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Robustness of Systems with Uncertainties in the Input*

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In B. R. Barmish (*IEEE Trans. Automat. Control* **AC-22**, No. 7 (1977), 123, 124; **AC-24**, No. 6 (1979), 921–926) and B. R. Barmish and Y. H. Lin ("Proceedings of the 7th IFAC World Congress, Helsinki, 1978") a new notion of "robustness" was defined for a class of dynamical systems having uncertainty in the input–output relationship. This paper generalizes the results in the above-mentioned references in two fundamental ways: (i) We make significantly less restrictive hypotheses about the manner in which the uncertain parameters enter the system model. Unlike the multiplicative structure assumed in previous work, we study a far more general class of nonlinear integral flows, (ii) We remove the restriction that the admissible input set be compact. The appropriate notion to investigate in this framework is seen to be that of *approximate robustness*. Roughly speaking, an approximately robust system is one for which the output can be *guaranteed* to lie "ε-close" to a prespecified set at some future time $T > 0$. This guarantee must hold for *all* admissible (possibly time-varying) variations in the values of the uncertain parameters. The principal result of this paper is a necessary and sufficient condition for approximate robustness. To "test" this condition, one must solve a *finite-dimensional* optimization problem over a compact domain, the unit simplex. Such a result is tantamount to a major reduction in the complexity of the problem; i.e., the original robustness problem which is infinite-dimensional admits a finite-dimensional parameterization. It is also shown how this theory specializes to the existing theory of Barmish and Barmish and Lin under the imposition of additional assumptions. A number of illustrative examples and special cases are presented. A detailed computer implementation of the theory is also discussed.

1. INTRODUCTION

In [1–3], a new notion of *robustness* was defined for a linear dynamical system whose impulse response matrix is uncertain. Loosely speaking, a linear system, according to this notion, is said to be *robust* if an (admissible)

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input exists which *guarantees* that the resulting output will lie in a desired target set at a specified time instant. The "guarantee" above must hold for all admissible variations in the uncertain impulse response matrix.

The first objective of this paper is to generalize the existing robustness theory. In lieu of the linear system functions of [1–3], our new framework can accommodate nonlinear coupling between the uncertainties and the input. Although the model is described in terms of an integral flow, our new results can nevertheless be interpreted within the context of target set reachability problems for perturbed state equation type models (see, for example, [4–7]).

The second equally important objective is to develop a more general robustness framework which enables one to handle a wider class of admissible inputs and uncertainties. To meet this end, we introduce the notion of *approximate robustness*: For cases when the inputs and/or uncertainty restraint sets are not necessarily closed and/or bounded, we seek to *guarantee* that the output is arbitrarily close to the target. In this regard, we might envision a system whose robustness properties improve as we permit larger and larger inputs. Despite the fact that infinite (impulsive) inputs are inadmissible, we might still hope to achieve *approximate robustness* using a bounded input. Our plan for the sequel is as follows:

The second section contains the basic assumptions about the dynamical system, the solution and uncertainty restraint sets and the target set. Also given is some preliminary notation and definitions. The third section provides the formal definitions of robustness and approximate robustness. In the fourth section, we present some preliminary lemmas which enable us to reformulate the robustness problem as a function space minimax problem. Our main results are then given in Sections 5 and 6. Namely, it is possible to transform the robustness problem into an equivalent problem involving the finite-dimensional unit simplex; i.e., the robustness problem admits a reduction in complexity. One need only search over a finite-dimensional Euclidean space in lieu of the underlying infinite-dimensional function space. This result clearly facilitates computations required to decide on the (approximate) robustness of a given system. In Section 7, the notions of approximate robustness and robustness are shown to be equivalent under the additional assumption of *compact* input restraint sets. In Section 8, the construction of a test function for approximate robustness is illustrated by means of examples. Also in Section 8, the above theory is illustrated using a model of aircraft flight dynamics. In Section 9, the paper is concluded with some remarks. The proofs of most results are given in Appendixes A–D.

2. DYNAMICS (S_p)

The dynamical system under consideration will be defined by an integral flow. Roughly speaking, if $x(\cdot)$ is the input (independent variable), then the output $y(\cdot)$ (dependent variable), at some future time $T < \infty$ will be given by

$$y(T) = \int_0^T A(x(\tau), q(\tau), \tau)\, d\tau, \tag{2.1}$$

where $A(\cdot)$ is the so-called system-function and $q(\cdot)$ is a time-varying vector of uncertain parameters. The imprecision in our knowledge about the behavior of $q(\cdot)$ serves as motivation for much of the formalism to follow. Throughout this paper, we shall take $T = 1$ without loss of generality.

More precisely, we describe the so-called *perturbed dynamical system* via the following data:

(D1) A non-empty σ-compact[1] set $X \subseteq R^m$, the *solution input restraint set*. The σ-compactness of X permits us to handle control sets which are rather general in nature (such as cones, balls, open sets, even all of R^m, etc.).

(D2) A non-empty bounded set $Q \subseteq R^k$, the *uncertainty restraint set.*

(D3) A mapping $A: R^m \times R^k \times [0, 1] \to R^n$, *the system function* which generates the system output $y(T)$ via (2.1).

2.1. *Notation.* If V is a subset of R^l, then $\bar{V}$ will denote the closure of V and $M(V)$ will denote the collection of essentially bounded Lebesgue measurable functions on $[0, 1]$ taking values almost everywhere (a.e.) in V. Furthermore, if Θ is a Lebesgue measurable subset of R^p, then a mapping $h: \Theta \times [0, 1] \to R$ is said to be a *Carathéodory function* if

(i) $h(\cdot, \tau)$ is continuous for almost all $\tau \in [0, 1]$;

(ii) $h(\theta, \cdot)$ is Lebesgue measurable for all $\theta \in \Theta$.

2.2. *Regularity.* In terms of this notation, our results, to be described in the sequel, will be valid under the following *regularity assumptions on A*:

(A1) For each fixed pair $x(\cdot) \in M(\bar{X}), q(\cdot) \in M(\bar{Q})$, each component $A_i(x(t), q(t), t)$, is Lebesgue measurable and integrable over $[0, 1]$.

(A2) For each fixed pair $x \in \bar{X}$ and $t \in [0, 1]$, $A(x, q, t)$ is continuous with respect to $q \in \bar{Q}$.

(A3) For each fixed pair $q \in \bar{Q}$ and $t \in [0, 1], A(x, q, t)$ is continuous with respect to $x \in \bar{X}$.

Regularity conditions (A1)–(A3) lead us to conclude that $A(x, \cdot, \cdot)$ and $A(\cdot, q, \cdot)$ are Carathéodory functions for each $x \in \bar{X}$ and $q \in \bar{Q}$, respectively.

[1] Countable union of compact sets. In particular, this permits X itself to be compact.

2.3. *Complete Description of* (S_p). To complete our description of (S_p) we also take as given the following:

(D4) A set of vectors $\{a_1, a_2,..., a_r\} \subseteq R^n$ and a set of real scalars $\{b_1, b_2,..., b_r\}$ which are used to describe the *desired solution* (*target*) *set*

$$B \triangleq \{y \in R^n : a_i' y \leqslant b_i \quad \text{for} \quad i = 1, 2,..., r\}.$$

We assume that B is non-empty. Furthermore, we note that *any* closed and convex target set in R^n can be approximated arbitrarily well by appropriate choice of a_i and b_i above.

Hence, the *perturbed dynamical system* (S_p) is defined by the 4-tuple (X, Q, A, B) together with Assumptions (D1)–(D3) and Regularity Conditions (A1)–(A3).

3. Robustness and Approximate Robustness

3.1. Definition. The m-dimensional real-valued vector function $x(\cdot)$ will be called a *robust solution* (*input*) *of* (S_p) if

(RS1) $x(\cdot) \in M(X)$, i.e., $x(\cdot)$ is admissible;

(RS2) $\int_0^1 A(x(\tau), q(\tau), \tau)\, d\tau \in B$ for *all* admissible perturbations $q(\cdot) \in M(Q)$. (S_p) is said to be *robust* if and only if it has a robust solution.

3.2. Definition. Let $\varepsilon > 0$ be a given *positive* real number. Then the m-dimensional real-valued vector function $x_\epsilon(\cdot)$ will be called an *ε-robust solution* of (S_p) if

(RS1′) $x_\epsilon(\cdot) \in M(X)$, i.e., $x_\epsilon(\cdot)$ is admissible;

(RS2′) $d(\int_0^1 A(x_\epsilon(\tau), q(\tau), \tau)\, d\tau, B) \leqslant \varepsilon$ for *all* admissible perturbations $q(\cdot) \varepsilon M(Q)$, where $d(y, B)$ is the distance[2] between the point $y \in R^n$ and the desired solution set B. In such a case, we say that (S_p) is *ε-robust*. Furthermore, (S_p) is said to be *approximately robust* if it has an ε-robust solution for *every* $\varepsilon > 0$.

3.3. *Remarks*. The fact that a system can be approximately robust without being robust is illustrated by the simple example described by

[2] Without loss of generality, we shall henceforth take this distance to be as follows: $d(y, B) = \max\{(a_i' y - b_i) \vee 0 : i \in \{1, 2,..., r\}\}$, where a_i, b_i, $i = 1, 2,..., r$ are the defining quantities of B as in D4 and $\vee$ denotes the maximum.

$X=(-\infty,\infty)$, $Q=[0,1]$, $B=[-1,0]$ and $A(x,q,t)=q/(1+x^2)$. Clearly, an essentially bounded $x(\cdot)\in M(X)$ *cannot* guarantee that

$$\int_0^1 \frac{q(\tau)}{1+x^2(\tau)}\,d\tau \in [-1,0]$$

for all $q(\cdot)\in M(Q)$. Nevertheless, by making $x(\cdot)$ sufficiently large, we *can* guarantee that

$$\int_0^1 \frac{q(\tau)}{1+x^2(\tau)}\,d\tau \in [-1,\varepsilon]$$

for all $q(\cdot)\in M(Q)$.

Also note that the existence of an ε–robust solution for a *particular* choice of $\varepsilon>0$ does not necessarily imply approximate robustness.

In [2], it was noted that (S_p) can be viewed as a dynamical system whose impulse response matrix is uncertain. With this interpretation, robustness of (S_p) refers to the ability to steer the system to the target set B at time $T=1$.

3.4. *Special Cases.*

Some important special cases of the approximate robustness problem are given below.

Special Case 1. When the sets Q and B consist of singletons $\{q_0\}$ and $\{b_0\}$, respectively, we are asking whether or not one can solve the nonlinear integral equation

$$\int_0^1 A(x(\tau),q_0,\tau)\,d\tau=b_0,$$

to *any* desired degree of accuracy, (or exactly, if possible), by choice of $x(\cdot)\in M(X)$.

Special Case 2. When the system function A is a linear form in $x(\cdot)$, viz.,

$$A(x(\tau),q(\tau),\tau)=\tilde{A}(q(\tau),\tau))\,x(\tau),$$

our new results will generalize those given in [1] to the extent that we now permit nonlinear interdependencies among the uncertainties in the entries of the $n\times m$ matrix $\tilde{A}(q(\tau),\tau)$. For example, the (i,j)th term of $\tilde{A}(\cdot)$ might be $q_1^2(\tau)\cos q_2(\tau)\,q_3(\tau)$.

Special Case 3. We can further strengthen the hypotheses given in Special Case 2; i.e., suppose that $\tilde{A}(\cdot)$ is a matrix of the form

$$\tilde{A}(q(\tau),\tau)=\tilde{A}_0(\tau)+\delta\tilde{A}(q(\tau)),$$

where the "uncertain matrix" $\delta\tilde{A}(\cdot)$ has nm independently varying entries $q_{11}(\tau)$, $q_{12}(\tau),..., q_{nm}(\tau)$. In this special case, our new results will be comparable to those given in [1].

Special Case 4. Our framework also encompasses the so-called unconstrained robustness problems. Such problems characterized by $X = R^m$ evidently satisfy the σ-compact requirement (D1).

3.4. *Further remarks.* In most engineering applications, it is reasonable to impose further structure over and above σ-compactness on the restraint sets X and Q. For example, polyhedral convexity may often be safely assumed. As we might anticipate, additional assumptions of this sort will be quite helpful from the point of view of numerical computation. We note, however, that such additional assumptions are not required in the derivation of the main results.

4. Preliminary Result—a Reformulation

In this section, our main objective is to show that the robustness problem can be reformulated as a saddle point problem over $M(X) \times S$, where S is the unit simplex in R^r; i.e., let

$$S \triangleq \left\{ z = (z_1, z_2,..., z_r) \in R^r : z_i \geqslant 0, \sum_{i=1}^{r} z_i = 1 \right\}.$$

4.1. *Remarks.* The set $M(X)$ appears to be the natural domain for Problem (P). Then why the introduction of the unit simplex S? It will be seen in the sequel that the introduction of S is the "secret" to reducing the complexity of the (approximate) robustness problem; i.e., when all is said and done, we shall remain with a problem over the set S in lieu of the original robustness problem which is formulated over the set $M(X)$.

4.2. *Preliminary Results*

We begin naively by considering some fixed input $x(\cdot) \in M(X)$ and a fixed $\varepsilon > 0$. Then according to Definition 3.2, $x(\cdot)$ is an ε-robust solution of (S_p) if and only if

$$d\left(\int_0^1 A(x(\tau), q(\tau), \tau)\, d\tau, B\right) \leqslant \varepsilon \qquad \text{for all} \quad q(\cdot) \in M(Q).$$

Recalling the definition of B, it follows that $x(\cdot)$ is an ε-robust solution of (S_p) if and only if

$$\int_0^1 a_i' A(x(\tau), q(\tau), \tau)\, d\tau - b_i \leqslant \varepsilon \tag{4.2.1}$$

for *all* $q(\cdot)\in M(Q)$ and for *all* $i\in\{1,2,\ldots,r\}$. Requirement (4.2.1) is equivalent to

$$\sup\left\{\int_0^1 a_i' A(x(\tau),q(\tau),\tau)\,d\tau: q(\cdot)\in M(Q)\right\}-b_i\leqslant\varepsilon$$

for all $i\in\{1,2,\ldots,r\}$. We are now in a position to state our first lemma.

4.2.1. LEMMA (see Appendix A for proof). (a) *For each fixed* $x(\cdot)\in M(X)$ *and* $i\in\{1,2,\ldots,r\}$, *the real valued function* $f_i(x(\cdot),\cdot)$: $[0,1]\to R$ *defined by*

$$f_i(x(\tau),\tau)\triangleq\sup\{a_i' A(x(\tau),q,\tau): q\in Q\} \tag{4.2.3}$$

is measurable and integrable over $[0,1]$.

(b) *For a given* $\varepsilon>0$, *the vector function* $x_\varepsilon(\cdot)\in M(X)$ *is an* ε*-robust solution of* (S_p) *if and only if*

$$\int_0^1 f_i(x_\varepsilon(\tau),\tau)\,d\tau-b_i\leqslant\varepsilon \tag{4.2.4}$$

for all $i\in\{1,2,\ldots,r\}$.

4.2.2. *Testing functionals.* Motivated by Lemma 4.2.1, we define a number of *testing functionals* F_i: $M(X)\to R$ for $i=1,2,\ldots,r$. $F_i(x(\cdot))$ will tell us whether or not the ith inequality in (4.2.4) is satisfied; i.e., let

$$F_i(x(\cdot))\triangleq\int_0^1 f_i(x(\tau),\tau)\,d\tau-b_i \tag{4.2.5}$$

for $i\in\{1,2,\ldots,r\}$. To determine whether all r inequalities are satisfied *simultaneously*, we define F:$M(X)\to R$ by

$$F(x(\cdot))\triangleq\max\{F_1(x(\cdot)),F_2(x(\cdot)),\ldots,F_r(x(\cdot))\}. \tag{4.2.6}$$

The following lemma is a consequence of Lemma 4.2.1 and the definition of $F(x(\cdot))$.

4.2.3. LEMMA. (S_p) *is approximately robust if and only if, given any* $\varepsilon>0$, *there exists an* $x_\varepsilon(\cdot)\in M(X)$ *such that* $F(x_\varepsilon(\cdot))\leqslant\varepsilon$. *Equivalently*, (S_p) *is approximately robust if and only if*

$$\inf(P)\triangleq\inf\{F(x(\cdot)): x(\cdot)\in M(x)\}\leqslant 0. \tag{P}$$

We shall henceforth refer to this "minimization" as *Problem* (P).

4.3. *Derivation of Problem* (P_s)

As mentioned earlier, the (approximate) robustness problem shall ultimately be solved over the simplex S instead of $M(X)$. The following observation is instrumental to the attainment of this goal:

$$\max\{F_1(x(\cdot)), F_2(x(\cdot)),..., F_r(x(\cdot))\} = \max\left\{\sum_{i=1}^{r} z_i F_i(x(\cdot)) : z \in S\right\} \quad (4.3.1)$$

holds for all $x(\cdot) \in M(X)$. This fact is evident by observing that the maximum on the right hand side is achieved by taking z to be a unit vector e_k in *any* direction such that

$$F_k(x(\cdot)) = \max\{F_1(x(\cdot)), F_2(x(\cdot)),..., F_r(x(\cdot))\}.$$

Using the observation above, we substitute (4.3.1) and (4.2.6) into Problem (P) and obtain

$$\inf\{F(x(\cdot)): x(\cdot) \in M(X)\} = \inf\left\{\max\left[\sum_{i=1}^{r} z_i F_i(x(\cdot)) : x \in S\right] : x(\cdot) \in M(X)\right\}. \quad (4.3.2)$$

This leads us to focus our future attention on the functional $V: M(X) \times S \to R$ given by

$$V(x(\cdot), z) \triangleq \sum_{i=1}^{r} z_i F_i(x(\cdot)). \quad (4.3.3)$$

Using (4.3.2), (4.3.3) and Lemma 4.2.3, we arrive at the following proposition.

PROPOSITION 4.3.1. *(S_p) is approximately robust if and only if*

$$\inf(P_s) \triangleq \inf\{\max[V(x(\cdot), z): z \in S]: x(\cdot) \in M(X)\} \leqslant 0. \quad (P_s)$$

Clearly, this new "minimization" problem, termed Problem (P_s), replaces Problem (P).

5. PROPERTIES OF $V(x(\cdot), z)$

Our main objective in this section is to show that the order of inf and max operations in Proposition 4.3.1 can be reversed. As we shall see later, this is indeed the first step towards reduction of the infinite-dimensional (function space) optimization problem (P_s) to an optimization problem over a finite-dimensional space. We also note in passing that such an interchange of order is tantamount to the existence of a so-called *partial saddle point of* $V(x(\cdot), z)$.

Since $A(\cdot, \cdot, \cdot)$ is allowed to be nonlinear, our method will differ substantially from the saddle point arguments given in [1–3]. Specifically, the following Lyapunov type result on set-valued integration will be of central importance.

5.1. PROPOSITION (see Appendix B for proof). *The set*

$$I(M(X)) \triangleq \left\{ \int_0^1 f(x(\tau), \tau)\, d\tau \colon x(\cdot) \in M(X) \right\}$$

is non-empty and convex in R^r. *(*$f(x(\tau), \tau)$ *above is the vector function having* ith *component* $f_i(x(\tau), \tau)$, *where* $f_i(\cdot)$ *is defined by* (4.2.3), *and the* ith *component* $I(x(\cdot))$ *is denoted by* $I_i(x(\cdot)$.)

Thus, despite the nonlinearity of $A(\cdot, \cdot, \cdot)$, the set of *worst case outputs* $I(M(X))$ still retains convexity, a property obviously crucial to any saddle point type investigation. This suggests shifting of attention from the set $M(X)$ to the set $I(M(X))$; i.e., we may treat $I(M(X))$ rather than $M(X)$ as the domain of $V(\cdot)$. These ideas lead to the following theorem.

5.2. THEOREM (see Appendix B for proof). (S_p) *is approximately robust if and only if*

$$\max\{\inf[V(x(\cdot), z) \colon x(\cdot) \in M(X)] \colon z \in S\} \leqslant 0. \tag{5.2}$$

5.3. *Remark.* It will be shown in the next section that the inner minimum in (5.2) above can be evaluated in closed form. Subsequently, we will remain with a finite-dimensional maximization over S.

6. MAIN RESULTS: PROBLEM (P_e)

Motivated by Remark 5.3, we look more carefully at

$$\begin{aligned}
&\inf\{V(x(\cdot), z) \colon x(\cdot) \in M(X)\} \\
&\quad = \inf \left\{ \sum_{i=1}^{r} z_i F_i(x(\cdot)) \colon x(\cdot) \in M(X) \right\} \qquad \text{(see (4.3.3)} \\
&\quad = \inf \left\{ \sum_{i=1}^{r} z_i \int_0^1 f_i(x(\tau), \tau)\, d\tau - z_i b_i \colon x(\cdot) \in M(X) \right\} \qquad \text{(by (4.2.5))} \\
&\quad = \inf \left\{ \int_0^1 \sum_{i=1}^{r} z_i f_i(x(\tau), \tau)\, d\tau \colon x(\cdot) \in M(X) \right\} - \sum_{i=1}^{r} z_i b_i.
\end{aligned} \tag{6.1}$$

6.1. *Remark.* In order to eliminate $M(X)$ from Eq. (6.1), it will be sufficient to show that the integration and infimum operations can be commuted above. This can be accomplished with the aid of the following two lemmas.

6.2. LEMMA (see Appendix C for proof). *Define the mapping* $\tilde{H}: X \times S \times [0, 1] \to R$ by

$$\tilde{H}(x, z, \tau) \triangleq \sum_{i=1}^{r} z_i f_i(x, \tau). \tag{6.2.1}$$

Then $\tilde{H}(\cdot)$ *has the following properties:*

(i) *For all* $z \in S$ *and almost all* $\tau \in [0, 1]$, $\tilde{H}(\cdot, z, \tau)$ *is continuous on* X.

(ii) *For all* $z \in S$ *and* $x \in X$, $\tilde{H}(x, z, \tau)$ *is measurable on* $[0, 1]$. *Equivalently* $\tilde{H}(\cdot, z, \cdot)$ *is a Carathéodory function on* $X \times [0, 1]$ *for all* $z \in S$.

6.3. LEMMA (see Appendix C for proof). Let $\tilde{H}: X \times S \times [0, 1] \to R$ *be as defined in* (6.2.1). *Then the following commutation holds for each* $z \in S$:

$$\inf \left\{ \int_0^1 \tilde{H}(x(\tau), z, \tau)\, d\tau : x(\cdot) \in M(X) \right\} = \int_0^1 \inf\{\tilde{H}(x, z, \tau): x \in X\}\, d\tau. \tag{6.3.1}$$

Using this equality, $M(X)$ can be eliminated from Eq. (6.1); i.e., for fixed $z \in S$, we now have

COROLLARY 6.3.1.

$$\inf\{V(x(\cdot), z): x(\cdot) \in M(X)\} = \int_0^1 \inf\{\tilde{H}(x, z, \tau): x \in X\}\, d\tau - \sum_{i=1}^{r} z_i b_i . \tag{6.3.2}$$

6.4. *Derivation of Problem* (P_e). Theorem 5.2 in conjunction with Lemma 6.2 motivates the definition of Problem (P_e); i.e., (S_p) is approximately robust if and only if

$$0 \geqslant \max\{\inf[V(x(\cdot), z): x(\cdot) \in M(X)]: z \in S\} \qquad \text{(by Theorem 5.2)}$$

$$= \max \left\{ -\sum_{i=1}^{r} z_i b_i + \int_0^1 \inf\{\tilde{H}(x, z, \tau): x \in X\}\, d\tau : z \in S \right\} \qquad \text{(by (6.3.2)}. \tag{6.4.0}$$

Looking closely at the maximand above, we notice that the function space

$M(X)$ has disappeared and $z \in S \subseteq R^r$ is the only variable. We now formally define *Problem* (P_e): Let $h\colon S \to R$ and $H\colon S \times [0, 1] \to R$ be given by

$$h(z) \triangleq -\sum_{i=1}^{r} z_i b_i; \tag{6.4.1}$$

$$H(z, \tau) \triangleq \inf\{\tilde{H}(x, z, \tau)\colon x \in X\}, \tag{6.4.2}$$

where $\tilde{H}(\cdot)$ is given in (6.2.1). Then we seek a maximum of the *parameterized* (concave) *function*

$$F_e(z) \triangleq h(z) + \int_0^1 H(z, \tau)\, d\tau \tag{6.4.3}$$

over the set $z \in S$.

Our first main result can now be stated.

6.5. THEOREM. *(S_p) is approximately robust if and only if*

$$\max(P_e) \triangleq \max\{F_e(z)\colon z \in S\} \leqslant 0.$$

Proof. Immediate from (6.4.0) and the definition of $F_e(z)$. ∎

6.6. ILLUSTRATION. Consider the perturbed system (S_p) given by $A(x, q, \tau) = x + q$, $B = [1, 2]$, $Q = [-0.5, 0.5]$ and $X = [-p, p]$. By inspection, we observe that (S_p) is robust (hence approximately robust) for all $p \geqslant 1.5$ since $x(\tau) \equiv 1.5$ is clearly a robust solution. On the other hand, the above system is neither robust nor approximately robust for $p < 1.5$. We now demonstrate that *formal* application of Theorem 6.5 does indeed confirm these obvious conclusions. Straightforward substitution into (6.4.1)–(6.4.3) yields the following expressions:

$$h(z) = -2z_1 + z_2,$$

$$\tilde{H}(x, z, \tau) = 0.5(z_1 + z_2) + x(z_1 - z_2),$$

$$H(z, \tau) = 0.5(z_1 + z_2) - p|z_1 - z_2|,$$

$$F_e(z) = -1.5(z_1 - z_2) - p|z_1 - z_2|.$$

It is not difficult to see that

$$\max\{F_e(z)\colon z \in S\} = F_e((0.5, 0.5)) = 0$$

provided $p \geqslant 1.5$; while on the other hand, if $p < 1.5$, we find that

$$\max\{F_e(z)\colon z \in S\} = F_e((0, 1)) = 1.5 - p > 0.$$

Notice that, although we know that (S_p) is robust, the above application of Theorem 6.5 only permits us to conclude that the system is *approximately* robust for $p \geqslant 1.5$. In order to draw the stronger conclusion that (S_p) is robust, we need the theory of Section 7 which provides conditions under which approximate robustness implies robustness. More elaborate examples are given in Section 8.

7. Existence of Robust Solutions

In the last section, a complete answer was supplied by Problem (P_e) as to the question of the existence of approximately robust solutions for (S_p). Recalling, however, that approximate robustness does not necessarily imply robustness, we are motivated to explore the "gap" between these two conditions. The following theorem addresses this issue.

7.1. Theorem (see Appendix D for proof).

(a) *A necessary condition for* (S_p) *to be robust is that*

$$\max(P_e) \leqslant 0.$$

(b) *A sufficient condition for* (S_p) *to be robust is that*

$$\max(P_e) < 0.$$

The "gap" between the necessary and sufficient conditions of Theorem 7.1 closes for those systems for which *approximate robustness implies robustness*. The example in Section 3.3 shows that this gap cannot be closed without additional assumptions on (S_p). The next theorem provides one such set of sufficient conditions for closing this gap.

7.2. Theorem (see Appendix D for proof). *In addition to Assumptions* (D1–D4) *and Regularity Conditions* (A1–A3), *suppose that X is a compact subset of* R^m. *Then,*

(a) (S_p) *is robust if and only if* $\max(P_e) \leqslant 0$.

(b) *Provided* (S_p) *is deemed robust, there is a robust solution* $x(\cdot) \in M(X)$ *solving* (P) *and a vector* $z_0 \in S$ *solving* (P_e) *which together satisfy the following necessary condition:*

$$\tilde{H}(x_0(\tau), z_0, \tau) = H(z_0, \tau) \qquad \text{for almost all } \tau \in [0, 1];$$

or equivalently,

$$x_0(\tau) \in \arg\min\{\tilde{H}(x, z_0, \tau): x \in X\} \qquad \textit{for almost all } \tau \in [0, 1].$$

7.3. *Remarks.* Theorem 7.2 first appeared without proof in [8] and covers the results of [3] as a special case. We note that the assumption of compactness (of solution restraint set X) it requires, is one commonly made in the existence theory for optimal controls and controllability of dynamical systems (see, for example, [9, 10]). Unlike [9], however, we have made no assumptions regarding continuity of system function $A(\cdot, \cdot, \cdot)$ with respect to time.

8. Illustrative Examples

In this final section, we present some interesting special cases, worked out examples and a digital computer implementation of the foregoing theory.

8.1. *Special Cases*

(i) Our first illustration deals with the application of the preceding theory to the study of robustness properties of the class of *linear state equations* with *input coupling uncertainty* and "amplitude constraints." Specifically, we consider the model

$$\dot{y}(t) = A(t)y(t) + [b(t) + q(t)]\, x(t), \qquad z(0) = 0, \tag{8.1.1}$$

where $y(t) \in R^n$ is the state vector, $x(t)$ is a scalar input, $A(\cdot)$ is a (continuous) $n \times n$ matrix time function, $b(\cdot)$ is a bounded measurable n-vector time function and $q(\cdot)$ is an n-dimensional uncertainty vector. The uncertainty restraint set Q is described componentwise by $|q_i| \leqslant \bar{q}_i$, where $\bar{q}_i \geqslant 0$ is a prespecified bound on q_i, the ith component of q. The control restraint set X will be simply $[-M, M]$, where $M > 0$ is given. Finally, the target set B will be assumed to be a rectangle given by $[\underline{b}_1, \bar{b}_1] \times [\underline{b}_2, \bar{b}_2] \times \cdots \times [\underline{b}_n, \bar{b}_n]$.

Under these conditions, it is well known (see, for example, pp. 65, 66 of [11]) that the state equation (8.1.1) has a unique solution for each $x(\cdot) \in M(X)$ and $q(\cdot) \in M(Q)$ which can be written as

$$y(t) = \int_0^t \phi(t, \tau)[b(\tau) + q(\tau)]\, x(\tau)\, d\tau, \tag{8.1.2}$$

where, $\phi(t, \tau) = [\phi_{ij}(t, \tau)]_{i,j=1,2,\ldots,n}$ is the so-called state-transition matrix associated with $A(\cdot)$. With T as the terminal time of interest, the system function $A: R^n \times R^n \times [0, T] \to R^n$ can now be identified as

$$A(x, q, t) = \phi(T, t)[b(t) + q]\, x. \tag{8.1.3}$$

We shall presently see that the various functions introduced in the last section take on fairly simple explicit forms for the class of problems above. Using the notation of Section 2, we let $r = 2n$, $b_i = \bar{b}_i$ for $1 \leqslant i \leqslant n$, $b_i = -\underline{b}_i$ for $n + 1 \leqslant i \leqslant 2n$; a_i is taken to be e_i, the unit vector in the ith-direction for $1 \leqslant i \leqslant n$ and we take $a_i = -e_{i-n}$ for $n + 1 \leqslant i \leqslant 2n$. We shall furthermore write $z = [\bar{z}\, \underline{z}]'$, where $\bar{z} = [z_1, z_2, \ldots, z_n]'$, $\underline{z} = [z_{n+1}, z_{n+2}, \ldots, z_{1n}]'$, and $b = [\bar{b}'\, \underline{b}'\,]'$ where $\bar{b} = [b_1, b_2, \ldots, b_4]'$ and $\underline{b} = [b_{n+1}, b_{n+2}, \ldots, b_{2n}]'$. Substitution into (6.2.1), (6.4.1) and (6.4.2) now yields

$$h(z) = \underline{z}'\underline{b} - \bar{z}'\bar{b}; \tag{8.1.4}$$

$$\tilde{H}(x, z, t) = (\bar{z} - \underline{z})' A^0(t)\, x(t) + (\bar{z} + \underline{z})' \delta A(t)\, x(t); \tag{8.1.5}$$

and

$$H(z, t) = M \cdot \min\{0, [(\bar{z} + \underline{z})' \delta A(t) - |(\bar{z} - \underline{z}) A^0(t)|]\}, \tag{8.1.6}$$

where $A^0(t)$ and $\delta A(t)$ are n-vectors given by

$$A^0(t) = \phi(T, t)\, b(t)$$

and

$$\delta A(t) = |\phi(T, t)|\bar{q}; \qquad \bar{q} \triangleq [\bar{q}_1, \bar{q}_2, \ldots, \bar{q}_n]'.$$

Above, we interpret the absolute value of a matrix entry by entry. The vectors $A^0(t)$ and $\delta A(t)$ may be interpreted respectively as the nominal impulse response vector and the vector of maximal allowable perturbations on the nominal impulse response. To illustrate the above computations, we consider the double integrator whose input vector is $b(t) \equiv [0\ 1]'$ and whose state transition matrix is

$$\phi(T, t) = \begin{bmatrix} 1 & (T - t) \\ 0 & 1 \end{bmatrix}.$$

It can be easily checked that the preceding expressions specialize for this example as follows:

$$A^0(t) = \begin{bmatrix} (T - t) \\ 1 \end{bmatrix};$$

$$A(t) = \begin{bmatrix} \bar{q}_1 + (T-t)\,\bar{q}_2 \\ \bar{q}_2 \end{bmatrix};$$

$$h(z) = -z_1\bar{b}_1 - z_2\bar{b}_2 + z_3\underline{b}_3 + z_4\underline{b}_4;$$

$$\tilde{H}(x, z, t) = [(z_1 - z_3)(T-t) + (z_2 - z_4)]\,x(t)$$
$$+\{(z_1 + z_3)\,\bar{q}_1 + [(z_1 + z_3)(T-t) + (z_2 + z_4)]\,q_2\}\,|x(t)|;$$

$$H(z, t) = M \cdot \min\{0, \bar{q}_2[(z_1 + z_3)(T-t) + (z_2 + z_4)]$$
$$+(z_1 + z_3)\,\bar{q}_1 - |(z_1 - z_3)(T-t) + (z_2 - z_4)|\}.$$

We observe that the "candidate" robust control suggested by Theorem 7.2 turns out in this class of problems to be of the "bang-off-bang" type. The preceding formulas are easily seen to generalize in a straightforward fashion when one is dealing with the multi-input case.

(ii) For this second special case, we start with the assumption that the so-called system function $A(\cdot)$ can be additively decomposed as follows:

$$A(x(\tau), q(\tau), \tau) = A_1(x(\tau), \tau) + A_2(q(\tau), \tau). \tag{8.1.7}$$

We find via application of (6.2.1), (6.4.1)–(6.4.3) that

$$F_e(z) = \int_0^1 \Bigg\{ \min \left[\sum_{i=1}^r z_i a_i' A_1(x, \tau) \colon x \in X \right]$$
$$+ \sum_{i=1}^r z_i \max[a_i' A_2(q, \tau) \colon q \in Q] \Bigg\}\, d\tau - \sum_{i=1}^r z_i b_i. \tag{8.1.8}$$

If $A_1(\cdot)$, $A_2(\cdot)$, X and Q are endowed with additional structure, then a more tractable closed form for $F_e(z)$ is possible. For example, suppose $A_1(\cdot)$ and $A_2(\cdot)$ are linear in x and q, respectively; i.e.,

$$A_1(x, \tau) = A_1(\tau)\, x; \tag{8.1.9}$$

$$A_2(q, \tau) = A_2(\tau)\, q; \tag{8.1.10}$$

and further assume that X and Q are closed balls centered at the origin having radii R_x and R_q, respectively. Then, with the aid of (8.1.9) and (8.1.10), $F_e(z)$ in (8.1.8) assumes the explicit form

$$F_e(z) = \int_0^1 \left[R_q \sum_{i=1}^r z_i \|a_i' A_2(\tau)\|_* - R_x \|A_1'(\tau)\, \Omega z\|_* \right] d\tau - \sum_{i=1}^r z_i b_i, \tag{8.1.11}$$

where $\|\cdot\|_*$ denote the norm which is dual to that chosen for X and Q and

$$\Omega \triangleq [a_1 a_2 \cdots a_r].$$

Once $F_e(z)$ in (8.1.11) has been minimized, we can apply the necessary condition of Theorem 7.2 to generate a candidate robust solution $x_0(\cdot)$. For example, suppose X is endowed with the max norm and $z_0 \in S$ has been found to uniquely solve Problem (P_e). Then there is a robust solution $x(\cdot) \in M(X)$ such that each component $x_0^i(\cdot)$ of $x_0(\cdot)$ satisfies

$$\begin{aligned} x_0^i(t) &= -R_x, & &\text{if} \quad (A_1'(t)\,\Omega z_0)_i > 0, \\ &= \in [-R_x, R_x], & &\text{if} \quad (A_1'(t)\,\Omega z_0)_i = 0, \\ &= R_x, & &\text{if} \quad (A_1'(t)\,\Omega z_0)_i < 0, \end{aligned} \tag{8.1.12}$$

where $(A_1'(t)\,\Omega z_0)_i$ denotes the ith conponent of $A_1'(t)\,\Omega z_0$.

8.2. *Remark.* If the norm on X and Q above is the standard Euclidean norm, then so is $\|\cdot\|_*$ and the problem $\max\{F_e(z): z \in S\}$ becomes a "smooth" nonlinear program. On the other hand, as in the case of amplitude constraints, $F_e(z)$ may turn out to be non-differentiable in z (see (8.6)). For this case (see example to follow) Veinott's adaptation [13] of Kelly's cutting hyperplane method [14] proved effective in numerical computation.

8.3. *Digital computer implementation.* Figure 1 shows the organization of a software routine developed at the University of Rochester. This program enabled us to investigate the robustness properties of the class of systems described by special case (ii) of Section 8.

The entire program is divided into three modules. The module, INPUT, which may vary from user to user, converts the generic problem (e.g., a state equation model) into the format required by the theory. The module ROBUST then decides on the robustness of (S_p) by solving Problem (P_e). This is accomplished by solving the required nonlinear program via Kelly's cutting hyperplane method [13, 14]. Note that, because of the finite precision and the various errors inherent in any digital computation, the distinction between approximate robustness and robustness is best left to the judgement of the user. Finally, if $\max\{F_e(z): z \in S\}$ is non-positive, the module SOLUTION is used to generate a robust solution $x_0(\cdot)$. This is done iteratively by using the necessary conditions of Theorem 7.2 to generate candidate robust solutions which are then tested for robustness via stochastic simulation.

8.4. EXAMPLE. The linearized, short period model for longitudinal dynamics of certain aircraft is given by the following state equations (see [12, Chap. 5]):

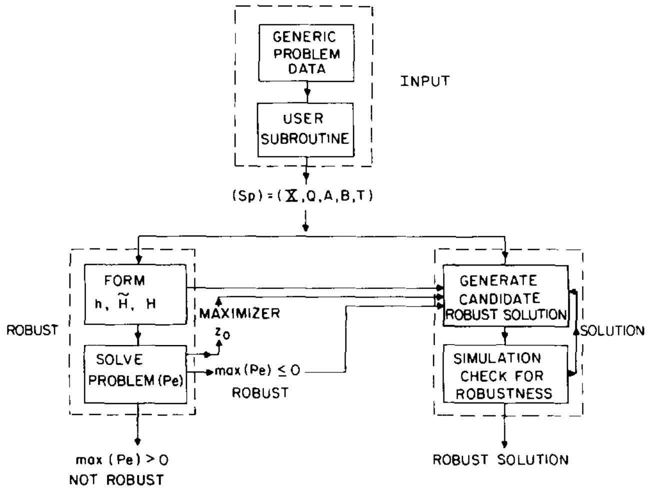

FIG. 1. Software for robustness investigations of perturbed linear systems.

$$\begin{bmatrix} \dot{w}(t) \\ \dot{p}(t) \end{bmatrix} = \begin{bmatrix} z_w & u_0 \\ M_w + M_{\dot{w}} z_w & u_0 M_{\dot{w}} + M_p \end{bmatrix} \begin{bmatrix} w(t) \\ p(t) \end{bmatrix} + \begin{bmatrix} z_{\delta e} \\ M_{\dot{w}} z_{\delta e} + M_{\delta e} \end{bmatrix} \delta_e(t), \tag{8.4.1}$$

where the state variables $w(t)$ and $p(t)$ represent vertical velocity (ft/sec) and pitch rate (rad/sec), respectively; the elevator angle $\delta_e(t)$ is the control variable; u_0 is the forward velocity (ft/sec); the model parameters z_w, M_w, M_p and $M_{\dot{w}}$ are the so-called stability derivatives; and finally, the parameters $z_{\delta e}$ and $M_{\delta e}$ are what are the so–called control derivatives. These parameters are functions of the nominal nonlinear trajectory about which the dynamics are linearized.

Using the data for F-89 (obtained from [12]) at 20,000 feet and 2.6 Mach, we arrive at the state equation

$$\begin{bmatrix} \dot{w}(t) \\ \dot{p}(t) \end{bmatrix} = \begin{bmatrix} -1.43 & 660 \\ -.022 & -2.78 \end{bmatrix} \begin{bmatrix} w(t) \\ p(t) \end{bmatrix} + \begin{bmatrix} -69.8 \\ -23.1 \end{bmatrix} \delta_e(t).$$

A coupling uncertainty (as in special case (i), Section 8) of 10% was introduced in the parameter $z_{\delta e}$. Hence, we take $Q = [-6.98, 6.98]$ with the elevator angle $\delta_e(t)$ restricted to the interval $X = [-0.0175, 0.0175]$. Starting in the zero state, it is desired to have the upward vertical velocity $w(T)$ inside the interval $B = [-16.5, -15.9]$ at the final time $T = 0.79$ seconds. With the above data as input, the two modules INPUT and ROBUST were used to compute $\max\{F_e(z) : z \in S\} = -0.284$ attained at the point (0.5014, 0.4986).

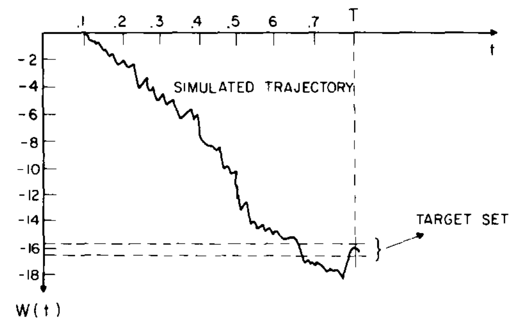

FIG. 2. Simulated trajectory for Example 8.3.

Hence, by Theorem 7.2, this system is robust. The module SOLUTION then generated the following candidate robust input:

$$\begin{aligned} \delta_e(t) &= 0.0175 \qquad \text{if} \quad t \in [0.1, 0.65]; \\ &= 0 \qquad \text{if} \quad t \in [0, 0.1) \cup (0.65, 0.79]. \end{aligned}$$

With this candidate robust input, we then simulated a trajectory $w(t)$ for a typical value of disturbance $q(t) \in Q$. As predicted by the theory $w(T)$ reaches the target at the terminal time (see Fig. 2).

The above experiment was repeated for a range of different terminal times. It was found that there is a critical threshold terminal time $T_h \approx 0.52$, such that for all $T < T_h$, the system is *not* robust, whereas for all $T > T_h$ the system *is* robust.

9. CONCLUDING REMARKS

The validity of the approximate robustness criterion "$\max(P_e) \leqslant 0$" of Theorem 6.5 depends crucially on the boundedness of Q. We conclude with an example which shows that a fallacious conclusion can be reached by naively applying Theorem 6.5 to systems having unbounded uncertainty. Indeed, let $n = m = r = 2$ and $k = T = 1$. The remaining data describing (S_p) are taken as follows: $X = [-1, 1] \times [0, \infty)$; $Q = [0, \infty)$; $A(x, q, \tau) = [x_1 \ |x_1| q - x_2]'$; $B = (-\infty, -\frac{1}{2}] \times (-\infty, 1]$. First, we argue that the above system is *not* approximately robust. To see this, note that we require $x_1(\tau) \equiv 0$ a.e. in order to have

$$\sup \left\{ \int_0^1 [|x_1(\tau)| q(\tau) - x_2(\tau)] \, d\tau : q(\cdot) \in M(Q) \right\} < \infty.$$

This being the case, it follows that

$$\int_0^1 x_1(\tau)\, d\tau = 0.$$

Hence, the system does not have an ε-robust solution for all $\varepsilon < \frac{1}{2}$. This precludes approximate robustness.

Now, we formally apply Theorem 6.5: Using (6.2.1), (6.4.1) and (6.4.2), we compute

$$\begin{aligned} H(z, \tau) &= \inf\{z_1 x_1 + z_2 \sup[|x_1|\, q - x_2 : q \geqslant 0] : -1 \leqslant x_1 < 1,\ x_2 \geqslant 0\} \\ &= -1 \qquad \text{if} \quad z_2 = 0 \\ &= -\infty \qquad \text{if} \quad z_2 \neq 0; \\ h(z) &= \tfrac{1}{2} z_1 - z_2. \end{aligned}$$

Hence, we conclude that

$$\max\{F_e(z) : z \in S\} = F_e(z)|_{\substack{z_1 = 1 \\ z_2 = 0}} = -\tfrac{1}{2} \leqslant 0.$$

Therefore, for this case of unbounded uncertainties, formal application of Theorem 6.5 would lead to the erroneous conclusion that (S_p) is robust.

The above example points to the need for a new approach to the robustness problem with unbounded uncertainties. This problem is currently under investigation.

APPENDIX A

Proof of Lemma 4.2. Let $x(\cdot) \in M(X)$ and $i \in \{1, 2, \ldots, r\}$ be fixed. Note that $\bar{Q}$ is compact and $a_i' A(x(\tau), q, \tau)$ is a Carathéodory function over $\bar{Q} \times [0, 1]$. Hence the so-called Measurable Selection Theorem (cf. p. 236 of [15]) enables one to choose $\bar{q}(\cdot) \in M(\bar{Q})$ such that

$$\sup\{a_i' A(x(\tau), q, \tau) : q \in \bar{Q}\} \equiv a_i' A(x(\tau), \bar{q}(\tau), \tau).$$

We notice that for fixed $\tau \in [0, 1]$, $A(x(\tau), \cdot, \tau)$ is continuous (Assumption (A3)). Hence,

$$\sup\{a_i' A(x(\tau), q, \tau) : q \in \bar{Q}\} \equiv \sup\{a_i' A(x(\tau), q, \tau) : q \in Q\}.$$

From the two preceding equalities and the definition of $f_i(x(\tau), \tau)$, it follows that

$$f_i(x(\tau), \tau) \equiv a_i' A(x(\tau), \bar{q}(\tau), \tau).$$

Having expressed $f_i(x(\tau), \tau)$ in this way, measurability and integrability of $f_i(x(\cdot), \cdot)$ is a consequence of regularity Assumption (A1). ∎

APPENDIX B

Proof of Proposition 5.1. Let $x_1(\cdot)$ and $x_2(\cdot)$ be any arbitrary elements in $M(X)$ and suppose that $\alpha \in [0, 1]$ is given. We must now exhibit an $x(\cdot) \in M(X)$ such that $I(x(\cdot)) = \alpha I(x_1(\cdot)) + (1-\alpha) I(x_2(\cdot))$. Towards this end, define the vector valued function $g(\cdot)$ on $[0, 1]$ by $g(\tau) \triangleq f(x_1(\tau), \tau) - f(x_2(\tau), \tau)$. Since $g(\cdot)$ is integrable (Lemma 4.2.1), we may use Lyapunov's Theorem[3] (cf. Lemma 4A on p. 163 of [11]) to extract a measurable set $\Lambda_\alpha \subseteq [0, 1]$ such that

$$\int_{\Lambda_\alpha} g(\tau)\, d\tau = \alpha \int_0^1 g(\tau)\, d\tau. \tag{B.1}$$

Letting Λ_α^c denote the complement of Λ_α in $[0, 1]$, we substitute for $g(\cdot)$ in (B.1) and re-arrange to obtain

$$\int_{\Lambda_\alpha} f(x_1(\tau), \tau)\, d\tau + \int_{\Lambda_\alpha^c} f(x_2(\tau), \tau)\, d\tau = \alpha I(x_1(\cdot)) + (1-\alpha) I(x_2(\cdot)).$$

The desired $x(\cdot) \in M(X)$ is now easily seen to be

$$x(\tau) \triangleq \begin{cases} x_1(\tau) & \text{if } \tau \in \Lambda_\alpha; \\ x_2(\tau) & \text{if } \tau \in \Lambda_\alpha^c. \end{cases} \quad ∎ \tag{B.2}$$

Proof of Theorem 5.2. Using the notation of Section 5.1, Problem (P_s) (described in Section 4.3) may be reformulated as

$$\inf(P_s) = \inf \left\{ \max \left[\sum_{i=1}^{r} z_i(I_i(x(\cdot)) - b_i) : z \in S \right] : x(\cdot) \in M(X) \right\}. \tag{B.3}$$

Define the function $\tilde{V}: \overline{I(M(X))} \times S \to R$ by

$$\tilde{V}(a, z) \triangleq \sum_{i=1}^{r} z_i(a_i - b_i), \tag{B.4}$$

where $a = [a_1 a_2 \ldots a_r]' \in \overline{I(M(X))}$ and $z = [z_1 z_2 \ldots z_r]' \in S$. We note that $\tilde{V}(a, z)$ is *continuous* and recall that S is compact. Consequently, the

[3] We can define a non-atomic vector valued measure μ on measurable subsets Λ of $[0, 1]$ by $\mu(\Lambda) = \int_\Lambda g(\tau)\, d\tau$. Lyapunov's Theorem asserts that the range of this measure is convex.

expression $\max[\tilde{V}(\alpha, z): z \in S]$ depends continuously on $\alpha \in \overline{I(M(X))}$ (cf. Berge's Maximum Theorem, pp. 115, 116 of [16]). Given this fact, we can replace $I(M(X))$ by $\overline{I(M(X))}$ in (B.3). This gives

$$\inf(P_s) = \inf\{\max[\tilde{V}(\alpha, z): z \in S]: \alpha \in \overline{I(M(X))}\}. \tag{B.5}$$

The next step of the proof involves interchange of the order of infimum and maximum in (B.5) above. To accomplish this, we apply a *partial saddle point* theorem (cf. Proposition 2.3 on p. 175 of [15].) The pre-conditions for this theorem are satisfied because

(i) S and $\overline{I(M(X))}$ are closed convex sets (recall Proposition 5.1);

(ii) S is compact, hence bounded;

(iii) The function $\tilde{V}(\alpha, z)$ is clearly convex (in fact affine) in α for fixed z and concave in z (in fact affine) for fixed α.

Thus, using Lemma 4.2.3, we conclude that (S_p) is approximately robust if and only if

$$\begin{aligned}\inf(P) = \inf(P_s) &= \inf\{\max[\tilde{V}(\alpha, z): z \in S]: \alpha \in \overline{I(M(X))}\}\\ &= \max\{\inf[\tilde{V}(\alpha, z): \alpha \in \overline{I(M(X))}]: z \in S\}\\ &= \max\{\inf[\tilde{V}(\alpha, z): \alpha \in I(M(X))]: z \in S\}\\ &= \max\{\inf[V(x(\cdot), z): x(\cdot) \in M(X)]: z \in S\} \leqslant 0. \quad \blacksquare\end{aligned}$$

APPENDIX C

The proof of Lemma 6.3 hinges crucially on the ability to interchange the order of integration and infimum operations. In light of the fact that the constraint set X is not necessarily closed, the "standard" interchange theorems (see [15] and/or [17]) do not directly apply. The theorem below will prove useful in a robustness context.

C.1. INTERCHANGE THEOREM. *Let V be a non-empty σ-compact subset of R^l and suppose that f: $V \times [0, 1] \to R \cup \{\pm\infty\}$ is a Carathéodory function such that*

$$\int_0^1 |f(v(\tau), \tau)|\, d\tau < \infty \qquad \textit{for all } v(\cdot) \in M(V).$$

We then conclude that

(a) *The function g: $[0, 1] \to \mathbf{R} \cup \{\pm\infty\}$ defined by $g(\tau) = \inf\{f(v, \tau)$: $v \in V\}$ is measurable and $\int_0^1 g(\tau)\, d\tau$ exists.*

(b)

$$\inf \left\{ \int_0^1 f(v(\tau), \tau)\, d\tau : v(\cdot) \in M(V) \right\} = \int_0^1 \inf\{f(v, \tau) : v \in V\}\, d\tau < \infty.$$

Proof of (a). Since V is *σ-compact*, there is an *increasing* sequence V_n, $n = 1, 2, \ldots$ of *compact* sets with $V = \bigcup_{n=1}^{\infty} V_n$. Define the sequence of functions $g_n : [0, 1] \to \mathbf{R} \cup \{\pm\infty\}$ by $g_n(\tau) \triangleq \inf\{f(v, \tau) : v \in V_n\}$. Clearly for each fixed $\tau \in [0, 1]$, $g_n(\tau)$ is non-increasing and hence has a well defined limit. We assert that $g(\tau) = \lim_{n \to \infty} g_n(\tau)$ for all $\tau \in [0, 1]$. To prove this assertion, let $\tau \in [0, 1]$ be fixed and observe that $g(\tau) \leqslant g_n(\tau)$ for all n. Hence, $g(\tau) \leqslant \lim_{n \to \infty} g_n(\tau)$ for all $\tau \in [0, 1]$. To prove the reverse inequality, *fix* a $\tau \in [0, 1]$ and a $v_* \in V$. Since $V = \bigcup_{n=1}^{\infty} V_n$, there is an integer N (dependent on v_*) such that $v_* \in V_N$. Consequently, $f(v_*, \tau) \geqslant \inf\{f(v, \tau) : v \in V_N\} = g_N(\tau) \geqslant \lim_{n \to \infty} g_n(\tau)$. Since v_* was arbitrarily chosen, we can take the infimum over $v_* \in V$ yielding

$$g(\tau) = \inf\{f(v_*, \tau) : v_* \in V\}.$$

This completes the proof of the assertion.

Now, to prove measurability of $g(\cdot)$, note that $f(\cdot)$ being a Carathéodory function implies that $f(\cdot)$ is a so-called *normal integrand* (see p. 234 of [15]). Since each of the sets V_n is compact, the *Measurable Selection* theorem (see p. 236 of [15]) then enables one to choose $v_n(\cdot) \in M(V_n)$ such that

$$\begin{aligned} f(v_n(\tau), \tau) &\equiv \inf\{f(v, \tau) : v \in V_n\} \\ &= g_n(\tau). \end{aligned} \tag{C.1}$$

Since $f(v_n(\tau), \tau)$ is Lebesgue measurable, it follows that each $g_n(\cdot)$ must also be Lebesgue measurable. Consequently $g(\cdot)$, being the monotone limit of $g_n(\cdot)$, is also measurable.

To complete the proof of (a), existence of the integral of $g(\cdot)$ must be proven. First we note that each $g_n(\cdot)$ is integrable as a consequence of the integrability assumption on $f(\cdot)$ and the existence of a measurable selection $v_n(\cdot)$ given in (C.1). Now, fix any $v_* \in V_1$ and notice that the function $h_*(\tau) \triangleq f(v_*, \tau)$ is integrable and majorizes (pointwise) each $g_n(\cdot)$ and $g(\cdot)$. The existence of the integral of $g(\cdot)$ now follows from a variant of the Monotone Convergence Theorem (see [18, p. 90, #13]).

Proof of (b). Pick any $v_* \in V$. Then using the integrability hypothesis on $f(\cdot)$, we have

$$\begin{aligned} \infty > \int_0^1 f(v_*, \tau)\, d\tau &\geqslant \inf \left\{ \int_0^1 f(v(\tau), \tau)\, d\tau : v(\cdot) \in M(V) \right\} \\ &\geqslant \int_0^1 \inf\{f(v, \tau) : v \in V\}\, d\tau. \end{aligned}$$

To prove the reverse inequality it suffices to establish the following: Given any $\varepsilon > 0$, there exists a $v_\epsilon(\cdot) \in M(V)$ such that

(i) $$\int_0^1 f(v_\epsilon(\tau), \tau)\, d\tau \leqslant \int_0^1 \inf\{f(v, \tau): v \in V\}\, d\tau + \varepsilon$$
$$\text{if} \quad \int_0^1 \inf\{f(v, \tau): v \in V\}\, d\tau > -\infty;$$

(ii) $$\int_0^1 f(v_\epsilon(\tau), \tau)\, d\tau \leqslant -\frac{1}{\varepsilon} \qquad \text{if} \int_0^1 \inf\{f(v, \tau): v \in V\}\, d\tau = -\infty.$$

As in part (a), a variant of the Monotone Convergence Theorem [18, p. 90, #13] is used; i.e.,

$$\lim_{n \to \infty} \int_0^1 g_n(\tau)\, d\tau = \int_0^1 g(\tau)\, d\tau. \tag{C.2}$$

Proof of Case (i). Let $\varepsilon > 0$ be given. Since the limit in (C.2) is monotone, there is an integer N (dependent on ε) such that

$$\int_0^1 g_N(\tau)\, d\tau \leqslant \int_0^1 g(\tau)\, d\tau + \varepsilon. \tag{C.3}$$

For this value of N, pick $v_N(\cdot)$ satisfying (C.1) and let $v_\epsilon(\tau) \equiv v_N(\tau)$. Hence, we now have

$$\begin{aligned}
\int_0^1 f(v_\epsilon(\tau), \tau)\, d\tau &= \int_0^1 f(v_N(\tau), \tau)\, d\tau \\
&= \int_0^1 \inf\{f(v, \tau): v \in V_N\} && \text{(by C.1))} \\
&= \int_0^1 g_N(\tau)\, d\tau \\
&\leqslant \int_0^1 g(\tau)\, d\tau + \varepsilon && \text{(by C.3))} \\
&= \int_0^1 \inf\{f(v, \tau): v \in V\}\, d\tau + \varepsilon.
\end{aligned}$$

Proof of Case (ii). Let $\varepsilon > 0$ be given. Once again by monotonicity of the limit in (C.2), we can pick an integer N (dependent on ε) such that

$$\int_0^1 g_N(\tau)\, d\tau \leqslant -\frac{1}{\varepsilon}.$$

As in Case (i), we select $v_N(\cdot)$ satisfying (C.1) and obtain the desired result by setting $v_\epsilon(\tau) \equiv v_N(\tau)$. This completes the proof of the theorem. ∎

Proof of Lemma 6.2. Since linear combinations of Carathéodory functions remain Carathéodory, it suffices to show that each of the $f_i(x, \tau)$ is Carathéodory over $X \times [0, 1]$. Measurability in τ is immediate from Lemma 4.2.1a. In order to prove continuity of $f_i(\cdot, \tau)$ for almost every $\tau \in [0, 1]$, note that continuity assumption (A2) enables us to write

$$f_i(x, \tau) = \max\{a_i' A(x, q, \tau) \colon q \in \bar{Q}\}.$$

Since $\bar{Q}$ is compact, Berge's Maximum Theorem (cf. pp. 115, 116 of [16]) implies continuity of $f_i(\cdot, \tau)$ for almost all $\tau \in [0, 1]$. ∎

Proof of Lemma 6.3. For fixed $z \in S$, Lemma 6.2 establishes that $\tilde{H}(\cdot, z, \cdot)$ is a Carathéodory function over $X \times [0, 1]$. Furthermore, for fixed $x(\cdot) \in M(X)$, the integrability of the functions $f_i(x(\tau), \tau)$ (Lemma 4.2.1) implies that $\tilde{H}(x(\tau), z, \tau) = \sum_{i=1}^{r} z_i f_i(x(\tau), \tau)$ must also be integrable. Hence, $\tilde{H}(\cdot, z, \cdot)$ satisfies the hypotheses of the Interchange Theorem C.1. ∎

Appendix D

Proof of Theorem 7.1. (a) If (S_p) is robust, then (S_p) is approximately robust. Now Theorem 6.5 requires $\max(P_e) \leqslant 0$.

(b) Examining the proof of Theorem 5.2 (Appendix B) and the derivation of Problem (P_e)(Section 6.4), it is clear that the trio of problems (P), (P_s) and (P_e) all have the same value. Call this value μ_0. Hence, if $\mu_0 < 0$, it follows that $\inf(P) < 0$ which implies the existence of an $x_0(\cdot) \in M(X)$ such that $F(x_0(\cdot)) \leqslant 0$. Hence, by (4.2.5) and (4.2.6)

$$\int_0^1 a_i' A(x_0(\tau), q(\tau), \tau)\, d\tau \leqslant b_i$$

for all $q(\cdot) \in M(Q)$ and $i \in \{1, 2, ..., r\}$; i.e., $x_0(\cdot)$ is a robust solution and therefore (S_p) is robust. ∎

Proof of Theorem 7.2. (a) Recalling that $\max[\tilde{V}(\alpha, z) \colon z \in S]$ is continuous in $\alpha \in I(M(X))$, we observe that the inf in the chain of equalities

$$\begin{aligned}\max(P_e) &= \inf(P_s) = \inf\{\max[\tilde{V}(\alpha, z) \colon z \in S] \colon \alpha \in I(M(X))\} \\ &= \inf\{F(x(\cdot)) \colon x(\cdot)) \colon x(\cdot) \in M(X)\}\end{aligned}$$

can be replaced by the min *provided* $I(M(X))$ is proven to be compact. This minimizer would then serve as a robust solution in case $\max(P_e) \leqslant 0$. Thus, compactness of $I(M(X))$ is sufficient to prove part (a) of the theorem. This will be achieved by invoking a theorem of Olech [19]. Accordingly, $I(M(X))$ is compact if the *integrand* $f: X \times [0, 1] \to R^r$ defined (as in 4.2.3) by

$$f_i(x, \tau) = \sup\{a_i' A(x, q, \tau): q \in Q\}$$

satisfies the following two conditions:

(i) $f(\cdot)$ is a Carathéodory function;

(ii) there is an integrable function $\mu: [0, 1] \to R$ such that $\|f(x, \tau)\| \leqslant \mu(\tau)$ for all $(x, \tau) \in X \times [0, 1]$.

Condition (i) is immediate from Lemma 6.2. It remains to verify (ii).

Verification of condition (ii). For this purpose, we shall take (without loss of generality, $\|\cdot\|$ to be the max norm on R^r; i.e., for $x = (x_1, x_2, \ldots, x_r) \in R^r$ $\|x\| \triangleq \max\{|x_i|: i \in \{1, 2, \ldots, r\}\}$. Now define $\tilde{\mu}: X \times [0, 1] \to R$ by

$$\tilde{\mu}(x, \tau) \triangleq \|f(x, \tau)\| \triangleq \max\{|f_i(x, \tau)|: i \in \{1, 2, \ldots, r\}\}.$$

Since we have already shown that each $f_i(\cdot)$ is a Carathéodory function, it follows easily that $\tilde{\mu}(\cdot)$ is also a Carathéodory function. Hence, we can make a *measurable selection* $x(\cdot):[0, 1] \to X$ (see p. 236 of [15]) such that

$$\tilde{\mu}(\tilde{x}(\tau), \tau) = \max\{\tilde{\mu}(x, \tau): x \in X\} = \max\{|f_i(\tilde{x}(\tau), \tau)|: i \in \{1, 2, \ldots, r\}\}.$$

We now claim that the function μ: $[0, 1] \to R$ defined by

$$\mu(\tau) \triangleq \tilde{\mu}(\tilde{x}(\tau), \tau)$$

will satisfy condition (iii).

Given any $x(\cdot) \in M(X)$, we have already shown that each $f_i(x(\tau), \tau)$ is integrable. Being a pointwise maximum of a finite collection of integrable functions $|f_1(x(\tau), \tau)|$, $|f_2(x(\tau)|, \ldots, |f_r(x(\tau), \tau)|$, we conclude that $\tilde{\mu}(x(\tau), \tau)$ is also integrable for all $x(\cdot) \in M(X)$. In particular for the measurable selection $\tilde{x}(\cdot) \in M(X)$ as above, $\tilde{\mu}(\tilde{x}(\tau), \tau) = \mu(\tau)$ is also integrable.

Finally, by construction, it follows that

$$\mu(\tau) = \tilde{\mu}(\tilde{x}(\tau), \tau) \geqslant \tilde{\mu}(x, \tau) = \|f(x, \tau)\|$$

holds for all $x \in X$ and all $\tau \in [0, 1]$.

(b) From the proof of part (a) of this theorem, it is clear that the function $V: M(X) \times S \to R$ defined by (see (4.3.3))

$$V(x(\cdot), z) \triangleq \sum_{i=1}^{r} z_i F_i(x(\cdot))$$

has at least one saddle point. We claim that any such saddle point pair $(x_0(\cdot), z_0)$ satisfies the requirements stated in the theorem.

By the definition of a saddle point, we have

$$V(x_0(\cdot), z) \leqslant V(x_0(\cdot), z_0) \leqslant V(x(\cdot), z_0) \tag{D.1}$$

for all $(x(\cdot), z) \in M(X) \times S$, and furthermore

$$V(x_0(\cdot), z_0) = \max\{V(x_0(\cdot), z): z \in S\} = F(x_0(\cdot)) \tag{D.2}$$

while

$$V(x_0(\cdot), z_0) = \min\{V(x(\cdot), z_0): x(\cdot) \in M(X)\} = F_e(z_0). \tag{D.3}$$

We also have

$$V(x(\cdot), z_0) \leqslant \max\{V(x(\cdot), z): z \in S\} = F(x(\cdot)) \tag{D.4}$$

for all $x(\cdot) \in M(X)$ and

$$V(x_0(\cdot), z) \geqslant \min\{V(x(\cdot), z): x(\cdot) \in M(X)\} = F_e(z(\cdot)) \tag{D.5}$$

for all $z \in S$. Equations (D.1,), (D.2) and (D.4) together establish that $x_0(\cdot)$ solves problem (P) while (D.1), (D.3) and (D.5) imply that z_0 solves (P_e).

It remains to verify the necessary condition. Looking at equality (D.3),

$$V(x_0(\cdot), z_0) = F_e(z_0)$$

which, upon straightforward substitution yields

$$\int_0^1 \tilde{H}(x_0(t), z_0, t)\, dt = \int_0^1 H(z_0, t)\, dt. \tag{D.6}$$

Equation (D.6) together with (6.4.2) yields the desired result. ∎

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References

1. B. R. Barmish, Robust solution of perturbed linear equations, *IEEE Trans. Automat. Control* **AC-22**, no. 7 (1977), 123–124.
2. B. R. Barmish and Y. H. Lin, A new notion of robustness for perturbed linear dynamical systems, "Proceedings of 7th IFAC World Congress, Helsinki, 1978"

3. B. R. Barmish, Robust solution of perturbed dynamical equations from within a convex restraint set, *IEEE Trans. Automat. Control* **AC-24**, No. 6 (1979), 921–926.
4. M. C. Delfour and S. K. Mitter, Reachability of perturbed systems and min-sup problems, *SIAM J. Control* **7**(1969), 521–533.
5. D. P. Bertsekas and I. B. Rhodes, On the minimax reachability of target sets and target tubes, *Automatica* **7** (1971), 233–247.
6. H. S. Witsenhausen, A minimax control problem for sampled linear systems, *IEEE Trans. Automat. Control* **AC-13** (1968), 5–21.
7. B. R. Barmish, A reachability theory for a class of uncertain dynamical systems, *in* "Proceedings 14th Allerton Conf. on Circuits and Systems," pp. 644–651, Monticello, Illinois, 1976.
8. B. R. Barmish, L. E. Blume, and S. D. Chikte, On a class of nonlinear robustness problems, *in* "Proceedings of the 17th Annual Allerton Conference on Communication Control and Computing," pp. 555–561, Monticello, Illinois, 1979.
9. L. W. Neustadt, The existence of optimal controls in the absence of convexity," *J. Math. Anal. Appl.* **7** (1963), 110–117.
10. C. Olech, Existence theory in optimal control, *in* "Control Theory and Topics in Functional Analysis," Vol. 1, pp. 291–328, International Atomic Energy Agency, Vienna, 1976,
11. E. B. Lee and L. Markus, "Foundations of Optimal Control Theory," Wiley, New York, 1976.
12. D. McRuer, I. Ashkenas, and D. Graham, "Aircraft Dynamics and Automatic Control," Princeton Univ. Press, Princeton, N.J., 1973.
13. A. F. Veinott, Jr., The supporting hyperplane method for unimodal programming, *Oper. Res.* **15**, No. 1 (1967), 147–152.
14. J. E. Kelley, Jr., The cutting plane method for solving convex programs, *SIAM Rev.* **8** (1960), 703–712.
15. I. Ekeland and R. Temam, "Convex Analysis and Variational Problems," North-Holland, Amsterdam, 1976.
16. C. Berge, "Topological Spaces," Oliver & Boyd, London, 1963.
17. R. T. Rockafellar, Integral functionals, normal integrands and measurable selections, *in* "Nonlinear Operators and Calculus of Variations," Lecture Notes in Mathematics, No. 543, Springer-Verlag, Berlin, 1976.
18. H. L. Royden, "Real Analysis," MacMillan & Co., London, 1968.
19. C. Olech, Extremal solutions of control systems, *J. Differential Equations* **2** (1966), 74–101.