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Nonparabolic Subgroups of the Modular Group

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1. INTRODUCTION

Neumann [9] (see also Magnus [7]) and later Tretkoff [12] have studied subgroups of the modular group $M = PSL(2, Z)$ that are maximal with respect to containing no parabolic elements. If P is a maximal parabolic subgroup of M, that is, with all nontrivial elements parabolic, and S is a complement of P in M, then S is a maximal nonparabolic subgroup. It was groups S of this type that were studied by Neumann and Tretkoff, and we call such groups *Neumann subgroups*; it is not known whether all maximal nonparabolic subgroups are of this sort (see Magnus [7, p. 121]).[1]

Neumann and Tretkoff showed that the Neumann subgroups are associated with what we call *transitive triples* (Ω, A, B): A and B are permutations of a (necessarily countable) infinite set Ω such that $A^2 = B^3 = 1$ and that $C = AB$ is transitive on Ω. A knowledge of all such triples is equivalent to a knowledge of all Neumann subgroups. Moreover, Tretkoff obtained, by the Reidemeister–Schreier process, a presentation for S expressed very simply in terms of an associated triple (Ω, A, B). From the well known fact that M is the free product of a group of order 2 with a group of order 3, it follows from the Kurosh Subgroup Theorem that S is the free product of r_2 groups of order 2, r_3 groups of order 3, and r_∞ infinite cyclic groups, for certain numbers r_i, $0 \leqslant r_i \leqslant \infty$ (we write ∞ for $\aleph_0$). Tretkoff obtained partial information about the numbers r_2, r_3, r_∞.

Tretkoff's work prompted us to study the set of transitive triples (Ω, A, B).

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[1] See note (2) on page 321

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where we now relax the condition that Ω be infinite. In another paper [1] we have associated each transitive triple with what we call an *Eulerian graph* G^*, that is, a graph in which each vertex has degree at most 3, and equipped with a path π that traverses each directed edge exactly once, and we have reduced the study of such graphs to cubic Eulerian graphs. Although we did not obtain a detailed description of the class of such graphs, our analysis suffices to determine the structure of S, that is, the numbers r_2, r_3, r_∞. We complete Tretkoff's results by establishing the following (and this without appeal to the Kurosh Subgroup Theorem).

(1.1) *r_2 is the number of fixed points of A and r_3 is the number of fixed points of B.*

(1.2) *r_∞ is the Betti number of the graph G^*.*

(1.3) $r_2 + r_3 + r_\infty = \infty$.

(1.4) *If r_∞ is finite then it is even.*

(1.5) *Every triple of numbers r_2, r_3, r_∞, where $0 \leqslant r_2, r_3, r_\infty \leqslant \infty$, that satisfies* (1.3) *and* (1.4) *is realized by a triple* (Ω, A, B) *associated with some Neumann subgroup S.*

Some of our arguments establish somewhat more general results concerning a generalization of Tretkoff's presentation associated with an arbitrary locally finite graph, in particular, with any finite graph, and with the Betti numbers of such graphs. We know of no application of these results beyond that given above.

2. Nonparabolic Subgroups

The *modular group* $M = PSL(2, Z)$ can be viewed as the group of all linear fractional transformations of the extended complex plane $C^* = C \cup \{\infty\}$, of the form

$$\alpha: \quad z \mapsto \frac{az+b}{cz+d}, \qquad \text{where } a, b, c, d \in Z \quad \text{and} \quad ad - bc = 1. \quad (2.1)$$

A transformation α is *parabolic* if it has a single (necessarily real or ∞) fixed point, or alternatively if it is nontrivial and has trace $a + d = \pm 2$. A *parabolic subgroup* of M is one whose nontrivial elements are all parabolic. A *nonparabolic subgroup* is one containing no parabolic elements.

It is well known that M is the free product of a group of order 2 with a group of order 3. Explicitly, M has a presentation

$$M = \langle \omega, \tau: \quad \omega^2 = 1, \ (\omega\tau)^3 = 1 \rangle, \qquad (2.2)$$

where ω and τ are the transformations

$$\omega: \quad z \mapsto -1/z, \qquad \tau: \quad z \mapsto z+1.$$

(2.3) LEMMA. *The maximal parabolic subgroups of M are exactly the conjugates of the translation group $T = \langle \tau \rangle$.*

Proof. M acts transitively on $Q^* = Q \cup \{\infty\}$. For, if $q \in Q$, writing $q = a/c$, where $(a, c) = 1$, there exist b and d such that $ad - bc = 1$, and hence an element α of M, as in (2.1), such that $\alpha(\infty) = a/c$. Now if P is a parabolic group containing a nontrivial element α with fixed point q, then, after replacing P by a conjugate in M, we may suppose that α has fixed point ∞. This means that α, as in (2.1), has $c = 0$: hence $ad = 1$, and we may take $a = d = 1$, whence $\alpha = \tau^b \in T$, $b \neq 0$. If α_1 is another element of P, then $\beta = \alpha_1 \alpha^{nb}$ is in P for all $n \in A$, and the trace $a_1 + d_1 + nbc_1$ must be ± 2 for all $n \in Z$, which implies that $c_1 = 0$ and hence that α is in T. Thus $P \subseteq T$, and if P is maximal, $P = T$. □

(2.4) DEFINITION. *A Neumann subgroup of M is a complement S of a maximal parabolic subgroup P; that is, $SP = M$ and $S \cap P = 1$.*

(2.5) LEMMA. *The following are equivalent*:

(2.5.1) *S acts transitively on Q^*.*

(2.5.2) *$SP = M$ for some maximal parabolic subgroup P.*

(2.5.3) *$SP = M$ for all maximal parabolic subgroups P.*

Proof. Since all maximal parabolic subgroups are conjugate in M, it suffices to show that (2.5.1) holds if and only if $ST = M$. If $ST = M$, then, since M is transitive on Q^*, $Q^* = M(\infty) = ST(\infty) = S(\infty)$, and S is transitive on Q^*. For the converse assume that S is transitive on Q^*, and let $\alpha \in M$. Then S contains some β such that $\beta(\infty) = \alpha(\infty)$, whence $\beta^{-1}\alpha(\infty) = \infty$, $\beta^{-1}\alpha \in T$, and $\alpha \in \beta T \subseteq ST$. □

(2.6) LEMMA. *If S is transitive on Q^* and $S \cap P = 1$ for some maximal parabolic subgroup P, then $S \cap P = 1$ for all maximal parabolic subgroups P.*

Proof. If P has a fixed point $p \in Q^*$, then $S \cap P = S_p$, the stabilizer of p in S. Since S is transitive on Q^*, all S_p for $p \in Q^*$ are conjugate in S. □

(2.7) COROLLARY. *A Neumann subgroup of M is a complement to every maximal parabolic subgroup of M.* □

(2.8) PROPOSITION. *A Neumann subgroup of M is a maximal nonparabolic subgroup of M.*

Proof. If S is a Neumann subgroup then $S \cap P = 1$ for some maximal parabolic subgroup P, and hence, by (2.5), for all. Thus S contains no parabolic elements; that is, S is nonparabolic. If $\alpha \notin S$ and $\alpha(\infty) = q$, then, since, by (2.5), S is transitive on Q^*, $\beta(\infty) = q$ for some $\beta \in S$. Now $\beta^{-1}\alpha \neq 1$ fixes ∞, and the group $\langle S, \alpha \rangle$ contains a parabolic element. This shows that S is maximal nonparabolic. □

By a *triple* (Ω, A, B) we shall always understand one where Ω is an infinite set, and where A and B are permutations of Ω such that $A^2 = B^3 = 1$ and that $C = AB$ is transitive on Ω.

(2.9) PROPOSITION. *The conjugacy classes of Neumann subgroups of M are in one-to-one correspondence with the isomorphism classes of triples.*

Proof. Let S be a Neumann subgroup, and hence a complement to T. Let Ω be the family of cosets $S\tau^k$, $k \in Z$. Then the action of M on Ω by right multiplication defines a map

$$\phi: \quad M \to \operatorname{Sym} \Omega. \tag{2.9.1}$$

Let

$$A = \omega\phi, \qquad B = (\omega\tau)\phi;$$

then $A^2 = B^3 = 1$ and $C = AB = \tau\phi$ is transitive on Ω; that is, (Ω, A, B) is a triple.

Since $ST = M$, every conjugate of S has the form $S' = \tau^{-h}S\tau^h$. The cosets of S' have the form $\tau^{-h}S\tau^{h+k}$, $k \in Z$. The correspondence $\tau^{-h}S\tau^{h+k} \mapsto S\tau^{h+k}$ from Ω' to Ω induces an isomorphism between the triples (Ω', A', B') and (Ω, A, B).

Now let a triple (Ω, A, B) be given. Since M has the presentation (2.2), Eqs. (2.9.2) define a map

$$\phi: \quad M \mapsto \operatorname{Sym} \Omega. \tag{2.9.1}$$

Choose an element $p \in \Omega$ and let $S = \{\alpha: \alpha \in M, p(\alpha\phi) = p\}$. Since $C = \tau\phi$ is transitive on infinite Ω, $p(\tau^k\phi) \neq p$ for $k \neq 0$, whence $S \cap T = 1$. Moreover, if $\alpha \in M$, then $p(\alpha\phi) = pC^k = p(\tau^k\phi)$ for some k, whence $p((\alpha\tau^{-k})\phi) = p$, $\alpha\tau^{-k} \in S$, and $\alpha \in S\tau^k \subseteq ST$. Thus $ST = M$ and S is a Neumann subgroup.

Finally, it is clear that if a different element p of Ω is chosen, then S will be replaced by a conjugate in M. □

3. ASSOCIATED GRAPHS

In [1] we associated with each triple (Ω, A, B) a pair of graphs G and G^*. The graph G has vertex set Ω. It has a directed edge, called an *A-edge*, from

p to q whenever $p \neq q$ and $pA = q$, and a *B-edge* from p to q whenever $p \neq q$ and $pB = q$, with inverse B^{-1}-edge from q to p. A B-orbit is represented either by a single vertex of degree 1, or by an oriented *triangle* of B-edges.

The graph G^* is obtained from G by contracting every triangle to a point. It is a *cuboid graph* in the sense that each vertex has degree at most 3. It possesses an *Eulerian path* π, that is, a path that is reduced except at vertices of degree 1, and which traverses each directed edge exactly once.

The main result of [1], restricted to infinite graphs, is as follows.

(3.1) THEOREM. *If* (Ω, A, B) *is a triple, then* G^* *has one of the following forms*:

(3.1.1) G^* *is a simply infinite tree, that is, a tree with exactly one infinite reduced path beginning at each point*;

(3.1.2) G^* *is obtained from a finite cubic graph* G_0^* *by attaching trees to* G_0^* *at new vertices introduced to subdivide certain edges; exactly one of these trees is simply infinite, and there are finitely many finite trees*;

(3.1.3) G^* *is obtained from an infinite cubic graph* G_0^* *by attaching a finite or infinite number of finite trees.*

From this we shall derive the main result of this paper.

(3.2) THEOREM. *Let* S *be a Neumann subgroup of the modular group, and let* (Ω, A, B) *be the associated triple, with* G, G^* *the associated graphs. Then, for certain numbers* r_2, r_3, r_∞, *where* $0 \leqslant r_2, r_3, r_\infty \leqslant \infty$. S *is the free product of* r_2 *groups of order* 2, r_3 *groups of order* 3, *and* r_∞ *infinite cyclic groups. Moreover,*

(3.2.1) r_2 *is the number of fixed points of* A *and* r_3 *is the number of fixed points of* B;

(3.2.2) r_∞ *is the Betti number of the graph* G^*;

(3.2.3) $r_2 + r_3 + r_\infty = \infty$;

(3.2.4) *if* r *is finite then it is even*;

(3.2.5) *every triple of numbers* r_2, r_3, r_∞, *where* $0 \leqslant r_2, r_3, r_\infty \leqslant \infty$, *that satisfies* (3.2.3) *and* (3.2.4) *is realized by a triple* (Ω, A, B) *associated with some Neumann subgroup of the modular group.*

We shall use also the following result of Tretkoff, obtained by the Reidemeister–Schreier Process.

(3.3) THEOREM (Tretkoff). *Let* S *be a Neumann subgroup of the modular group, and let* (Ω, A, B) *be the associated triple. Then* S *has a* *presentation with Ω as set of generators, and with defining relations $p(pA) = 1$ and $p(pB)(pB^2) = 1$ for all p in Ω.*

We shall use a modification of Tretkoff's presentation, which we first describe in the simplest case. Suppose the permutations A and B in the triple (Ω, A, B) are without fixed point, so that the associated graph $H = G^*$ is a cubic graph. As generators for S we replace each vertex v by the unique (directed) edge e_v beginning at v; thus the presentation will have as set of generators the set E of all edges of H. If $vA = v'$, then $e_{v'} = e_v^{-1}$, whence the "edge relations" $v(vA) = 1$ take the form $ee^{-1} = 1$. Let p_1, p_2, p_3 be, in order, the three elements of a B-orbit in G, corresponding to a single vertex v in H, and let e_1, e_2, e_3 be the three A-edges beginning at these points. Then the "vertex relation" $p_1(p_1B)(p_1B^2) = 1$, or $p_1p_2p_3 = 1$ takes the form r_v: $e_1e_2e_3 = 1$.

(3.4) Proposition. *Let S be a Neumann subgroup of the modular group such that, in the associated triple (Ω, A, B), neither A nor B has a fixed point, whence the associated graph $H = G^*$ is cubic. Then S has a presentation*

$$S = \langle E\colon \quad \{r_e = 1\}, \{r_v = 1\} \rangle,$$

where E is the set of edges of H, where, for each edge e of H, $r_e = ee^{-1}$, and where, for each vertex v of H, $r_v = e_1e_2e_3$, e_1, e_2, e_3 being the three edges at v, in the order dictated by the Eulerian path on H.

Next suppose that H is merely cuboid (but not necessarily cubic). Reference to the graph G establishes the following.

(3.5a) *If v is a vertex of H of degree 2, with edges e_1 and e_2 at v, then there is associated with v an additional generator f and a pair of relations $e_1e_2 = f$ and $f^2 = 1$.*

(3.5b) *If v is a vertex of H of degree 1, with edge e_1 beginning at v, then there is associated with v either a relation $e_1^3 = 1$, or else two additional generators f_1 and f_2 and three relations $e_1 = f_1f_2$, $f_1^2 = 1$, $f_2^2 = 1$.*

In the sequel we shall use both presentations for S.

We begin the proof of (3.2) by studying the contribution to the group S of an attached tree T. Suppose that H is obtained by attaching a tree T to the remainder K of H at a root v_0. Figure (3.6) shows schematically, in the case that T is not trivial, the configuration in H and the corresponding configuration in G.

By $S_1(T)$ we understand the group associated with T, but *excluding* the relation $r_{v_0} = 1$ associated with the vertex v_0. (If $T = \{v_0\}$, trivial, then v_0 is a

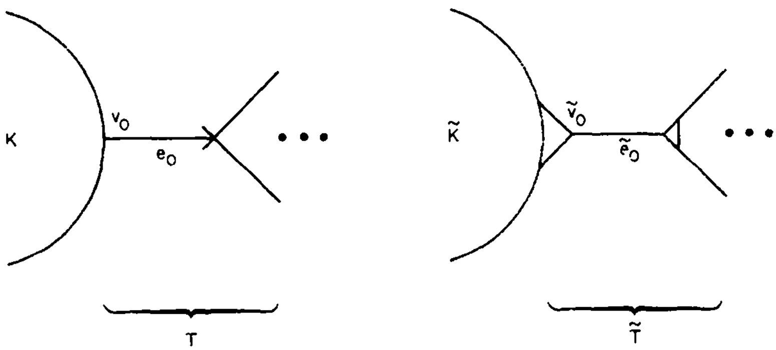


FIGURE 3.6

fixed point of A, and we take $S_1(T) = \langle v_0: v_0^2 = 1\rangle$.) We take $S^*(T)$ to be the free product of all the groups $\langle v: v^2 = 1\rangle$ and $\langle v: v^3 = 1\rangle$ generated by v in T, a fixed point of A or B, excluding the case that K is trivial and v_0 is a fixed point of B.

(3.7) LEMMA. *If T is finite, then $S_1(T) = S^*(T)$; in particular, if T is trivial, with a single vertex v, then $S_1(T) = \langle v_0: v_0^2 = 1\rangle$.*

Proof. If an edge e of T has one end of degree 1, then $e \in S^*(T)$ by (3.5a,b). Inductively, if e is the last edge on the path from v_0 to v, then the remaining edges at v lie in $S^*(T)$, whence by the relation $r_i = 1$ we have $e \in S^*(T)$. □

(3.8) LEMMA. *If T is a simply infinite tree with root v_0, then $S_1(T) = S^*(T) * \langle e_0\rangle$, where e_0 is the edge of T at v_0, and $\langle e_0\rangle$ is infinite cyclic.*

Proof. We may suppose that T consists of vertices v_0, v_1, v_2, with edges e_i from v_i to v_{i+1}, and with additional finite trees $T_1, T_2,\ldots$ attached at v_1, $v_2,\ldots$. We show, as in the proof of (3.7), that all the edges of the T_i are in $S^*(T)$, whence at each vertex $v_i = v_1, v_2,\ldots$ there is a relation $e_i = e_{i-1}f_i$ or $e_i = f_i e_{i-1}$ with $f_i \in S^*(T)$. These relations just suffice to eliminate recursively $e_1, e_2,\ldots$, yielding $S_1(T) = S^*(T) * \langle e_0\rangle$, $\langle e_0\rangle$ being an infinite cyclic group. □

(3.9) COROLLARY. *If H is a simply infinite tree, then* (3.2) *holds.*

Proof. We have $H = T$ for T as in (3.8). S is obtained from $S_1(T)$ by adding relations associated with the vertex v_0 in accordance with (3.5b): that is, either $e_0^2 = 1$ or $e_0 = f_1 f_2$, where $f_1^2 = 1$ and $f_2^3 = 1$. Thus $S = S^*$, the free product of the groups of order 2 and 3 corresponding to the fixed points of A and B. □

(3.10) *If H is obtained by attaching (possibly infinitely many) finite trees T^i to an infinite cubic graph K, then* (3.2) *holds.*

Proof. Let K' be obtained from K by subdividing certain edges of K by the points v_0^i of attachment of the finite trees T^i. Let K'' be obtained from K' by attaching a new edge e_0^i at each v_0^i. (By (3.7), the relations in $S_1(T^i)$ give $e_0^i \in S^*(T^i)$.) Let $S^* = \ast\, S^*(T^i)$. Then the group S is obtained from S^* by adding as new generators the set E of all edges e of K', with the relations $ee^{-1} = 1$, and also the relations $r_i = 1$ at the vertices of K'. Note that at v_0^i one has a relation of the form $ee'e_0^i = 1$, where $e, e' \in E$ and e_0^i is a given element of S^*.

Let M be a maximal tree in K', and E_1 the set of edges of K' not in M. If e is the only edge of M at a vertex v, we can use the relation at v to express e in terms of E_1 and S^*, that is, as an element of $\langle E_1, S^* \rangle$. We can repeat this process on M' obtained from M by deleting e and v, and, continuing thus, we can delete any finite branch of M. Now either M is simply infinite, or every finite branch of M is contained in a maximal finite branch. In either case, after deleting finite branches, we can replace M by an infinite tree M' with a base point v_0, such that M' has no finite branch not containing v_0.

It follows that at every vertex v of M' there is an edge e_v leading away from v_0. We choose such an e_v for each vertex v, and let E_2 be the set of all edges of M' not of the form e_v, e_v^{-1} for any v. The relation at v_0 enables us to express e_0 in terms of E_2, E_1, and S^*, that is, as an element of $\langle E_1 \cup E_2, S^* \rangle$. By induction on the distance from v_0 to v, we can use the relation at v to express e_v as an element of $\langle E_1 \cup E_2, S^* \rangle$. In this way we use up all the relations associated with vertices of M' to express all the e_v in terms of $\langle E_1 \cup E_2, S^* \rangle$. In short, we have shown that $S = F * S^*$, where $F = \langle E_1 \cup E_2 \rangle$ is a free group.

To show that F has infinite rank, it is enough to prove that K, and thus also H, have infinite Betti number, for then, E_1 is infinite. We suppose that the Betti number of K is finite, and derive a contradiction.

Suppose that K_0, obtained from K by deleting a finite number of edges, is simply connected. Since K is infinite, K_0 also is infinite, and therefore has a connected component K_1 that is infinite and simply connected, that is, an infinite tree. Moreover, since K was cubic, K_1 will be cuboid, and indeed with only finitely many vertices of degree less than 3. If v_0 is any vertex of K_1, we may choose a vertex v_1 farther from v_0 than any vertex of degree less than 3. Then there will be an infinite branch K_2 at v_1, all of whose vertices except v_1 have degree 3—that is, K_2 is a "binary tree," as shown in Fig. (3.11). But K_2 also occurs as a branch in K. Since K_2 is not simply infinite, this contradicts the fact that K has an Eulerian path. □

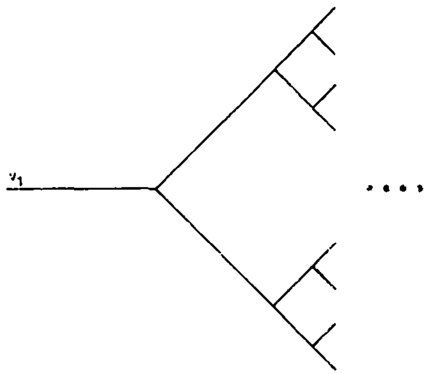


FIGURE 3.11

(3.12) PROPOSITION. *If H is obtained by attaching one infinite tree T^0 and a finite set $T^1, \ldots, T^f$ of finite trees to a finite cubic graph K, then* (3.2) *holds.*

Proof. Let K', K'', v_0^0, $v_0^1, \ldots, v_0^f$, and e_0^0, $e_0^1, \ldots, e_0^f$ be as in the proof of (3.10), except that we now have $e_0^1, \ldots, e_0^f \in S^*$, and S is obtained from $S^* * \langle e_0^0 \rangle$ by adjoining the set E of generators with relations as before. Again let M be a maximal tree in K' and E_1 the set of edges of K' not in M. We emphasize that in this presentation for S, the generator e_0^0 appears in only one relation, that given by the vertex v_0^0. If e is any edge of M at a vertex v of M other than v_0^0 of degree 1, we may use the relation at v to eliminate e in terms of $\langle E_1, S^* \rangle$. Iterating this elimination on the finite tree M, we finally come to a point where the only remaining relation at v_0^0 is of the form $ee'e_0^0 = 1$, and where e and e' have already been expressed as elements of $\langle E_1, S^* \rangle$. We use this last relation to eliminate e_0^0. This gives a presentation $S = F * S^*$, where $F = \langle E_1 \rangle$ is a free group. Now $|E_1|$ is twice the rank of the free group F, while, by construction, $|E_1|$ is twice the Betti number of K', and hence twice the Betti number of K. It follows that the rank of F is the Betti number of K. □

4. COMPLEMENTARY REMARKS

An obvious generalization of Tretkoff's presentation of S, as given in (3.3), pertains to an arbitrary set of permutations of the set Ω. We have not pursued this generalization.

A second obvious generalization is based on the alternative form of

Tretkoff's presentation, as given in (3.4). We discuss it here because a number of the arguments used to establish (3.2) do go through in this more general situation. Although we have found no application for these more general results, we believe they help to put the discussion above in perspective.

(4.1) DEFINITION. Let H be any graph. A group S will be called a *Tretkoff group* of H if it has a presentation of the form

$$S = \langle E: \quad \{r_e = 1\}, \{r_v = 1\} \rangle,$$

where

(i) E is the set of edges of H;

(ii) for each edge e there is a relation $r_e = 1$, with $r_e = ee^{-1}$:

(iii) for each vertex v of finite degree d there is a relation $r_v = 1$, with $r_v = e_1 \cdots e_d$, where $e_1, \ldots, e_d$ are the edges at v in some order.

Evidently the study of such groups reduces to the case that H is connected and locally finite.

(4.2) THEOREM. *Let H be a connected and locally finite graph, and S a Tretkoff group of H. Then*

(4.2a) *If H is infinite, then S is a free group.*

(4.2b) *If H is finite, then, for some finite f, $g \geqslant 0$, one has a presentation for S of the form*

$$S = \langle x_1, \ldots, x_g, y_1, \ldots, y_g, z_1, \ldots, z_f: \quad [x_1, y_1] \cdots [x_g, y_g] = 1 \rangle.$$

(4.3) THEOREM. *Let H be a finite graph with an Eulerian path π, that is, a reduced closed path that traverses each edge of H exactly once in each direction. Let S be the Tretkoff graph of H in which, in the relations $r_v = 1$, the factors of $r_v = e_1 \cdots e_d$ appear in the (cyclic) order induced by τ (that is, for subscripts modulo d, e_{i+1} follows e_i^{-1} in π). Then S has a presentation*

$$S = \langle x_1, \ldots, x_g, y_1, \ldots, y_g: \quad [x_1, y_1] \cdots [x_g, y_g] = 1 \rangle,$$

where $2g$ is the Betti number of H.

(4.4) THEOREM. *Let H be a finite cuboid graph with an Eulerian path π, associated with a triple (Ω, A, B), where Ω is finite. Let S be the Tretkoff graph of H associated with (Ω, A, B) in the manner of* (3.5a,b), *and with the factors in the $r_v = e_1 \cdots e_d$ in the order induced by π. Then*

$$S = G_0 * G_2 * G_3,$$

where

(i) $G_0 = \langle x_1, ..., x_g, y_1, ..., y_g : [x_1, y_1] \cdots [x_g, y_g] = 1 \rangle$, *and* $2g$ *is the Betti number of* H;

(ii) G_2 *is the free product of* r_2 *groups of order* 2, r_2 *being the number of fixed points of* A;

(iii) G_3 *is the free product of* r_3 *groups of order* 3, r_3 *being the number of fixed points of* B.

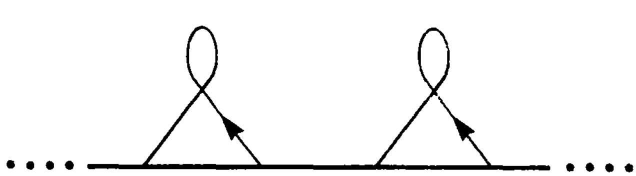
We do not give proofs of these results. In part they can be obtained by essentially the same methods as those that were used in the proof of (3.2). However, a substantial simplification can be obtained by appealing to the theory of quadratic systems of words as studied by Hoare *et al.* [2, 3]. (See also [5, p. 58].)

For example, we note that all the presentations of S given in (4.1), (4.2), (4.3) are quadratic, and, indeed, may be taken as alternating. Now (4.1) and (4.2) follow from the general theory of quadratic presentations.

The proof of (4.4) contains a new element, beyond the ideas used in the proof of (3.2), which appears clearly in the case that H is cubic, that is, that $r_2 = r_3 = 0$. Here one can observe that the cyclic order in which the edges e appear in the Eulerian path π is precisely the order in which these edges (in the role of "letters") appear as vertices in the cycles of the coinitial graph (or star graph). It follows that this graph is connected; that is, it is a single cycle. Thus the system of relations is minimal under automorphism (that is, Nielsen transformations), and from this it follows in turn that S has a presentation of the form (4.2b), in which all generators appear in the defining relator, that is, in which $f = 0$.

Postscript. (1) It has come to our attention that Stothers [13] (see also [14]) has clearly anticipated us in proving Theorem 3.2, by essentially the same method.

(2) We have recently obtained, by these same methods, maximal nonparabolic subgroups of the modular group that are not Neumann subgroups. The simplest of these is the free product of the 2 element groups generated by the $C^{n}BAB^{-1}C^{-n}$, for all integers n; the coset graph for this group is shown in the accompanying scheme.



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