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Generalized Classical Albert–Zassenhaus Lie Algebras

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Two very large classes *GCAZ* and *CAZK* of Lie algebras are introduced, which contain all sums of classical, Albert–Zassenhaus, generalized Witt algebras of Kaplansky and associated *holomorphs*. Their rootsystems R are classified up to isomorphism. The group Aut L of automorphisms of L is shown to contain extensions of the *Weyl group* of R and the inner automorphism groups of *classical Lie algebra complements* of the *Witt subalgebra* of L. The *Weyl group* extension in Aut L acts transitively by conjugation on the classical complements, under general conditions.

1. INTRODUCTION

In Block [1], the class of (finite dimensional) Lie algebras $L=\sum_{a\in R}L_a$ over a field k such that

(1) $L=L^2$ and Center $L=0$;
(2) $\dim L_a=1$ and $a([L_a, L_{-a}])\neq 0$ for $a\in R-\{0\}$

is determined for characteristic $p>5$ as follows.

1.1. THEOREM (Block [1]). *A Lie algebra $L=\sum_{a\in R}L_a$ of characteristic $p>5$ satisfies conditions* (1) *and* (2) *if and only if $L=L_1\oplus\cdots\oplus L_n$ (direct sum of ideals) and $R=R_1\cup\cdots\cup R_n$ where $L_i=\sum_{a\in R_i-\{0\}}([L_aL_{-a}]+L_a)$ and either L_i is classical or L_i is Albert–Zassenhaus and R_i is a subgroup of ka_i ($a_i\in R_i-\{0\}$) for each $1\leqslant i\leqslant n$.*

1.2. DEFINITION. Accordingly, we refer to a Lie algebra $L=\sum_{a\in R}L_a$ satisfying conditions (1) and (2) as a *classical Albert–Zassenhaus* (CAZ) Lie algebra with *CAZ Cartan subalgebra* L_0. And we call a subset R of a vector space V over k a *CAZ rootsystem* if $R=R_1\cup\cdots\cup R_n$ where

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1. each R_i is a *classical rootsystem* (isomorphic to the rootsystem of a classical Lie algebra) or an *Albert–Zassenhaus rootsystem* (additive subgroup of ka_i for some $a_i \in R$) for $1 \leqslant i \leqslant n$;
2. $R_i \cap R_j = \{0\}$ for $i \neq j$;
3. $a, b \in R - \{0\}$ with $a \sim b$ implies that $a, b \in R_i$ for some $1 \leqslant i \leqslant n$, where $a \sim b$ if and only if $a + b \in R$ or $a - b \in R$.

The purpose of this paper is to study *generalized classical Albert–Zassenhaus* GCAZ *Lie algebras* and *GCAZ rootsystems* in the sense of Definition 1.3 below. Throughout the paper, k is a field of characteristic $p > 3$.

Note that the class of GCAZ Lie algebras contains the class of CAZ Lie algebras. It also contains the generalized Witt algebras of Kaplansky, as well as algebras of derivations of extensions of classical Lie algebras. In the definition, L_a^1 is the eigenspace $\{x \in L \mid [h, x] = a(h)x \text{ for all } h \in L_0\}$ and a root is *Witt* if $a, 2a, \ldots, (p-1)a$ are all roots.

1.3. DEFINITION. A Lie algebra $L = \sum_{a \in R} L_a$ is a *GCAZ Lie algebra* with *GCAZ Cartan subalgebra* L_0 if

1. $a(L_a^1, L_{-a}^1]) \neq 0$ for $a \in R - \{0\}$;
2. $[L_a, L_b] = L_{a+b}$ if $a, b, a + b \in R - \{0\}$ where $a \neq b$ and either a or b (or both) is Witt.

A subset R of a vector space V is a *GCAZ rootsystem* if $R = R \cup \cdots \cup R_n$ where

1. each R_i is $G_i + S_i$ where S_i is a classical rootsystem or $\{0\}$ and G_i is a *Kaplansky rootsystem* (additive subgroup of V) ($1 \leqslant i \leqslant n$);
2. $R_i \cap R_j = \{0\}$ for $i \neq j$;
3. $a \sim b$ implies $a, b \in R_i$ for some i for all $a, b \in R - \{0\}$.

We also consider the class of CAZK Lie algebras of Definition 1.4. Note that CAZ $\subset$ CAZK $\subset$ GCAZ, and that CAZK contains all direct sums of classical, Albert–Zassenhaus, Kaplansky algebras by Proposition 2.1.

1.4. DEFINITION. A Lie algebra $L = \sum_{a \in R} L_a$ is a *classical Albert–Zassenhaus–Kaplansky* (*CAZK*) *Lie algebra* with *CAZK Cartan subalgebra* L_0 if

1. $a([L_a^1, L_{-a}^1]) \neq 0$ for $a \in R - \{0\}$;
2. $[L_a, L_b] = L_{a+b}$ for all $a, b \in R - \{0\}$ with $a \neq \pm b$.

A *CAZK rootsystem* is a union $R = R_1 \cup \cdots \cup R_n$ (irreducible component decomposition) of classical rootsystems and Kaplansky rootsystems R_i.

For this paper, we need the following results on *symmetric Lie algebras*, those Lie algebras $L=\sum_{a\in R}L_a$ satisfying condition (1) of Definition 1.3; and on *Lie rootsystems*, a class of vector subsets containing the class of rootsystems of symmetric Lie algebras. A *Lie rootsystem* is a finite subset R of a vector space V over k such that:

1. $0\in R=-R$;

2. for $a\in R-\{0\}$, there exists $a^0\in \mathrm{Hom}_k(V,k)$ such that $a^0(a)=2$ and the corresponding *reflection* $r_a(c)=c-a^0(c)a$ stabilizes all rootstrings $R_b(a)=\{b-ra,\ldots,b+qa\}\subseteq R$ $(b\in R,\ q,r$ maximal with $0\leqslant q,\ r\leqslant p-1)$ having fewer than p elements;

3. $R=R^{\mathrm{W}}\cup R^{\mathrm{c}}$ where $R^{\mathrm{W}}=\{a\in R\mid a$ is Witt$\}$ and $R^{\mathrm{c}}=\{a\in R\mid a$ is classical$\}$; where a root a is *classical* if $R\cap\mathbb{Z}a=\{-a,0,a\}$;

4. $|R_b(a)|=1, p-1$, or p for $a\in R^{\mathrm{W}}-\{0\}$, $b\in R$.

In the following theorems, we assume that $L=\sum_{a\in R}L_a$ is a symmetric Lie algebra and/or R is a Lie rootsystem over a field k of characteristic $p>3$. Some of these and later theorems are concerned with *sections* $Ra=R\cap\mathbb{Z}a$, $Rab=R\cap(\mathbb{Z}a+\mathbb{Z}b)$, $Rabc=R\cap(\mathbb{Z}a+\mathbb{Z}b+\mathbb{Z}c)$, etc., of R and *sections* $La=\sum_{d\in Ra}L_d$, $Lab=\sum_{d\in Rab}L_d$, $Labc=\sum_{d\in Rabc}L_d$, etc., of L. The *irreducible* components of R are the $R_i=(R_i-\{0\})\cup\{0\}$ $(1\leqslant i\leqslant n)$ where the $R_i-\{0\}$ are the equivalence classes of $R-\{0\}$ of the equivalence relation on $R-\{0\}$ generated by the *adjacency relation* "$a\sim b$" defined by the condition "$a+b\in R$ or $a-b\in R$." In particular, the irreducible components of a GCAZ rootsystem $R=R_1\cup\cdots\cup R_n$ with $R_i=G_i+S_i$ are those R_i for which $G_i\neq\{0\}$, together with the irreducible components of those $R_j=G_j+S_j=S_j$ for which $G_j=\{0\}$.

1.5. THEOREM (Winter [5]). *R is a classical rootsystem (cf. Definition 1.2) if and only if all roots in R are classical.*

1.6. THEOREM (Winter [5]). *The Lie rootsystem R is classical if and only if the symmetric Lie algebra $L=\sum_{a\in R-\{0\}}([L_a,L_{-a}]+L_a)$ is classical.*

We call the $\mathbb{Z}_p$-dimension of $\mathbb{Z}R$ the prime rank *of R.*

1.7. THEOREM (Winter [5]). *The sections Ra, Rab are isomorphic to*:

1. $A=\{-1,0,1\}$, $W=\mathbb{Z}_p$ *(irreducible, prime rank* 1*)*;

2. $A\vee A$, $A\vee W$, $W\vee W$ *(reducible, prime rank* 2*)*;

3. A_2, B_2, G_2, $W\oplus W=W_2$, $A\oplus W$, $S_2=\{i\oplus j\mid i+j\neq 0$ or $i=j=0\}$, $T_2=S_2\cup A=\{i\oplus j\mid i\oplus j\neq 0$ or $i=-j=\pm 1$ or $0\}$ *(irreducible, prime rank* 2*)*.

1.8. THEOREM (Winter [5]). *Let R be a Lie rootsystem which has no* *section of type* T_2. *Then* R^{W} *is a* Witt rootsystem (*Lie rootsystem whose roots are all Witt*) *and for each* $b \in R^{c} - \{0\}$:

1. *R has a* splitting *f at b*, $f: R \to R$ *such that* $f(R) \ni b$ and

(i) $f(a)=0$ *if and only if* $a \in R^{\mathrm{W}}$;

(ii) $f^2 = f$;

(iii) $f(a)^0(f(c)) = a^0(c)$ *for all* $a \notin R^{\mathrm{W}}$, $c \in R$;

2. *For any splitting f of R*, $S = f(R)$ *is a classical rootsystem, called a* classical complement *of* R^{W}, *and* $R \subset R^{\mathrm{W}} + S$;

3. *Any two splittings* f, f' *are* uniquely isomorphic, *that is, there exists a unique isomorphism* $W: f(R) \to f'(R)$ *of clasical rootsystems such that* $W(c) - c \in R^{\mathrm{W}}$ *for all* $c \in f(R)$.

Although Part 1 of Theorem 1.8 follows from a corresponding extrinsic result of Winter [5] stated in terms of a universal *closure mapping* $R \to \hat{R}$, Parts 2 and 3 are not proved there. Accordingly, Parts 2 and 3 are proved in Section 6, which is concerned with the action of Aut R on the set of classical complements of R^{W}.

1.9. THEOREM (Winter [5]). L_a *is one dimensional if a is classical.*

1.10. THEOREM (Winter [4]). *Let R be a Witt rootsystem and let* $a, b, c \in R - \{0\}$ *with* $a \sim b \sim c$ *and* $a \nsim c$. *Then Rab and Rac are both of type* W_2. *Moreover Rabc is either* $Rb \oplus (Ra \cup Rc)$ *of type* $W \oplus (W \vee W)$, *or* $S_3(Ra \cup Rc)$ *of type* $S_3(W \vee W)$ *where* $S_3(Ra \cup Rc) = \{rx + sy + tz \mid (r+s+t \neq 0)$ *or* $(r+s+t=0$ *and* $rt=0)\}$ *with* $a = x - y$, $b = y - z$.

1.11. THEOREM (Winter [4]). *The irreducible sections Rabc of a Witt rootsystem are of types* W, W_2, S_2, $W \oplus W_2 = W_3$, $W \oplus (W \vee W)$, $W \oplus S_2$, $S_3 = \{r \oplus s \oplus t \in k^3 \mid r+s+t \neq 0$ *or* $r=s=t=0\}$, $S_3(W \vee W) = \{r \oplus s \oplus t \in k^3 \mid r+s+t \neq 0$ *or* $(r+s+t=0$ *and* $rt=0)\}$, $S_3(S_2) = r \oplus s \oplus t \in k^3 \mid r + s + t \neq 0$ or $(r+s+t=0$ *and* $(s+t \neq 0$ *or* $s=t=0))\}$.

In this paper, we generalize Theorems 1.10 and 1.11 as follows, and use them to classify the rootsystems of GCAZ Lie algebras and CAZK Lie algebras. These results are valid for characteristic $p > 3$.

THEOREM 4.6. *Let R be a Lie rootsystem having no section of type* T_2 *and let* $a, b, c \in R - \{0\}$ *with* $a \sim b \sim c$ *and* $a \nsim c$. *Then b is a Witt root and Rabc is either* $R_b \oplus (Ra \vee Rc)$, *of type* $W \oplus (W \vee W)$ *or* $W \oplus (W \vee A)$, *or* $S_3(\mathbb{Z}a \vee \mathbb{Z}_b) = \{rx + sy + tz \mid (r+s+t \neq 0)$ *or* $(r+s+t=0$ *and* $rt=0)\}$ *with* $a = x - y$, $b = y - z$ *of type* $S_3(W \vee W)$.

THEOREM 4.7. *The irreducible sections Rabc of a Lie rootsystem R having no section of type T_2 are of types A, W, A_2, B_2, G_2, W_2, S_2, $A \oplus W$, A_3, B_3, C_3, $W \oplus A_2$, $W \oplus B_2$, $W \oplus G_2$, $W \oplus (A \vee A)$, $W_2 \oplus A$, $W \oplus (W \vee A)$, W_3, $W \oplus (W \vee W)$, $W \oplus S_2$, S_3, $S_3(W \vee W)$, $S_3(T_2)$.*

THEOREM 3.1. *R is the rootsystem of a GCAZ Lie algebra over a field k of characteristic $p > 3$ if and only if R is a GCAZ rootsystem over k.*

THEOREM 3.2. *R is the rootsystem of a CAZK Lie algebra if and only if R is a CAZK rootsystem.*

In Section 6, we then prove the following theorems about Weyl and automorphism groups of R and L.

THEOREM 6.8. *Let R be a Lie rootsystem with no section of type T_2. Let S be a classical complement of R^W with base π, and let $W(R)$, $W(S)$, $U(S)$ be the subgroups of* Aut R *generated by $\{r_b \mid b \in R^c - \{0\}\}$, $\{r_b \mid b \in S - \{0\}$, $\{r_b r_{b+a} \mid b \in S - \{0\}$, $a \in R^W$, $a + b \in R\}$. Then:*

1. *$W(R) = W(S)U(S)$ (semidirect product with $U(S)$ normal);*
2. *$W(S)$ is isomorphic to the Weyl group of S under restriction to S;*
3. *$W(R)$ acts transitively on the set of classical complements S' of R^W in R, provided that S have no irreducible component of type $A_n(p \mid n+1)$ and either $p > 5$ or $p = 3$ and E_8 is not an irreducible component of R.*

THEOREM 6.9. *Let $L = \sum_{a \in R} L_a$ be a symmetric Lie algebra with no section of type T_2. Then* Aut L *has a subgroup $N = \langle W_b(t) \mid b \in R^c - \{0\}$, $t \in k - \{0\} \rangle$ and a surjective homomorphism $N \to W(R)$ given by $w \mapsto (w \mid_{L^0})^{*-1} \mid_R$ $(w \in N)$. For each classical complement S of R^W in R,* Aut L *has a closed subgroup* $\exp S$ *such that* $\exp S \mid_{L_S} = (\mathrm{Aut}\, L_S)_0$, *the inner automorphism group of $L_S = \sum_{b \in S - \{0\}} ([L_b L_{-b}] + L_b)$. Any two* $\exp S$, $\exp S'$ *are conjugate under N (S, S′ classical complements of R^W in R), provided that S have no irreducible component of type A_n $(p \mid n+1)$ and either $p > 5$ or $p = 3$ and E_8 is not an irreducible component of R.*

2. KAPLANSKY ALGEBRAS L_G AND HOLOMORPHS $L_{G \oplus S}$

Kaplansky [2] introduced the Lie algebra L_G with basis $\{(i, g) \mid i \in I, g \in G\}$ over k and multiplication

$$[(i, g), (j, h)] = h_i(j, g+h) - g_j(i, g+h)$$

for any additive group G of functions from a set I to the field k. These algebras are the generalized Witt algebras of Kaplansky, which we refer to as the *Kaplansky algebras*.

We regard G as any additive subgroup of the vector space k^I of functions from I to k. Letting $A_G = k[x^G]$ be the group algebra of the multiplication group x^G isomorphic to G under $g \mapsto x^g$, we may regard L_G as the Lie subalgebra $A_G T$ (A_G-span of T in Der A_G) of the derivation algebra Der A_G of A_G, where T is the k-span of t_i ($i \in I$), t_i being the derivation $t_i(x^g) = g_i x^g$ of A_G ($i \in I$). In fact, $\{x^g t_i \mid i \in I, g \in G\}$ is a basis for $A_G T$ and $(i, g) \mapsto x^g t_i$ is an isomorphism from L_G to $A_G T$:

$$[x^g t_i, x^h t_j] = x^{g+h}(h_i t_j - g_j t_i).$$

A Cartan subalgebra of $L_G = A_G T$ is kT, and the corresponding rootspace decomposition of $L_G = L$ is $L = \sum_{g \in G} L_g$ where $L_g = x^g T$ and $g(t_i) = g_i$ ($i \in I, g \in G$). We call this the *symmetric* Cartan decomposition of the Kaplansky algebra L, since L together with this Cartan decomposition is a symmetric Lie algebra.

2.1. PROPOSITION. *Let $L = \sum_{g \in G} L_g$ be the symmetric Cartan decomposition of a Kaplansky Lie algebra. Then $[L_g, L_h] = L_{g+h}$ ($g, h \in G, g \neq h$). In particular, L is a CAZK algebra.*

Proof. Let $g \neq h$. Then $g_i \neq h_i$ for som i. For such an i, we have $[x^g t_i, x^h t_i] = (h_i - g_i) x^{g+h} t_i$ with $h_i - g_i \neq 0$. It follows that $[L_g, L_h]$ contains $x^{g+h} t_i$ for all i such that $g_i \neq h_i$. Next, suppose that $g_j = h_j$ and choose i such that $g_i \neq h_i$. Then $g_i + g_j \neq h_i + h_j$ and $[x^g(t_i + t_j), x^h(t_i + t_j))] = (h_i + h_j - (g_i + g_j)) x^{g+h}(t_i + t_j)$ with $h_i + h_j - (g_i + g_j) \neq 0$. It follows that $[L_g, L_h]$ contains $x^{g+h}(t_i + t_j)$. Since $[L_g, L_h]$ contains $x^{g+h} t_i$ as well, $[L_g, L_h]$ contains $x^{g+h} t_j$. We conclude that $[L_g, L_h] = L_{g+h}$. ▌

We next construct the *holomorph* $L_{G \oplus S}$ corresponding to a finite subgroup G of a vector space V over k and a classical rootsystem S over k. We begin with a classical Lie algebra $L_S = \sum_{a \in S} L_a$ with rootsystem S; and with a Kaplansky algebra $L_G = A_G T = \sum_{d \in G} x^g T$ with rootsystem G. We view L_G as an algebra of derivations of the algebra $A_G = k[x^G]$, and $L_G \otimes 1$ as an algebra of derivations of $A_G \otimes L_S = k[x^G] \otimes L_S = \sum_{g \oplus a \in G \oplus S} x^g \otimes L_a$ where $(d \otimes 1)(f \otimes x) = d(f) \otimes x$ for $d \in L_G$ (viewed as derivation of A_G), $f \in A_G$, $x \in L_S$. Finally, we define $L = L_{G \oplus S} = L_G \otimes 1 + A_G \otimes L_S = \sum_{G \oplus S} L_{g \oplus a}$ where $L_{0 \oplus 0} = T \otimes 1 + \otimes L_0$, $L_{g \oplus 0} = x^g T \otimes 1 + x^g \otimes L_0$, $L_{g \oplus a} = x^g \otimes L_a$ ($g \in G$, $a \in R - 0$). Letting the product $[x, y]$ be $[x, y] = xy - yx$ for $x, y \in L_G \otimes 1$ (an algebra of derivations of $A_G \otimes L_S$), $[x, y] = \sum a_i b_j \otimes [u_i, v_j]$ for $x = \sum a_i \otimes u_i, y = \sum b_j \otimes v_j \in A_G \otimes L_S$ and $[x, y] = x(y) = -[y, x]$ for $x \in L_G \otimes 1$, $y \in A_G \otimes L_S$, $L_{G \oplus S} = L_G \otimes 1 + A_G \otimes L_S$ is the desired *holomorph* of the Lie algebra $A_G \otimes L_S$ with the subalgebra $L_G \otimes 1$ of the derivation algebra of $A_G \otimes L_S$.

Proposition 2.1 leads directly to Proposition 2.2 below.

2.2. PROPOSITION. *$L_{G\oplus S}$ is a GCAZ Lie algebra with rootsystem $G\oplus S$.*

Note that $L_{G\oplus 0}/\text{Solv}\, L_{G\oplus 0}$ is the Kaplansky algebra L_G and $L_{0\oplus S}/$ Solv $L_{0\oplus S}$ is the classical Lie algebra L_S, where Solv L denotes the solvable radical (maximal solvable ideal) of a Lie algebra L.

3. ROOTSYSTEMS OF GCAZ AND CAZK LIE ALGEBRAS

Throughout this section, $L=\sum_{a\in R} L_a$ denotes a GCAZ Lie algebra. We show in Theorem 5.2 that a Lie rootsystem R is GCAZ if and only if R has no section of type S_2, T_2, $W\oplus(W\vee W)$, $W\oplus(W\vee A)$. Using this result, we now determine the rootsystems of GCAZ Lie algebras as follows.

3.1. THEOREM. *R is the rootsystem of a GCAZ Lie algebra over a field k of characteristic $p>3$ if and only if R is a GCAZ rootsystem over k.*

Proof. Suppose first that R is a GCAZ rootsystem over k and $R=R_1\cup\cdots\cup R_n$ (irreducible component decomposition) with $R_i=G_i+S_i$ where G_i is a subgroup of a vector space V over k and S_i is a classical rootsystem or $\{0\}$ $(1\leqslant i\leqslant n)$. Then R is the rootsystem of the GCAZ Lie algebra $\sum\oplus L_{G_i+L_i}$ where $L_{G_i+S_i}$ is the holomorph discussed in Proposition 2.2.

Suppose, conversely, that $L=\sum_{a\in R} L_a$ is a GCAZ Lie algebra with rootsystem R. We claim that R is a GCAZ rootsystem. By Theorem 5.2, it suffices to show that R has no section of type S_2, T, $W\oplus(W\vee W)$, $W\oplus(W\vee A)$. Suppose first that R has a section Rab of type S_2 or T_2. Then Rab contains Witt roots v, w such that

1. $v-w\notin R$ and $2(v-w)\notin R$;
2. $2v-w$, $3v-w$, $3v-2w\in R$.

Define $u=w+2(v-w)$ and note that u, v, $u+v$, $u+v-w$ are all Witt roots. Conditions (1) and (2) on v, w together with the hypothesis $[L_c, L_d]=L_{c+d}$ $(c, d, c+d\in R-\{0\}$, $c\neq d$, c or d Witt) implies that $L_{u+v-w} = [L_{u+v}, L_{-w}] = [[L_u, L_v], L_{-w}] \subset [[L_u, L_{-w}], L_v] + [L_u, [L_v, L_{-w}]] \subset [L_{u-w}, L_v] + [L_u, L_v-w]$. But the extreme right-hand term is $\{0\}$, because $v-w\notin R$ and $u-w=2(v-w)\notin R$. It follows that $\{0\}= L_{u+v-w}=L_{3v-2w}$, which is impossible since $3v-2w\in R$ by condition (2). We conclude that R has no section of type S_2 or T_2.

Next, consider a section $Labc= \sum_{d\in Rabc} L_d$ corresponding to a section $Rabc=R\cap(\mathbb{Z}a+\mathbb{Z}b+\mathbb{Z}c)$. Suppose that $Rabc$ is of type $W\oplus(W\vee W)$ or type $W\oplus(W\vee A)$. Then we may assume with no loss of generality that the generators a, b, c are chosen such that $Rabc$ is $\mathbb{Z}_p(a+b)+(\mathbb{Z}_p a\vee\mathbb{Z}_p c)$ or

$\mathbb{Z}_p(a+b)+(\mathbb{Z}_p a \vee \{-c, 0, c\})$. In both cases, we have $a, b, c, a+b, a+b+c \in R$ with $a+b \neq 0$, $a+c \notin R$, $b+c \notin R$. It follows that:

$$[[L_a, L_b], L_c] \subset [[L_a, L_c], L_b] + [L_a, [L_b, L_c]] = \{0\}$$

so that $[[L_a, L_b], L_c] = 0$. It follows that $a = \pm b$ or $c = \pm(a+b)$, since otherwise the conditions $[L_a, L_b] = L_{a+b}$, $[L_{a+b}, L_c] = L_{a+b+c}$ imply that $\{0\} = [[L_a, L_b], L_c] = [L_{a+b}, L_c] = L_{a+b+c}$ and $L_{a+b+c} = \{0\}$, which contradicts $a+b+c \in R$. Note here that the conditions assumed in the foregoing argument, namely, $a, b, a+b \in R - \{0\}$ with $a \neq b$ and a or b Witt and $a+b$, c, $a+b+c \in R - \{0\}$ with $a+b \neq c$ and $a+b$ or c Witt, are indeed met.

We conclude that R has no section of type S_2, T_2, $W \oplus (W \vee W)$, $W \oplus (W \vee A)$, so that R is GCAZ by Theorem 5.2. ▌

3.2. THEOREM. *R is the rootsystem of a CAZK Lie algebra if and only if R is a CAZK rootksystem.*

Proof. One direction follows from Proposition 2.1. For the other, let $L = \sum_{a \in R} L_a$ be a CAZK Lie algebra with rootsystem R. By Theorem 3.1, the irreducible components of R are $R_i = G_i + S_i$ $(1 \leqslant i \leqslant n)$. It suffices, therefore, to show that $G_i = \{0\}$ or $S_i = \{0\}$ for all i. Suppose, to the contrary, that $G_i \ni g \neq 0$, $S_i \ni s \neq 0$ for some i. Then $Rgs = R \cap (\mathbb{Z}g + \mathbb{Z}s)$ is of type $W \oplus A$: $Rgs = (G_i + S_i) \cap (\mathbb{Z}g + \mathbb{Z}s) = \mathbb{Z}g + \{-s, 0, s\}$. Since $\pm g = s \pm g - s$, we have $L_g = [L_{s+g}, L_{-s}]$, $L_{-g} = [L_{s-g}, L_{-s}]$ so that $[L_{-g}, L_g] \subset [[L_{s-g}, L_{-s}], [L_{s+g}, L_{-s}]] \subset [L_s, L_{-s}] + [L_{s-g}, L_{-s+g}]$. Taking $h_a \in [L^1_{-a}, L^1_a]$ with $a(h_a) = 2$, and noting that $[L_{-a}, L_a] = kh_a$ for a classical ($a \in R^c - \{0\}$), by Theorem 1.9, we have $h_g \in kh_s + kh_{s-g}$, since s and $s-g$ are classical. But $g(h_g) = 2$; whereas $g(h_s) = s^0(g) = 0$, $g(h_{s-g}) = (s-g)^0(g) = 0$, since s, $s-g$ are classical and g Witt in $\mathbb{Z}_p g + \{-s, 0, s\}$. by examinination of the strings $R_g(s)$ and $R_g(s-g)$, stable under r_s and r_{s-g}, respectively. Thus, $2 = 0$, a contradiction. It follows that $G_i = \{0\}$ or $S_i = \{0\}$ for $1 \leqslant i \leqslant n$. ▌

4. LIE ROOTSYSTEMS OF PRIME RANK AT MOST 3

Throughout Section 4, R is a Lie rootsystem having no section of type T_2. It follows from the the rank 2 classification that $Rab = \mathbb{Z}a \vee \{-b, 0, b\}$ or $\mathbb{Z}a + \{-b, 0, b\}$ for any nonzero roots a, b with a Witt and b classical. For such a, b, it follows that $b^0(a) = 0$, since $R_a(b) = \{a\}$ or $R_a(b) = \{a-b, a, a+b\}$ in the above two cases, and $r_b(a) = a$ in each case. In particular, $a+b$ is classical for any nonzero roots a, b with a Witt, b classical, and $a+b \in R$, since $Ra(a+b) = Rab =$

$\mathbb{Z}a + \{-(a+b), 0, a+b\}$. And, consequently, we then also have $(a+b)^0(a) = 0$ by our remarks above.

We claim for $a \in R^W - \{0\}$, $b \in R^c - \{0\}$ and $a+b \in R$ that $r_b r_{a+b}(c) = c - b^0(c)a$ for all $c \in R$. To see this, we consider the $\mathbb{Z}$ – closure map $a \mapsto \hat{a}$ of Winter [5, Sect. 4]; and we first observe that $\hat{a} + \hat{b} = \hat{b} \neq \hat{0}$ and $(a+b)^0(c) = (\hat{a}+\hat{b})^0(\hat{c}) = \hat{b}^0(\hat{c}) = b^0(c)$, by Theorem 1.8. It follows that $R_b(c) = c - b^0(c)b$, $r_{a+b}(c) = c - (a+b)^0(c)\,(a+b) = c - b^0(c)(a+b) = c - b^0(c)b - b^0(c)a = r_b(c) - b^0(c)a$. But then $r_b r_{a+b}(c) = r_b(r_b(c) - b^0(c)a) = c - b^0(c)a$. We now formulate these observations for future reference.

4.1. Proposition. *Let a and b be nonzero Witt and classical roots, respectively, in a Lie rootsystem R excluding T_2. Then*:

1. $Rab = \mathbb{Z}a \vee \{-b, 0, b\}$ *or* $\mathbb{Z}a + \{-b, 0, b\}$;
2. $b^0(a) = 0$;
3. $r_b r_{a+b}(c) = c - b^0(c)a$ *for all* $c \in R$, *provided that* $a + b \in R$.

Let $a \in R^W - \{0\}$, $b \in R^c - \{0\}$, $a+b \in R$, $c \in R$ with $b^0(c) \neq 0$. Then R contains $r_b r_{a+b}(c) = c - b^0(c)a$, by Proposition 4.1, so that $a \sim c$ by the rank 2 classification and the observation that $b^0(c) \neq 0$. We state this for future reference.

4.2. Proposition. *$a \in R^W - \{0\}$, $b \in R^c - \{0\}$, $a+b \in R$, $c \in R$ with $b^0(c) \neq 0$. Then $a \sim c$.*

We let $b \approx c$ represent the condition $b^0(c) \neq 0$. Note that $b \approx c$ implies $b \sim c$, by the rank 2 classification, so that $\approx$ is a strong version of $\sim$. Then Proposition 4.2 above shows that $a \sim b \approx c$ implies $a \sim c$ for $a \in R^W - \{0\}$, $b \in R^c - \{0\}$, $c \in R - \{0\}$.

Since R has no section of type T_2, we know from Theorem 1.7 that $R \subset R^W + S$ where S is any classical complement of R^W. Moreover, any two such S are isomorphic.

Let $a_1, \ldots, a_n$ be a base for a classical rootsystem S over k.

4.3. Definition. We let det S be the determinant of the Cartan matrix $(a_j^0(a_i))$ over k. We say that S is *nonsingular* if det $S \neq 0$.

The following proposition is evident from examination of the Cartan matrices of the irreducible classical rootsystems A_n, B_n, C_n, D_n, E_n, F_n, G_n.

4.4. Proposition. *A classical rootsystem S is nonsingular if and only if no irreducible component of S is of type A_n $(p \mid n+1)$ and either $p > 5$ or $p = 5$ and E_8 is not an irreducible component of S.*

4.5 PROPOSITION. *Let S be a nonsingular classical subsystem of R. Then:*

1. $kR^{W} \cap kS = \{0\}$ *where* kR^{W}, kS *are the k-spans of* R^{W}, S, *respectively;*
2. Prime Rank $R' \geqslant$ Prime Rank R^{W} + Prime Rank S *and* Rank $R' \geqslant$ Rank R^{W} + Rank S *where R' is the subsystem of R generated by* R^{W}, S.

Proof. Clearly 1 implies 2. For 1, let $a_1,...,a_n$ be a base for S and let $d = \sum_{i=1}^{n} d_i a_i$ be in kR^{W}. We must show that $d = 0$. Since R excludes T_2, we have $a_j^0(R^{W}) = 0$, by Proposition 4.1, so that $a_j^0(d) = 0$ for $1 \leqslant i \leqslant n$. But then $\sum_{i=1}^{n} d_i a_j^0(a_i) = 0$ for $1 \leqslant j \leqslant n$. Since det $S \neq 0$, it follows that $d_i = 0$ for $1 \leqslant i \leqslant n$, so that $d = 0$, as was to be shown. ∎

It is not known whether there exists a Lie rootsystem R with no section of type T_2 having a classical subsystem S with $kR \cap kS \neq \{0\}$.

We now can extend the transitivity Theorem 1.10 for Witt rootsystems to Lie rootsystems having no section of type T_2.

4.6. THEOREM. *Let a, b, c be nonzero elements of R. Assume that a is a Witt root and that $a \sim b \sim c$ with $a \sim c$. Then b is a Witt root and $Rabc$ is either $Rb \oplus (Ra \vee Rc)$, of type $W \oplus (W \vee W)$ or $W \oplus (W \vee A)$, or $S_3(\mathbb{Z}a \vee \mathbb{Z}b) = \{ix + jy + kz \mid (i+j+k \neq 0)$ or $(i+j+k=0$ and $ik=0)\}$ with $a = x - y$, $b = y - z$, of type $S_3(W \vee W)$.*

Proof. Suppose first that b is classical. Then $b^0(c) = 0$, by Proposition 4.2. Since b is classical and $b \sim c$, Rbc is classical or $Rbc = \{-b, 0, b\} \oplus \mathbb{Z}d$ for some $d \in R$, by the rank 2 classification. This together with the condition on $b^0(c)$ imply in all cases that $R_c(b) = \{c-b, c, b+c\}$. We then have $a \sim b \sim (b+c)$ with $b^0(b+c) = b^0(b) + b^0(c) = 2 + 0 \neq 0$, so that $a \sim (b+c)$ by Proposition 4.2. It follows that $a \sim (b+c) \sim c$. Since $b^0(b+c) \neq 0$ implies that $b+c$ is classical, by Proposition 4.1, we conclude that $(b+c)^0(c) = 0$; for otherwise $(b+c)^0(c) \neq 0$ implies that $a \sim c$, by Proposition 4.2, a contradiction. Replacing b by $b+c$ in the foregoing arguments, we may then conclude similarly that $R_c(b+c) = \{-b, c, b+c\}$, $a \sim (b+c) \sim (b+2c)$ with $(b+c)^0(b+2c) = 2$ and $b+2c$ classical. Thereby, we have now generated classical roots $b-c$, b, $b+c$, $b+2c$. Moreover, we may continue to repeat the argument: $a \sim (b+2c)$, by Proposition 4.2, so that $a \sim (b+2c) \sim$ and $a \not\sim c$ implies that $(b+2c)^0(c) = 0$ and $R_c(b+2c) = \{-b, -c, c, b+3c\}$ by Proposition 4.2. But this implies that $R_b(c) \supset \{b-c, b, b+c, b+2c, b+3c\}$. (In fact, continuing repetition of the argument leads to continuing increase in estimated length—which is an observation which enables one to draw the conclusion in the next sentence also for generalized Lie rootsystems corresponding to nonsplit versions of symmetric Lie algebras.) We conclude that c is a Witt root, by inspection of the possibilities for Rbc given in the rank 2 classification. (Classical orbits have at most four elements.)

We have $Rabc \supset Rab \cup Rbc = (\mathbb{Z}a + \{-b, 0, b\}) \cup (\{-b, 0, b\} + \mathbb{Z}c) = \{-b, 0, b)\} + (\mathbb{Z}a \vee \mathbb{Z}c)$, since b is classical and $a, c \in R^W - \{0\}$. Note that $\{-b, 0, b\} + \{\mathbb{Z}a \vee \mathbb{Z}c)$ is not a Lie rootsystem:

1. $(a+b)^0(b) = 0$ by Proposition 4.2;
2. $2 = (a+b)^0(a+b) = (a+b)^0(a)$, by 1;
3. $R_a(a+b) = \{-b, a, 2a+b\}$, so that $r_{a+b}(a) = a$, $(a+b)^0(a) = 0$,

in contradiction to 2.

It follows that R has an element $d = ra + sb + tc \in Rac$ with $r, s, t \neq 0$, since $Rac = Ra \vee Rc \subset Rab \cup Rbc$. Suppose first that some such d is a Witt root. Then Rad is of type S_2 or W_2, by the rank 2 classification, since the other possibility $W \oplus A$ cannot contain two linearly independent Witt roots a, d.

Since $b \sim c$ with b classical and c Witt, $Rbc = \{-b, 0, b\} + \mathbb{Z}c$, so that $b + (t/s)c$ is a classical root of R. But this is impossible, since then Rad, which is of type S_2 or W_2, contains the classical root $b + (t/s)c = -(r/s)a + (1/s)(ra + sb + tc)$.

Since the assumption that b is classical leads to the contradiction above, we conclude that b is a Witt root. Suppose first that c is also a Witt root. Then a, b, c are in the Witt subsystem $R^W = \{x \in R \mid \hat{x} = \hat{0}\}$ and $Rabc$ is a Witt rootsystem, by Theorem 1.8. But then the assertion of Theorem 4.6 follows form Theorem 1.10.

We may now assume that $a, b \in R^W - \{0\}$ and $c \in R^c - \{0\}$, and we take $R = Rabc$. Since $a, b \in R^W$, R^W has prime rank $\geqslant 2$. Taking a classical complement S containing c, by Theorem 1.8, we have $R \subset R^W + S$ with $3 \geqslant$ prime rank $R^W +$ prime rank S, by Theorem 1.8. We conclude that $R^W = Rab$ and $S = \{-c, 0, c\}$. Suppose $a^0(b) \neq 0$. We have $b + c \in Rbc$, since $b \sim c$ implies that Rbd is of type $W \oplus A$. Since $a^0(b+c) = a^0(b) + 0 \neq 0$, we have $a \sim b + c \sim c$. In Rbc, we have $(b+c)^0(c) \neq 0$, since $Rbc = \mathbb{Z}b + \{-c, 0, c\}$ and $R_c(b+c) = \{-2b-c, -b, c\}$. Since $b+c$ is classical, we then have $a \sim c$ by Proposition 4.2, a contradiction. We conclude that $a^0(b) = 0$. Since $a \sim b$, it follows from the rank 2 classification that $R^W = Rab = \mathbb{Z}a + \mathbb{Z}b$.

We have $R \supset (\mathbb{Z}a + \mathbb{Z}b) \cup (\mathbb{Z}b + \{-c, 0, c\}) = \mathbb{Z}b \oplus (\mathbb{Z}a \vee \{-b, 0, b\})$. If the conclusion is equality, then R is of type $W \oplus (W \vee A)$ and the proof is complete. Suppose, to the contrary, there exists a root $d = ra + sb + tc \in R$ with $r, t \neq 0$. Then $s \neq 0$, since $a \not\sim c$; and we may take $t = 1$, since $d \in R^W \cup \{-c, 0, c\}$. But then we have d classical, since $d \notin R^W$, and $d \sim c$ since $d - c \in \mathbb{Z}a + \mathbb{Z}b$. Thus, Rcd is of type $W \oplus A$, so that $Rcd = \mathbb{Z}(ra + sb) + \{-c, 0, c\}$ and $R_c(d) = \{-2(ra+sb) - c, -(ra+sb), c\}$. It follows that $d^0(c) \neq 0$ and $d \approx c$ with d classical. Since

$a^0(b)=a^0(c)=0$ and $a^0(d)=ra^0(a)+sa^0(b)+a^0(b)=2r$, we then have $a\sim d\approx c$ with d classical, so that $a\sim c$, a contradiction. Thus c cannot be classical, and the proof is complete. ∎

We next extend the classification of Witt rootsystems of prime rank three to Lie rootsystems with no section of type T_2.

4.7. THEOREM. *Let R be an irreducible Lie rootsystem excluding T_2 of prime rank* 3. *Then R is isomorphic to exactly one of the following*:

R^W of rank 0. A_3, B_3, C_3 (*classical rootsystems*);
R^W of rank 1. $W\oplus A_2$, $W\oplus B_2$, $W\oplus G_2$, $W\oplus(A\vee A)$;
R^W of rank 2. $W_2\oplus A$, $W\oplus(W\vee A)$;
R^W of rank 3. W_3, $W\oplus(W\vee W)$, $W\oplus S_2$, S_3, $S_3(W\vee W)$, $S_3(S_2)$ (*Witt rootsystems*).

Proof. For $R=R^W$, our assertions follow from Theorem 1.10. For $R=R^c$, R is a classical rootsystem, by Theorem 1.5, and our assertions are again clear. Thus, we may assume that $R^W\neq\{0\}$ and $R^W\neq R$.

Since R has no section of type T_2, we have $R\subset R^W\cup S$ where S is a classical complement, by Theorem 1.8. By Proposition 4.5, we have $3\geqslant$ prime $R\geqslant$ prime rank R^W+ rank $S\geqslant 1+1=2$. Thus, R^W and S are both of prime rank 1 or 2. It follows from Theorem 1.7 that R^W is one of W, $W\vee W$, W_2, S_2. We now consider each of these cases.

Suppose first that $R=W=\mathbb{Z}a$. Then rank $S=1$ or 2. If rank $S=1$ and $S=\{-b,0,b\}$, we have $R=Rab=\mathbb{Z}a\vee S$ or $\mathbb{Z}a+S$, by the rank 2 classification. Let rank $S=2$. We claim that $R=\mathbb{Z}a\oplus S$. For this, it suffices to show that $Rac=\mathbb{Z}a+\{-c,0,c\}$ for any $c\in R-R^W$. By Theorem 4.6, the latter is equivalent to showing that $a\sim c$ for all $c\in R-R^W$. Suppose, to the contrary, that $c\in R-R^W$ and $a\not\sim c$. By the irreducibility of R, we can find $b_i\in R-\{0\}$ $(1\leqslant i\leqslant n)$ such that $a\sim b_1\sim\cdots\sim b_n\sim c$. From among all possible choices of $c\in R-R^W$ and consequent b_i with $a\not\sim c$, take c so that n is minimal. By the minimality of n, no b_i is in $\mathbb{Z}a$ $(1\leqslant i\leqslant n)$, for otherwise we could use the shorter chain $a\sim b_{i+1}\sim b_{i+2}\sim\cdots\sim b_n\sim c$; and $a\sim b_i$ for $1\leqslant i\leqslant n$, for otherwise we would have $a\not\sim b_i$ for some i, and we could take $b_i=c$ with shorter chain $a\sim b_1\sim\cdots\sim b_{i-1}\sim c$. Finally, we now have $a\sim b_n\sim c$, that is, $n=1$. Writing $b=b_n$, we have $a\sim b\sim c$ with b, c clasical and $a\not\sim c$. But this is impossible, by Theorem 4.6. We must conclude that $a\sim c$, as asserted.

Suppose next that $R^W=W\vee W=\mathbb{Z}a\vee\mathbb{Z}c$. Then $3\geqslant$ prime rank R^W+rank $S+2+$rank S implies that rank $S=1$ and $S=A=\{-b,0,b\}$ for some $b\in R$. By the irreducibility of R, it follows that $a\sim b\sim c$. Since a is Witt and b classical, we then have $a\sim c$, by Theorem 4.6, contrary to our hypothesis that $R^W=Rac=\mathbb{Z}a\vee\mathbb{Z}c$.

Suppose next that $R^{W} = W_2 = \mathbb{Z}A + \mathbb{Z}b$ for any $\mathbb{Z}_p$-independent $a, b \in R^{W}$. Since $3 \geqslant$ prime rank R^{W} + rank $S = 2$ + rank S, we have rank $S = 1$ and $S = A = \{-c, 0, c\}$. Thus, $R \subset R^{W} + S = W_2 + A$, by Theorem 1.8. By the irreducibility of R, we have $b \sim c$ for some $b \in R^{W}$. If $a \sim c$ for all $a \in R^{W} - \{0\}$, then $R^{W} + \{-c, 0, c\} \subset R$ since $Rac = \mathbb{Z}a + \{-c, 0, c\}$ for all $a \in R^{W} - \{0\}$ such that $a \sim c$. In this case, we then have $R = R^{W} + \{-c, 0, c\}$ and R is of type $W_2 \oplus A$. Otherwise, there exists $a \in R^{W} - \{0\}$ with $a \sim b \sim c$ and $a \not\sim c$, b being the element of R^{W} chosen above. But then R is of type $W \oplus (W \vee A)$ by Theorem 4.6.

Finally, suppose that $R^{W} = S_2$. Then $3 \geqslant$ rank R^{W} + rank $S = 2$ + rank S, so that rank $S = 1$ and $S = A = \{-c, 0, c\}$ for some $c \in R - R^{W}$. By irreducibility of R, we have $b \sim c$ for some $b \in R^{W} - \{0\}$.

Suppose that there exists $a \in R^{W} - \{0\}$ such that $a \not\sim c$. Then $a \sim b$ since $a, b \in R^{W} - \{0\}$ and R^{W} is of type S_2, so that $a \sim b \sim c$ and $a \not\sim c$. It again follows from Theorem 4.6 that R is of type $W \oplus (W \vee A)$. But this is impossible, since R^{W} is of type S_2. We conclude, therefore, that $a \sim c$ for all $a \in R^{W} - \{0\}$. Since $Rac = \mathbb{Z}a + \{-c, 0, c\}$ for all such a, it follows that $R = R^{W} + \{-c, 0, c\} = S_2 + A$ where $A = \{-c, 0, c\}$ and $S_2 = \{ia + jb \mid i + j \neq 0\}$. Since R has no section of type T_2, we have $\mathbb{Z}_p S_2 \cap \mathbb{Z}_p A = \{0\}$, by Proposition 4.5. (Note, however, that there is a Lie rootsystem $\{ia + jb \mid i + j \neq 0\} + \{-d, 0, d\}$ with $d = a - b$, namely, the Lie rootsystem T_2.) Thus, we may take a, b, c to be linearly independent over $\mathbb{Z}_p$, and we write $R = S_2 \oplus A$ to indicate the independence. But then R is not a Lie rootsystem. To see this, note that if R is a Lie rootsystem, then b is a Witt root and $a + c$ is classical with $b + a + c \in R$ and $-b + a + c \notin R$. But this cannot happen in a Lie rootsystem, since the rank two classification implies that $Rxy = \mathbb{Z}x + \{-y, 0, y\}$ for $x \in R^{W} - \{0\}$, $y \in R^{c} - \{0\}$, and $x + y \in R$. We therefore have reached a contradiction, so that the assumed case $R^{W} = S_2$ never occurs for irreducible Lie rootsystems excluding T_2 of prime rank 3. ▮

5. Characterization of GCAZ and CAZK Rootsystems

The proofs of Theorems 3.1 and 3.2, which classify the rootsystems of GCAZ and CAZK Lie algebras, depend on Theorem 5.2 below, which characterizes GCAZ rootsystems. We now establish Theorem 5.2.

We begin by noting that Theorem 4.6 has the following direct consequence.

5.1. Theorem. *Let R be a Lie rootsystem having no sections of type S_2, T_2, $W \oplus (W \vee W)$, $W \oplus (W \vee A)$, and let $a, b, c \in R - \{0\}$ with a Witt and $a \sim b \sim c$. Then $a \sim c$.*

5.2. THEOREM. *A Lie rootsystem R is a GCAZ rootsystem if and only if R has no section of type S_2, T_2, $W\oplus(W\vee W)$, $W\oplus(W\vee A)$.*

Proof. One direction is trivial. For the other, assume that R has no section of type S_2, T_2, $W\oplus(W\vee W)$, $W\oplus(W\vee A)$. By Theorem 1.8, we have $R\subset G+S$ where $G=R^{\mathrm{W}}$ and S is a classical complement of R^{W}. By Theorem 5.1, $g\sim h$ is an equivalence relation for $G-\{0\}$ with equivalence classes $G_1-\{0\},\cdots,G_m-\{0\}$. Since $Rgh=\mathbb{Z}_p g+\mathbb{Z}_p h$ for $g,h\in G_i-\{0\}$, $G_i=G_i-\{0\}\cup\{0\}$ is a group for $1\leqslant i\leqslant m$. Let $S_i=\{s\in S-\{0\}\mid g\sim s$ for some $g\in G_i-\{0\}\}\cup\{0\}$ $(1\leqslant i\leqslant m)$. Note that $s\in S_i-\{0\}$, $t\in S-\{0\}$ and $s\sim t$ implies $t\in S_i-\{0\}$, by Theorem 5.2. Thus, S_i is a union of irreducible components of S and S_i is a classical rootsystem $(1\leqslant i\leqslant m)$. Consequently, $S=S_1\cup\cdots\cup S_m\cup S_{m+1}\cup\cdots\cup S_n$ where $S_{m+1},\ldots,S_n$ are the irreducible components of S not contained in any S_i $(1\leqslant i\leqslant m)$. Take $g\in G_i-\{0\}, s\in S_i-\{0\}$. Note that $g\sim s$, by Theorem 5.2, so that $Rgs=\mathbb{Z}_g+\{-s,0,s\}$, by Theorem 1.7. It follows that $g+s\in R$ for $g\in G_i$, $s\in S_i$, that is, R contains G_i+S_i. But then $R=G_1+S_1\cup\cdots\cup G_m+S_m\cup S_{m+1}\cdots\cup S_n$, which completes the proof. ∎

6. THE WEYL GROUP OF L: CONJUGACY OF CLASSICAL COMPLEMENTS UNDER THE WEYL GROUP

Throughout this section R denotes a Lie rootsystem which has no section of type T_2 and L denotes a symmetric Lie algebra with rootsystem R.

By Theorem 1.8, Part 1, we know that R has a *splitting* f at any $b\in R^{\mathrm{c}}-\{0\}$, that is, a homomorphism $f\colon R\to R$ such that $b\in f(R)$ and:

(i) $f(a)=0$ if and only if $a\in R^{\mathrm{W}}$;
(ii) $f^2=f$;
(iii) $f(a)^0(f(c))=a^0(c)$ for all $a\notin R^{\mathrm{W}}$, $c\in R$.

We now prove Parts 2 and 3 of Theorem 1.8. For this, let $S=f(R)$ and consider any $f(a),f(c)\in S-\{0\}$. Then we define $r_{f(a)}(f(c))=f(c)-a^0(c)f(a)=f(c-a^0(c)a)=f(r_a(c))$. It follows easily that $r_{f(a)}$ stabilizes all rootstrings $R_{f(c)}(f(a))$ of fewer then p elements. To show that S is a classical rootsystem, it therefore remains only to show that all roots in S are classical, by Theorem 1.5. But if $f(b)\in S$ is not classical, then $f(b)\in R^{\mathrm{W}}$ implies that $0=ff(b)=f^2(b)=f(b)$ and $f(b)=0$. We claim next that $R\subset R^{\mathrm{W}}+S$. Take $b\in R-R^{\mathrm{W}}$ and let $a=f(b)$. Then $f(a)=f^2(b)=f(b)\neq 0$, so that $a\notin R^{\mathrm{W}}$ and $2=f(a)^0(f(a))=f(a)^0(f(b))=a^0(b)$. It follows that $b-a\in R$. But then $f(b-a)=f(b)-f(a)=0$, so that $b-a\in R^{\mathrm{W}}$. This implies that $b=(b-a)+a\in R^{\mathrm{W}}+f(R)=R^{\mathrm{W}}+S$. This proves Part 2 of Theorem 1.8. For Part 3, let $f\colon R\to R$ and $f'\colon R\to R$ be splittings and consider the corresponding classical complements $S=f(R)$

and $S' = f'(R)$ of R^W. Consider the $\mathbb{Z}$-closure mapping $\wedge : R \to \hat{R}$ of Winter [5]. It is shown there that $R^W = \{a \in R \mid \hat{a} = \hat{0}\}$. Since $R^* = \text{Hom}(R, \mathbb{Z})$ contains $S^0 = \{a^* \mid a \in S - \{0\}\}$, the set of Cartan integer functions $a^*(b)$ $(b \in S)$ of the classical rootsystem S, it follows as in the proof of Theorem 1.5 that the restriction of $\wedge$ to S is injective. Since $R^W = \hat{0}$, it follows that $\hat{S} = \hat{R}$ and $\hat{}$ maps S isomorphically to $\hat{R}$, also as in the proof of Theorem 1.5. Similarly, $\hat{}$ maps S' isomorphically to $\hat{R}$. It follows that there is an isomorphism $w: S \to S'$ such that $\hat{b} = \widehat{w(b)}$ for all $b \in S$. But then $b^0(\widehat{w(b)}) = \hat{b}(w(b)) = \hat{b}^0(\hat{b}) = 2$, so that $w(b) - b \in R$. It follows that $w(b) - b \in R^W$, since $\widehat{w(b)} - \hat{b} = \hat{0}$. To see that w is unique, take a base $s_1,...,s_n$ for S and let $s_i' = w(s_i)$ $(1 \leqslant i \leqslant n)$. Let $\bar{w}: S \to S'$ be an isomorphism with $\bar{w}(b) - b \in R^W$ $(b \in S)$ and let $s_i'' = \bar{w}(s_i)$ $(1 \leqslant i \leqslant n)$. Then $\hat{s}_i' = \hat{s}_i''$ implies that $s_i' = s_i''$ $(1 \leqslant i \leqslant n)$, so that $w = \bar{w}$. This proves Part 3 of Theorem 1.8.

Our hypothesis that R have no section of type T_2 implies that for any $b \in R - \{0\}$, $a \in R$ with b classical, the Cartan integer $b^*(a) = q - r$ defined by the a-orbit $R_b(a) = \{b - ra,...,b + qa\}$ of b is 0, -1, -1, -1, -2, -3, 0, 0, -2 in all possible types $A \vee A$, A_2, B_2 long, G_2 long, B_2 short, B_2 short, $w \vee A$, $w + A$ mixed, $w + A$ classical. Here, we take the values $b^*(a)$ from Fig. 1 of Winter [5]. It follows that $R_b(a)$ has fewer them p elements for all $a \in R$, so that $r_b \in \text{Aut } R$ with r_b uniquely determined by b for $b \in R - \{0\}$, b classical. This makes possible the following definition.

6.1. Definition. The *Weyl group* of R is the group $W(R)$ generated by $\{r_b \mid b \in R - \{0\},\ b \text{ classical}\}$.

Taking $p > 7$ so that $(p-1)/2 \geqslant 4 > |R_b(a)|$ for all $b \in R - \{0\}$ classical and all $a \in R$, and taking $x \in L_b$, $b \in R - \{0\}$, b classical, the series $\exp \text{ad } x = 1 + \text{ad } x + (\text{ad } x)^2/2! + \ldots$ terminates at or before the $(p-1)/2$ power. Since $\text{ad } x$ is a derivation, it follows that $\exp \text{ad } x \in \text{Aut } L$. We let $w_b(t) = \exp \text{ad } te_b \exp - \text{ad } t^{-1} e_{-b} \exp \text{ad } teb$ where $t \in k - \{0\}$, $e_b \in L_b$, $e_{-b} \in L_{-b}$, $b([e_b, e_{-b}]) = 2$. Since $\dim L_{\pm b} = 1$ for $b \in R - \{0\}$, b classical, by Theorem 1.9, all possible choices for e_b, e_{-b} are accounted for by including the parameter $t \neq 0$: $L_b = \{t\, e_b \mid t \in k\}$.

6.2. Definition. The *Weyl group* of $L = \sum_{a \in R} L_a$ is $W(L) = N(L_0)/N_0(L_0)$ where $N(L_0)$ is the group generated by $\{W_b(t) \mid b \in R - \{0\},\ b$ classical, $t \in k - \{0\}\}$ and $N_0(L_0) = \{w \in N(L_0) \mid wL_a = L_a \text{ for all } a \in R\}$.

We have generated a group $N(L_0)$ of automorphisms of L, and it is easily checked that the generators $w_b(t)$ satisfy:

1. $W_b(t)$ normalizes L_0;
2. the adjoint $W_b(t)^*$ of $W_b(t)|_{L_0}$ is r_b, that is, $W_b(t)^*(a) = r_b(a) = a - b^0(a)b$ $(b \in R - \{0\},\ b \text{ classical})$;

3. $WL_a = L_{w^{*-1}a}$ $(w \in N(L_0))$ and $W_b(s)^* W_b(t)^* L_a = L_a$ $(b \in R - \{0\}$, b classical, $a \in R)$.

Consequently, the mapping $W \mapsto W^{*-1}|_R$ is a homomorphism from $N(L_0)$ onto $W(R)$ with kernel $N_0(L_0)$, that is, $W(L)$ and $W(R)$ are isomorphic.

6.3. THEOREM. *The Weyl group $W(L) = N(L_0)/N_0(L_0)$ of L is isomorphic to the Weyl group $W(R)$ of R, where the isomorphism is induced by $w \mapsto w^{*-1} | R$ $(w \in N(L_0))$.*

Since the Weyl group $W(R)$ lifts to the group $N(L_0)$ of automorphisms of L, by Theorem 6.3, conjugacy of subsets of R under $W(R)$ implies conjugacy of associated subsets of L under Aut L. We now pursue such conjugacy where the subsets of R are the classical compliments S of R^{W} of Theorem 1.8 and the associated subsets of L are the *classical Lie algebra complements* $L_s = \sum_{b \in S - \{0\}} ([L_b, L_{-b}] + L_b)$ *of the Witt Lie subalgebra* $L^{\mathrm{W}} = \sum_{a \in R^{\mathrm{W}}} L_a$, R^{W} being the Witt rootsystem $\{a \in R | a \text{ is Witt}\}$.

We now take two classical complements S, S', of R^{W} in R. Let $s_1, \ldots, s_n$ be a basis for S (as classical rootsystem). By Part 3 of Theorem 1.8, proved earlier in this section, there exists an isomorphism $w: S \to S'$ such that $w(b) - b \in R^{\mathrm{W}}$ for all $b \in S$. Let $s_i' = w(s_i)$ $(1 \leqslant i \leqslant n)$, so that $a_i =_{\mathrm{def}} s_i' - s_i$ is in R^{W} for $1 \leqslant i \leqslant n$. It follows that $a_1 + s_1, \ldots, a_n + s_n$ is a basis for S'. Conversely, if $a_1, \ldots, a_n \in R^{\mathrm{W}}$ and $s_1 + a_1, \ldots, s_n + a_n \in R$, then it can be shown as in Winter [5] that there is a classical complement S'' with base $a_1 + s_1, \ldots, a_n + s_n$ and isomorphism from S to S'' mapping s_i to $a_i + s_i$ $(1 \leqslant i \leqslant n)$. Thus, relative to one fixed classical complement S with base $s_1, \ldots, s_n$, the other classical complements S' are determined by

1. elements $a_1, \ldots, a_n \in R^{\mathrm{W}}$ such that $a_1 + s_1, \ldots, a_n + s_n \in R$;
2. the corresponding defining condition $s' \in S'$ if and only if $s' = \sum_{i=1}^n m_i(a_i + s_i)$ where $s = \sum_{i=1}^n m_i s_i$ is in S.

We now discuss passage from the base $\pi = \{s_1, \ldots, s_n\}$ of S to the base $\pi' = \{s_1', \ldots, s_n'\}$ $(s_1' = a_1 + s_1, \ldots, s_n' = a_n + s_n)$ for a given S' by an element of $W(R)$. Our starting point is the pair of equations of Proposition 4.1 for Witt roots $a \in R^{\mathrm{W}}$ and nonzero classical roots $b \in R^{\mathrm{c}} - \{0\}$ with $a + b \in R$:

$$r_b r_{a+b}(c) = c - b^0(c)a \qquad (c \in R)$$

$$b^0(c) = 0 \qquad (c \in R^{\mathrm{W}}).$$

6.4. DEFINITION. $r_{b,a}(c) = c - b^0(c)a$ $(a, b, c \in R, b \neq 0)$.

A base $\pi = (s_1, \ldots, s_n)'$ of a classical complement S of R^{W} is called a *base complement* of R^{W}. Given two base complements $\pi = (s_1, \ldots, s_n)'$, $\pi' = (s_1', \ldots, s_n')'$ of R^{W}, π and π' are *adjacent* if $\pi' = a + \pi$ for some

$a=(a_1,\dots,a_n)'$ with $a_i \in R^W$ for $1 \leqslant i \leqslant n$. For such adjacent base complements π, π' of R^W, we define:

$$r_{\pi\pi'} = r_{s_n,a_n} \cdots r_{s_1,a_1}.$$

The equations $r_b r_{a+b}(c) = c - b^0(c)a = r_{b,a}(c)$ lead to the recursive family of equations:

$$r_{a_1,s^1}(_i) = s_i - s_1^0(s_i)a_1$$

$$r_{a_2,s_2} r_{a_1}, s_1(s_i) = s_i - s_1^0(s_i)a_1 - s_1^0(s_i)a_2$$

$$\vdots$$

$$r_{a_n,s_n} \cdots r_{a_2,s_2} r_{a_1 s_1}(s_i) = s_i - s_1^0(s_i)a_1 - s_2^0(s_i)a_2 - \dots - s_n^0(s_i)a_n.$$

Replacing s_i by the column vector $\pi=(s_1,\dots,s_n)'$ in the last of these equations, we have

$$r_{\pi\pi'}\begin{pmatrix} s_1 \\ \cdot \\ \cdot \\ \cdot \\ s_n \end{pmatrix} = \begin{pmatrix} s_1 \\ \cdot \\ \cdot \\ \cdot \\ s_n \end{pmatrix} - \begin{pmatrix} s_1^0(s_1)\, s_2^0(s_1) \cdots s_n^0(s_1) \\ \cdot \\ \cdot \\ \cdot \\ s_1^0(s_n) s_2^0(s_n) \cdots s_n^0 s_n) \end{pmatrix} \begin{pmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{pmatrix}$$

where $r_{\pi\pi'}$ acts coordinate-wise on the entries s_i. We rewrite this as

$$r_{\pi\pi'}(\pi) = \pi - (s_j^0(s_i))(\pi' - \pi)$$

where $(s_j^0(s_i))$ is the *Cartan Matrix* of the base π for the classical rootsystem S.

We need the following proposition, which applies to Lie rootsystems R having no sections of type T_2 and nonzero classical roots b of R.

6.5. PROPOSITION. *Let R be a Lie rootsystem, $f \in$ Aut R, $b \in R - \{0\}$. Suppose that all b-orbits $R_c(b)$ $(c \in R)$ have fewer than p elements. Then*:

1. $f(b)^0(f(c)) = b^0(c)$;
2. $f r_b f^{-1} = r_{f(b)}$.

Proof. $R_b(c) = \{c - rb, \dots, c + qb\}$ uniquely determines $b^0(c)$ as $r - q$. Thus, 1 follows from the equation $f(R_b(c)) = R_{f(b)}(f(c))$. And 2 then follows from the equations $f(r_b(c)) = f(c - b^0(c)b) = f(c) - b^0(c)f(b) = f(c) - f(b)^0(f(c))f(b) = r_{f(b)}(f(c))$. ∎

6.6. COROLLARY. *Let R be Lie rootsystem having no section of type T_2. Let $a \in R^W$, $b \in R^c - \{0\}$; and let π, π' be adjacent base complements of R^W. Then*

1. $f r_{b,a} f^{-1} = r_{f(b), f(a)}$; *and*
2. $f r_{\pi\pi'} f^{-1} = r_{f(\pi) f(\pi')}$

for any $f \in \operatorname{Aut} R$.

We are now ready to prove the main theorem of Section 6, which we state using the following definition.

6.7. DEFINITION. *For any $X \subset R^c - \{0\}$, $r_X = \{r_b \mid b \in X\}$, $W(X)$ is the group $< r_X >$ generated by r_X and $U(X)$ is the group generated by $\{r_{b,a} \mid b \in X,\ a \in R^W,\ a + b \in R\}$.*

Note that $W(\pi) = < r_\pi >$ normalizes $U(\pi)$ for any base complement π of R^W, since

1. $r_b r_{b',a} r_b^{-1} = r_{r_b(b'),a}$ and $r_{b(b')} \in S_\pi$ where S is the classical complement of R^W with base π and $b, b' \in S_\pi$; and
2. the restrictions $W(\pi)|_{S_\pi}$, $W(S_\pi)|_{S_\pi}$ both coincide with the Weyl group of S_π (by the "Theorem of Generation of the Weyl Group by Simple Reflections" for rootsystems of complex semisimple Lie algebras).

It follows that $W =_{\text{def}} W(\pi)U(\pi)$ is a subgroup of Aut R where $U(\pi)$ is a normal subgroup of W and $W(\pi)|_{S_\pi}$ is the Weyl group of the classical rootsystem S_π.

Recall from Proposition 4.4 that a classical rootsystem S is nonsingular if and only if no irreducible component of S is of type A_n with $p \mid n+1$ and either $p > 5$ or $p = 3$ and E_8 is not an irreducible component of S.

6.8. THEOREM. *Let R be a Lie rootsystem with no section of type T_2. Let S be a classical complement of R^W with base π. Then*:

1. *$W(\pi) = W(S)$ and restriction $W(S) \to W(S)|_S$ is an isomorphism from $W(S)$ to the Weyl group of the classical rootsystem S;*
2. *$U(\pi) = U(S)$, $U(S)$ is normalized by $W(S)$, $W(S) \wedge U(S) = 1$, and $W(R) = W(S) \cup (S)$ (semidirect product);*
3. *if S is nonsingular, $W(R)$ acts simply transitively on the set of base complements π' of R^W and, therefore, transitively on the set of all classical complements of R^W.*

Proof. For 1, it suffices, by the discussion preceding the theorem, to show that $W(S) \to W(S)|_S$ is injective. Thus, let $w \in W(S)$ with $w(s) = s$ for all $s \in S$, since $w \in W(S)$ and $W(S) = W(\pi)$ is generated by r_b $(b \in \pi)$ with $r_b(a) = a$ for all $a \in R^W$. Since $R \subset R^W + S$, by Theorem 1.8, it follows that

$w(a+s)=w(a)+w(s)=a+s$ for all $a+s\in R$. Thus, $w=1$, as asserted. For 2, observe that $U(\pi)$ is normalized by $W(\pi)=w(S)$, by the discussion preceding the theorem. Let $b\in S-\{0\}$, $a\in R^{\mathrm{W}}$ with $a+b\in R$. By a theorem on rootsystem of complex semisimple rootsystems, there exists $f\in W(S)=W(\pi)$ such that $f(b)\in\pi$. But then $f(a)=a$, by Proposition 4.1, and $fr_{b,a}f^{-1}=r_{f(b),a}\in U(\pi)$, by Corollary 6.6. This proves that $U(\pi)=U(S)$. To see that $W(R)=W(S)\cup(S)$, observe that the generators $r_{b'}$ $(b'\in R^{\mathrm{c}}-\{0\})$ for $W(R)$ can be written as $r_{b'}=r_{b+a}$ $(b'=a+b$ with $a\in R^{\mathrm{W}}$, $b\in R^{\mathrm{c}})$, by Theorem 1.8, so that $r_br_{b'}=r_br_{b+a}=r_{b,a}\in U(S)$ and $r_{b'}\in W(S)\cup(S)$. Thus, $W(R)=W(S)\cup(S)$. Finally, let $w\in W(S)\cap U(S)$ and let $s\in S$. Then $w(s)\in S$, since $w\in W(S)$, and $w(\hat{s})=\hat{s}$ since $w\in U(S)$. Since $S\to\hat{S}$ is injective, it follows that $w(s)=s$ for $s\in S$. By Proposition 4.1, $w(a)=a$ for all $a\in R^{\mathrm{W}}$. Since $R\subset R^{\mathrm{W}}+S$, it follows that $w=1$ and $W(S)\wedge U(S)=1$. For 3, suppose that S is nonsingular, that is, $\det(S_j^0(s_i))\neq 0$ (modulo p). We claim firstly that $U(S)$ acts transitively on the set of base complements π' of R^{W} which are adjacent to π. Letting $a=\pi'-\pi=(s_1',\ldots,s_n')^t-(s_1,\ldots,s_n)^t=(a_1,\ldots,a_n)^t$ with $a_i\in R^{\mathrm{W}}$ $(1\leqslant i\leqslant n)$, we know from our earlier discussion of $r_{\pi\pi'}$ that the base complement π' adjacent to π leads to a new base complement $r_{\pi\pi'}(\pi)=\pi-(s_j^0(s_i))a$. The Cartan matrix $C=(s_j^0(s_i))$ is, by our hypothesis, a nonsingular element of the finite group $g1_n\mathbb{Z}_p$. Letting $C(\pi)=r_{\pi\pi'}(\pi)=\pi-ca$ for $\pi'=\pi+a$, and iterating, we have $C^i(\pi')=\pi+(-1)^iC^ia$, where $C^i(\pi')$ is "ith iterate" and C^ia is "matrix product." Letting m be the order of C in $G1_n\mathbb{Z}_p$, we conclude that a base complement $\pi\pm C^{-1}a$ of R^{W} adjacent to π is reached after $m-1$ iterations $C, C(\pi),\ldots, C^{m-1}(\pi)$. Since $b+a\in R$ if and only if $b-a\in R$ for $b\in R^{\mathrm{c}}-\{0\}$ and $a\in R^{\mathrm{W}}$, by the rank 2 classification Theorem 1.7, it follows that $\pi''=\pi-C^{-1}a$ exists as a base complement of R^{W} adjacent to π, that is, the entry differences all are roots! But then we have:

$$r_{\pi\pi''}(\pi)=\pi-C(-C^{-1}a)=\pi+a=\pi'.$$

Since $r_{\pi\pi''}\in U(\pi)=U(S)$, it follows that $U(S)$ acts transitively on the set of base complements π' adjacent to π. To see that $W(S)\cup(S)$ (hence $W(R)$) acts transitively on all base complements π' of R, recall from our discussion of base complements early in the section that for any classical complement S' of R^{W} with base π', S has some base $\bar{\pi}$ adjacent to π'. Taking $f\in W(S)$ such that $f(\pi)=\bar{\pi}$, and $r_{\pi\pi''}\in U(S)$ such that $r_{\pi\pi''}(\bar{\pi})=\pi'$, we have $r_{\pi\pi'}$, $f(\pi)\pi'$ with $r_{\pi\pi'}f\in U(S)W(S)=W(R)$. Finally, we show that $W(R)$ acts simply transitively on the set of base complements π of R^{W}. For this, let $f\in W(R)$ with $f\pi=\pi$. We must show that $f=1$. Let S be the classical complement with base π. Write f as $f=f_1f_2$ in $W(R)=W(S)\cup(S)$ $(f_1\in W(S)$, $f_2\in U(S))$. Then $fS=S$, $f_1S=S$ implies that $f_2S=S$. But $s, f_2(s)\in S$ with $\widehat{f_2(s)}=\hat{s}$ implies $f_2(s)=s$. Thus, $f_2=1$ and $f=f_1\in W(S)$. But then $f\pi=\pi$

implies that $f=1$ by the "Theorem on Simple Transitivity of the Weyl Group on Simple Systems" for rootsystems of complex semisimple Lie algebras.

We now consider the group Aut L of $L=\sum_{a\in R} L_a$. Since no section of L is of type T_2, each classical complement S of R^W in R determines the group of automorphisms exp S of L generated by $\{\exp t \operatorname{ad} e_b \mid b\in R^c-\{0\}, t\in k\}$ where $L_b=k\, e_b$ (cf. the discussion leading to Definition 6.2). By the theory of algebraic groups, exp S is a closed connected subgroup of Aut L. By Seligman [3], $\exp S|_{L_S}=(\operatorname{Aut} L_S)_0$, the connected component of the identiy of Aut L_S, where L_S is the classical Lie algebra $\sum_{b\in S-\{0\}}([L_{b'-b}]+L_b)$ (cf. Theorem 1.6). We observed in Theorem 6.3 that Aut L has a subgroup $N=N(L_0)$ and surjective homomorphism $N\to W(R)$ given by $w\mapsto (w\mid L_0)^{*-1}|_R$. It follows from Theorem 6.8 that N acts transitively on the set of classical complements S of R^W in R. This implies, in turn, that N acts transitively on the subgroups exp S of Aut L (S a classical complement of R^W in R). We collect these conclusions in the following theorem.

6.9. THEOREM. *Let $L=\sum_{a\in R} L_a$ be a symmetric Lie algebra with no section T_2. Let $N=N(L_0)$ and exp S ($S\in \mathbf{S}$, $\mathbf{S}$ the set of classical complements of R^W in R) be as constructed above. Then N maps surjectively to $W(R)$ under $w\to (w\mid_{L_0})^{*-1}|_R$. Any two subgroups* exp S, exp S' ($S, S'\in\mathbf{S}$) *are conjugate by some element of N, provided that S have no irreducible component of type A_n $(p\mid n+1)$ and either $p>5$ or $p=3$ and E_8 is not an irreducible component of R.*

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