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Asymptotic Estimates of Sums Involving the Moebius Function

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Let $p(n)$ denote the smallest prime factor of an integer $n > 1$ and let $p(1) = \infty$. We study the asymptotic behavior of the sum $M(x, y) = \sum_{1 \leqslant n \leqslant x, p(n) > y} \mu(n)$ and use this to estimate the size of $A(x) = \max_{|f| \leqslant 1} |\sum_{2 \leqslant n < x} \mu(n) f(p(n))|$, where $\mu(n)$ is the Moebius function. Applications of bounds for $A(x)$, $M(x, y)$ and similar quantities are discussed.

1. INTRODUCTION

For an integer $n > 1$ let $p(n)$ denote its least prime factor and put $p(1) = \infty$. Our object is to determine the size of the quantity

$$A(x) = \max_{f} \left| \sum_{2 \leqslant n \leqslant x} \mu(n) f(p(n)) \right|, \tag{1.1}$$

where $\mu(n)$ is the Moebius function and the maximum is over all functions f for which $|f(n)| \leqslant 1$ for all $n \geqslant 1$. Previously [1], I gave a crude bound for $A(x)$ and noted some consequences of such bounds. Clearly

$$\sum_{2 \leqslant n \leqslant x} \mu(n) f(p(n)) = \sum_{\substack{p \neq x \\ p = \text{prime}}} f(p) \sum_{\substack{2 \leqslant n \leqslant x \\ p(n) = p}} \mu(n),$$

and on writing $n = m \cdot p$, we see that this is

$$-\sum_{p \leqslant x} f(p) \sum_{\substack{1 \leqslant m \leqslant x/p \\ p(m) > p}} \mu(m). \tag{1.2}$$

Hence

$$A(x) = \sum_{p \leqslant x} \left| M\left(\frac{x}{p}, p\right) \right|, \tag{1.3}$$

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where

$$M(x, y) = \sum_{\substack{1 \leqslant m \leqslant x \\ p(m) > y}} \mu(m), \tag{1.4}$$

and we are led to consider the size of this latter sum.

Let

$$\Psi(x, y) = \sum_{\substack{1 \leqslant n \leqslant x \\ P(n) \leqslant y}} 1,$$

where $P(n)$ is the largest prime factor of n if $n > 1$ and $P(1) = 1$. It is well known [2] that for any fixed $\alpha > 0$

$$\Psi(x, x^{1/\alpha}) \sim x\rho(\alpha), \qquad x \to \infty,$$

where $\rho(\alpha)$ is defined by the relations

$$\begin{aligned} \rho(\alpha) &= 1 && \text{for} \quad 0 < \alpha \leqslant 1 \\ &= 1 - \int_1^\alpha \frac{\rho(t-1)\, dt}{t} && \text{for} \quad \alpha > 1. \end{aligned} \tag{1.5}$$

Similarly we derive

THEOREM 1. *If* $y \geqslant x$, *then* $M(x, y) = 1$. *If* $y = x^{1/\alpha}$, *then*

$$M(x, y) = \frac{x\rho'(\alpha)}{\log y} + \frac{y}{\log y} + O\left(\frac{x \cdot \alpha^2}{\log^2 y}\right)$$

uniformly for $2 \leqslant y < x$.

From the analysis of de Bruijn [2] we see that if $\alpha > 3$, then

$$\rho'(\alpha) = -\exp\{-\alpha \log \alpha - \alpha \log\log \alpha + O(\alpha)\}. \tag{1.6}$$

Thus the main terms of Theorem 1 are smaller than the error term when α is large. For large values of α we can obtain bounds for $M(x, y)$ that are superior to Theorem 1 by using the following results of de Bruijn [3]:

$$\begin{aligned} &\Psi(x, y) \ll x \exp\{-\alpha \log \alpha\} && \text{for} \quad y \geqslant \frac{\log^2 x}{16} \\ \text{and} \\ &\Psi(x, y) \ll x^{2/3} && \text{for} \quad y < \frac{\log^2 x}{16}. \end{aligned} \tag{1.7}$$

In fact we use de Bruijn's bounds in an elementary way to demonstrate

THEOREM 2. *Suppose that* $\alpha \geqslant 2$ *and* $y = x^{1/\alpha}$. *Then*

$$M(x,y) \ll x(\log\log x)^2 \exp\left\{-\frac{\alpha}{2}\log\alpha\right\} + \frac{x}{\log^2 x}$$

uniformly for $2 \leqslant y \leqslant \sqrt{x}$.

We use Theorems 1 and 2 to get our main result:

THEOREM 3. *Let* $A(x)$ *be defined by* (1.1). *Then*

$$A(x) \sim \frac{2x}{\log x}$$

as $x \to \infty$.

Finally, in Section 5 we discuss some applications of upper bounds for $A(x)$, $M(x,y)$ and similar quantities.

Note that if $y = 1$, then, $M(x,y)$ is the well-known sum $M(x) = \sum_{1 \leqslant n \leqslant x} \mu(n)$. Apart from this special case, the function $M(x,y)$ has not been studied in detail, though it has been implicit in the literature for some time. There is a paper of Levin and Fainleib [5] where various results are established for functions which generalize $\Psi(x,y)$. Some of their results do apply to $M(x,y)$. But there are some mathematical errors in [5] and so we preferred to study the function $M(x,y)$ independently. Moreover, many results and applications presented here are new.

Throughout we let $\alpha = \log x/\log y$, where x, $y > 1$. By c_1, c_2, c_3... we mean absolute positive constants. Implicit constants are absolute unless otherwise indicated.

2. PROOF OF THEOREM 1

From (1.5) it follows that $\rho'(\alpha)$ satisfies

$$\begin{aligned} \rho'(\alpha) &= -\frac{1}{\alpha} && \text{for} \quad 1 < \alpha \leqslant 2 \\ &= -\frac{1}{\alpha} - \frac{1}{\alpha}\int_2^{\alpha} \rho'(u-1)\,du && \text{for} \quad \alpha > 2. \end{aligned} \tag{2.1}$$

Let $\sqrt{x} < y < x$. Then from (1.4) and the quantitative form of the Prime Number Theorem (see Prachar [6, p. 61]) we get

$$M(x,y) = 1 - \sum_{y < p \leqslant x} 1 = \frac{-x}{\log x} + \frac{y}{\log y} + O\left(\frac{x}{\log^2 y}\right). \tag{2.2}$$

From (2.1) and (2.2) we see that Theorem 1 is true when $1 < \alpha \leqslant 2$.

Next, let $\alpha > 2$. So $y < \sqrt{x}$. We note that

$$M(x,y) = M(x, \sqrt{x}) + \sum_{\substack{1 \leqslant n \leqslant x \\ y < p(n) \leqslant \sqrt{x}}} \mu(n)$$

and on letting $n = m \cdot p$ we see that

$$M(x,y) = -\frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right) - \sum_{y < p \leqslant \sqrt{x}} M\left(\frac{x}{p}, p\right). \tag{2.3}$$

By Stieltjes integration and the quantitative form of the Prime Number Theorem we have

$$\sum_{y < p \leqslant \sqrt{x}} M\left(\frac{x}{p}, p\right) - \int_y^{\sqrt{x}} M\left(\frac{x}{t}, t\right) \frac{dt}{\log t} \ll \frac{x\alpha}{\log^2 y}. \tag{2.4}$$

We will prove Theorem 1 from (2.3) by induction on $[\alpha]$, the largest integer $\leqslant \alpha$.

For $x > t > 1$ let $u = \log x / \log t$. For $\alpha > 2$ we assume that there exists $\phi(u)$, a positive increasing function of u, such that for all $u \leqslant \alpha - 1$ and $x > t > 1$ we have

$$\left| M(x, t) - \frac{x\rho'(u)}{\log t} - \frac{t}{\log t} \right| < \frac{x\phi(u)}{\log^2 t}. \tag{2.5}$$

For instance (2.2) shows that (2.5) is valid for $1 < u \leqslant 2$. Therefore

$$\left| \int_y^{\sqrt{x}} M\left(\frac{x}{t}, t\right) \frac{dt}{\log t} - \int_y^{\sqrt{x}} \frac{x}{t \log^2 t} \rho'\left(\frac{\log x - \log t}{\log t}\right) dt - \int_y^{\sqrt{x}} \frac{t\,dt}{\log^2 t} \right|$$
$$< \int_y^{\sqrt{x}} \frac{x\phi(\alpha - 1)\,dt}{t \log^3 t}. \tag{2.6}$$

By change of variables we see that the second integral on the left of (2.6) is

$$\frac{-x}{\log x} \int_2^{\alpha} \rho'(u-1)\,du$$

since $t = x^{1/u}$. So from (2.3), (2.4), (2.5) and (2.6) we get

$$\left| M(x, y) - \frac{x}{\log y} \left\{ -\frac{1}{\alpha} - \frac{1}{\alpha} \int_2^{\alpha} \rho'(u-1)\,du \right\} \right|$$
$$< \frac{c_1 \alpha x}{\log^2 y} + \frac{x\phi(\alpha - 1)}{\log^2 y}. \tag{2.7}$$

We will choose ϕ to satisfy (2.5) for $1 < u \leqslant 2$ and the inequality

$$\phi(\alpha - 1) + c_1 \alpha < \phi(\alpha) \qquad \text{for} \quad \alpha > 2.$$

Clearly $\phi(\alpha) = c_2 \cdot \alpha^2$ satisfies these conditions. Therefore by (2.1) and (2.7) we have

$$M(x, y) = \frac{x\rho'(\alpha)}{\log y} + O\left(\frac{x \cdot \alpha^2}{\log^2 y}\right) \tag{2.8}$$

holds uniformly for $2 \leqslant y < \sqrt{x}$. Since $\alpha > 2$, the term $y/\log y$ can be added to (2.8) without effect. Theorem 1 for the case $2 \leqslant y < x$ follows from (2.8) and (2.2). The case $y \geqslant x$ is trivial. Theorem 1 is proved.

From (1.6) it follows that Theorem 1 fails to yield an asymptotic estimate for $M(x, y)$ if $\alpha \to \infty$ with x. We now discuss briefly a method which enables us to estimate $M(x, y)$ asymptotically, even for large α, provided y is also large. This method was initiated by de Bruijn [3] while considering the problem of obtaining an asymptotic estimate of $\Psi(x, y)$ for long ranges of α. We note that $M(x, y)$ satisfies the recurrence

$$M(x, y) = M(x, y^h) - \sum_{y < p \leqslant y^h} M\left(\frac{x}{p}, p\right) \tag{2.9}$$

when $1 < y < y^h < x$. We want a continuous function $A(x, y)$ that satisfies a recurrence similar to (2.9) and is close to $M(x, y)$ when $\sqrt{x} \leqslant y < x$.

If we let

$$A(x, y) = \frac{x}{\log y} \int_1^{x/y} \rho'\left(\frac{\log x - \log t}{\log y}\right) \frac{dt}{t^2}, \tag{2.10}$$

we see from (2.1) that

$$A(x, y) = -\int_y^x \frac{dt}{\log t} \tag{2.11}$$

when $\sqrt{x} \leqslant y < x$. From (2.2) and (2.11) we deduce that the difference $M(x, y) - A(x, y)$ is relatively small when $1 < \alpha \leqslant 2$. Also from (2.10), (2.1) and (1.6) we deduce by change of variables that

$$A(x, y) = A(x, y^h) - \int_y^{y^h} A\left(\frac{x}{t}, t\right) \frac{dt}{\log t} + O\left(\frac{y^h}{\log y}\right) \tag{2.12}$$

when $1 < y < y^h < x$. Therefore $A(x, y)$ satisfies our requirements.

From (2.2), (2.11), (2.9) and (2.12) we expect that the difference $M(x, y) - A(x, y)$ can be bounded in terms of α and a monotonic decreasing function $R(y)$ that bounds the relative error in the Prime Number Theorem. More precisely, for $y \geqslant 2$ we want $R(y)$ to satisfy the inequality

$$|\pi(y) - \ell i(y)| < \frac{y}{\log y} \cdot R(y),$$

where

$$\ell i(y) = \int_2^y \frac{dt}{\log t}.$$

Then from (2.2), (2.11), (2.9) and (2.12) we can show by induction on $[\alpha]$ that

$$M(x, y) - A(x, y) \ll x\alpha^2 R(y) \tag{2.13}$$

holds uniformly for $2 \leqslant y \leqslant x$.

In most applications of (2.13) it suffices to choose $R(y)$ to be

$$\ll \exp\{-c_3 \sqrt{\log y}\}. \tag{2.14}$$

If we integrate (2.10) by parts and use a result of de Bruijn [3] that

$$\rho''(\alpha) \ll \rho'(\alpha) \log(\alpha + 2),$$

we get

$$A(x, y) = \frac{x\rho'(\alpha)}{\log y} + \frac{y}{\log y} + O\left(\frac{x\rho'(\alpha)\log(\alpha + 2)}{\log^2 y}\right). \tag{2.15}$$

It follows from (2.13), (2.14) and (2.15) that as $x \to \infty$

$$M(x, y) \sim \frac{x\rho'(\alpha)}{\log y} \qquad \text{if } \exp\{(\log x)^{2/3+\epsilon}\} < y = o(x). \tag{2.16}$$

This is an improvement of Theorem 1 for large α and y.

In fact repeated integration by parts of (2.10) shows that

$$A(x, y) = \frac{x\rho'(\alpha)}{\log y} - \frac{x\rho''(\alpha)}{\log^2 y} + \frac{x\rho'''(\alpha)}{\log^3 y} \cdots + (-1)^{n-1} \frac{x\rho^{(n)}(\alpha)}{\log^n y}$$
$$+ O_n\left(\frac{x\rho(\alpha)\log^n(\alpha + 2)}{\log^{n+1} y}\right), \tag{2.17}$$

where $n \leqslant [\alpha] - 1$ and $\rho^{(n)}(\alpha)$ is the nth derivative of $\rho(\alpha)$. From (2.13) and (2.17) we get a series expansion for $M(x, y)$ of any desired length for large y, provided $\alpha \to \infty$ with x.

3. Proof of Theorem 2

We note that for $\sigma > 1$

$$\sum_{\substack{n=1 \\ p(n) > y}}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)} \cdot \prod_{p \leqslant y} \left(1 - \frac{1}{p^s}\right)^{-1},$$

where $\zeta(s)$ is the Riemann zeta function. Thus the coefficients of the series on the left is a Dirichlet convolution of the coefficients of the Dirichlet series on the right. Therefore

$$M(x, y) = \sum_{\substack{md \leqslant x \\ P(m) \leqslant y}} \mu(d). \tag{3.1}$$

Let t be a real number satisfying $1 \leqslant t \leqslant \sqrt{x}$. We break up (3.1) as follows:

$$M(x, y) = \sum_{d \leqslant t} \mu(d) \sum_{\substack{m \leqslant x/d \\ P(m) \leqslant y}} 1 + \sum_{\substack{m \leqslant x/t \\ P(m) \leqslant y}} \sum_{t \leqslant d \leqslant x/m} \mu(d). \tag{3.2}$$

The upper bounds (1.7) show that the first term on the right of (3.2) is

$$\ll \sum_{d \leqslant t} \Psi\left(\frac{x}{d}, y\right) \ll \frac{x}{\log^2 x} + x \log t \exp\left\{-\frac{\log(x/t)}{\log y} \log\left(\frac{\log(x/t)}{\log y}\right)\right\}. \tag{3.3}$$

From the quantitative form of the Prime Number Theorem for $M(x)$ (see Prachar [6, p. 71]) we see that the last term on the right of (3.2) is

$$\ll \sum_{m \leqslant x/t} \frac{x}{m} \exp\left\{-c_4 \sqrt{\log\left(\frac{x}{m}\right)}\right\} \ll x \log x \exp\{-c_4 \sqrt{\log t}\}. \tag{3.4}$$

Theorem 2 follows from (3.2), (3.3) and (3.4) on choosing

$$t = \exp\left\{\frac{9}{c_4^2} (\log \log x)^2\right\}.$$

For certain purposes $t = \sqrt{x}$ proves useful. In this case we can prove that

$$M(x, y) \ll x \log x \exp\left\{-\frac{\alpha}{2} \log\left(\frac{\alpha}{3}\right)\right\} + x \exp\{-c_5 \sqrt{\log x}\}. \tag{3.5}$$

We can prove a slightly sharper form of (3.5) for large α and y by using

$$M(x, y) = \frac{1}{2\pi i} \int_{a-iT}^{a+iT} \frac{x^s \prod_{p \leqslant y}(1 - 1/p^s)^{-1}}{s\zeta(s)}\, ds + O\left(\sum_{n=1}^{\infty} \frac{x^a}{Tn^a |\log(x/n)|} \right),$$

where $a > 1$, $T > 3$ and x is half an odd integer. We estimate the integral by contour integration, using well-known results on zero free regions of $\zeta(s)$ and bounds for $\zeta(s)$ in such regions. For instance, when $x^{1/3} > y > \exp\{c_6 \sqrt{\log x}$ $\log\log x\}$, this method yields

$$M(x, y) \ll x \log^k x \exp\{-\alpha \log \alpha - \alpha \log\log \alpha + O(\alpha)\},$$

where k is an absolute positive constant.

4. Proof of Theorem 3

We begin with a

Lemma. *There exist a constant $x_0 > 0$ such that if $x > x_0$ and $x^{1/\log\log x} < y < x(1 - 1/\log^2 x)$, then $M(x, y) \leqslant 0$.*

Proof. From the quantitative form of the Prime Number Theorem it follows that for all large x, there is a prime in the interval $(x\{1 - 1/\log^2 x\}, x)$. So if x is large enough and $\sqrt{x} < y < x\{1 - 1/\log^2 x\}$, then

$$M(x, y) = 1 - \pi(x) + \pi(y) \leqslant 0. \tag{4.1}$$

Next from Theorem 1 and (1.6) we deduce that if x is large and $x^{1/\sqrt{\log\log x}} < y < \sqrt{x}$, then

$$M(x, y) \leqslant \frac{x\rho'(\alpha)}{2 \log y} < 0. \tag{4.2}$$

The lemma follows from (4.1) and (4.2).

Proof of Theorem 3. We begin by splitting the sum in (1.3) into

$$\sum_{p < \sqrt{x}} + \sum_{\sqrt{x} \leqslant p \leqslant x} = \Sigma_1 + \Sigma_2. \tag{4.3}$$

The Prime Number Theorem implies that

$$\Sigma_2 = \sum_{\sqrt{x} \leqslant p \leqslant x} \sim \frac{x}{\log x}. \tag{4.4}$$

To estimate Σ_1 we note that there exists $x_1 > 0$ such that if $x > x_1$ and $x^{1/\sqrt{\log\log x}} < p < \sqrt{x}\{1 - 3/\log^2 x\}$, then

$$\frac{x}{p} > x_0$$

and

$$\left(\frac{x}{p}\right)^{1/\sqrt{\log\log(x/p)}} < p < \frac{x}{p}\left(1 - \frac{1}{\log^2(x/p)}\right).$$

Next we write

$$\Sigma_1 = \sum_{p \leqslant z_0} + \sum_{z_0 < p \leqslant z_1} + \sum_{z_1 < p \leqslant z_2} + \sum_{z_2 < p \leqslant \sqrt{x}} = \Sigma_3 + \Sigma_4 + \Sigma_5 + \Sigma_6, \quad (4.5)$$

where

$$z_0 = x^{1/\log\log x}, \qquad z_1 = x^{1/\sqrt{\log\log x}}$$

and

$$z_2 = \sqrt{x}\left(1 - \frac{3}{\log^2 x}\right).$$

First by Theorem 2 we get

$$\Sigma_3 \ll \frac{x \log\log x}{\log^2 x}. \quad (4.6)$$

To estimate Σ_4 we use Theorem 1, (1.6) and elementary results on prime numbers. That is, we have

$$\begin{aligned}\Sigma_4 &\ll \sum_{z_0 < p \leqslant z_1} \left\{ \frac{x}{p \log p} \rho' \left(\frac{\log x}{\log p} - 1\right) + \frac{x(\log\log x)^2}{p \log^2 p} \right\} \\ &\ll \frac{x \exp\{-\sqrt{\log\log x}\}}{\log x} \Bigg\} + \frac{x(\log\log x)^4}{\log^2 x} = o\left(\frac{x}{\log x}\right). \end{aligned} \quad (4.7)$$

On the other hand it is clear that

$$\Sigma_6 \ll \sum_{z_2 < p \leqslant \sqrt{x}} \frac{x}{p} \ll \sqrt{x}(\sqrt{x} - z_2) \ll \frac{x}{\log^2 x}. \quad (4.8)$$

Finally the lemma shows that

$$\Sigma_5 = \sum_{z_1 < p \leqslant z_2} \left| M\left(\frac{x}{p}, p\right) \right| = - \sum_{z_1 < p \leqslant z_2} M\left(\frac{x}{p}, p\right) = -\Sigma_5'. \quad (4.9)$$

So we combine (4.3)–(4.9) and arrive at

$$A(x)=\Sigma_1+\Sigma_2=\frac{x}{\log x}-\Sigma_5'+o\left(\frac{x}{\log x}\right). \qquad (4.10)$$

Now consider

$$M(x)-1=\sum_{2\leqslant n\leqslant x}\mu(n).$$

If we take $f\equiv 1$ in (1.2), then we get from (1.4)

$$M(x)-1=-\sum_{2\leqslant p\leqslant x}M\left(\frac{x}{p},p\right). \qquad (4.11)$$

From the quantitative form of the Prime Number Theorem for $M(x)$ we get

$$M(x)=o\left(\frac{x}{\log x}\right). \qquad (4.12)$$

Therefore if we split the sum in (4.11) as in (4.3) and (4.5), then the above method yields

$$M(x)-1=-\Sigma_2-\Sigma_5'+o\left(\frac{x}{\log x}\right)=-\frac{x}{\log x}-\Sigma_5'+o\left(\frac{x}{\log x}\right). \qquad (4.13)$$

Thus from (4.13) and (4.12) we deduce that

$$-\Sigma_5'=\frac{x}{\log x}+o\left(\frac{x}{\log x}\right). \qquad (4.14)$$

Theorem 3 now follows from (4.14) and (4.10).

With greater care we can improve Theorem 3 by using (2.16) and (3.5). First we show that $M(x,y)\leqslant 0$ for $x\geqslant x_0(\varepsilon)$ and $\exp\{(\log x)^{2/3+\epsilon}\}<y<x(1-\exp\{-c_7\sqrt{\log x}\})$. This is an improvement of the lemma. Then by choosing z_0, z_1 and z_2 properly we can use the above method and prove

$$A(x)=2\ell i(x)+O_\epsilon(x\cdot\exp\{-(\log x)^{1/3-\epsilon}\}),$$

where $\varepsilon>0$ is arbitrary.

5. Applications

We first consider an application to the Sieve.

Let $\lambda(n)$ be Liouville's function. The asymptotic results stated so far for $M(x, y)$ also hold for the function

$$L(x, y) = \sum_{1 \leqslant n \leqslant x, p(n) \geqslant y} \lambda(n).$$

The proofs are almost identical with $\lambda(n)$ replacing $\mu(n)$.

Let $1 < y < x$ and

$$\mathscr{S}(x, y) = \{n \mid 1 \leqslant n \leqslant x, p(n) \geqslant y\}.$$

We now consider the distribution of $\Omega(n) = \sum_{p^v \| n} v$ modulo 2 for $n \in S(x, y)$. So we define

$$\mathscr{S}_i(x, y) = \{n \in \mathscr{S}(x, y) \mid \Omega(n) \equiv i \pmod 2\}, \qquad i = 0, 1.$$

It is known by the use of sieve methods (see Halberstam and Richert [4, pp. 225–240]) that

$$|\mathscr{S}_i(x, y)| = \frac{|\mathscr{S}(x, y)|}{2} \{1 + O(\exp\{-\alpha \log \alpha\}) + O((\log x)^{-1/14})\} \quad (5.1)$$

for $i = 0, 1$.

We note that

$$L(x, y) = |\mathscr{S}_0(x, y)| - |\mathscr{S}_1(x, y)|. \quad (5.2)$$

From Theorem 1, (3.5) (now for $L(x, y)$) and (1.6) it follows that

$$L(x, y) \ll \frac{x}{\log y} \left(\exp\left\{-\frac{\alpha}{2} \log \frac{\alpha}{4}\right\} + \exp\{-c_8 \sqrt{\log x}\}\right). \quad (5.3)$$

On the other hand, if $y > \exp\{(\log x)^{2/3+\epsilon}\}$, then by (2.16), Theorem 1 (now for $L(x, y)$) and (1.6) we have

$$L(x, y) \ll_\epsilon \frac{x}{\log y} \exp\{-\alpha \log \alpha\}. \quad (5.4)$$

From the Prime Number Theorem and the Linear Sieve (see Halberstam and Richert [4, pp. 225–240]) it follows that

$$\frac{x}{\log y} \ll |\mathscr{S}(x, y)| \ll \frac{x}{\log y} \qquad \text{for} \quad y \leqslant \sqrt{x}. \quad (5.5)$$

Therefore from (5.2), (5.3), (5.4) and (5.5) we have the following improvement of (5.1):

THEOREM 4. *If $x > y > 1$, then for $i = 0, 1$ we have*

$$|\mathcal{S}_i(x,y)| = \frac{|\mathcal{S}(x,y)|}{2}\left\{1 + O\left(\exp\left\{-\frac{\alpha}{2}\log\frac{\alpha}{4}\right\}\right) + O(\exp\{-c_8\sqrt{\log x}\})\right\}. \tag{5.6}$$

If $\varepsilon > 0$ is arbitrary, then for $i = 0, 1$ we have

$$|\mathcal{S}_i(x,y)| = \frac{|\mathcal{S}(x,y)|}{2}\{1 + O(\exp\{-\alpha\log\alpha\}) + O_\epsilon(\exp\{-(\log x)^{1/3-\epsilon}\})\}. \tag{5.7}$$

Theorem 4 or the weaker (5.1) shows that $\Omega(n)$ is uniformly distributed modulo 2 (u.d. mod 2) in $\mathcal{S}(x,y)$, provided $\alpha \to \infty$ with x. The function $\Omega(n)$ does not have this property if $\alpha \not\to \infty$ with x. A necessary and sufficient condition for $\Omega(n)$ to be u.d. mod 2 in $\mathcal{S}(x,y)$ is that $L(x,y) = o(|\mathcal{S}(x,y)|)$, and this clearly does not hold when $1 \leqslant \alpha \leqslant 2$. If $\alpha > 2$ remains bounded as $x \to \infty$, then by Theorem 1 (for $L(x,y)$) and (5.5) we have $L(x,y) = o(|\mathcal{S}(x,y)|)$ if and only if $\rho'(\alpha) = 0$. But this never happens because $\rho'(\alpha) < 0$ for all $\alpha > 1$.

The function $M(x,y)$ is the basic computational tool to evaluate sums of the form

$$M_f(x) = \sum_{2\leqslant n\leqslant x} \mu(n) f(p(n))$$

because we can use (1.2) and (1.4) to rewrite this as

$$M_f(x) = -\sum_{p\leqslant x} f(p) M\left(\frac{x}{p}, p\right). \tag{5.8}$$

When $f(x)$ is a simple differentiable function it is not usually difficult to estimate $M_f(x)$. The contribution due to "large p" in (5.8) can be estimated with the aid of (2.16) and well-known results on primes, while the contribution due to small p can be bounded by using (3.5). For example, when $f(n) = \log n$ we can show that

$$\sum_{2\leqslant n\leqslant x} \mu(n)\log p(n) = C\cdot x + O\left(\frac{x}{\log x}\right),$$

where

$$C = \int_1^{\infty} \frac{\rho'(s)\, ds}{s}.$$

Previously [1], I observed that non-trivial upper bounds for $A(x)$ have some interesting consequences. I used a result weaker than Theorem 3 and showed for that if f is a bounded function, then

$$\lim_{x \to \infty} \frac{1}{x} \sum_{n \leqslant x} f(P(n)) = l$$

exists if and only if

$$\sum_{n=2}^{\infty} \frac{\mu(n)\, f(p(n))}{n} = -l.$$

This equivalence has an interesting application to the Prime Number Theorem for Arithmetic Progressions.

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