Int. J. Engng Sci. Vol. 19, pp. 1009–1014, 1981
Printed in Great Britain.

0020-7225/81/071009-06$02.00/0

ON A CLASS OF EXACT SOLUTIONS TO THE EQUATIONS OF MOTION OF A SECOND GRADE FLUID

K. R. RAJAGOPAL and A. S. GUPTA†
Department of Mechanical Engineering and Applied Mechanics, The University of Michigan, Ann Arbor, MI 48109, U.S.A.

(Communicated by S. I. PAI)

Abstract—A class of exact solutions to the equations of motion of a second grade fluid is exhibited wherein the non-linearities which occur in the equations of motion are self-cancelling though individually non-vanishing. These flows are those in which the vorticity and the Laplacian of the vorticity remain constant along stream lines.

1. INTRODUCTION

THE NON-LINEARITIES which occur in the Navier–Stokes equations severely restrict the class for which exact solutions can be established. However, there exists a general class of flows wherein these non-linearities are self-cancelling though the individual terms are non-vanishing. The first such solution where the non-linearities are self-cancelling was exhibited by Taylor[1] in his investigations of the decay of a double array of vortices. Another such situation where a similar cancellation of the non-linearities occur is the steady flow behind a 2-dimensional grid which was studied by Kovasznay[2]. Wang[3] found a general class of flows where these non-linearities are self-cancelling and showed that the results of Taylor[1] and Kovasznay[2] belonged to that class.

For a certain class of non-Newtonian fluids, namely the homogenous incompressible Rivlin–Ericksen fluids of second grade (Coleman and Noll[4]), it was shown by Gupta[5] that the higher order non-linearities which occur in the equations of motion are also self-cancelling for the specific problems studied by Taylor and Kovasznay mentioned above.

In this note, following Wang[3], we exhibit exact solutions for a general class of flows of a homogenous incompressible second grade fluid. It is found that these exact solutions form a sub-class of the solutions exhibited by Wang[3] for the Navier–Stokes equation. It turns out that these exact solutions exist for plane flows where the vorticity and the Laplacian of the vorticity remain constant along stream lines.

Thus, it is found that the homogenous incompressible second grade fluid displays a truly remarkable property in that, in addition to sharing the same unique solution for the velocity field as the classical linearly viscous fluid for slow steady flows (Tanner[6], Huilgol[7], Fosdick and Rajagopal[8]), it also admits exact solutions in the case of certain class of unsteady flows which are not necessarily slow. In fact, the second grade fluid shares yet another remarkable property in common with the classical linearly viscous fluid in that it is completely characterized by viscometric flows (Truesdell[9]), which is not true for non-Newtonian fluids in general. It should, however, be noted that a thermodynamically compatible Rivlin–Ericksen fluid of grade three also shares the above property (Rajagopal[10]).

2. EQUATIONS OF MOTION

The Cauchy stress $\mathbf{T}$ in a homogenous incompressible Rivlin–Ericksen fluid of second grade is related to the fluid motion in the following form

$$\mathbf{T} = -p\mathbf{1} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{2.1}$$

where μ is the coefficient of viscosity, α_1 and α_2 are the normal stress moduli and $-p\mathbf{1}$ denotes

†Permanent address: Department of Mathematics, Indian Institute of Technology, Kharagpur, India.

the indeterminate spherical stress. Here $\mathbf{A}_1$ and $\mathbf{A}_2$ are the kinematic tensors defined through

$$\mathbf{A}_1 = \text{grad } \mathbf{v} + (\text{grad } \mathbf{v})^T,$$

and

$$\mathbf{A}_2 = \dot{\mathbf{A}}_1 + \mathbf{A}_1 L + \mathbf{L}^T \mathbf{A}_1, \tag{2.2$_{1,2}$}$$

where $\mathbf{v}$ denotes the velocity field and the dot represents material time differentiation.

Substitution of (2.1) into the balance of linear momentum gives in the absence of body forces

$$p\Delta\mathbf{v} + \alpha_1\Delta\mathbf{v}_t + \alpha_1(\Delta\mathbf{w} \times \mathbf{v})(\alpha_1 + \alpha_2)\{\mathbf{A}_1\Delta\mathbf{v} + 2 \text{ div } [(\text{grad } \mathbf{v})(\text{grad } \mathbf{v})^T]\} - \rho\mathbf{v}_t - \rho(\mathbf{w} \times \mathbf{v}) = \text{grad } \hat{p}, \tag{2.3}$$

where Δ denotes the Laplacian operator, the subscript t denotes partial derivative with respect to time, and

$$\mathbf{w} = \text{curl } \mathbf{v}, \tag{2.4}$$

$$\hat{p} = p - \alpha_1\mathbf{v} \cdot \Delta\mathbf{v} - \frac{1}{4}(2\alpha_1 + \alpha_2)|\mathbf{A}_1|^2 + \frac{1}{2}\rho|\mathbf{v}|^2. \tag{2.5}$$

In the above equation $|\mathbf{v}|^2$ denotes the usual norm for vectors and $|\mathbf{A}_1|^2$ denotes the usual trace norm for tensors.

If one takes the curl of (2.3), in the case of plane flows it is found that

$$\mu\Delta w + \alpha_1\Delta w_t + \alpha_1\left\{u\frac{\partial(\Delta w)}{\partial x} + v\frac{\partial(\Delta w)}{\partial y}\right\} - \rho w_t - \rho\left\{u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y}\right\} = 0. \tag{2.6}$$

We now use Δ to denote the 2-dimensional Laplacian operator, and u and v are the x and y components of the velocity, and

$$\mathbf{w} = w\mathbf{k},$$

$\mathbf{k}$ being the unit vector normal to the plane of flow.

For plane flows, we introduce the stream function $\psi(x, y)$ through

$$u = \frac{\partial\psi}{\partial y}, \tag{2.7$_1$}$$

and

$$v = -\frac{\partial\psi}{\partial x}, \tag{2.7$_2$}$$

and rewrite eqn (2.6) in the following form

$$\mu(\Delta^2\psi) + \alpha_1\Delta^2\psi_t + \alpha_1\left\{\frac{\partial\psi}{\partial y}\frac{\partial\Delta^2\psi}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\Delta^2\psi}{\partial y}\right\} - \rho\Delta\psi_{,t} - \rho\left\{\frac{\partial\psi}{\partial y}\frac{\partial\Delta\psi}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\Delta\psi}{\partial y}\right\} = 0, \tag{2.8}$$

where Δ^2 denotes the usual biharmonic operator.

We now observe from (2.8) that if the stream function ψ is such that

$$\Delta\psi = f(\psi), \tag{2.9}$$

and

$$\Delta^2\psi = g(\psi), \tag{2.10}$$

where f and g are arbitrary functions, then the non-linearities which occur in (2.8) are self-cancelling and the equation becomes linear. Equations (2.9) and (2.10) imply that the vorticity and the Laplacian of vorticity remain constant along stream lines. If (2.9) and (2.10) hold, then (2.8) reduces to

$$\mu(\Delta^2\psi)+\alpha_1(\Delta^2\psi_t)-\rho\Delta\psi_t=0. \tag{2.11}$$

The constitutive assumption (2.1) can be considered to be the second order approximation to the response functional of a simple fluid in the sense of retardation (Coleman and Noll[4]). On the other hand, since the constitutive model is properly invariant it could be thought of as an exact model, as is done, for example in the case when $\alpha_1=\alpha_2=0$, the classical linearly viscous theory. If this exact model represented by (2.1) is required to be compatible with thermodynamics in the sense that all arbitrary motions of the fluid satisfy the Clausius–Duhem inequality and the assumption that the specific Helmholtz free energy ψ of the fluid be a minimum when the fluid is locally at rest, i.e.

$$\psi(\theta,\mathbf{A}_1,\mathbf{A}_2)\geq\psi(\theta,0,0),$$

for all symmetric tensors $\mathbf{A}_1$ and $\mathbf{A}_2$, then it follows that[11]

$$\mu\geq 0,\ \alpha_1\geq 0 \text{ and } \alpha_1+\alpha_2=0. \tag{A}$$

Of course, this does not preclude the possibility of an approximate model with $\alpha_1<0$ and $\alpha_1+\alpha_2\neq 0$. The results expressed in (A) are the subject of much controversy and this involves the work of Coleman, Dunn, Fosdick, Mizel, Rajagopal, Ting and Truesdell. We refer the reader to Dunn and Fosdick[11] for a brief review of the same. Recently, the investigations of Fosdick and Rajagopal[12] and Rajagopal and Wineman[13] have also addressed themselves to questions regarding the validity of eqn (A). However, eqn (A) has not been employed in obtaining (2.6). It just happens that in plane flows the terms multiplied by $(\alpha_1+\alpha_2)$ are self-cancelling.

In this note, without restricting ourselves to either point of view we shall investigate the implications and the consequences of both these differing viewpoints.

We proceed to show that the class of problems which are solutions to the eqns (2.9)–(2.11) includes all the interesting examples considered by Wang [3] in the case of the Navier–Stokes equation. Our problem is different from Wang's in that in addition to the extra term $\Delta^2\psi_t$ in eqn (2.11), we have an additional eqn (2.10) to be met. It so happens that the examples found by Wang[3] are such that (2.10) is met automatically since in all his examples $f(\psi)$ is a linear function of ψ. We also consider an example where $f(\psi)$ is not a linear function of ψ.

Analysis

We wish to construct solutions for physically meaningful problems wherein the stream function ψ satisfies (2.9)–(2.11). First, we shall consider the class of flows where

$$\Delta\psi=f(\psi)=A\psi, \tag{2.12}$$

where A is a constant. It follows that

$$\Delta^2\psi=A\Delta\psi=A^2\psi=g(\psi),$$

and hence a choice of $f(\psi)$ of the form shown in (2.12) meets the conditions expressed by (2.9) and (2.10).

Equations (2.9)–(2.11) with $f(\psi)=A(\psi)$ yield solutions of the form

$$\psi=Re\left\{\sum_n C_n\exp(\alpha_n x)\exp(\beta_n y)\exp(\lambda t)\right\}, \tag{2.13}$$

where C_n, α_n and β_n are complex constants and Re stands for the real part of the expression within the parenthesis.

The following interesting flows belong to the class of exact solutions represented by (2.13)

(1) The decay of vortices represented by

$$\psi = A \cos mx \cos ny \, e^{-\lambda t}. \tag{2.14}$$

Equation (2.14) represents a system of eddies in which each is rotating in the opposite direction to that of its four neighbors. Now, eqn (2.14) is compatible with eqn (2.11) if

$$\frac{1}{\lambda} = \frac{\alpha_1}{\mu} + \frac{\rho}{\mu(m^2+n^2)}.$$

The case $m = n$ was studied by Gupta[4] for $\alpha_1 \neq 0$ and by Taylor[1] for $\alpha_1 = 0$. Also the case $m \neq n$ was studied by Rajagopal[14] for a fluid for which (2.1) is regarded as an exact model. Since the characteristic time of decay is $0(1/\lambda)$, the above expression shows the smaller the size of vortices, the faster is the decay in a given fluid. Further, for vortices of given size the decay is hastened for $\alpha_1 < 0$ (provided λ remains positive) and is delayed if $\alpha_1 > 0$.

If one takes the point of view that the model represented by (2.1) is exact and that the restrictions based on eqn (A) hold, then it follows that $\lambda > 0$. On the other hand, if the constitutive relation (2.1) considered as an approximate model up to second order in the time scale to the flow of a simple fluid with fading memory, for the validity of such a model, the characteristic time scale $1/|\lambda|$ of motion in this problem should be sufficiently large compared with the time scale $|\alpha_1/\mu|$ characterizing the memory of the fluid. This demands that $\rho(m^2+n^2)^{-1} \gg |\alpha_1|$ so that $\lambda > 0$. Thus, in either case there is decay of vortices.

(2) Direct impingement of two rotational flows represented by

$$\psi = A \sinh ax \sinh by \, e^{\lambda t}. \tag{2.15}$$

If the stream function ψ given by (2.15) is to satisfy (2.9) and (2.11), it then follows from a straightforward computation that

$$\lambda = \frac{\mu(a^2+b^2)}{\rho - \alpha_1(a^2+b^2)}. \tag{2.16}$$

If one is interested in solutions to the problem of direct impingement of two rotational flows of increasing strength, i.e. $\lambda > 0$, then it follows from (2.16) that for the exact model which meets the restrictions due to eqn (A), such flows are possible only if

$$\alpha_1 < \frac{\rho}{(a^2+b^2)}.$$

If the model is considered as an approximate one with $\alpha_1 < 0$, then $\lambda > 0$. If one is interested in the physical problem of the direct impingement of two rotational flows of decreasing strength, i.e. $\lambda < 0$, then such a flow is not possible since λ given by (2.16) is always positive.

(3) A viscoelestic analog to Kelvin's "Cat's eye" vortices represented by

$$\psi = A \cosh ax \cos by \, e^{\lambda t}. \tag{2.17}$$

Substitution of eqn (2.17) in eqn (2.11) gives

$$\lambda = \frac{\mu(a^2-b^2)}{\rho - \alpha_1(a^2-b^2)}. \tag{2.18}$$

Once again as before one can obtain conditions under which a solution of the form of (2.17) can exist for the exact or approximate model.

All the above exact solutions are obtained when the vorticity $f(\psi)$ is a linear function of ψ, namely $f(\psi) = A\psi$. A common feature of all these exact solutions is that the stream function ψ is of the separable form.

We next investigate the problem when $f(\psi)$ is not a linear function of ψ. We show that in this case the exact solution cannot be of the separable form. Suppose, if possible†

$$\psi(x, y) = X(x)Y(y),$$

then it follows from eqn (2.9) that

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = \frac{f(XY)}{XY}. \tag{2.19}$$

This implies that

$$X'' = C_1X, \quad Y'' = C_2Y \text{ and } f = (C_1 + C_2)\psi, \tag{2.20}$$

where C_1 and C_2 are constants. The last of eqns (2.20) contradicts the fact that f is not linear in ψ. Hence our assertion is proved.

Thus, to consider a problem when $f(\psi)$ is non-linear we have to look for a solution which is not of the separable form. Consider, for instance the following expression for the stream function.

$$\psi(x, y) = (x + y)^3. \tag{2,21}$$

It follows trivially that

$$\Delta\psi = f(\psi) = 12(x + y) = 12\psi^{1/3},$$

and clearly $f(\psi)$ is not a linear function of ψ. Also

$$\Delta^2\psi = g(\psi) = 0.$$

Hence the conditions required by eqns (2.9) and (2.10) are met and a stream function of the form (2.21) is a solution to the equations of motion for the steady plane flow of a second grade fluid. Physically the above problem represents a situation wherein both the vorticity and velocity are constant along stream lines represented by $x + y = \text{constant}$, which constitute a 1-parameter family of parallel stream lines.

It is worth observing that by virtue of the chain rule, (2.9) and (2.10) can be replaced by the equivalent conditions

$$\Delta\psi = f(\psi)$$

and

$$\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2 = \hat{f}(\psi).$$

Thus, all the problems considered thus far are problems wherein the vorticity and velocity are constant along stream lines.

There are several other physically meaningful problems where "exact solutions" can be exhibited due to the non-linearities in the equations being self-cancelling when $f(\psi)$ is not linear. The example expressed by eqn (2.21) is valid for steady flows wherein $f(\psi)$ is a non-linear function of ψ.

†We consider the form of the stream function corresponding to a steady problem. Similar results can be established if the problem is not steady.

REFERENCES

[1] G. I. GAYLOR, *Philosophical Magazine.* Series 6, **46**, 671 (1923).
[2] L. I. KOVASZNAY, *Cambridge Philos. Soc.* **44**, 58 (1948).
[3] C. Y. WANG, *J. Appl. Mec.*, **33**, 696 (1966).
[4] B. D. COLEMAN and W. NOLL, *Arch. Ratl. Mech. Anal.* **6**, 355 (1960).
[5] A. S. GUPTA, *Meccanica* **7**, 232 (1972).
[6] R. I. TANNER, *Phys. Fluids* **9**, 1246 (1966).
[7] R. R. HUILGOL, *SIAM J. Appl. Math* **24**, 226 (1973).
[8] R. L. FOSDICK and K. R. RAJAGOPAL, *Int. J. Non-Linear Mech.* **13**, 131 (1978).
[9] C. TRUESDELL, *Phys. Fluids* **7**, 1134 (1964).
[10] K. R. RAJAGOPAL, *Mech. Res. Commun.* **7**, 21 (1980).
[11] J. E. DUNN and R. L. FOSDICK, *Arch. Ratl. Mech. Anal.* **56**, 191 (1974).
[12] R. L. FOSDICK and K. R. RAJAGOPAL, *Arch. Ratl. Mech. Anal.* **70**, 145 (1979).
[13] K. R. RAJAGOPAL and A. S. WINEMAN, *J. Engng Sci.* **19**, 237 (1981).
[14] K. R. RAJAGOPAL, *On the Decay of Vortices in a Second Grade Fluid.* To be published.

(*Received* 21 *July* 1980)