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Notes, Comments, and Letters to the Editor

The Definition of Stability in Models with Perfect Foresight

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Frequently the stationary states of decentralized models with perfect foresight and more than one asset are saddlepoints. In this note we develop a distinction between "historical" and "nonhistorical" variables. Then we use the distinction to define economic concepts of "weak" and "strong" local asymptotic stability—which apply even for some saddlepoints. We conclude with three examples. *Journal of Economic Literature* Classification Number: 111.

The phase diagrams of many interesting decentralized growth models reveal saddlepoint configurations about one or more of the stationary solutions. This is particularly true for perfect foresight models with heterogeneous assets—see, for instance, Shell *et al.* [5], Shell and Stiglitz [4], and Stiglitz [6]. Faced with such a saddlepoint, we can either forsake perfect foresight or reconsider the conventional definition of stability. This note takes the latter approach—we present two new concepts of stability for economic models, and then apply them to the three examples listed above.

STABILITY AND DETERMINACY

Consider the class of models which can be summarized with a single equation of motion

$$x_{t+1} = g(x_t) \qquad \text{or} \qquad \dot{x}_t = G(x_t), \tag{1}$$

where $g(\cdot)$ or $G(\cdot)$ is continuously differentiable and $x_t \in R^n$.[1] Assume x_t has

[1] Some economic models lead to equations of a more general form—such as $f(x_{t+1}, x_t) = 0$. The concluding section of this paper comments on the possible analysis of such specifications.

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been normalized in such a way that $g(\cdot)$ or $G(\cdot)$ is time autonomous and that $0 \in R^n$ is a stationary solution—

$$0 = g(0) \qquad \text{or} \qquad 0 = G(0). \tag{2}$$

The conventional test of the local asymptotic stability of the latter solution involves restrictions on all n eigenvalues of $[\partial g^i(0)/\partial x_j]$ or $[\partial G^i(0)/\partial x_j]$. We now argue that in many economic contexts only a smaller set of eigenvalues need obey the constraints.

Consider the vector x_t for an economic model. Some components may consist of variables which are fixed from history. A capital stock or a variable measuring the usage to date of a natural resource endowment would be examples. (Each context will require separate judgments.) Renumbering indices if necessary, let the first p components of x_t consist of such "historical" variables—where $n \geqslant p \geqslant 0$—and let these components constitute a subvector u_t.

The remainder of x_t will consist of variables having time-t values dependent only on current and future developments. Let these "nonhistorical" variables constitute a subvector v_t with $q = n - p$ components. We have

$$x_t = (u_t, v_t).$$

The quintessential example of a "nonhistorical" variable would be a stock argument's price: at time t, in a perfect foresight model such a price will depend solely on capital gains prospects and on present supply and demand conditions.

Given our dichotomy for x_t, we next turn to a consideration of the local stability of $0 \in R^n$. Suppose our model is new at time t or an exogenous shock had shifted the system away from $x_s = 0$ just prior to time t. Then history may dictate a value

$$u_t = u \neq 0 \in R^p.$$

On the other hand, history cannot fix v_t. Hence, if $q \geqslant 1$, the first step in studying the asymptotic stability of $0 \in R^n$ is to ask: Is there any solution $\phi(s)$ of line (1) such that $\phi(t) = (u_t, v_t) = (u, v_t)$ and $\lim_{s \to \infty} \|\phi(s)\| = 0$? If the answer is "yes" for each u "close" to $0 \in R^p$, a fundamental part of the usual notion of stability holds.

To be precise,

DEFINITION. *We say $0 \in R^n$ is "weakly" locally asymptotically stable if there exists an $\varepsilon > 0$ such that for each $u \in R^p$ with $\|u\| < \varepsilon$ we can find*

$v \in R^q$ and a solution $\phi(s)$ of line (1) *for all $s \geqslant t$ with $\phi(t) = (u, v)$ and* $\lim_{s\to\infty} \|\phi(s)\| = 0$.

If $0 \in R^n$ is not "weakly" stable, it will not deserve special attention as an interesting steady state. If, however, it is weakly locally asymptotically stable, arguably a detailed steady-state analysis is warranted: convergence from every u fixed by history and "close" to $0 \in R^p$ is mathematically possible—and whether convergence will occur in practice presumably depends upon elements beyond the scope of the model of line (1).

In some cases we may be able to verify weak stability using linear approximations:

PROPOSITION 1. *Let the model of lines* (1) *and* (2) *be continuously differentiable. Suppose $[\partial G^i(0)/\partial x_j]$ or $[\partial g^i(0)/\partial x_j]$ has n distinct eigenvalues of which m have negative real parts in the case of $G(\cdot)$, or modulus < 1 in the case of $g(\cdot)$. Let $a_1, \ldots, a_m \in R^n$ generate the subspace of convergent solutions for the linearized model. Then $0 \in R^n$ is weakly locally asymptotically stable if*

(i) *$m \geqslant p$; and,*

(ii) *for some p-by-p matrix Z having columns consisting of the first p elements of a_j for p distinct indices $j \in \{1, \ldots, m\}$, we have* $\det[Z] \neq 0$.

The proof follows directly from Theorem 4 in Chapter 4 of Bellman [1] and from Lemma 1, Lemma 2, and Step 1 in the proof of Proposition 3 in Laitner [2]. Condition (ii) has a simple geometric interpretation: Suppose $p = q = 1$, $\dot{x}_t = G(x_t)$, and $G(\cdot)$ is linear. Then in Diagram 1, even for values of $u_t \neq 0 \in R^p$ very close to 0, we cannot find a vector v_t which puts us on a convergent trajectory—a situation which condition (ii) rules out.

Ideally each stationary state of potential interest would manifest two properties in addition to weak stability: properties which we call "strong" stability and local "determinacy." We will say $x_t \in R^n$ is "feasible" if there

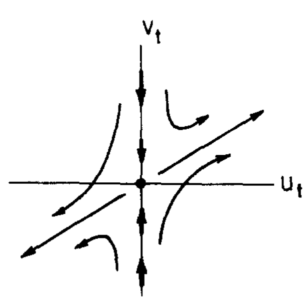

DIAGRAM 1. Phase diagram for $\dot{x}_t = G(x_t)$.

exists a solution $\phi(s)$ of line (1) for all $s \geqslant t$ having $\phi(t) = x_t$. For any $u \in R^p$ let

$$\Psi(u) = \{v \in R^q : x_t = (u, v) \text{ is feasible}\}.$$

(Note that perfect foresight compels an $x_t \in (u_t, \Psi(u_t))$ for any historically given u_t.[2]) Then

DEFINITION. *We say $0 \in R^n$ is "strongly" locally asymptotically stable if*

(i) *it is weakly locally asymptotically stable; and,*

(ii) *there exists an $\varepsilon > 0$ such that $u \in R^p$ and $\|u\| < \varepsilon$ imply $\lim_{s \to \infty} \|\phi(s)\| = 0$ for every solution $\phi(s)$ to line (1) all $s \geqslant t$ having $\phi(t) = x_t = (u, v_t) \in (u, \Psi(u))$.*

DEFINITION. *We say $0 \in R^n$ is locally "determinate" if there exists an $\varepsilon > 0$ such that $u \in R^p$ and $\|u\| < \varepsilon$ imply $\Psi(u)$ contains one and only one vector.*

Notice that if $0 \in R^n$ is weakly stable and locally determinate, then it is also strongly stable. Strong stability does not, on the other hand, imply local determinacy.

Unfortunately, neither new property can be established without a global analysis of $g(\cdot)$ or $G(\cdot)$—which will often be prohibitively difficult in practice. Even if we have n eigenvalues at $0 \in R^n$ with negative real parts (or moduli < 1 in the difference-equation case), our first example below illustrates that we may not have either strong stability or determinacy.

In the notation of Proposition 1, Bellman's Theorem 4 shows that $m > p$ implies indeterminacy. In practice, therefore, we may want to check that $m = p$ and that condition (ii) of Proposition 1 holds. If both are true, even if a global analysis is out of the question, we will have verified weak stability and left open the possibility of the most desired outcome.

EXAMPLES

We now briefly examine the three well-known papers alluded to in our introduction.

First, consider the Shell *et al.* [5, Sect. 3] model with a physical capital stock and a government debt. The labor force and debt grow at exogenously specified exponential rates. Let k_t be the capital-to-labor ratio and b_t the value-of-government-bond-to-labor ratio. Then in our notation

$$x_t = (u_t, v_t) = (k_t, b_t),$$

[2] See p. 606 of [4].

where k_t depends (endogenously) on the capital stock, and, hence, is an "historical" variable; and, where b_t depends (endogenously) on the price of bonds and, hence, is a "nonhistorical" variable.

Diagram 2 reproduces the model's phase diagram. As can be seen, both stationary points are weakly locally asymptotically stable, but neither is locally determinate or strongly stable. (For the right-hand point—the stationary state of interest for the model—the latter two results would not be apparent from a local analysis alone.)

Second, consider the Shell and Stiglitz [4] model with two capital goods. Let p_i, $i = 1, 2$, be the goods' prices, and let k_i, $i = 1, 2$, be the two capital-to-labor ratios. The model's single consumption commodity is the numeraire, and labor grows at an exogenous, exponential rate.

In our notation,

$$u_t = (k_{1t}, k_{2t}) \qquad \text{and} \qquad v_t = (p_{1t}, p_{2t}).$$

Although the model is not continuously differentiable, the authors show that it displays saddlepoint behavior, and they prove that its unique stationary solution is strongly stable and determinate.

Third, consider Stiglitz's [6] model with one capital good and one depletable natural resource. Stiglitz establishes the existence of a unique steady-state growth path. Let z_t be the price of resource units used at time t normalized by an exponential factor reflecting steady-state growth, let k_t be the physical capital stock similarly normalized, and let s_t be the stock of unused resource similarly normalized.

We can derive equations for z_t and k_t which are independent of s_t:

$$\dot{z}_t/z_t = A \cdot (z_t)^a (k_t)^b + B,$$
$$\dot{k}_t/k_t = C \cdot (z_t)^a (k_t)^b + D,$$

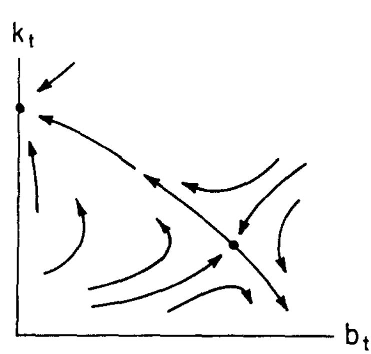

DIAGRAM 2. The Shell–Sidrauski–Stiglitz phase diagram.

where, using Stiglitz's α_i, $i = 1,\ldots, 3$,

$$a = -\alpha_3/(1-\alpha_3) < 0,$$
$$b = \alpha_1 - 1 + \alpha_1\alpha_3/(1-\alpha_3) < 0.$$

Let (k^*, z^*, s^*) be the unique stationary solution. Diagram 3 shows the behavior of z_t and k_t. Stiglitz proves that convergence to a point southeast of (k^*, z^*) implies $s_T < 0$ some finite T. Convergence to points northwest of (k^*, z^*) is possible, however. If $(k_t, z_t) \to (k^*, z^*)$, then $s_t \to s^*$ necessarily.

In our terminology,

$$u_t = (k_t, s_t) \qquad \text{and} \qquad v_t = z_t.$$

Thus, Stiglitz's steady state is weakly locally asymptotically stable, but not strongly stable and not locally determinate.

Conclusion

We have defined a minimal stability property for economically interesting stationary states—"weak" local asymptotic stability. We have also developed concepts of "strong" stability and local "determinacy." Our examples hint that weak stability may be the most that we can hope for in general.

Proposition 3 in Laitner [2] presents a way of checking what we now call weak stability for a class of models not discussed here—models of the form

$$f(x_T, \ldots, x_S) = x_t$$

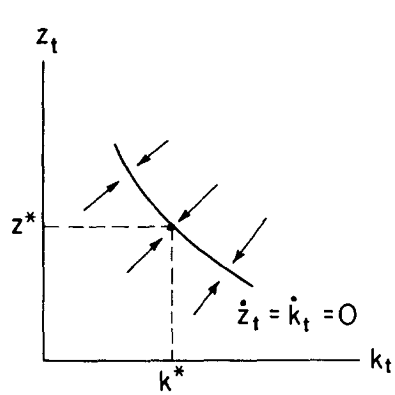

Diagram 3. Phase diagram for the Stiglitz model.

with $T \geqslant t \geqslant S$ and $x_t \in R^n$. Samuelson's [3] consumption-loan model with money would fall into this class, for instance. Our concepts of "historical" and "nonhistorical" variables, and of local "determinacy" and "strong" stability could be applied to such models in a straightforward fashion.

If we change our subject from perfect foresight to adaptive expectations models, on the other hand, the issues discussed in this note cease to arise: with adaptive expectations, price expectations become historical variables, our q shrinks to 0, strong and weak stability both coincide with conventional local stability, and determinacy is generally guaranteed.

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