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Symmetry and Symmetry Breaking in Generalized Parastatistics

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An analysis is made of the characteristics of internal symmetry and symmetry breaking in a quantum field theory with generalized parastatistics, defined by either double commutation relations or single commutation relations. The connection between the two statistics is clarified. We develop a formalism in which statistics is viewed as a dynamical or phase variable of quantum systems. It is shown that the types of Higgs phases possible depend upon statistics. Relationships between physical amplitudes implied by internal symmetry with normal statistics are violated in the case of generalized parastatistics.

I. INTRODUCTION

A physical system is described by a set of equations of motion. In quantum field theory further appropriate commutation relations among fields are imposed as extra conditions. While the general axioms of quantum field theory imply the standard spin-statistics commutation relations for normal bosons and fermions, those used for existing particles, there is no a priori reason not to have a more general set of commutation relations appearing in nature. In fact, starting with only Heisenberg's equations of motion, Green proposed a general set of commutation relations known as parastatistics [1]. In this theory the fundamental relations are double commutation relations with fields, $\phi(x)$, satisfying the relations, being expressed in terms of Green components, $\phi_i(x)$, as

$$\phi(x) = \sum_{i=1}^{p} \phi_i(x), \qquad p = \text{integer}, \tag{1.1}$$

where each $\phi_i(x)$ satisfies canonical relations for the same index and

$$[\phi_i(x), \phi_j(y)]_\pm = 0, \qquad (i \neq j) \begin{cases} + \text{ bosons} \\ - \text{ fermions} \end{cases} \tag{1.2}$$

Recently, the authors proposed a more general set of parastatistical commutation relations as a solution of Heisenberg's equation of motion, hereafter called generalized parastatistics [2] or general statistics. This formulation allows the elimination of the Green Ansatz constraint, Eq. (1.1), resulting in the identification of

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an internal symmetry index with the Green indices. In Ref. [2] it is noted that the generalized parastatistical commutation relations are covariant under internal symmetry transformations instead of invariant as in the case of normal commutation relations.

In this article we show it is the covariant nature of the commutation relations which breaks internal symmetry existing in the case of normal statistics. Starting with a Lagrangian, symmetric under an internal symmetry group transformation of classical fields, diffeent generalized parastatistical commutation relations can be assumed for the quantum theory provided locality is satisfied. By locality we mean interaction terms commuting at spacelike separation. Each set of commutation relations breaks the symmetry existing with normal statistics in a unique manner. This can be viewed as a new symmetry-breaking mechanism or *symmetry transmutation* which, unlike the Higgs mechanism, has no classical analogue. This formalism has possible applications to symmetry breaking in particle theory, such as the quark–lepton generation problem or supersymmetry, as well as in statistical mechanical phase transitions between systems of different statistics.

In Section II we briefly recapitulate the characteristics of generalized parastatistics. In Section III we develop the field theory for generalized parastatistics in a functional integral form and discuss possible physical realizations of different statistics. In Section IV we exhibit the canonical formalism for a generalized parastatistical field theory. Section V contains a discussion of symmetry transmutation with a model $SO(N)$ ϕ^4 theory. In Section VI the constraint of locality on possible interaction terms with an internal symmetry is discussed. A general summary and discussion follows in Section VII. In Appendices A, B, and C the most general form of statistics allowed for real and complex scalar fields, obeying simple commutation relations, is derived. In Appendix D it is shown that momentum-dependent commutation relations exist for which an $SU(N)$ invariant interaction term is local.

II. Generalized Parastatistics

Starting only with the requirement of Heisenberg's equation of motion

$$i[H, \phi_j(x)] = \dot{\phi}_j(x), \tag{2.1a}$$

where j represents an internal symmetry index, and

$$H_0 = \frac{1}{4} \sum_j \int d^3x \left(\{\dot{\phi}_j(x), \dot{\phi}_j(x)\} + \{\vec{\nabla}\phi_j(x), \vec{\nabla}\phi_j(x)\} + m^2 \{\phi_j(x), \phi_j(x)\} \right) \tag{2.1b}$$

one can assume general commutation relations for real parabosons,

$$[\{\phi_i(x), \dot{\phi}_j(x')\}, \phi_l(x'')]|_{t=t'=t''} = -2i\delta(\vec{x}' - \vec{x}'') \sum_n \gamma_n^{ijl} \phi_n(x), \tag{2.2a}$$

$$[\{\dot{\phi}_i(x), \dot{\phi}_j(x')\}, \phi_l(x'')]|_{t=t'=t''} = -2i\delta(\vec{x}' - \vec{x}'') \sum_n \gamma_n^{ijl} \dot{\phi}_n(x) \\ -2i\delta(\vec{x} - \vec{x}'') \sum_n \gamma_n^{jil} \dot{\phi}_n(x') \quad (2.2b)$$

$$[\{\phi_i(x), \phi_j(x')\}, \dot{\phi}_l(x'')]|_{t=t'=t''} \\ = 2i\delta(\vec{x}' - \vec{x}'') \sum_n \gamma_n^{ijl} \phi_n(x) + 2i\delta(\vec{x} - \vec{x}'') \sum_n \gamma_n^{jil} \phi_n(x'), \quad (2.2c)$$

$$[\{\phi_i(x), \phi_j(x')\}, \phi_l(x'')]|_{t=t'=t''} = 0, \quad (2.2d)$$

with constraints

$$\sum_i \gamma_n^{iil} = \delta_{ln} \quad (2.3)$$

and

$$\gamma_n^{ijl} = \gamma_n^{ilj}. \quad (2.4)$$

Here the generalized matrix γ_n^{ijl} is real for neutral scalar fields $\phi_j(x)$. Equations (2.3) and (2.4) derive from Eqs. (2.2) and the Jacobi identity involving $\phi_i(x)$, $\dot{\phi}_j(x')$ and $\phi_l(x'')$ at equal time, respectively. Similar equations can be assumed for parafermions with the anticommutators in Eqs. (2.2) replaced by commutators.

A special solution for γ_n^{ijl} is gotten by assuming, in Eqs. (2.2),

$$\phi_i(x) = \sum_{j=1}^{N} \alpha_{ij} \phi_j^g(x), \quad (2.5)$$

where $\phi_j^g(x)$ are Green components satisfying canonical commutation relations and Eq. (1.2), and where α is an orthogonal matrix. For Eq. (2.5) substituted into Eqs. (2.2), the resulting form for γ_n^{ijl} is

$$\gamma_n^{ijl} = \sum_m \alpha_{im} \alpha_{jm}^* \alpha_{lm} \alpha_{nm}^*. \quad (2.6)$$

In Ref. [2] it was proven that the form in Eq. (2.6) follows from Eqs. (2.2), (2.3), (2.4) and the symmetry relation

$$\gamma_n^{ijl} = \gamma_n^{lji}, \quad (2.7)$$

with appropriate properties of the physical vacuum. It is important to note that different vilinear commutation relations for $\phi_j^{(g)}$ other than Eq. (1.2) may result in a different form for γ_n^{ijl} satisfying neither Eq. (2.6) nor Eq. (2.7).

The commutation relations in Eq. (2.2) are covariant in the sense that an orthogonal transformation on $\phi_j(x)$,

$$\phi_i'(x) = \sum_j \Lambda^{ij} \phi_j(x), \quad (2.8)$$

induces the transformation

$$(\gamma')^{ijl}_n = \sum_{i'j'l'n'} A_{ii'} A^*_{jj'} A_{ll'} A^*_{nn'} \gamma^{i'j'l'}_{n'}. \tag{2.9}$$

That is, the set $\{\phi'_i\}$ satisfies Eqs. (2.2) with γ^{ijl}_n replaced by $(\gamma')^{ijl}_n$. Note that the canonical commutation relations, those associated with normal statistics, are invariant under such transformations.

Given a set of real boson fields (an extension to complex fields is in Appendix C), $\{\phi_\alpha\}$, and fermion fields, $\{\psi_i\}$, define a bilinear statistics, $[\sigma]$, to be a set of numbers $\{\rho_{\alpha\beta}, \rho'_{ij}, \rho''_{i\alpha}\}$ satisfying

$$\{\rho_{\alpha\beta}, \rho'_{ij}, \rho''_{i\alpha}\} = \{\pm 1\}, \tag{2.10a}$$

$$\rho_{\alpha\beta} = \rho_{\beta\alpha}, \rho'_{ij} = \rho'_{ji}, \rho''_{i\alpha} = \rho''_{\alpha i}, \tag{2.10b}$$

and

$$\rho_{\alpha\alpha} = \rho'_{ii} = 1. \tag{2.10c}$$

The corresponding fields satisfy the equations

$$(\phi_\alpha(x)\, \dot{\phi}_\beta(x') - \rho_{\alpha\beta} \dot{\phi}_\beta(x')\, \phi_\alpha(x))_{t=t'} = i\delta(\vec{x} - \vec{x}')\, \delta_{\alpha\beta}, \tag{2.11a}$$

$$(\phi_\alpha(x)\, \phi_\beta(x') - \rho_{\alpha\beta} \phi_\beta(x')\, \phi_\alpha(x))_{t=t'} = 0, \tag{2.11b}$$

$$(\psi_i(x)\, \psi^\dagger_j(x') + \rho'_{ij} \psi^\dagger_j(x')\, \psi_i(x))_{t=t'} = \delta(\vec{x} - \vec{x}')\, \delta_{ij}, \tag{2.11c}$$

$$(\psi_i(x)\, \psi_j(x') + \rho'_{ij} \psi_j(x')\, \psi_i(x))_{t=t'} = 0, \tag{2.11d}$$

and

$$(\psi_i(x)\, \phi_\alpha(x') - \rho_{\alpha i} \phi_\alpha(x')\, \psi_i(x))_{t=t'} = 0, \qquad \text{etc.} \tag{2.11e}$$

Note that fields $\{\phi_\alpha\}$, $\{\psi_i\}$, satisfying Eqs. (2.10) and (2.11), are solutions of Heisenberg's equation of motion, Eq. (2.1), with H_0 the kinetic energy part of the Hamiltonian.

The two types of statistics represented by double commutators which are covariant, Eqs. (2.2), and those by single commutators, Eqs. (2.11), both satisfy Heisenberg's equation of motion, Eq. (1.2). Defining Γ_d to be the set of statistics represented by double commutators and Γ_s the set represented by single commutators, we show the relation between $\Gamma_s$and Γ_d in Fig. 1; that is, neither set is included in the other and $\Gamma_s \cap \Gamma_d$ is not empty. Namely, there are sets of fields satisfying single commutation relations but not double and vice versa. We derive a relationship

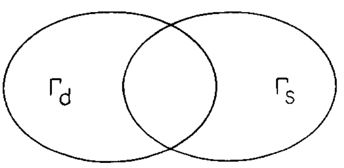

FIG. 1. Sets of theories, Γ_d and Γ_s, which satisfy the double and single commutation relations, Eqs. (2.2) and Eqs. (2.11), respectively.

between $\{\rho_{\alpha\beta}\}$ and $\{\gamma_\mu^{\alpha\beta\kappa}\}$ for statistics in the intersection $(\Gamma_s \cap \Gamma_d)$ in the case of real bosons, as well as a constraint on $\{\rho_{\alpha\beta}\}$ to be in the intersection.

Assume $\{\phi_\alpha\}$ satisfies Eqs. (2.2) and (2.11) $([\sigma] \in \Gamma_s \cap \Gamma_d)$. Then, substituting directly Eqs. (2.11a) and (2.11b) into Eq. (2.2a), one has the equation for $\vec{x} \neq \vec{x}' \neq \vec{x}''$, $t = t' = t''$,

$$[\{\phi_\alpha(x), \dot{\phi}_\beta(x')\}, \phi_\kappa(x'')] = (1 + \rho_{\alpha\beta})(1 - \rho_{\alpha\kappa}\rho_{\kappa\beta}) \times \phi_\alpha(x)\, \dot{\phi}_\beta(x')\, \phi_\kappa(x'') = 0, \tag{2.12}$$

which implies the constraint

$$(1 + \rho_{\alpha\beta})(1 - \rho_{\alpha\kappa}\rho_{\kappa\beta}) = 0. \tag{2.13}$$

From Eqs. (2.10) the result is that if

$$\rho_{\alpha\beta} = 1, \tag{2.14}$$

then

$$\rho_{\alpha\kappa} = \rho_{\kappa\beta} \tag{2.15}$$

for all κ. That is, if ϕ_α and ϕ_β commute at spacelike separated points, they must have the same commutation relations (commute or anticommute) with other fields. As shown in Fig. 2, this implies the grouping of fields into equivalence classes defined by mutual commutivity such that elements of different equivalence classes anticommute.

In the case that Eq. (2.13) is satisfied, we derive a relation between $\gamma_\mu^{\alpha\beta\kappa}$ and $\rho_{\alpha\beta}$ by substitution of Eqs. (2.11a) and (2.11b) into Eq. (2.2), yielding

$$\gamma_\mu^{\alpha\beta\kappa} = \tfrac{1}{2}(1 + \rho_{\alpha\beta})\, \rho_{\alpha\kappa}\, \delta_{\beta\kappa}\, \delta_{\alpha\mu}. \tag{2.16}$$

The same equations, (2.13) and (2.16), hold for fermions with $\alpha, \beta, \kappa, \mu$ replaced by i, j, l, n. In this case the equivalence classes in $\Gamma_s \cap \Gamma_d$ are defined by mutual anticommutation between fermion fields with commutivity between different equivalence classes. With the choice of appropriate relative double commutation relations between bosons and fermions compatible with single relative commutators, $\{\rho''_{i\alpha}\}$, an equivalence class structure exists. In this case normal statistics exists for bosons and fermions in each class with abnormal statistics between the classes.

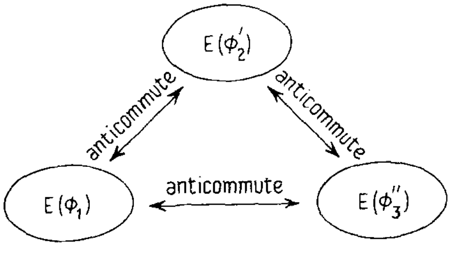

FIG. 2. Equivalence class of commuting boson fields, $E(\phi_\alpha) = \{\phi_\beta | \rho_{\alpha\beta} = 1\}$, with statistics satisfying both single and double commutation relations, $\Gamma_s \cap \Gamma_d$.

In the remainder of this paper we consider mostly field theories built on statistics satisfying single commutation relations, Γ_s. An interesting problem is the construction and characteristics of field theories with any statistics in $\Gamma_s \cup \Gamma_d$, especially in light of the non-invariance of these statistics under unitary transformations. Due to this property, generalized parastatistics $(\Gamma_s \cup \Gamma_d)$ intrinsically breaks an internal symmetry existing with normal statistics and assumed in the classical Lagrangian. By intrinsic symmetry breaking we mean the relationships between various amplitudes implied by internal symmetry, such as isospin invariance, are replaced by a new set of relations; each depending uniquely upon the commutation relations assumed. We will discuss this further in Section V.

In Ref. [2] an analysis was made of the statistics implied by various invariant interaction terms due to the constraint of locality; that is, commutivity at spacelike separation. For example, the $SU(2)$ invariant Yukawa interaction,

$$\varepsilon_{ijk} \bar{\psi}^i \psi^j \phi^k, \tag{2.17}$$

where ψ^i and ϕ^i are in the adjoint representation, satisfies locality with any set of statistics $\{\rho\}$ including normal or *maximal* parastatistical commutation relations. By maximal parastatistical commutation relations we mean equal-time commutation relations for fields with different indices given by

$$[\psi_i, \psi_j] = 0, \qquad \{\phi_i, \phi_j\} = 0, \qquad \{\psi_i, \phi_j\} = 0, \qquad (i \neq j). \tag{2.18}$$

On the other hand, the interaction terms invariant under $SO(N)$ or $SU(N)$ transformations,

$$\bar{\psi}_i \psi_j \phi^{ij} \tag{2.19}$$

and

$$(\phi_i^\dagger \phi_i)^2, \tag{2.20}$$

are local with the appropriate assignment of generalized bilinear statistics between different fields. In particular, note that Eq. (2.20) allows arbitrary sign assignment, $\{\rho_{\alpha\beta}\}$, due to repeated indices while Eq. (2.19) allows arbitrary signs among fermions as long as ϕ^{ij} satisfies commutation relations of a composite-rule type [2]. In Section VI we discuss further the locality properties of invariant interaction terms, as well as conditions on statistics for fields which take vacuum expectation values. A *normal* Higgs mechanism is defined by fields taking VEV which are real numbers. In this case such fields must commute with all other fields for consistency.

III. Statistics as Dynamical or Phase Variable

A quantum system can be described by a functional integral over field fluctuations weighted by the exponentiated action which is called the generating functional. The statistics of the integration variables is imposed as an extra condition in the form of the commutivity of field variables, external fields, and their differentiation. In light of Section II, statistics is not determined by equations of motion nor locality and therefore may be viewed as a dynamical variable. The observed statistics may then be a result of physical principles such as energy minimalization.

In order to formulate this idea consider the generating functional extended to include statistics,

$$Z[\{J_i, j_\alpha\}, [\sigma]] = \int \mathscr{D}(\bar{\psi}, \psi, \phi) \exp\left[\int (\mathscr{L}(\psi, \phi) + \bar{J}_i \psi_i + \bar{\psi}_i J_i + j_\alpha \phi_\alpha)\, d^4x\right], \quad (3.1)$$

where J_i, j_α are external sources for spin $\frac{1}{2}$ and spin 0 fields with Green index (i) and (α), respectively. The new variable $[\sigma]$ represents a set of commutation relations, Eq. (2.11), which are consistent with locality of interactions in $\mathscr{L}(\phi, \psi)$. In other words, each $[\sigma]$ represents generalized parastatistical bilinear commutation rules which are reflected in the set of numbers $\{\rho_{\alpha\beta}, \rho'_{ij}, \rho''_{i\alpha}\}$ defined earlier, and is used in the exchange of the order of differentiation:

$$\frac{\delta}{\delta j_\alpha} \frac{\delta}{\delta j_\beta} = \rho_{\alpha\beta} \frac{\delta}{\delta j_\beta} \frac{\delta}{\delta j_\alpha}, \quad (3.2a)$$

$$\frac{\delta}{\delta J_i} \frac{\delta}{\delta J_j} = -\rho'_{ij} \frac{\delta}{\delta J_j} \frac{\delta}{\delta J_i}, \quad (3.2b)$$

$$\frac{\delta}{\delta J_i} \frac{\delta}{\delta j_\alpha} = \rho''_{i\alpha} \frac{\delta}{\delta j_\alpha} \frac{\delta}{\delta J_i}, \qquad \text{etc.} \quad (3.2c)$$

For the Legendre transformed generating functional, $\Gamma(\phi_\alpha, \psi_i, \bar{\psi}_j)$, one needs rules for differentiation given by

$$\frac{\delta}{\delta \phi_\alpha} \frac{\delta}{\delta \phi_\beta} = \rho_{\alpha\beta} \frac{\delta}{\delta \phi_\beta} \frac{\delta}{\delta \phi_\alpha}, \quad (3.3a)$$

$$\frac{\delta}{\delta \psi_i} \frac{\delta}{\delta \psi_j} = -\rho'_{ij} \frac{\delta}{\delta \psi_j} \frac{\delta}{\delta \psi_i}, \quad (3.3b)$$

where ϕ_α, ψ_i are the expectation values of the fields. Note that the spin-statistics relationship, derivable from microcausality and positive energy requirements, implies

$$\rho_{\alpha\alpha} = \rho'_{ii} = 1. \quad (3.4)$$

The formulation, represented by Eq. (3.1), is an entirely new way of viewing the meaning of statistics in that $[\sigma]$ plays the role of a dynamical or phase variable as a state label, as well as a boundary condition. Different values of $[\sigma] = [\sigma_1], \ldots, [\sigma_n]$ all correspond to physically allowed and available states as long as locality is statisfied. (The latter is required for consistency of the theory.) Then the solution, $[\sigma_0]$, corresponding to the observed statistics in nature, is that which gives the minimum vacuum energy. A model $SO(3)$ ϕ^4 theory (utilizing the Higgs mechanism) can be constructed in which exactly this condition determines the statistics [3].

Deferring details of the formulation to later sections, some general remarks are in order. In the non-interacting theory all propagators are diagonal and independent of $[\sigma]$, therefore the Feynman rules for interacting systems differ only in the vertex rules. It will be shown explicitly that a choice of statistics other than the normal canonical type leads to the breaking of internal symmetry relations such as isospin invariance. This symmetry breaking is intrinsically quantum mechanical and intimately related to the choice of statistics which minimizes the vacuum energy. As a preliminary consideration the canonical formalism for bilinear generalized parastatistics will be shown in the next section.

IV. Canonical Formalism for Generalized Parastatistics

Generalized parastatistics requires a modification of the standard operator definitions and the LSZ reduction in the canonical formalism of quantum field theory. For simplicity in what follows we will consider only real scalar bosons.

Given $\phi_\alpha(x)$, $\alpha = 1, 2, \ldots, N$, define the time-ordered product

$$T(\phi_\alpha(x)\,\phi_\beta(y)) \equiv \theta(x^0 - y^0)\,\phi_\alpha(x)\,\phi_\beta(y) + \rho_{\alpha\beta}\,\theta(y^0 - x^0)\,\phi_\beta(y)\,\phi_\alpha(x), \tag{4.1}$$

where $\rho_{\alpha\beta}$ is an element of the set associated with $[\sigma]$ defined in the previous section. This operator is generalized immediately,

$$T(\phi_{\alpha_1}(x_1) \cdots \phi_{\alpha_n}(x_n)) = \sum_{\text{perm.}} \rho \begin{pmatrix} \alpha_1 \cdots \alpha_n \\ \alpha_1' \cdots \alpha_n' \end{pmatrix} \theta(x_1'^0 - x_2'^0) \cdots \theta(x_{n-1}'^0 - x_n'^0) \cdot \phi_{\alpha_n'}(x_1') \cdots \phi_{\alpha_n'}(x_n'), \tag{4.2}$$

where $\rho(\begin{smallmatrix} \alpha_1 \cdots \alpha_n \\ \alpha_1' \cdots \alpha_n' \end{smallmatrix})$ is the product of ρ-factors associated with each successive permutation defining the overall permutation $(\begin{smallmatrix} \alpha_1 \cdots \alpha_n \\ \alpha_1' \cdots \alpha_n' \end{smallmatrix})$.

Due to the fact that the non-interacting theory is independent of $[\sigma]$, the field operators in the interaction representation are identical to the canonical field operators with normal statistics. Therefore, they have the same positive and negative frequency decomposition,

$$\phi_\alpha(x) = \phi_\alpha^+(x) + \phi_\alpha^-(x), \tag{4.3}$$

where $\phi_\alpha^\pm$ denote positive and negative frequency parts. Define the normal-ordered product,

$$:\phi_\alpha(x)\,\phi_\beta(y): = \phi_\alpha^+(x)\,\phi_\beta^+(y) + \phi_\alpha^-(x)\,\phi_\beta^-(y) + \phi_\alpha^-(x)\,\phi_\beta^+(y) + \rho_{\alpha\beta}\phi_\beta^-(y)\,\phi_\alpha^+(x), \tag{4.4}$$

with obvious generalization to $:\phi_{\alpha_1}(x_1) \cdots \phi_{\alpha_n}(x_n):$.

Define further c-number functions, $f_{+-}(x, y)$ and $f_{-+}(x, y)$, by equations

$$\rho_{\alpha\beta}\phi_\alpha^+(x)\,\phi_\beta^-(y) = \phi_\beta^-(y)\,\phi_\alpha^+(x) + \delta_{\alpha\beta} f_{+-}(x, y), \tag{4.5}$$

$$\rho_{\alpha\beta}\phi_\alpha^-(x)\,\phi_\beta^+(y) = \phi_\beta^+(y)\,\phi_\alpha^-(x) + \delta_{\alpha\beta} f_{-+}(x, y). \tag{4.6}$$

From Eqs. (2.10) it is easy to show

$$f_{+-}(x, y) = -f_{-+}(y, x). \tag{4.7}$$

Note that

$$:\phi_\alpha\phi_\beta: = \rho_{\alpha\beta}\, :\phi_\beta\phi_\alpha:, \tag{4.8a}$$

and

$$T(\phi_\alpha\phi_\beta) = \rho_{\alpha\beta}\, T(\phi_\beta\phi_\alpha), \tag{4.8b}$$

with obvious generalization to an arbitrary number of fields in the product. From Eqs. (4.1), (4.2), (4.5), (4.6) and (2.10), one has

$$T(\phi_\alpha\phi_\beta) - :\phi_\alpha\phi_\beta: = -\delta_{\alpha\beta}[f_{-+}(x, y)\,\theta(x^0 - y^0) + f_{-+}(y, x)\,\theta(y^0 - x^0)]. \tag{4.9}$$

Therefore, contracting with $|0\rangle$,

$$\langle 0|\, T(\phi_\alpha\phi_\beta)|0\rangle = -\delta_{\alpha\beta}[f_{-+}(x, y)\,\theta(x^0 - y^0) + f_{-+}(y, x)\,\theta(y^0 - x^0)], \tag{4.10}$$

and finally,

$$T(\phi_\alpha\phi_\beta) = :\phi_\alpha\phi_\beta: + \langle 0|\, T(\phi_\alpha\phi_\beta)|0\rangle. \tag{4.11}$$

Starting from Eq. (4.11), and following the induction proof of Wick's theorem, we have the generalized Wick formula:

$$\begin{aligned} T(\phi_{\alpha_1}(x_1)\cdots\phi_{\alpha_n}(x_n)) &= :\phi_{\alpha_1}(x_1)\cdots\phi_{\alpha_n}(x_n): \\ &+ \sum_{ij} \rho\{\alpha_i, \alpha_j\}\, :\phi_{\alpha_1}(x_1)\cdots\overline{\phi_{\alpha_i}(x_i)\cdots\phi_{\alpha_j}}(x_j)\cdots: \\ &+ \sum_{ijkl} \rho\{\alpha_i, \alpha_k; \alpha_j, \alpha_l\} \\ &\quad :\phi_{\alpha_1}(x_1)\cdots\overline{\phi_{\alpha_i}(x_i)\cdots\phi_{\alpha_j}(x_j)\cdots\phi_{\alpha_k}}(x_k)\cdots\phi_{\alpha_l}(x_l)\cdots: \\ &+ \cdots, \end{aligned} \tag{4.12}$$

where the contractions are accompanied by the ρ-factor necessary to bring contracted fields next to one another, e.g.,

$$\rho\{\alpha_i, \alpha_j\} = \prod_{m=i+1}^{j-1} \rho_{\alpha_i\alpha_m} = \prod_{m=i+1}^{j-1} \rho_{\alpha_m\alpha_j} \qquad (\text{for} \quad \alpha_i \neq \alpha_j), \tag{4.13}$$

and

$$\overbrace{\phi_{\alpha_i}(x_i)\,\phi_{\alpha_j}}(x_j)=\langle 0|\,T(\phi_{\alpha_i}(x_i)\,\phi_{\alpha_j}(x_j))|0\rangle. \tag{4.14}$$

Again, due to the independence on statistics $[\sigma]$ for fields without interaction, the definitions of asymptotic fields, $\phi_\alpha^{\rm in}$ and $\phi_\alpha^{\rm out}$, as well as the asymptotic condition, are the same as normal statistics. The only difference that occurs in construction of the S-matrix is the permutation symmetry of states and the appearance of ρ-factors in the LSZ reduction formula.

Defining $|j_1\cdots j_n\text{ in(out)}\rangle$ to be the properly normalized in(out) Fock states, we derive the LSZ reduction formula:

$$\begin{aligned}&\langle j_1\cdots j_n\,\text{out}\,|\,i_1\cdots i_m\,\text{in}\rangle\\ &\quad=\langle j_1\cdots j_n\,\text{out}|\,a^\dagger_{i_1}(p_1)^{\rm out}\,|\,i_2\cdots i_m\,\text{in}\rangle\\ &\qquad+i\int d^4x f_{p_i}(x)\overrightarrow{(\Box_x+m^2)}\\ &\qquad\times\langle j_2\cdots j_n\,\text{out}\,|\,\phi_{i_1}(x)\,a_{j_1}(q_1)\,\text{in}\,|\,i_2\cdots i_m\,\text{in}\rangle\,\rho_{i_1j_1}\\ &\qquad-\int d^4x d^4y f_{p_i}(x)\overrightarrow{(\Box_x+m^2)}\\ &\qquad\times\langle j_2\cdots j_n\,\text{out}|\,T(\phi_{j_1}(y)\,\phi_{i_1}(x))|\,i_2\cdots i_m\,\text{in}\rangle\overleftarrow{(\Box_y+m^2)}f^*_{q_i}(y).\end{aligned} \tag{4.15}$$

The first two terms on the right-hand side of Eq. (4.15) can be cast in the form

$$\begin{aligned}\langle j_2\cdots j_n\,\text{out}|\,\delta_{i_1j_1}+(a^{\dagger\,\rm out}_{i_1}a^{\rm in}_{j_1}+a^{\dagger\,\rm out}_{i_1}a^{\rm out}_{j_1}\\ -a^{\dagger\,\rm in}_{i_1}a^{\rm in}_{j_1})\,\rho_{i_1j_1}\,|\,i_2\cdots i_m\,\text{in}\rangle,\end{aligned} \tag{4.16}$$

which has the direct diagrammatic interpretation of partially forward scattering. For two-body scattering this simplifies to

$$\langle j_2\,\text{out}|\,\delta_{i_1j_1}+a^{\dagger\,\rm out}_{i_1}\,a^{\rm in}_{j_1}\rho_{i_1j_1}|\,i_2\,\text{in}\rangle, \tag{4.17}$$

which indicates the appearance of ρ-factors in the amplitude. Note that ρ-factors are embedded in the non-forward part of the S-matrix, Eq. (4.15), due to the definition of the T-product, Eq. (4.1). The appearance of ρ-factors in the S-matrix signifies internal symmetry breaking because ρ is not an invariant object except for normal statistics. This is shown in the next section.

Other properties of the S-matrix, such as unitarity and Bogoliubov causality, are defined in the same manner as normal statistics and have to be investigated in each theory separately. For example, we would anticipate the same problem of non-renormalizability and unitarity for a massive vector meson theory regardless of the relative statistics $[\sigma]$ for fields. The lowest-order diagrams in a scalar ϕ^4 theory, investigated in the next section, satisfy the unitarity condition in the form of cut equations.

V. Transmutation of Internal Symmetry

It has been suggested that general statistics $\{\sigma\}$ induces symmetry breaking due to the appearance of non-invariant ρ-factors in the S-matrix. We show this in a scalar $SO(3)$ ϕ^4 theory by deriving amplitude relations unique for each statisics. These results are generalized in a straightforward manner to $SO(N)$.

Consider the classically $SO(N)$-symmetric Lagrangian,

$$\mathscr{L} = -\sum_{\alpha}^{N} ((\partial_\mu \phi_\alpha)(\partial_\mu \phi_\alpha) + m^2 \phi_\alpha \phi_\alpha) - g \sum_{\alpha\beta}^{N} (\phi_\alpha \phi_\alpha)(\phi_\beta \phi_\beta). \tag{5.1}$$

The lowest-order appearance of symmetry breaking occurs in the four-point function at the one loop level. We show that the counterterms preserve the original classical symmetry which allows us to write down an all orders S-matrix element with appropriate ρ-factors.

Consider the four-point function $G_{\alpha\beta\kappa\mu}$,

$$G_{\alpha\beta\kappa\mu}(x_1, x_2, x_3, x_4) = \langle 0 |\, T(\phi_\alpha(x_1)\, \phi_\beta(x_2)\, \phi_\kappa(x_3)\, \phi_\mu(x_4)) | 0 \rangle. \tag{5.2}$$

Figure 3 contains the diagrams to one loop order. From the contraction rules in Section IV one has the following results for the Γ-function corresponding to diagrams A, B, and C in Fig. 3;

$$\Gamma_A = g(\delta_{\alpha\beta}\delta_{\kappa\mu} + \rho_{\alpha\beta}\delta_{\alpha\kappa}\delta_{\beta\mu} + \delta_{\alpha\mu}\delta_{\beta\kappa}) \tag{5.3a}$$

and

$$\Gamma_{B+C} = g^2((\delta_{\alpha\beta}\delta_{\kappa\mu} + \rho_{\alpha\beta}\delta_{\alpha\kappa}\delta_{\beta\mu} + \delta_{\alpha\mu}\delta_{\beta\kappa})\, I(k_1, k_2, k_3, k_4), \tag{5.3b}$$

where k_i are momentum associated with α, β, κ, μ, and I is the Feynman integral associated with the diagrams. As can be seen from Eq. (5.3b) the counterterm associated with $(B + C)$ has the same factor as the Born term. Therefore, symmetry is not broken by counterterms. In an $SO(N)$ theory, this property holds in higher order regardless of the subtraction procedure because all diagrams contributing to the four-point function have the form of Eq. (5.3b). In other theories, such as $SU(N)$ ϕ^4 theory or a theory with Yukawa interactions, ϕ^4 counterterms do not break the symmetry as long as symmetric subtraction are used.

Due to the all orders invariance of the counterterms and the general form of the reduction formula for the S-matrix elements, the two-body scattering amplitude can be expressed as

$$S_{\alpha\beta\to\kappa\mu} = A\delta_{\alpha\beta}\delta_{\kappa\mu} + B\rho_{\alpha\beta}\delta_{\alpha\kappa}\delta_{\beta\mu} + C\delta_{\alpha\mu}\delta_{\beta\kappa}, \tag{5.4}$$

where A, B, and C are invariant amplitudes. In the case of scalars with an $SO(N)$ symmetry, $A = B = C$, and for $SU(N)$, $A = 0$. Equation (5.4) holds with Yukawa interactions (in the following analysis we allow this possibility by not constraining A,

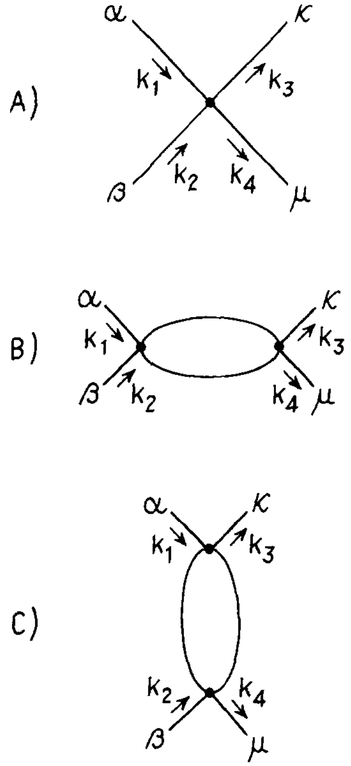

FIG. 3. Diagrams corresponding to Eqs. (5.3) for $SO(3)$ scalar theory.

B, and C.), however, locality requirements exist on $\{\rho_{\alpha\beta}, \rho''_{\alpha i}\}$, which we will explain in a later section. We consider only the $SO(3)$ case with adjoint representation, $\alpha, \beta, \kappa, \mu = \{1, 2, 3\}$, in order to show how transmutation of internal symmetry occurs. Define the physical states $\pi^{\pm} = (\phi_1 \mp i\phi_2)/\sqrt{2}$, and $\pi^0 = \phi_3$. The resulting two-body scattering amplitudes for arbitrary $\{\rho\}$, derivable from Eq. (5.4), are given in Table I. Also included are the cases of normal statistics ($\rho_{\alpha\beta} = 1$ for all α, β), maximal parastatistics ($\rho_{\alpha\beta} = -1$, $\alpha \neq \beta$), and mixed parastatistics.

For normal statistics isospin invariance results, with

$$f_2 = B + C, \tag{5.5a}$$

$$f_1 = B - C, \tag{5.5b}$$

and

$$f_0 = 3A - B - C, \tag{5.5c}$$

where f_I is the amplitude corresponding to isospin I. Also, relationships among the amplitudes are given by

TABLE I

Two-Body Scattering Amplitudes among Physical States for Various Statistics

	$f(A, B, C, \rho)$	Normal statistics $\rho_{12}=\rho_{23}=\rho_{13}=1$	Mixed statistics $\rho_{12}=1$, $\rho_{23}=\rho_{13}=-1$	Maximal statistics $\rho_{12}=\rho_{23}=\rho_{13}=-1$
$\pi^+\pi^+ \to \pi^+\pi^+$	$\left(\frac{1+\rho_{12}}{2}\right) B + C$	$B+C$	$B+C$	C
$\pi^+\pi^0 \to \pi^+\pi^0$	$\left(\frac{\rho_{12}+\rho_{23}}{2}\right) B$	B	$-B$	$-B$
$\pi^+\pi^0 \to \pi^0\pi^+$	C	C	C	C
$\pi^+\pi^- \to \pi^+\pi^-$	$A + \left(\frac{1+\rho_{23}}{2}\right) B$	$A+B$	$A+B$	A
$\pi^+\pi^- \to \pi^-\pi^+$	$A + \left(\frac{1-\rho_{12}}{2}\right) B + C$	$A+C$	$A+C$	$A+B+C$
$\pi^0\pi^0 \to \pi^0\pi^0$	$A+B+C$	$A+B+C$	$A+B+C$	$A+B+C$
$\pi^0\pi^0 \to \pi^+\pi^-$ ($=\pi^0\pi^0 \to \pi^-\pi^+$)	A	A	A	A

$$f(\pi^+\pi^0 \to \pi^+\pi^0) + f(\pi^+\pi^0 \to \pi^0\pi^+) = f(\pi^+\pi^+ \to \pi^+\pi^+), \tag{5.6a}$$

$$f(\pi^+\pi^0 \to \pi^+\pi^0) - f(\pi^+\pi^0 \to \pi^0\pi^+) = f(\pi^+\pi^- \to \pi^+\pi^-) - f(\pi^+\pi^- \to \pi^-\pi^+), \tag{5.6b}$$

$$f(\pi^+\pi^- \to \pi^+\pi^-) + f(\pi^+\pi^- \to \pi^-\pi^+) = f(\pi^0\pi^0 \to \pi^+\pi^-) + f(\pi^0\pi^0 \to \pi^0\pi^0), \tag{5.6c}$$

and

$$f(\pi^0\pi^0 \to \pi^0\pi^0) - f(\pi^+\pi^- \to \pi^0\pi^0) = f(\pi^+\pi^+ \to \pi^+\pi^+). \tag{5.6d}$$

These relations of isospin invariance are all violated for general statistics. New relations involving ρ-factors exist, such as

$$\begin{aligned} &f(\pi^+\pi^0 \to \pi^+\pi^0) + \tfrac{1}{2}(\rho_{13}+\rho_{23}) f(\pi^+\pi^0 \to \pi^0\pi^+) \\ &\quad = \tfrac{1}{2}(\rho_{13}+\rho_{23})[f(\pi^0\pi^0 \to \pi^0\pi^0) - f(\pi^0\pi^0 \to \pi^+\pi^-)], \end{aligned} \tag{5.7}$$

and three others. Note that Eq. (5.6c) holds for all statistics. In the case of mixed and maximal parastatistics ($\frac{1}{2}(\rho_{13}+\rho_{23}) = -1$), Eq. (5.7) gives

$$f(\pi^+\pi^0 \to \pi^+\pi^0) - f(\pi^+\pi^0 \to \pi^0\pi^+) = -f(\pi^0\pi^0 \to \pi^0\pi^0) + f(\pi^0\pi^0 \to \pi^+\pi^-). \tag{5.8}$$

The right-hand side of Eq. (5.8) is symmetric in isospin as well as space coordinates, therfore it contains only even parity functions. This corresponds on the left-hand side to antisymmetric isospin and symmetric space coordinates. Of course, this is the result of antisymmetric commutation relations between ϕ_1, ϕ_2, and ϕ_3 in these statistics.

The breaking of isospin invariance can be shown explicitly. Defining $|I, I_z\rangle$ ($I = 0, 1, 2$) states,

$$|2, 2\rangle = |\pi^+, \pi^+\rangle, \tag{5.9a}$$

$$|2, 1\rangle = \frac{1}{\sqrt{2}} (|\pi^+\pi^0\rangle + |\pi^0\pi^+\rangle), \tag{5.9b}$$

one has amplitudes,

$$\langle 2, 2 | 2, 2\rangle = \left(\left(\frac{1+\rho_{12}}{2}\right) B + C\right) \tag{5.10a}$$

and

$$\langle 2, 1 | 2, 1\rangle = \left(\left(\frac{\rho_{12}+\rho_{23}}{2}\right) B + C\right), \tag{5.10b}$$

which are equal only for $\rho_{23} = 1$.

The replacement of the isospin invariance relations, Eq. (5.6), by a new set of ρ-dependent relations, Eq. (5.7), is what we call transmutation of the symmetry. It results from non-invariance of the ρ-factors under the unitary transformation necessary to rotate to a physical basis.

Note that in the quark model the pion, corresponding to a bispinor bound state in $SU(2)$, must obey normal statistics independent of the quark statistics because of the composite rule [2]. Therefore, the above example is not applicable to physical pions but exemplifies a situation which may exist for $SO(N)$ or $SU(N)$ ϕ^4 or Yukawa models.

VI. Locality, Composite Rule, and Higgs Mechanism

In general, locality requirements constrain the statistics allowed for various interactions [2]. We consider the cases of (a) $SO(N)$ ϕ^4, (b) $SU(2)$ Yukawa, and (c) $SU(N)$ Yukawa interactions.

(a) The $SU(N)$ or $SO(N)$ ϕ^4 interaction,

$$\sum_\alpha (\phi_\alpha^\dagger \phi_\alpha)^2, \tag{6.1}$$

allows any statistics $[\sigma]$ because of the repeated index (a more detailed study of $SU(N)$ locality is shown in Appendices C and D).

(b) The $SU(2)$ adjoint representation Yukawa interaction,

$$\varepsilon^{ijk}\bar{\psi}_i\psi_j\phi_k, \tag{6.2}$$

where ϕ_k is a scalar or vector meson, allows any statistics $[\sigma]$. In this case each term has fields of different index and the interaction term contains all combinations of the indices, which makes locality in these statistics obvious. Note that this is the only invariant form for the adjoint representation in $SO(N)$ and indicates the uniqueness of the $SU(2)$ triplet representation.

(c) The $SU(N)$ Yukawa interaction in the adjoint representation for scalar or vector mesons,

$$\bar{\psi}^i\psi^j\phi_{ij}, \tag{6.3}$$

requires that ϕ_{ij} satisfy a composite rule $[\sigma]$ dictated by the statistics of ψ_i. In other words, the commutation rules of ϕ_{ij} are the same as a product $\psi_i\psi_j$. In the case of $SU(2)$, the composite rule results in normal statistics for the meson ϕ_{ij} $(i, j = 1, 2)$ independent of fermion statistics, as mentioned earlier. Note that with the composite rule the diagonal element ϕ_i^i commutes with any fields.

As was mentioned in Ref. [2], generalized parastatistics can exist among Green components satisfying a local symmetry as well as a global symmetry. In both cases the Goldstone–Higgs mechanism can be formulated in generalized parastatistics. However, there is a significant difference in that only scalars which commute (as opposed to anticommute) with all other fields can have a vacuum expectation value unless extended number systems, which contain anticommuting c-numbers, are introduced. This means, for example, that in the standard $SU(2) \times U(1)$ model the Higgs particles must obey normal statistics. Therefore, from note (c) above, in generalized parastatistics with a composite rule candidates for vacuum expectation values are the diagonal elements of an internal symmetry tensor, ϕ_i^i or $\phi_{ij\cdots}^{ij\cdots}$.

In Section II we derived a stringent constraint on $\{\rho_{\alpha\beta}\}$ to be in the intersection $\Gamma_s \cap \Gamma_d$, i.e., satisfying both double and single commutation relations. The constraint Eq. (2.13) results in the set of equivalence classes of commuting fields described in Fig. 2. In particular, the usual Higgs mechanism is stringently constrained by Eq. (2.13) and the resulting equivalence class structure. A field having a vacuum expectation value must commute with all other fields in order to shift the field by a constant which is an ordinary number commuting with all fields. Therefore, allowing ϕ_α to have a vacuum expectation value implies

$$\rho_{\alpha\beta} = 1, \tag{6.4}$$

for all β, which implies, from Eqs. (2.14) and (2.15),

$$\rho_{\delta\beta} = 1, \tag{6.5}$$

for all β and δ. In other words, ϕ_β and ϕ_δ commute for all β and δ. This implies that only normal statistics allows a standard Higgs mechanism for the class of theories $[\sigma]$ in $\Gamma_s \cap \Gamma_d$. This can also be seen directly from the equivalence class structure in Fig. 2 due to the fact that the field with a vacuum expectation value, commuting with all other fields, cannot belong to any class unless the theory contains one class only. In a forthcoming article [3] the authors consider the Higgs mechanism with arbitrary statistics by the introduction of new number systems.

VII. Summary and Discussion

In this paper we have presented a new formulation of quantum theory in which generalized statistics is a phase variable. The concept of generalized statistics was introduced in an earlier paper in which its incorporation with internal symmetries was considered. This formulation is realized by taking the generating functional to be a function of statistics $[\sigma]$ as well as external fields. Given the Lagrangian, the statistics $[\sigma_0]$ realized in nature is decided by minimalization of the vacuum energy as a function of statistics $[\sigma]$. This leads to the possibility of a new type of phase transition between domains of different statistics.

This view of statistics allows the statistics realized in nature to be different depending on internal symmetry, interaction type, and initial conditions. Statistics may well be time, coupling constant, and/or temperature dependent.

It is an interesting possibility that the statistics of the early universe may be different from those of the present. Another possibility is a temperature- or coupling constant-dependent phase transition between domains of different statistics $[\sigma]$. The intimate relationship between classical statistical systems at the critical point and quantum field theories, in particular $0(N)$ ϕ^4 theories, suggests this possibility.

We also defined and developed the relation between statistics of single and covariant double commutation relations. Restricting to the case of single commutations, a canonical formalism for quantum field theory was presented. Based on this, it is shown how original isospin invariance is violated when the physical basis is transformed relative to the Green–type canonical fields in the theory. New relations among amplitudes, involving non-invariant ρ-factors, exist; a process which we defined to be symmetry transmutation. This is a result of non-invariance of arbitrary single commutation relations under internal symmetry transformations.

The usual Higgs mechanism is constrained by statistics in that fields taking vacuum expectation values must commute with all other fields. This implies, for $[\sigma]$ satisfying double and single commutation relations $(\Gamma_s \cap \Gamma_d)$, that only normal statistics allows a Higgs mechanism. In the case of only single commutation relations $(\Gamma_s - (\Gamma_s \cap \Gamma_d))$ other statistics may have a Higgs mechanism. Note, however, because normal statistics allows more directions of Higgs symmetry breaking, that it is conjectured that with the Higgs mechanism normal statistics is favored in the functional integral as statistics is varied. This may explain the predominance of normal statistics at low energies. However, special circumstances, such as in dynamical symmetry breaking, may favor abnormal statistics.

An open question remains of the formulation of quantum field theory in the case where double commutators, but not single commutators ($\Gamma_d - (\Gamma_s \cap \Gamma_d)$), define the statistics. In this case all vacuum expectation values of a given number of fields are defined recursively in terms of vacuum expectation values of a smaller number. S-matrix elements are then defined in terms of the γ-matrices appearing in Eq. (2.2). An interesting possibility is dynamical symmetry breaking (à la Coleman and Weinberg) with this statistics.

Finally, it is interesting to consider possible applications of this formulation to the generation problem of the quark–lepton system, supersymmetry breaking, and phase transitions in statistical systems.

APPENDIX A: General Form of $\rho_{\alpha\beta}$

In this appendix the most general single commutation relation are derived consistent with Heisenberg's equation of motion and the spin-statistics relationship. For simplicity we consider only spin 0 boson operators.

Assume commutation relations for boson operators, $A_{\vec{k}}^{\alpha}$, $\alpha = 1, 2, \ldots, N$, and $\vec{k}$ = three momentum,

$$A_{\vec{k}}^{\alpha\dagger} A_{\vec{k}'}^{\beta\dagger} = \rho^{\alpha\beta}(\vec{k}, \vec{k}') A_{\vec{k}'}^{\beta\dagger} A_{\vec{k}}^{\alpha\dagger}, \tag{A.1}$$

$$A_{\vec{k}}^{\alpha} A_{\vec{k}'}^{\beta\dagger} = \xi^{\alpha\beta}(\vec{k}, \vec{k}') A_{\vec{k}'}^{\beta\dagger} A_{\vec{k}}^{\alpha} + \delta_{\alpha,\beta} \delta_{\vec{k},\vec{k}'}, \tag{A.2}$$

where $\rho^{\alpha\beta}(\vec{k}, \vec{k}')$ and $\xi^{\alpha\beta}(\vec{k}, \vec{k}')$, $\alpha, \beta = 1, 2, \ldots, N$, are complex-valued functions with the spin-statistics constraint for bosons,

$$\rho^{\alpha\alpha}(\vec{k}, \vec{k}') = \xi^{\alpha\alpha}(\vec{k}, \vec{k}') = 1, \tag{A.3}$$

for all $\vec{k}$ and $\vec{k}'$. Equation (A.1) and the hermitian conjugate of Eq. (A.2) imply the equations

$$\rho^{\alpha\beta}(\vec{k}, \vec{k}') = \frac{1}{\rho^{\beta\alpha}(\vec{k}', \vec{k})} \tag{A.4}$$

and

$$\xi^{\alpha\beta}(\vec{k}, \vec{k}') = \xi^{\beta\alpha}(\vec{k}', \vec{k})^*. \tag{A.5}$$

From Eqs. (A.1)–(A.5) further commutation relations result from hermitian conjugation,

$$A_{\vec{k}}^{\alpha} A_{\vec{k}'}^{\beta} = \rho^{\beta\alpha}(\vec{k}', \vec{k})^* A_{\vec{k}'}^{\beta} A_{\vec{k}}^{\alpha} = \frac{1}{\rho^{\alpha\beta}(\vec{k}, \vec{k}')^*} A_{\vec{k}'}^{\beta} A_{\vec{k}}^{\alpha} \tag{A.6}$$

and

$$A_{\vec{k}}^{\alpha\dagger} A_{\vec{k}'}^{\beta} = \frac{1}{\xi^{\alpha\beta}(\vec{k}, \vec{k}')^*} A_{\vec{k}'}^{\beta} A_{\vec{k}}^{\alpha\dagger} - \delta_{\alpha,\beta} \delta_{\vec{k},\vec{k}'}. \tag{A.7}$$

Consider the Jacobi identity

$$[[A_{\vec{k}}^{\alpha}, A_{\vec{k}'}^{\beta}], A_{\vec{k}''}^{\delta\dagger}] + [[A_{\vec{k}'}^{\beta}, A_{\vec{k}''}^{\delta\dagger}], A_{\vec{k}}^{\alpha}] + [[A_{\vec{k}''}^{\delta\dagger}, A_{\vec{k}}^{\alpha}], A_{\vec{k}'}^{\beta}] = 0. \tag{A.8}$$

Taking $\alpha \neq \beta$, $\beta = \delta$ in Eq. (A.8) and using Eqs. (A.1)–(A.7), one obtains

$$\xi^{\alpha\beta}(\vec{k}, \vec{k}') = \rho^{\alpha\beta}(\vec{k}, \vec{k}')^{*}. \tag{A.9}$$

Then, from Eqs. (A.4) and (A.5),

$$\xi^{\alpha\beta}(\vec{k}, \vec{k}') = \xi^{\beta\alpha}(\vec{k}', \vec{k})^{*} = \rho^{\beta\alpha}(\vec{k}', \vec{k}) = \frac{1}{\rho^{\alpha\beta}(\vec{k}, \vec{k}')} = \frac{1}{\xi^{\alpha\beta}(\vec{k}, \vec{k}')^{*}}, \tag{A.10}$$

or

$$|\xi^{\alpha\beta}(\vec{k}, \vec{k}')|^2 = |\rho^{\alpha\beta}(\vec{k}, \vec{k}')|^2 = 1. \tag{A.11}$$

Due to Eq. (A.11) define real-valued functions $\Lambda_{\alpha\beta}(\vec{k}, \vec{k}')$ by the equations

$$\rho_{\alpha\beta}(\vec{k}, \vec{k}') = e^{i\Lambda_{\alpha\beta}(\vec{k}, \vec{k}')} \tag{A.12}$$

and

$$\xi^{\alpha\beta}(\vec{k}, \vec{k}') = e^{-i\Lambda_{\alpha\beta}(\vec{k}, \vec{k}')}, \tag{A.13}$$

where Eqs. (A.4) and (A.5) imply

$$\Lambda_{\alpha\beta}(\vec{k}, \vec{k}') = -\Lambda_{\beta\alpha}(\vec{k}', \vec{k}) + 2\pi n, \qquad n = \text{integer}, \tag{A.14}$$

and

$$\Lambda_{\alpha\alpha}(\vec{k}, \vec{k}') = 0. \tag{A.15}$$

Rewriting the commutation relations with Eqs. (A.12) and (A.13),

$$A_{\vec{k}}^{\alpha\dagger} A_{\vec{k}'}^{\beta\dagger} = e^{i\Lambda_{\alpha\beta}(\vec{k}, \vec{k}')} A_{\vec{k}'}^{\beta\dagger} A_{\vec{k}}^{\alpha\dagger}, \tag{A.16}$$

$$A_{\vec{k}}^{\alpha} A_{\vec{k}'}^{\beta\dagger} = e^{-i\Lambda_{\alpha\beta}(\vec{k}, \vec{k}')} A_{\vec{k}'}^{\beta\dagger} A_{\vec{k}}^{\alpha} + \delta_{\alpha,\beta}\delta_{\vec{k},\vec{k}'}, \tag{A.17}$$

$$A_{\vec{k}}^{\alpha} A_{\vec{k}'}^{\beta} = e^{i\Lambda_{\alpha\beta}(\vec{k}, \vec{k}')} A_{\vec{k}'}^{\beta} A_{\vec{k}}^{\alpha}, \tag{A.18}$$

$$A_{\vec{k}}^{\alpha\dagger} A_{\vec{k}'}^{\beta} = e^{-i\Lambda_{\alpha\beta}(\vec{k}, \vec{k}')} A_{\vec{k}'}^{\beta} A_{\vec{k}}^{\alpha\dagger} - \delta_{\alpha,\beta}\delta_{\vec{k},\vec{k}'}. \tag{A.19}$$

Note that for $\alpha \neq \beta$ and all $(\vec{k}, \vec{k}')$, Eqs. (A.15)–(A.18) imply

$$[\{A_{\vec{k}}^{\alpha\dagger}, A_{\vec{k}}^{\alpha}\}, A_{\vec{k}'}^{\beta}] = 0 \qquad (\alpha \neq \beta), \tag{A.20}$$

which is required for validity of Heisenberg's equation of motion,

$$i[H_0, A_{\vec{k}}^{\alpha}] = -\omega_{\vec{k}} A_{\vec{k}}^{\alpha}, \tag{A.21}$$

where

$$H_0 = \tfrac{1}{2} \sum_{\vec{k},\alpha} \omega_{\vec{k}} \{A_{\vec{k}}^{\alpha\dagger}, A_{\vec{k}}^{\alpha}\} \tag{A.22}$$

and

$$\omega_{\vec{k}} = \sqrt{\vec{k}^2 + m^2}. \tag{A.23}$$

APPENDIX B: REAL SCALAR FIELDS

In this appendix we derive the general form of $\rho_{\alpha\beta}$ for operators forming a real scalar field. In order to have simple commutation relations at spacelike separation for real fields, the momentum dependence of ρ must be dropped and, in fact, we will show

$$\rho_{\alpha\beta} = \pm 1, \tag{B.1}$$

for all α, β.

Define hermitian $\phi_\alpha(x)$, $\alpha = 1, 2, \ldots, N$, by the momentum space expansion

$$\phi_\alpha(x) = \sum_{\vec{k}} N_{\vec{k}} [A_{\vec{k}}^{\alpha} e^{ik\cdot x} + A_{\vec{k}}^{\alpha\dagger} e^{-ik\cdot x}], \tag{B.2}$$

where $k \cdot x = \vec{k} \cdot \vec{x} - \omega t$, $\omega = (\vec{k}^2 + m^2)^{1/2}$, and $N_{\vec{k}} = (1/2V\omega_{\vec{k}})^{1/2}$. The operators $A_{\vec{k}}^{\alpha}$ and $A_{\vec{k}}^{\alpha\dagger}$ are assumed to satisfy commutation relations, Eqs. (A.15)–(A.18). Consider the products ($\alpha \neq \beta$)

$$\phi_\alpha(x)\, \phi_\beta(y) \tag{B.3a}$$

and

$$\phi_\beta(y)\, \phi_\alpha(x), \tag{B.3b}$$

at spacelike separation, $(x-y)^2 > 0$. The product $\phi_\alpha(x)\, \phi_\beta(y)$ for $t_x = t_y$ is given by

$$\sum_{\vec{k},\vec{k}'} [A_{\vec{k}}^{\alpha} A_{\vec{k}'}^{\beta} e^{ik\cdot x} e^{ik'\cdot y} + A_{\vec{k}}^{\alpha\dagger} A_{\vec{k}'}^{\beta} e^{-ik\cdot x} e^{ik'\cdot y} + A_{\vec{k}}^{\alpha} A_{\vec{k}'}^{\beta\dagger} e^{ik\cdot x} e^{-ik'\cdot y} + A_{\vec{k}}^{\alpha\dagger} A_{\vec{k}'}^{\beta\dagger} e^{-ik\cdot x} e^{-ik'\cdot y}] \times N_{\vec{k}} N_{\vec{k}'}. \tag{B.4}$$

For $\alpha \neq \beta$, using Eqs. (A.16) and (A.17), this can be written

$$\sum_{\vec{k},\vec{k}'} [e^{i\Lambda_{\alpha\beta}} A_{\vec{k}'}^{\beta} A_{\vec{k}}^{\alpha} e^{ik'\cdot y} e^{ik\cdot x} + e^{-i\Lambda_{\alpha\beta}} A_{\vec{k}'}^{\beta} A_{\vec{k}}^{\alpha\dagger} e^{ik'\cdot y} e^{-ik\cdot x} + e^{-i\Lambda_{\alpha\beta}} A_{\vec{k}'}^{\beta\dagger} A_{\vec{k}}^{\alpha} e^{-ik'\cdot y} e^{ik\cdot x} + e^{i\Lambda_{\alpha\beta}} A_{\vec{k}'}^{\beta\dagger} A_{\vec{k}}^{\alpha\dagger} e^{-ik'\cdot y} e^{-ik\cdot x}] \times N_{\vec{k}} N_{\vec{k}'}. \tag{B.5}$$

It is immediately apparent that single commutation relations of the form Eqs. (2.11) are possible only for

$$\Lambda_{\alpha\beta}(\vec{k}, \vec{k}') = \Lambda_{\alpha\beta}, \tag{B.6}$$

independent of $(\vec{k}, \vec{k}')$, and

$$e^{i\Lambda_{\alpha\beta}} = e^{-i\Lambda_{\alpha\beta}}, \tag{B.7}$$

for all (α, β). This implies $\rho_{\alpha\beta} = \pm 1$, a condition that was used in Section III to define general statistics, $[\sigma]$.

APPENDIX C: Complex Scalar Fields—Analytic Continuation of Statistics

In this appendix we derive the general form of $\rho_{\alpha\beta}$ for operators forming complex scalar fields. Again the momentum dependence may be dropped in order to have simple field commutation relations. Consider boson operators, $A_{\vec{k}}^{\alpha}$ and $B_{\vec{k}}^{\alpha}$, used to form complex fields and assume commutation relations from Eqs. (A.16)–(A.19) given by

$$A_{\vec{k}}^{\alpha\dagger} A_{\vec{k}'}^{\beta\dagger} = e^{i\Lambda_{\alpha\beta}} A_{\vec{k}'}^{\beta\dagger} A_{\vec{k}}^{\alpha\dagger}, \tag{C.1}$$

$$A_{\vec{k}}^{\alpha} A_{\vec{k}'}^{\beta\dagger} = e^{-i\Lambda_{\alpha\beta}} A_{\vec{k}'}^{\beta\dagger} A_{\vec{k}}^{\alpha} + \delta_{\alpha,\beta}\delta_{\vec{k},\vec{k}'}, \tag{C.2}$$

$$A_{\vec{k}}^{\alpha} A_{\vec{k}'}^{\beta} = e^{i\Lambda_{\alpha\beta}} A_{\vec{k}'}^{\beta} A_{\vec{k}}^{\alpha}, \tag{C.3}$$

$$A_{\vec{k}}^{\alpha\dagger} A_{\vec{k}'}^{\beta} = e^{-i\Lambda_{\alpha\beta}} A_{\vec{k}'}^{\beta} A_{\vec{k}}^{\alpha\dagger} - \delta_{\alpha,\beta}\delta_{\vec{k},\vec{k}'}, \tag{C.4}$$

$$B_{\vec{k}}^{\alpha\dagger} B_{\vec{k}'}^{\beta\dagger} = e^{i\xi_{\alpha\beta}} B_{\vec{k}'}^{\beta\dagger} B_{\vec{k}}^{\alpha\dagger}, \tag{C.5}$$

$$B_{\vec{k}}^{\alpha} B_{\vec{k}'}^{\beta\dagger} = e^{-i\xi_{\alpha\beta}} B_{\vec{k}'}^{\beta\dagger} B_{\vec{k}}^{\alpha} + \delta_{\alpha,\beta}\delta_{\vec{k},\vec{k}'}, \tag{C.6}$$

$$B_{\vec{k}}^{\alpha} B_{\vec{k}'}^{\beta} = e^{i\xi_{\alpha\beta}} B_{\vec{k}'}^{\beta} B_{\vec{k}}^{\alpha}, \tag{C.7}$$

$$B_{\vec{k}}^{\alpha\dagger} B_{\vec{k}'}^{\beta} = e^{-i\xi_{\alpha\beta}} B_{\vec{k}'}^{\beta} B_{\vec{k}}^{\alpha\dagger} - \delta_{\alpha,\beta}\delta_{\vec{k},\vec{k}'}, \tag{C.8}$$

$$B_{\vec{k}}^{\alpha\dagger} A_{\vec{k}'}^{\beta} = e^{-i\kappa_{\alpha\beta}} A_{\vec{k}'}^{\beta} B_{\vec{k}}^{\alpha\dagger}, \tag{C.9}$$

$$B_{\vec{k}}^{\alpha} A_{\vec{k}'}^{\beta\dagger} = e^{-i\kappa_{\alpha\beta}} A_{\vec{k}'}^{\beta\dagger} B_{\vec{k}}^{\alpha}, \tag{C.10}$$

$$B_{\vec{k}}^{\alpha\dagger} A_{\vec{k}'}^{\beta\dagger} = e^{i\lambda_{\alpha\beta}} A_{\vec{k}'}^{\beta\dagger} B_{\vec{k}}^{\alpha\dagger}, \tag{C.11}$$

$$B_{\vec{k}}^{\alpha} A_{\vec{k}'}^{\beta} = e^{i\lambda_{\alpha\beta}} A_{\vec{k}'}^{\beta} B_{\vec{k}}^{\alpha}, \tag{C.12}$$

where, in general, $\lambda_{\alpha\beta} = \lambda_{\alpha\beta}(\vec{k}, \vec{k}')$, $\kappa_{\alpha\beta} = \kappa_{\alpha\beta}(\vec{k}, \vec{k}')$, $\xi_{\alpha\beta} = \xi_{\alpha\beta}(\vec{k}, \vec{k}')$, and

$\Lambda_{\alpha\beta} = \Lambda_{\alpha\beta}(\vec{k}, \vec{k}')$. The spin-statistics relationship and the results from Appendix A (Eq. (A.14) with $n = 0$) imply

$$\Lambda_{\alpha\alpha}(\vec{k}, \vec{k}') = \xi_{\alpha\alpha}(\vec{k}, \vec{k}') = \kappa_{\alpha\alpha}(\vec{k}, \vec{k}') = \lambda_{\alpha\alpha}(\vec{k}, \vec{k}') = 0, \tag{C.13a}$$

$$\Lambda_{\alpha\beta}(\vec{k}, \vec{k}') = -\Lambda_{\beta\alpha}(\vec{k}', \vec{k}), \tag{C.13b}$$

$$\xi_{\alpha\beta}(\vec{k}, \vec{k}') = -\xi_{\beta\alpha}(\vec{k}', \vec{k}). \tag{C.13c}$$

Note that Eqs. (C.10) and (C.12) follow from Eqs. (C.9) and (C.11), respectively, by hermitian conjugation.

Taking the Jacobi identity,

$$[[B^{\alpha}_{\vec{k}}, B^{\beta\dagger}_{\vec{k}'}], A^{\delta}_{\vec{k}''}] + [[B^{\beta\dagger}_{\vec{k}'}, A^{\delta}_{\vec{k}''}], B^{\alpha}_{\vec{k}}] + [[A^{\delta}_{\vec{k}''}, B^{\alpha}_{\vec{k}}], B^{\beta\dagger}_{\vec{k}'}] = 0, \tag{C.14}$$

with condition $\alpha = \beta$, $\beta \neq \delta$, one derives the relation

$$e^{i\Lambda_{\alpha\beta}(\vec{k}, \vec{k}')} = e^{i\kappa_{\alpha\beta}(\vec{k}, \vec{k}')}, \tag{C.15}$$

or

$$\lambda_{\alpha\beta}(\vec{k}, \vec{k}') = \kappa_{\alpha\beta}(\vec{k}, \vec{k}') + 2\pi n \qquad (n = \text{integer}). \tag{C.16}$$

Define the complex scalar field, $\phi^{\alpha}(x)$, $\alpha = 1, 2, \ldots, N$, involving $A^{\alpha}_{\vec{k}}$ and $B^{\alpha}_{\vec{k}}$,

$$\phi^{\alpha}(x) = \sum_{\vec{k}} N_{\vec{k}}(A^{\alpha}_{\vec{k}}\, e^{ik\cdot x} + B^{\alpha\dagger}_{\vec{k}}\, e^{-ik\cdot x}), \tag{C.17}$$

where $k \cdot x = \vec{k} \cdot \vec{x} - \omega t$ and the operators, $A^{\alpha}_{\vec{k}}$ and $B^{\alpha}_{\vec{k}}$, are assumed to satisfy commutation relations Eqs. (C.1)–(C.13) and (C.16). We consider the products

$$\phi_{\alpha}(x)\, \phi_{\beta}(y), \qquad \phi_{\beta}(y)\, \phi_{\alpha}(x) \tag{C.18}$$

and

$$\phi^{\dagger}_{\alpha}(x)\, \phi_{\beta}(y), \qquad \phi_{\beta}(y)\, \phi^{\dagger}_{\alpha}(x), \tag{C.19}$$

at spacelike separation, $(x-y)^2 > 0$, for $\alpha \neq \beta$. The product, $\phi_{\alpha}(x)\, \phi_{\beta}(y)$, yields

$$\sum_{\vec{k}, \vec{k}'} [A^{\alpha}_{\vec{k}} A^{\beta}_{\vec{k}'}\, e^{ik\cdot x} e^{ik'\cdot y} + A^{\alpha}_{\vec{k}} B^{\beta\dagger}_{\vec{k}'}\, e^{ik\cdot x} e^{-ik'\cdot y} \tag{C.20}$$
$$+ B^{\alpha\dagger}_{\vec{k}} A^{\beta}_{\vec{k}'} e^{-ik\cdot x} e^{ik'\cdot y} + B^{\alpha\dagger}_{\vec{k}} B^{\beta\dagger}_{\vec{k}'} e^{-ik\cdot x} e^{-ik'\cdot y}] \times N_{\vec{k}} N_{\vec{k}'},$$

which, from Eqs. (C.1)–(C.12), equals

$$\sum_{\vec{k}, \vec{k}'} [e^{i\Lambda_{\alpha\beta}} A^{\beta}_{\vec{k}'} A^{\alpha}_{\vec{k}} e^{ik\cdot x} e^{ik'\cdot y} + e^{i\kappa_{\beta\alpha}} B^{\beta\dagger}_{\vec{k}'} A^{\alpha}_{\vec{k}}\, e^{ik\cdot x} e^{-ik'\cdot y}$$
$$+ e^{-i\kappa_{\alpha\beta}} A^{\beta}_{\vec{k}'} B^{\alpha\dagger}_{\vec{k}}\, e^{-ik\cdot x} e^{ik'\cdot y} + e^{i\xi_{\alpha\beta}} B^{\beta\dagger}_{\vec{k}'} B^{\alpha\dagger}_{\vec{k}}\, e^{-ik\cdot x} e^{-ik'\cdot y}]$$
$$\times N_{\vec{k}} N_{\vec{k}'}. \tag{C.21}$$

The product, $\phi_\alpha^\dagger(x)\,\phi_\beta(y)$, is given by (for $\alpha \neq \beta$)

$$\sum_{\vec{k},\vec{k}'} [A_{\vec{k}}^{\alpha\dagger} A_{\vec{k}'}^{\beta} e^{-k\cdot x} e^{ik'\cdot y} + A_{\vec{k}}^{\alpha\dagger} B_{\vec{k}'}^{\beta\dagger} e^{-ik\cdot x} e^{-ik'\cdot y} + B_{\vec{k}}^{\alpha} A_{\vec{k}'}^{\beta} e^{ik\cdot x} e^{ik'\cdot y} + B_{\vec{k}}^{\alpha} B_{\vec{k}'}^{\beta\dagger} e^{ik\cdot x} e^{-ik'\cdot y}] \times N_{\vec{k}} N_{\vec{k}'}, \tag{C.22}$$

which equals, using Eqs. (C.1)–(C.11) and (C.16),

$$\sum_{\vec{k},\vec{k}'} [e^{-i\Lambda_{\alpha\beta}} A_{\vec{k}'}^{\beta} A_{\vec{k}}^{\alpha\dagger} e^{-ik\cdot x} e^{ik'\cdot y} + e^{-i\kappa_{\beta\alpha}} B_{\vec{k}'}^{\beta\dagger} A_{\vec{k}}^{\alpha\dagger} e^{-ik\cdot x} e^{-ik'\cdot y} + e^{i\kappa_{\alpha\beta}} A_{\vec{k}'}^{\beta} B_{\vec{k}}^{\alpha} e^{ik\cdot x} e^{ik'\cdot y} + e^{-i\xi_{\alpha\beta}} B_{\vec{k}'}^{\beta\dagger} B_{\vec{k}}^{\alpha} e^{ik\cdot x} e^{-ik'\cdot y}] \times N_{\vec{k}} N_{\vec{k}'}. \tag{C.23}$$

As in Appendix B, it is apparent from Eq. (C.23) that the momentum dependence of $\Lambda_{\alpha\beta}$, $\xi_{\alpha\beta}$, and $\kappa_{\alpha\beta}$ must be dropped in order to have simple real space commutation for fields. Furthermore, taking

$$\Lambda_{\alpha\beta} = \kappa_{\beta\alpha} = -\kappa_{\alpha\beta} = \xi_{\alpha\beta}, \tag{C.24}$$

one has, for $\alpha \neq \beta$, from Eqs. (C.22) and (C.23),

$$\phi^\alpha(x)\,\phi^\beta(y) = e^{i\Lambda_{\alpha\beta}} \phi^\beta(y)\,\phi^\alpha(x) \tag{C.25}$$

and

$$\phi^{\alpha\dagger}(x)\,\phi^\beta(y) = e^{-i\Lambda_{\alpha\beta}} \phi^\beta(y)\,\phi^{\alpha\dagger}(x). \tag{C.26}$$

The result is that for complex fields simple commutation relations constrain $\rho_{\alpha\beta}$ to be momentum independent, but not necessarily ± 1 as in the case of real scalar fields. Note that (C.1)–(C.13), (C.25), and (C.26) satisfy Heisenberg's equation of motion, as well as make the combination

$$\phi_\alpha^\dagger(x)\,\phi_\alpha(x) \tag{C.27}$$

commute with all fields at spacelike separation; a property used to prove locality of $SU(N)$ interaction terms of this form.

Note that Eqs. (C.1)–(C.13), (C.16), or the real space analogues, (C.25) and (C.26), may be viewed as an analytic continuation of normal quantum statistics. Unlike the case for real scalar fields, the normal statistics limit,

$$\Lambda_{\alpha\beta} \to 0, \tag{C.28}$$

for all (α, β), can be taken.

Similarly, the classical limit of general statistics can be taken by

$$\Lambda_{\alpha\beta} \to 0 \tag{C.29}$$

and

$$\hbar \to 0. \tag{C.30}$$

APPENDIX D: Momentum–Dependent Statistics and Exchange Oscillations

In the previous appendix we showed that complex scalar fields could carry complex relative statistics and still be consistent with Heisenberg's equation of motion and locality of the $SU(N)$ invariant interaction term. In this appendix momentum–dependent relative statistics are shown to satisfy these assumptions.

Assume $A^{\alpha}_{\vec{k}}$ and $B^{\alpha}_{\vec{k}}$, $\alpha = 1, 2,..., N$, defined with statistics in Eqs. (C.1)–(C.12), (C.16) with $n = 0$, and complex fields, $\phi_\alpha(x)$, defined in Eq. (C.17). The $SU(N)$ invariant interaction term,

$$H_{\text{int}}(x) = \left(\sum_{\alpha} \phi^{\dagger}_{\alpha}(x)\, \phi_{\alpha}(x)\right)^2 = \sum_{\alpha\beta} \phi^{\dagger}_{\alpha}(x)\, \phi_{\alpha}(x)\, \phi^{\dagger}_{\beta}(x)\, \phi_{\beta}(x), \tag{D.1}$$

is assumed to satisfy the locality condition

$$[H_{\text{int}}(x), H_{\text{int}}(y)] = 0, \qquad (x-y)^2 > 0 \text{ (spacelike)}. \tag{D.2}$$

A set of conditions which imply Eq. (D.1) will be used to define the relative statistics. Define operators, $O^{\alpha}_{i}(\vec{k}, \vec{k}')$, $i = 1, 2, 3, 4$ and $\alpha = 1, 2,..., N$,

$$O^{\alpha}_{1}(\vec{k}, \vec{k}') = A^{\alpha\dagger}_{\vec{k}} A^{\alpha}_{\vec{k}'}, \tag{D.3}$$

$$O^{\alpha}_{2}(\vec{k}, \vec{k}') = B^{\alpha}_{\vec{k}} B^{\alpha\dagger}_{\vec{k}'}, \tag{D.4}$$

$$O^{\alpha}_{3}(\vec{k}, \vec{k}') = B^{\alpha}_{\vec{k}} A^{\alpha}_{\vec{k}'}, \tag{D.5}$$

$$O^{\alpha}_{4}(\vec{k}, \vec{k}') = A^{\alpha\dagger}_{\vec{k}} B^{\alpha\dagger}_{\vec{k}'}, \tag{D.6}$$

which occur in the product $\phi^{\alpha\dagger}(x)\, \phi^{\alpha}(x)$. We require the condition

$$[O^{\alpha}_{i}(\vec{k}, \vec{k}'), O^{\beta}_{j}(\vec{k}'', \vec{k}''')] = 0, \tag{D.7}$$

for $\alpha \neq \beta$, $(i,j) = 1, 2, 3, 4$, and all $\vec{k}$, $\vec{k}'$, $\vec{k}''$, $\vec{k}'''$. From the form of Eq. (D.1) it is obvious that Eqs. (D.7) imply Eq. (D.2). Equations (D.7) can be used as condition to deduce the form of $\Lambda_{\alpha\beta}$, $\xi_{\alpha\beta}$, and $\kappa_{\alpha\beta}$ in Eqs. (C.1)–(C.12) consistent with locality. From Eqs. (C.1)–(C.12) and Eqs. (D.3)–(D.6) we have the relations (for clarity of notation use $\vec{k}_i \to i$)

$$O_1^\alpha(1,2)\,O_1^\beta(3,4) = e^{iX_{\alpha\beta}(1,2,3,4)} O_1^\beta(3,4)\,O_1^\alpha(1,2), \tag{D.8}$$

$$O_1^\alpha(1,2)\,O_2^\beta(3,4) = e^{iY_{\alpha\beta}(1,2,3,4)} O_2^\beta(3,4)\,O_1^\alpha(1,2), \tag{D.9}$$

$$O_1^\alpha(1,2)\,O_3^\beta(3,4) = e^{iZ_{\alpha\beta}(1,2,3,4)} O_3^\beta(3,4)\,O_1^\alpha(1,2), \tag{D.10}$$

$$O_2^\alpha(1,2)\,O_2^\beta(3,4) = e^{iU_{\alpha\beta}(1,2,3,4)} O_2^\beta(3,4)\,O_1^\alpha(1,2), \tag{D.11}$$

$$O_2^\alpha(1,2)\,O_3^\beta(3,4) = e^{iV_{\alpha\beta}(1,2,3,4)} O_3^\beta(3,4)\,O_2^\alpha(1,2), \tag{D.12}$$

$$O_3^\alpha(1,2)\,O_3^\beta(3,4) = e^{iW_{\alpha\beta}(1,2,3,4)} O_3^\beta(3,4)\,O_3^\alpha(1,2), \tag{D.13}$$

$$O_3^\alpha(1,2)\,O_4^\beta(3,4) = e^{iR_{\alpha\beta}(1,2,3,4)} O_4^\beta(3,4)\,O_3^\alpha(1,2), \tag{D.14}$$

where the arguments of the exponentials are set to zero by Eq. (D.7). These arguments are given by

$$X_{\alpha\beta}(1,2,3,4) = -\Lambda_{\alpha\beta}(2,3) + \Lambda_{\alpha\beta}(1,3) + \Lambda_{\alpha\beta}(2,4) - \Lambda_{\alpha\beta}(1,4) = 0, \tag{D.15}$$

$$U_{\alpha\beta}(1,2,3,4) = -\xi_{\alpha\beta}(2,3) + \xi_{\alpha\beta}(1,3) + \xi_{\alpha\beta}(2,4) - \xi_{\alpha\beta}(1,4) = 0, \tag{D.16}$$

$$Y_{\alpha\beta}(1,2,3,4) = -\kappa_{\beta\alpha}(3,2) + \kappa_{\beta\alpha}(3,1) + \kappa_{\beta\alpha}(4,2) - \kappa_{\beta\alpha}(4,1) = 0, \tag{D.17}$$

$$Z_{\alpha\beta}(1,2,3,4) = -\kappa_{\beta\alpha}(3,2) + \kappa_{\beta\alpha}(3,1) + \Lambda_{\alpha\beta}(2,4) - \Lambda_{\alpha\beta}(1,4) = 0, \tag{D.18}$$

$$V_{\alpha\beta}(1,2,3,4) = -\xi_{\alpha\beta}(2,3) + \xi_{\alpha\beta}(1,3) - \kappa_{\alpha\beta}(2,4) + \kappa_{\alpha\beta}(1,4) = 0, \tag{D.19}$$

$$W_{\alpha\beta}(1,2,3,4) = -\kappa_{\beta\alpha}(3,2) + \xi_{\alpha\beta}(1,3) + \Lambda_{\alpha\beta}(2,4) + \kappa_{\alpha\beta}(1,4) = 0, \tag{D.20}$$

$$R_{\alpha\beta}(1,2,3,4) = -\Lambda_{\alpha\beta}(2,3) - \kappa_{\alpha\beta}(1,3) + \kappa_{\beta\alpha}(4,2) - \xi_{\alpha\beta}(1,4) = 0. \tag{D.21}$$

Equations (D.15), (D.16), and (D.17) are satisfied by separation of variables in Λ, ξ, and κ. Also, Eqs. (C.13b) and (C.13c) are used to write solutions

$$\Lambda_{\alpha\beta}(\vec{k},\vec{k}') = a_{\alpha\beta}(\vec{k}) - a_{\beta\alpha}(\vec{k}'), \tag{D.22}$$

$$\xi_{\alpha\beta}(\vec{k},\vec{k}') = c_{\alpha\beta}(\vec{k}) - c_{\beta\alpha}(\vec{k}'), \tag{D.23}$$

$$\kappa_{\alpha\beta}(\vec{k},\vec{k}') = b^1{}_{\alpha\beta}(\vec{k}) - b^2{}_{\alpha\beta}(\vec{k}'), \tag{D.24}$$

where

$$a_{\alpha\alpha} = c_{\alpha\alpha} = b^i{}_{\alpha\alpha} = 0 \qquad (i = 1, 2), \tag{D.25}$$

from the spin-statistics constraint Eqs. (C.13a). Equations (D.18) and (D.19) then imply, upon substitution of Eqs. (D.22)–(D.24),

$$b^1_{\alpha\beta} = -c_{\alpha\beta}, \tag{D.26}$$

$$b^2_{\alpha\beta} = -a_{\beta\alpha}, \tag{D.27}$$

With the identifications made in Eqs. (D.22)–(D.27), Eqs. (D.20) and (D.21) hold automatically for arbitrary $\{a_{\alpha\beta}\}$ and $\{c_{\alpha\beta}\}$.

Summarizing, the functions Λ, ξ, and κ are given in terms of arbitrary vectors $a_{\alpha\beta}$ and $c_{\alpha\beta}$ by

$$\Lambda_{\alpha\beta}(\vec{k}, \vec{k}') = (a_{\alpha\beta}(\vec{k}) - a_{\beta\alpha}(\vec{k}')), \tag{D.28}$$

$$\xi_{\alpha\beta}(\vec{k}, \vec{k}') = (c_{\alpha\beta}(\vec{k}) - c_{\beta\alpha}(\vec{k}')), \tag{D.29}$$

$$\kappa_{\alpha\beta}(\vec{k}, \vec{k}') = (-c_{\alpha\beta}(\vec{k}) + a_{\beta\alpha}(\vec{k}')). \tag{D.30}$$

Equations (D.28)–(D.30) substituted into Eqs. (C.1)–(C.12) define a momentum–dependent statistics for which the $SU(N)$ invariant interaction term is local. The canonical formalism for fields constructed from operators satisfying these statistics can be developed analogous to the discussion in Section IV.

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