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SHORT NOTE

A NONITERATIVE NUMERICAL SOLUTION FOR STEP-HEATED SEMI-INFINITE SOLID WITH TEMPERATURE DEPENDENT THERMAL PROPERTIES

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INTRODUCTION

The classical problem of transient conduction into a step-heated semi-infinite solid having temperature dependent thermal properties is often chosen as a test problem to assess the efficacy of new techniques that are developed from time to time. For example it has been used to test the heat balance integral[1], the optimal linearisation[2, 3] and the perturbation[4] methods. The basis of comparison used in all four papers is the solution reported by Yang[5] which itself is based on the method of successive approximation. Moreover, results in these and other papers on the subject are often given for a specific value of the variable property parameter. It seems that direct numerical results covering a range of variable property parameter is not available. The present note is intended to provide this information. The results cover two cases of variable property: (i) linear thermal conductivity-temperature variation (ii) linear heat capacity-temperature variation. In each case the range of variable property parameter chosen is such that it covers all likely applications.

The numerical solutions are achieved by a noniterative scheme using the method of transformation groups[6]. This technique transforms the boundary value problems into initial value problems which are subsequently integrated numerically without any iteration. Compared to iterative schemes such as finite-difference and shooting, the present scheme is faster and more efficient. The scheme is particularly attractive when the boundary value problem contains a parameter. In chemical engineering its usage appears to be limited to the contribution of Lin & Fan[7] who applied it to the boundary values problems describing the flow of power-law fluids over a flat plate and the axial diffusion in tubular flow reactors. The present work should serve to provide further exposure to the technique.

BOUNDARY VALUE PROBLEMS

Let the surface at $x = 0$ of a semi-infinite solid initially (time $t = 0$) at uniform temperature T_i be suddenly raised to temperature T_o. We consider two cases of temperature dependent property: (i) linear-thermal conductivity-temperature variation and (ii) linear heat capacity-temperature variation. For case (i) we write

$$k = k_i[1 + \beta(T - T_i)] \tag{1}$$

and for case (ii) we write

$$C = C_o[1 + \nu(T - T_o)]. \tag{2}$$

Introducing Eqs. (1) and (2) into the one-dimensional transient conduction equation and using the well known similarity transformation, the pertinent boundary problems in dimensionless form become

Case (*i*)

$$\frac{d}{d\eta}\left[(1 + H\theta)\frac{d\theta}{d\eta}\right] + 2\eta\frac{d\theta}{d\eta} = 0 \tag{3}$$

$$\eta = 0;\quad \theta = 1;\quad \eta = \infty;\quad \theta = 0 \tag{4}$$

where

$$\theta = (T - T_i)/(T_0 - T_i),\ \eta = x/2\sqrt{(\alpha t)},\ H = \beta(T_0 - T_i)$$

and

$\alpha = k_i/C$ is the thermal diffusivity.

It is possible to introduce the well known Kirchoff transformation into the original heat equation and obtain Eq. (3) in a slightly different form.

Case (*ii*)

$$\frac{d^2\theta}{d\eta^2} + 2\eta\frac{d\theta}{d\eta} + 2\,G\,\eta\,\theta\frac{d\theta}{d\eta} = 0 \tag{5}$$

$$\eta = 0;\quad \theta = 0;\quad \eta = \infty,\quad \theta = 1 \tag{6}$$

where

$$\theta = (T - T_o)/(T_i - T_o),\ \eta = x/2\sqrt{(\alpha t)},\ G = \nu(T_i - T_o)$$

and

$$\alpha = = k/C_o$$

is the thermal diffusivity. It should be noted that H is

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positive for thermal conductivity increasing with temperature and negative for thermal conductivity decreasing with temperature. The reverse is the case for G.

SOLUTION METHOD

The solutions of Eqs. (3)–(6) are sought for a range of values of parameters H and G.

Case (i)

Let us consider Eqs. (3) and (4) and introduce a new variable ϕ such that

$$\phi = 1 + H\theta \tag{7}$$

to obtain

$$\frac{d}{d\eta}\left(\phi\frac{d\phi}{d\eta}\right) + 2\eta\frac{d\phi}{d\eta} = 0 \tag{8}$$

$$\eta = 0, \quad \phi = 1 + H; \quad \eta = \infty, \quad \phi = 1. \tag{9}$$

To reduce Eqs. (8) and (9) to an initial value problem we introduce the linear transformation

$$\eta = A^{\alpha_1}\bar{\eta}, \quad \phi = A^{\alpha_2}\bar{\phi} \tag{10}$$

where α_1, α_2 are constants and A is taken as the missing initial derivative, i.e.

$$A = \pm\frac{d\phi}{d\eta}\bigg|_{\eta=0}. \tag{11}$$

In Eq. (11) the minus and plus signs apply respectively for positive and negative values of H. Introducing (10) into Eqs. (8) and (11) one obtains

$$A^{2(\alpha_2-\alpha_1)}\frac{d}{d\bar{\eta}}\left(\bar{\phi}\frac{d\bar{\phi}}{d\bar{\eta}}\right) - A^{\alpha_2}2\bar{\eta}\frac{d\bar{\phi}}{d\bar{\eta}} = 0 \tag{12}$$

and

$$A^{\alpha_2-\alpha_1}\frac{d\bar{\phi}}{d\bar{\eta}}\bigg|_{\bar{\eta}=0} = \mp A. \tag{13}$$

Imposing the restriction that Eqs. (12) and (13) must be independent of A, we have

$$\alpha_2 - 2\alpha_1 = 0, \quad \alpha_2 - \alpha_1 = 1$$

from which

$$\alpha_1 = 1, \quad \alpha_2 = 2. \tag{14}$$

Equations (12) and (13) now become

$$\frac{d}{d\bar{\eta}}\left(\bar{\phi}\frac{d\bar{\eta}}{d\bar{\eta}}\right) + 2\bar{\eta}\frac{d\bar{\phi}}{d\bar{\eta}} = 0 \tag{15}$$

$$\frac{d\bar{\phi}}{d\bar{\eta}}\bigg|_{\bar{\eta}=0} = \mp 1. \tag{16}$$

From Eq. (9) the boundary condition at $\eta = 0$ now appears as

$$\bar{\eta} = 0, \quad \bar{\phi} = H^* = (1 + H)/A^2. \tag{17}$$

To find the parameter A we invoke the boundary condition at $\eta = \infty$ from Eq. (9) to obtain

$$A = [\bar{\phi}(\infty)]^{-1/2}. \tag{18}$$

Thus, the solution procedure is as follows:

(1) Choose a value of H^*.

(2) Integrate Eq. (15) as an initial value problem with the initial conditions given by Eqs. (16) and (17) until $\bar{\phi}$ approaches a constant value which may be taken as $\bar{\phi}(\infty)$.

(3) Calculate A from Eq. (18).

(4) Using the value of A, in Eq. (17) obtain the corresponding value of H.

(5) With the aid of Eqs. (7) and (10) the solution can be transformed to the original variables θ and η.

Case (ii)

Since the procedure is essentially the same as in case (i) we give only the details which are different. We introduce the linear transformations

$$\eta = A^{\alpha_1}\bar{\eta}, \quad \theta = A^{\alpha_2}\bar{\theta} \tag{19}$$

with A defined as

$$A = \pm\frac{d\theta}{d\eta}\bigg|_{\eta=0} \tag{20}$$

where the plus and the minus signs correspond respectively to positive and negative values of G. The constants α_1 and α_2 are obtained as

$$\alpha_1 = 0, \quad \alpha_2 = 1. \tag{21}$$

The transformed equations become

$$\frac{d^2\bar{\theta}}{d\bar{\eta}^2} + 2\bar{\eta}\frac{d\bar{\theta}}{d\bar{\eta}} + G^*2\bar{\eta}\,\bar{\theta}\frac{d\bar{\theta}}{d\bar{\eta}} = 0 \tag{22}$$

$$\bar{\eta} = 0, \quad \bar{\theta} = 0; \quad \bar{\eta} = 0, \quad \frac{d\bar{\theta}}{d\bar{\eta}} = \pm 1 \tag{23}$$

where

$$G^* = GA, \quad A = [\bar{\theta}(\infty)]^{-1}. \tag{24}$$

NUMERICAL RESULTS

To illustrate the solution procedure let us consider and select $H^* = 5$. The result of step 2 (using a fourth-order Runge–Kutta scheme) appear in the first two columns of Table 1. It is seen that $\bar{\phi}$ approaches a constant value of 2.8392 which is taken as $\bar{\phi}(\infty)$. Fro Eq. (18), $A = 0.5935$ and the corresponding value of H from Eq. (17) follow as 0.7611. The solution in terms of original variables θ and η can now be obtained from Eqs. (7) and (10). This result appears in the last two columns of Table 1.

By assigning different values to H^* a set of solutions can be generated. The question of chosing the range of H^* to cover the desired range of H is easily resolved with a few trials. Since the result frequently needed is the wall temperature gradient, $d\theta/d\eta|_{\eta=0}$ we present in Table 2 its values for a range of values of H.

The corresponding results for case (ii) appear in Table 3.

Table 1. Sample solution for $H^* = 5$

$\bar{\eta}$	$\bar{\phi}$	η	θ
0.00	5.0000	0.0000	1.0000
0.55	4.4316	0.3264	0.7371
1.21	3.7539	0.7181	0.4234
2.03	3.1557	1.2048	0.1466
3.02	2.8831	1.7924	0.0204
4.23	2.8407	2.5105	0.0008
5.70	2.8392	3.3830	0.0000
7.50	2.8392	4.4513	0.0000
9.69	2.8392	5.7510	0.0000

Table 2. Wall temperature gradient for case (i)

H^*	H	$\frac{d\theta}{d\eta}\Big\|_{\eta=0}$	H^*	H	$\frac{d\theta}{d\eta}\Big\|_{\eta=0}$
0.01	-0.8345	-4.8752	10.00	0.4221	-0.8934
0.05	-0.7292	-3.1916	5.00	0.7611	-0.7798
0.10	-0.6786	-2.6419	4.00	0.9587	-0.7299
0.20	-0.6130	-2.2692	3.00	1.3636	-0.6509
0.40	-0.5404	-1.9835	2.00	2.7831	-0.4942
0.60	-0.4959	-1.8483	1.75	4.0155	-0.4216
0.80	-0.4639	-1.7646	1.50	7.8454	-0.3095

Table 3. Wall temperature gradient for case (ii)

G^*	G	$\frac{d\theta}{d\eta}\Big\|_{\eta=0}$	G^*	G	$\frac{d\theta}{d\eta}\Big\|_{\eta=0}$
0.66	-0.9021	0.7316	0.1	0.0863	1.1589
0.65	-0.8613	0.7547	1.0	0.7321	1.3659
0.60	-0.7272	0.8251	2.0	1.3096	1.5272
0.50	-0.5501	0.9089	4.0	2.2691	1.7628
0.40	-0.4131	0.9684	6.0	3.0901	1.9417
0.30	-0.2951	1.0168	8.0	3.8286	2.0896
0.20	-0.1890	1.0583	10.0	4.5100	2.2173
0.10	-0.0913	1.0951			

DISCUSSION

The results presented in Tables 2 and 3 cover a very wide range of property variation. It is unlikely that any practical application will fall beyond this range. The only drawback of the noniterative numerical results is the lack of systematic spacing of values of H and G. However, this does not pose any difficulty because a numerical or graphical interpolation can be readily carried out.

The solutions of Eqs. (3)–(6) exist for all positive values of H and G but for negative values of H and G, physical considerations dictate $H > -1$, $G > -1$. In fact, numerical experiments indicated that solutions could be generated only down to $H = -0.8345$ and $G = -0.9021$. These constitute the lower limits in Tables 2 and 3.

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