_i): x_{i+1})$$

For if $cx_{i+1} = s + t$, where $s \in (x, x_2,..., x_i)$ and $t \in (0: x)$, then

$$t \in (x, x_2,..., x_n) \cap (0: x) = (0).$$

Hence $cx_{i+1} \in (x, x_2,..., x_i)$. Thus $R/(x, (0: x), J^{n-k})$ has a filtration whose factors are isomorphic to R/K where K is a related ideal of $x_1,..., x_n$. This proves Theorem 4.1.

COROLLARY 4.1. *Suppose R is a commutative ring and I is an ideal such that I^n/I^{n+1} are free R/I modules. Let $J = (x_1,..., x_n)$ be an ideal generated by elements $x_1,..., x_n$ which form a d-sequence modulo I. Let M be an ideal and let $t = \min_{0 \leqslant i \leqslant n-1} \operatorname{depth}_M R/((I, x_1,..., x_i): x_{i+1}), x_{i+1},..., x_n)$. Then*

$$\inf_n \operatorname{depth}_M(R/(I + J)^n) \geqslant t.$$

Proof. Theorem 2.1 shows $J^m \cap I \subseteq J^{m-1}I$ and now we may apply Theorem 3.1 to conclude for all n

$$\operatorname{depth}_M R/(I + J)^n \geqslant \inf_k \operatorname{depth}_M(R/(I, J^k)).$$

By the theorem above, this infimum is at least t.

Note the conditions of the corollary are satisfied if I is generated by an R-sequence.

We now apply these results to some of the examples of Section 1.

EXAMPLE 4.1. Let $X=(x_{ij})$ be a generic n by $n+1$ matrix and let μ_k be the minor determined by deleting the kth column of x. Set $J=(\mu_1,..., \mu_{n+1})R$, where $R=k[x_{ij}]_{(x_{ij})}$ with k a field. Then

$$\inf_f \operatorname{depth}(R/J^k)=n^2-1.$$

Proof. We apply Theorem 4.1. Let us find depth R/I where I is a related ideal to J. By the calculations in Section 1,

$$I=((\mu_1,..., \mu_{i-1}:\mu_i), \mu_i,..., \mu_{n+1})$$

equals the ideal generated by all the $n\times n$ minors of X together with the maximal minors of the matrix obtained by deleting the first $i-1$ columns of X. Hochster and Eagon [15] have shown such ideals are perfect; that is R/I is Cohen–Macaulay. But observe all such related ideals are contained in

$$(((\mu_1,..., \mu_n): \mu_{n+1}), \mu_{n+1})=(\mu_{n+1}, x_{1n+1}, x_{2n+1},..., x_{nn+1}).$$

As all the ideals are perfect and all are contained in this ideal, it is enough to find depth $R/(\mu_{n+1}, x_{1n+1},..., x_{nn+1})$. But this is clearly $n(n+1)-(n+1)=(n-1)(n+1)=n^2-1$. We conclude $\inf_k \operatorname{depth}(R/J^k)\geqslant n^2-1$. But as is well known ([7]) and also follows from Theorem 2.2, μ_i are analytically independent. L. Burch [6] proved the basic theorem relating the depths of R/J^k to the analytic spread.

THEOREM [4]. *Let R, m be local and let I be an ideal. Denote by $l(I)$ the analytic spread of $I=$ the degree $+1$ of the polynomial which gives the* dim I^n/mI^n *for large n. Then*

$$l(I)\leqslant \dim R-\inf_n \{\operatorname{depth} R/I^n\}.$$

We may conclude $n+1=l(J)\leqslant \dim R-\inf_k \{\operatorname{depth} R/J^k\}$, or, $\inf_k\{\operatorname{depth} R/J^k\}\leqslant n(n+1)-(n+1)=n^2-1$. Putting the two inequalities together, we conclude the $\inf_k\{\operatorname{depth} R/J^k\}=n^2-1$.

In fact even more can be shown. Brodmann [3] has shown that

$$l(I)\leqslant \operatorname{Dim}(R)-\lim\inf(\operatorname{depth}(R/I^n))$$

Thus whenever equality occurs in Burch's inequality, we see

$$\begin{aligned}\operatorname{Dim}(R)-\inf(\operatorname{depth} R/I^n)=l(I)&\leqslant \operatorname{Dim}(R)-\lim\inf(\operatorname{depth} R/I^n)\\ &\leqslant \operatorname{Dim}(R)-\inf(\operatorname{depth} R/I^n)\end{aligned}$$

and so

$$\liminf(\operatorname{depth} R/I^n) = \inf(\operatorname{depth} R/I^n)$$

Consequently the minimum depth is taken infinitely many times.

EXAMPLE 4.2. Let

$$X = \begin{pmatrix} 0 & x_{12} & \cdots & x_{15} \\ -x_{12} & 0 & & \vdots \\ \vdots & & \ddots & x_{45} \\ -x_{45} & \cdots & -x_{45} & 0 \end{pmatrix}$$

be as in Example 1.8. As in that example, let J be the ideal generated by the 4×4 Pfaffians. Then, if $R = k[x_{ij}]_{(x_{ij})}$ where

$$x_{ij} = 0 \quad \text{if} \quad i \geqslant j, \quad \inf_k \operatorname{depth}(R/J^k) = 5.$$

Proof. We apply Theorem 4.1. It is enough to find min depth R/I where I runs through the related ideals of the d-sequence $p_1,..., p_5$ (notation as in Example 1.8).

There are three types of related ideals for the Pfaffians $(p_1,..., p_5)$, $((p_1, p_2, p_3, : p_4), p_4, p_5)$ and $((p_1,..., p_4): p_5), p_5)$.

Now, $(p_1,..., p_5)$ is Cohen–Macaulay of codimension 3 [5]. As $\dim R = 10$, we obtain $\operatorname{depth} R/(p_1,..., p_5) = 7$. By the calculations in Example 1.8, $((p_1, p_2, p_3): p_4) = (p_1, p_2, p_3, x_{45})$.

Hence $(((p_1, p_2, p_3): p_4), p_4, p_5) = (p_1,..., p_5, x_{45})$ and as $(p_1,..., p_5)$ is a prime perfect ideal, we see

$$\operatorname{depth} R/(p_1,..., p_5, x_{45}) = 6.$$

Finally, $(((p_1,..., p_4): p_5), p_5) = (x_{15},..., x_{45}, p_5)$.

Hence $\operatorname{depth} R/(x_{15},..., x_{45}, p_5) = 10 - 5 = 5$ and this gives the result.

Note as for maximal minors of Example 4.1, since the Pfaffians are analytically independent we will obtain equality in Burch's Theorem.

EXAMPLE 4.3. Let $X = (x_{ij})$ be $r \times s$ matrix of indeterminates as in Example 1.3 and let I be the ideal generated by all the $m \times m$ minors of X, with $0 \leqslant m < r$. Set $S = (k[x_{ij}]/I)_{(x_{ij})}$ k a field. It was shown that $\bar{x}_{11},.., \bar{x}_{1s}$ for a d-sequence in S. Let $J = (\bar{x}_{11},..., \bar{x}_{1s})$. Then,

$$\inf \operatorname{depth}_k(S/J^k) = (r + s - m + 1)(m - 1) - s.$$

Again, by Theorem 4.1 it is enough to check the depths of the related ideals. As was shown in Example 1.3, the related ideals have the form

$$B = (C, \bar{x}_{1j+1}, ..., \bar{x}_{1s}),$$

where C is generated in S by the $m-1 \times m-1$ minor of the matrix

$$\begin{pmatrix} \bar{x}_{11} & \cdots & \bar{x}_{1j} \\ \vdots & & \vdots \\ \bar{x}_{r1} & \cdots & \bar{x}_{rj} \end{pmatrix}.$$

To calculate depth S/J, it is clearly enough to lift B back to R to an ideal B' and calculate depth R/B'.

These depths were computed in [15]. For B' as above, we obtain from [15]

$$\begin{aligned} \operatorname{depth} R/B' &= (r+s)(m-1) - (m-1)^2 - j \\ &= (m-1)(r+s-m+1) - j \end{aligned}$$

for $j < n$ and for $j = n$, the depth in question is

$$(r+s-m+1)(m-1) - s.$$

Now comparing these we see that

$$(m-1)(r+s-m+1) - j \geqslant (r+s-m+1)(m-1) - s$$

if $j \leqslant s$. This holds in our case and this implies the minimum depth is

$$(r+s-m+1)(m-1) - s.$$

From [15] we may compute $\dim S$. It is $(m-1)(r+s-m+1)$. Hence

$$s = l(J) \leqslant \dim S - \inf \operatorname{depth}_k(S/J^k) \leqslant s$$

and we obtain equality in Burch's equation.

We now consider when equality occurs in Burch's equation.

THEOREM 4.2. *Suppose R, m is a local Cohen–Macaulay ring and $x_1, ..., x_n$ is a d-sequence such that*

$$\dim R - \operatorname{depth} R/(x_1, ..., x_k) \leqslant k$$

for $1 \leqslant k \leqslant n$. Then

$$\dim R - \inf_m\{\operatorname{depth} R/(x_1, ..., x_n)^m\} = n = l((x_1, ..., x_n)).$$

Remark. The condition that R be Cohen–Macaulay is necessary; for if $x \in R$, not a zero divisor, then we would need

$$\dim R - \inf \operatorname{depth} R/(x^n) = 1.$$

But depth $R/(x^n) = \operatorname{depth} R - 1$ in this case so equality occurs if and only if R is Cohen–Macaulay.

Finally, if $\dim R - \operatorname{depth}(R/(x_1,\ldots,x_n)) > n$ then there is little chance of obtaining equality. But this should not be regarded as serious; where Burch's inequality becomes bad is in the high powers of the ideal in question, and it is not too much to assume the ideal itself is well behaved with regard to depth.

Proof of Theorem 4.2. Set $x = x_1$. We induct on n. First, we claim $R/((0:x), x)$ is Cohen–Macaulay. For, consider the exact sequence

$$0 \to R \to R/Rx \oplus R/(0:x) \to R/((0:x), x) \to 0,$$

which is exact since $(0:x) = (0:x^2)$. Let m be the maximal ideal of R and apply $\operatorname{Hom}_R(R/m, -)$ to this exact sequence. We obtain

$$\begin{aligned}&\to \operatorname{Ext}_R^i(R/m, R) \to \operatorname{Ext}_R^i(R/m, R/Rx) \oplus \operatorname{Ext}_R^i(R/m, R/(0:x))\\ &\to \operatorname{Ext}_R^i(R/m, R/(0:x), x)) \to \operatorname{Ext}_R^{i+1}(R/m, R) \to \cdots.\end{aligned}$$

Now $\operatorname{Ext}_R^i(R/m, R) = 0$ for $i < d = \dim R$. Also depth $R/((0:x), x) = \operatorname{depth}(R/(0:x)) - 1$ as x is not a zero divisor modulo $(0:x)$. By assumption depth $R/Rx \geqslant d-1$. Hence $\operatorname{Ext}_R^i(R/m, R/Rx) = 0$ if $i < d-1$. If $i < d-1$ we obtain the sequence

$$0 \to \operatorname{Ext}_R^i(R/m, R/(0:x)) \to \operatorname{Ext}_R^i(R/m, R/(0:x), x) \to 0.$$

If either of these were nonzero, then both would have to be nonzero. As the depth is characterized by the first nonvanishing Ext, this would contradict the depth inequality between $R/(0:x)$ and $R/((0:x), x)$. Hence all these are zero if $i < d-1$. When $i = d-1$ we obtain

$$\begin{aligned}0 &\to \operatorname{Ext}_R^{d-1}(R/m, R/Rx) \oplus \operatorname{Ext}_R^{d-1}(R/m, R/(0:x))\\ &\to \operatorname{Ext}_R^{d-1}(R/m, R/((0:x), x)) \to \operatorname{Ext}_R^d(R/m, R) \to .\end{aligned}$$

Now as $\dim R/((0:x)) = d-1$ and the depth cannot be larger than $d-1$, we see $\operatorname{Ext}_R^{d-1}(R/m, R/((0:x), x)) \neq 0$. In any case $R/((0:x), x)$ is Cohen–Macaulay.

Set $S = R/((0:x), x)$. We wish to apply the induction to S. To do this it is necessary to show

$$\dim S - \operatorname{depth} S/(\bar{x}_2,\ldots,\bar{x}_k) \leqslant k-1. \qquad (*)$$

Suppose (*) has been shown; we complete the proof of the Theorem. The induction allows us to conclude

$$n-1=\dim S-\inf_m \operatorname{depth} S/(\bar{x}_2,\dots,\bar{x}_n)^m.$$

But the proof of Theorem 4.1 shows

$$\begin{aligned}&\inf_m \operatorname{depth} R/(x_1,\dots,x_n)^m.\\ &\quad\geqslant \inf_m \operatorname{depth} R/((0:x_1),x_1,(x_2,\dots,x_n)^m)\\ &\quad= \inf_m \operatorname{depth} S/(\bar{x}_2,\dots,\bar{x}_n)^m.\end{aligned}$$

As $\dim S=\dim R-1$, we obtain,

$$\begin{aligned}n-1&=\dim R-1-\inf_m \operatorname{depth} S/(\bar{x}_2,\dots,\bar{x}_n)^m\\ &\geqslant \dim R-1-\inf_m \operatorname{depth} R/(x_1,\dots,x_n)^m\end{aligned}$$

and so

$$n\geqslant \dim R-\inf_m \operatorname{depth} R/(x_1,\dots,x_n)^m.$$

But, $x_1,\dots,x_n$ are analytically independent by Theorem 3.2 and so Burch's Theorem gives the opposite inequality which proves the theorem if we can demonstrate (*).

We need to prove that

$$\dim R-1-\operatorname{depth} R/((0:x_1),x_1,\dots,x_k)\leqslant k-1,$$

i.e., that

$$\dim R-\operatorname{depth} R/((0:x_1),x_1,\dots,x_k)\leqslant k.$$

We know by assumption that

$$\dim R-\operatorname{depth} R/(x_1,\dots,x_k)\leqslant k. \tag{1}$$

Let $J=(x_1,\dots,x_k)$ and $x_1=x$. Then by Proposition 3.1, $J\cap(0:x)=0$ mnd so there is an exact sequence

$$0\to R\to R/J\oplus R/(0:x)\to R/(J,(0:x))\to 0.$$

As above, apply $\operatorname{Hom}_R(R/m,-)$ to this sequence. As by above $R/(0:x)$ is Cohen–Macaulay and as its dimension is $d=\dim R$, we see that for $i<d$ the long exact sequence for Ext degenerates to

$$0\to \operatorname{Ext}^i_R(R/m,R/J)\to \operatorname{Ext}^i_R(R/m,R/(J,(0:x)))\to 0.$$

This shows depth $R/J = \text{depth}\, R/(J, (0{:}\, x))$ and this, together with the inequality (1) prove (*) and finish the proof of Theorem 4.2.

By discussion following Example 4.1 we see that under the conditions of Theorem 4.2 the minimum depth is taken infinitely many times.

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