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EXISTENCE OF SOLUTIONS TO NONSELFADJOINT BOUNDARY VALUE PROBLEMS FOR ORDINARY DIFFERENTIAL EQUATIONS

L. CESARI
Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109, U.S.A.

and

T. T. BOWMAN
Department of Mathematics, University of Florida, Gainesville, Florida 32611, U.S.A.

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1. INTRODUCTION

IN THE present paper we prove existence theorems for nonselfadjoint real boundary value problems for semilinear ordinary differential equations

$$x^{(n)} + p_1(t)x^{(n-1)} + \ldots + p_n(t)x = f(t) + g\,(t, x(t)), 0 < t < a, \tag{1}$$

$$\sum_{j=0}^{n-1} a_{hj}x^{(j)}(0) + b_{hj}x^{(j)}(a) = 0, h = 1, \ldots, m, \tag{2}$$

where the coefficients $p_s(t)$ are continuous functions on $[0, a]$, $p_0(t) > 0$, and the a_{hj}, b_{hj} are constants.

Our results show that, even for nonselfadjoint boundary value problems, a limited quantitative form of the Landesman and Lazer sufficient condition for existence holds. Moreover, further extensions are obtained of the recent form proposed by Shaw of the Landesman and Lazer theorem.

The point of departure of the present paper is the theorem of Landesman and Lazer [11], which represents a necessary and sufficient condition in order that a selfadjoint real elliptic partial differential equation of order 2, $Ex = f(t) + g(x)$, $t \in G$, with $x = 0$ on ∂G, has a weak solution $x \in W_0^{1,2}(G)$, where G is a bounded domain in $\mathbb{R}^n$, g: $\mathbb{R} \to \mathbb{R}$ is continuous with finite limits $g(-\infty) \neq g(+\infty)$, and $f \in L_2(G)$.

This remarkable theorem was extended by Williams [15] and by De Figueiredo [10] to elliptic problems of order $2n$ and nonlinearity g depending on derivatives of orders $<2n - 1$. The same theorem was then extended by Shaw [14] to nonselfadjoint boundary value problems for partial differential equations with $p = q < \infty$, $p = \dim \ker E$, $q = \dim \ker E^*$, where E^* is the adjoint of E, provided corresponding elements of $\ker E$ and $\ker E^*$, share the same regions of positivity and negativity. In particular, for ordinary differential equations with $n = 1$, $p = q = 1$, this condition is always satisfied, and the Landesman and Lazer theorem holds. For their results, Landesman and Lazer, as well as Williams and Shaw made use of ideas from the alternative method (see, e.g. [6]).

In the present paper, we first analyze further the alternative method, for the case where $\infty > p \geqslant q \geqslant 0$, and there is an orthonormal basis $(\omega_1, \ldots, \omega_q)$ of ker E^* and orthornormal elements $\phi_1, \ldots, \phi_q$ in ker E such that the $q \times q$ matrix $M = (\omega_s, \phi_i)$, $s, i = 1, \ldots, q$, is not singular. Thus, if in particular $\phi_i = \omega_i$, $i = 1, \ldots, q$, then M is the identity matrix and this condition is satisfied. Also, for $p = q = 1$, if ω_1 and ϕ_1 share the same regions of positivity and negativity (Shaw's condition), then M is a positive scalar.

Now the decompositions often used in the alternative method with $X = Y = L_2[0, a]$, say, $X = \ker E + X_1$, $Y = \ker E^* + Y_1$ are replaced by a further decomposition $X = X_{01} + X_{02} + X_1$, $Y = Y_0 + Y_1$, $Y_0 = \ker^* E = \mathrm{sp}\langle\omega_1, \cdots, \omega_q\rangle$, $X_{01} = \mathrm{sp}\langle\phi_1, \cdots, \phi_q\rangle$, $\ker E = X_{01} + X_{02}$, $X_1 = (\ker E)^\perp$, $Y_1 = (\ker E^*)^\perp$, with usual projection operator $Q: Y \to Y$, $QY = Y_0 = \ker E^*$, $P: X \to X$, $PX = X_0 = X_{01} + X_{02}$.

Under these assumptions, the map $S: \ker E^* \to \ker E$ with $S^{-1}(0) = \{0\}$, which is variously defined in the alternative method, can be so chosen that the map $S: \ker E^* \to X_{01}$ is linear, $1 - 1$, and onto, and $SQ: Y \to X_{01}$, restricted to X_{01}, is the identity.

On the basis of this particular map S, we present a new argument in dealing with the bifurcation equation $QNx = 0$. We then obtain a new existence theorem for problem (1), (2), with $\|f\|_\infty < c$, a suitable constant, which allows a remarkable freedom for the continuous Lipschitzian function g.

2. THE LINEAR OPERATOR E IN $L_2[0, a]$

Here J stands for a closed interval $[0, a]$ of the real line, and $D(T)$, $R(T)$, ker T denote the domain, the range, and the null space or kernel of a linear operator T, respectively. Also $\mathrm{sp}(\omega_1, \cdots, \omega_m)$ stands for the linear space spanned by $\omega_1, \cdots, \omega_m$, and $T|D_0$ denotes the restriction of the operator T on a given set $D_0 \subset D(T)$.

We consider the boundary value problem

$$\tau x \equiv x^{(n)} + p_1(t)x^{(n-1)} + \cdots + p_n(t)x = f(t) + g(t, x(t)), \qquad t \in (0, a), \tag{1}$$

$$B_i x \equiv \sum_{j=0}^{n-1} a_{ij}x^{(j)}(0) + b_{ij}x^{(j)}(a) = 0, \qquad i = 1, \cdots, m, \tag{2}$$

where each coefficient $p_i(t)$ is of class C^{n-i} in $[0, a]$, $f \in L_2[0, a]$, and $g: [0, a] \times \mathbb{R} \to \mathbb{R}$ is a continuous function of x for every t, and measurable in t for every x. We shall also assume that $g(t, x)$ is bounded in $[0, a] \times \mathbb{R}$. In (2) we assume that the coefficients a_{ij}, b_{ij} are real constants, and that the m forms B_i are linearly independent.

Let $S = L_2\,[0, a]$ with usual inner product (,) and norm $\|\ \|_2$. Let $S^{(n)}$ denote the space of all functions $x(t)$, $0 \leqslant t \leqslant a$, which are AC in $[0, a]$ together with $x', \cdots, x^{(n-1)}$, and $x^{(n)} \in L_2[0, a]$, and we take in $S^{(n)}$ the norm

$$\|x\|_2^{(n)} = \|x\|_\infty + \|x'\|_\infty + \cdots + \|x^{(n-1)}\|_\infty + a^{-1}\left(\int_0^a (x^{(n)}(t))^2\, dt\right)^{1/2}, \qquad x \in S^{(n)}.$$

Let E denote the operator defined by $Ex = \tau x$ with domain $D(E) = [x \in S^{(n)}, B_i x = 0, i = 1, \ldots, m]$ so that $E: D(E) \subset S \to S$.

The following statements are well known:

$D(E)$ is dense in S,
E is a closed linear operator,
$R(E)$ is a closed linear subspace of S.

We assume $\infty > p \geq q \geq 0$ where $p = \dim \ker E$ and $q = \dim \operatorname{coker} E = \dim \ker E^*$. Let $\phi_1, \cdots, \phi_p$ be an orthonormal basis for $\ker E$. Now the restriction of E to the space $D(E) \cap (\ker E)^\perp$ is a $1 - 1$ closed linear operator whose closed range is still $R(E)$. Hence, by the closed graph theorem, the inverse map $H = [E|D(E) \cap (\ker E)^\perp]^{-1}$ is a $1 - 1$ continuous linear operator with domain $R(E)$ and range $D(E) \cap (\ker E)$. Also,

$$EHy = y \qquad \text{for all} \qquad y \in R(E),$$

$$HEx = x - \sum_{i=1}^{p} (x, \phi_i)\phi_i \qquad \text{for all} \qquad x \in D(E),$$

where (u, v) denote the usual inner product in $L_2[0, a]$ and then $\|u\|_2 = (u, u)^{1/2}$. By Fredholm alternative theorem we also know that $y \in R(E)$ if and only if $y \perp \ker E^*$, or $\ker E^* = R(E)^\perp$.

We have just stated that the linear operator H: $R(E) \to D(E) \cap (\ker E)^\perp$ is continuous, hence bounded, in the norm L_2 in $R(E)$ and the norm $\|x\|_2^{(n)}$ in $D(E) \cap (\ker E)^\perp$, and the statement holds that for any $f \perp \ker E^*$ the only solution Hf of (1), (2) which is orthogonal to $\ker E$ satisfies

$$\|Hf\|^{(n)} \leq \gamma \|f\|_2 \tag{3}$$

where γ is a suitable constant independent of f. We note that, since $\|x\|_2^{(n)}$ generates a stronger topology on $D(E) \cap (\ker E)^\perp$ than the norm $\|x\|_2$, then H can be viewed also as a continuous map of $R(E)$ to S with both having the L_2 norm.

We denote by N the Nemitsky operator defined by $Nx = f(t) + g(t, x(t))$ for $x \in S, f \in L_2$, so that $g(t, x(t))$ is measurable and bounded, and $N: S \to S$, $D(N) = S \supset S^{(n)} = D(E)$. Problem (1), (2) now takes the form

$$Ex = Nx, x \in D(E). \tag{4}$$

Let $P : S \to S$ and $Q : S \to S$ denote the orthogonal projections of S onto $\ker E$ and $\ker E^*$ respectively, namely

$$(Px)(t) = \sum_{i=1}^{p} (x, \phi_i)\phi_i(t) \qquad \text{for} \quad x \in S,$$

$$(Qy)(t) = \sum_{s=1}^{q} (y, \omega_s)\omega_s(t) \qquad \text{for} \quad y \in S,$$

so that

$$PP = P, QQ = Q, PS = \ker E, QS = \ker E^* = \operatorname{coker} E,$$
$$(I - P)S = (\ker E)^\perp, (I - Q)S = (\ker E^*)^\perp = R(E).$$

It is well known that equation $Ex = Nx$ splits into the system of auxiliary and bifurcation equations

$$x = Px + H(I - Q)Nx, QNx = 0, x \in S. \tag{5}$$

3. A PARTICULARIZATION IN $C[0, a]$

We consider now a particularization of (1), (2), namely we assume that $f : [0, a] \to \mathbb{R}$ and g: $[0, a] \times \mathbb{R} \to \mathbb{R}$ are continuous functions, and that g is bounded in $[0, a] \times \mathbb{R}$. We further assume that the coefficients $p_1, \cdots, p_n$ in (1) are continuous functions in $[0, a]$.

Let X and Y be copies of $C = C[0, a]$. Then the Nemitsky operator $Nx = f(t) + g(t, x(t))$ is defined on all of X with values in Y, or $N : X \to Y$. We may restrict $D(E)$ to the new set

$$D_0(E) = [x \in S^{(n)}, x^{(n)} \text{ continuous}, B_i x = 0, i = 1, \cdots, m],$$

so that $E : D_0(E) \subset X \to Y$,

Let us prove that

$$E : D_0(E) \cap (\ker E)^{\perp} \to R(E) \cap C, \qquad H : R(E) \cap C \to D_0(E) \cap (\ker E)^{\perp} \tag{6}$$

and that $R(E) \cap C$ is closed in C. Indeed $R(E)$ is closed in L_2, convergence in C implies convergence in L_2, and hence $R(E) \cap C$ is closed in C. Also, $E : D(E) \to R(E) = S \cap (\ker E^*)^{\perp}$, and by the definition of E, Ex is continuous in $[0, a]$ since $x, x', \cdots, x^{(n)}, p_1, \cdots, p_n$ are all continuous, and thus $E : D_0(E) \to S \cap C \cap (\ker E^*)^{\perp} = R(E) \cap C$. Finally, for $y \in R(E) \cap C$, then $x = Hy \in D(E) \cap (\ker E)^{\perp}$, certainly $x, x' \cdots, x^{(n-1)}$ are AC, and $x^{(n)} = y - p_n x - \cdots - p_1 x^{(n-1)}$, where $p_1, \cdots, p_n$ are continuous; hence $x^{(n)}$ is continuous and $x = Hy \in D_0(E)$. Thereby, (6) is proved.

As usual we take now $X_0 = PX$, $X_1 = (I - P)X$, $Y_0 = QY$, $Y_1 = (I - Q)Y$, where $X = Y = S$. Since $\phi_1, \cdots, \phi_p, \omega_1, \cdots, \omega_q$ are continuous functions, we have decompositions $C = X = X_0 + X_1$, $C = Y = Y_0 + Y_1$, and $Y_1 = R(E) \cap C$, $X_1 = C \cap (\ker E)^{\perp}$, $X_0 = \ker E$, $Y_0 = \ker E^*$, and equation $Ex = Nx$, $x \in D_0(E)$, still splits into the two equations

$$x = Px + H(I - Q)Nx, \qquad QNx = 0, \qquad x \in X.$$

Any solution $x \in X$ of this system is an element $x = x_0 + x_1$, $x_0 = Px \in \ker E$, $x_1 \in D_0(E) \cap (\ker E)^{\perp}$ and is a solution of problem (1), (2). The uniform topology in $C = X = Y$ is defined by the usual Sup norm in $[0, a]$, or $\| \|_\infty$, and, in this norm, we cannot expect P and Q to have norm one, but necessarily ≥ 1:

$$\|P\| = c_1, \quad \|I - P\| = c_2, \quad \|Q\| = c_3, \quad \|I - Q\| = c_4, \quad \|H\| = L. \tag{7}$$

Actually, we shall denote by these letters constants such that $\|Px\|_\infty \leq c_1\|x\|_\infty$, $\|(I - P)x\|_\infty \leq c_2\|x\|_\infty$ for all $x \in X$, $\|Qy\|_\infty \leq c_3\|y\|_\infty$, $\|(I - Q)y\|_\infty \leq c_4\|y\|_\infty$ for all $y \in Y$, and $\|Hy\|_\infty \leq L\|y\|_\infty$ for all $y \in Y_1$.

Now

$$X_0 = \ker E = \mathrm{sp}(\phi_1, \cdots, \phi_p), Y_0 = \ker E^* = \mathrm{sp}(\omega_1, \cdots, \omega_q), p \geq q,$$

and we split X_0 into a direct sum of a space X_{01} of dimension q and its complement in X_0. Actually, by possibly changing bases and their indexing, we can always assume that

$$X_{01} = \mathrm{sp}(\phi_1, \cdots, \phi_q), X_{02} = \mathrm{sp}(\phi_{q+1}, \cdots, \phi_p), X_0 = X_{01} + X_{02}.$$

Let M denote the $q \times q$ matrix $M = [(\omega_s, \phi_i), s, i = 1, \cdots, q]$, and we assume that M is nonsingular. For instance, if $\omega_i = \phi_i$, $i = 1, \cdots, q$, then M is the identity matrix.

We shall use the notational convention to denote briefly by σ any given vector $\sigma = \mathrm{col}(\sigma_1, \cdots, \sigma_q)$. We now define a linear map $S : Y_0 \to X_{01}$. Indeed, for every $y \in Y_0$ we have $y = \sum_{s=1}^{q} d_s^{\#} \omega_s$ with $d_s^{\#} = (y, \omega_s)$, since the ω_s are orthogonal, and we take $Sy = \sum_{i=1}^{q} d_i \phi_i$ with $d = M^{-1} d^{\#}$.

(3.i) $S^{-1}(0) = \{0\}$. Indeed, if $y \in S^{-1}(0)$, then $Sy = 0$, that is, $d = 0$, hence $d^{\#} = 0$ since M is nonsingular, and $y = 0$.

We note now that $X_{01} \subset X = Y$; hence, $Q : Y \to Y$ is defined on X_{01}.

(3.ii) $SQx = x$ for all $x \in X_{01}$. In other words, $SQ = I$ is the identity on X_{01}.

Indeed, if $x \in X_{01}$ then $x = \sum_{i=1}^{q} c_i \phi_i$, $c = (x, \phi_i)$, and

$$Qx = \sum_{s=1}^{q} (x, \omega_s)\omega_s = \sum_{s=1}^{q} \left(\sum_{i=1}^{q} c_i \phi_i, \omega_s \right) \omega_s = \sum_{i=1}^{q} \left(\sum_{i=1}^{q} (\omega_s, \phi_i) c_i \right) \omega_s = \sum_{s=1}^{q} c_s^{\#} \omega_s$$

with $c^{\#} = Mc$, and then $SQx = \sum_{i=1}^{q} c_i' \phi_i$ with $c' = M^{-1}(Mc) = c$.

Now SQ, considered as a linear operator $SQ : Y \to X_0$ has a norm (in the uniform topologies of Y and X) we shall denote by c_5. Actually, we shall denote by c_5 a constant such that $\|SQy\|_\infty \leq c_5 \|y\|_\infty$ for all $y \in Y$.

4. THE EXISTENCE THEOREM

Let us consider the set

$$\Omega = S_0 \times S_1, S_0 = [x_{01} \in X_{01} \mid \|x_{01}\|_\infty \leq R_0], S_1 = [x_1 \in X_1 \mid \|x_1\|_\infty \leq r] \tag{8}$$

for given numbers R_0, $r > 0$. Let $x_{02} \in X_{02}$ be arbitrary, take $x = x_{01} + x_{02} + x_1$, $x_{01} \in S_0$, $x_1 \in S_1$, and let us consider the transformation $T : x_{01} + x_{02} + x_1 \to \bar{x}_{01} + \bar{x}_{02} + \bar{x}_1$ defined by

$$T : \begin{cases} \bar{x}_1 = H(I - Q)[f(t) + g(t, x_{01} + x_{02} + x_1)] \\ \bar{x}_{01} = x_{01} - kSQ[f(t) + g(t, x_{01} + x_{02} + x_1)], \qquad \bar{x}_{02} = x_{02}. \end{cases} \tag{9}$$

Since we shall treat x_{02} as an arbitrary element of X_{02}, it is convenient to consider T as a transformation $\bar{x} \to \bar{\bar{x}}$ with $\bar{x} = (x^*, x_1)$, $\bar{\bar{x}} = (\bar{x}^*, \bar{x}_1^*)$. With this convention, it is immediate that T maps Ω into $X_{01} + X_1$.

We assume now that for some constants R_0, r, k, c, C, D, $\rho > 0$ and $r' \geq 0$, we have

$$|f(t)| \leq c \quad \text{for } t \in J = [0, a], \tag{10}$$

$$|g(t, z)| \leq C \quad \text{for } (t, z) \in J \times \mathbb{R}, |z| \leq R_0 + r + r', \tag{11}$$

$$|g(t, y) - g(t, z)| \leq D|y - z| \quad \text{for all } t \in J, |y|, |y|, |z| \leq R_0 + r + r', \tag{12}$$

$$|z - kg(t, z)| \leq \rho R_0 \quad \text{for } t \in J, |z| \leq R_0, \tag{13}$$

$$Lc_4(c + C) \leq r, \tag{14}$$

$$kc_5(c + D(r + r')) \leq (1 - \rho c_5)R_0, r' = 0 \quad \text{if } p = q. \tag{15}$$

We have already assumed before that $q \times q$ matrix $M = [(\omega_s, \phi_i), s, i = 1, \cdots, q]$ is nonsingular, that $f : [0, a] \to R$ and $g : [0, a] \times \mathbb{R} \to \mathbb{R}$ are continuous, and that the coefficients $p_1, \cdots, p_n$ in (1) are also continuous.

(4.i) For continuous functions f and g satisfying relations (10)–(15), problem (1), (2) has at least one solution $x(t)$, $0 \leq t \leq a$, with $\|x\|_\infty \leq R_0 + r + r'$. Actually, for $p > q$, $r' > 0$, at least one solution for every element $x_{02} \in X_{02}$, $\|x_{02}\|_\infty \leq r'$.

Proof. We understand that if $p = q$ then $r' = 0$. For $p \geq q$ and $r' = 0$ we take $x_{02} = 0$, $x_{02} \in X_{02}$. If $r' > 0$ we take any element $x_{02} \in X_{02}$ with $\|x_{02}\|_\infty \leq r'$. Thus, $x = x_{01} + x_{02} + x_1$, $x_{01} \in S_0$, $x_{02} \in X_{02}$, $x_1 \in X_1$. Let us show that T maps Ω into itself. We note that, for $(x_{01}, x_1) \in \Omega$, hence $\|x_{01}\|_\infty \leq R_0$, $\|x_1\|_\infty \leq r$, we certainly have, by (9) and (14),

$$\|\bar{x}_1\|_\infty = \|H(I - Q)[f(t) + g(t, x(t))]\|_\infty \leq Lc_4(c + C) \leq r,$$

and $\bar{x}_1 \in S_1$.

For $x_{01} \in S_0$, that is, $\|x_{01}\|_\infty \leq R_0$, and $x_1 \in S_1$, we have now, by (9),

$$\begin{aligned}\bar{x}_{01}(t) = {} & [x_{01}(t) - SQx_{01}(t)] + SQ[x_{01}(t) - kg(t, x_{01}(t))] \\ & - kSQf(t) + kSQ[g(t, x_{01}(t)) - g(t, x_{01}(t) + x_{02}(t) + x_1(t))],\end{aligned}$$

where $x_{01} \in X_0$, hence $SQx_{01} = x_{01}$, and the first bracket is zero. Moreover $|x_{01}(t)| \leq R_0$ and, by (13), $|x_{01}(t) - kg(t, x_{01}(t)| \leq \rho R_0$. We have now, by (15), $|\bar{x}_{01}(t)| \leq 0 + \rho c_5 R_0 + kc_5 c + kc_5 D(r + r') \leq R_0$, and $\bar{x}_{01} \in S$. To complete the proof of (4.i) we must show that $T\Omega$ is compact in the topology of $C[0, a]$. Indeed, for $(x_{01}, x_1) \in \Omega$, $(\bar{x}_{01}, \bar{x}_1) = T(x_{01}, x_1) \in T\Omega$, then $\bar{x}_1$ is in X, and by (3) $\|\bar{x}_1'\|_\infty \leq \gamma\|f + g\|_2 = M_1$ if $n > 1$, and $\|\bar{x}_1'\|_2 \leq \gamma\|f + g\|_2 = M_1$ if $n = 1$. Thus, for $n > 1$, x_1 is continuous, bounded by $\|\bar{x}_1\|_\infty \leq r$, and uniformly Lipschitzian of fixed constant M_1. For $n = 1$ then x_1 is again continuous and bounded, and equiabsolutely continuous, as the following usual argument shows. Indeed, for any finite system of non-overlapping intervals in $[0, a]$, say

$$(\alpha_i, \beta_i),\ i = 1, \ldots, N, \text{ with } \sum_{i=1}^{N} (\beta_i - \alpha_i) \leq \delta, \text{ take } F = \bigcup_{i=1}^{N} (\alpha_i, \beta_i), \text{ meas } F \leq \delta,$$

and then

$$\sum_{i=1}^{N} |\bar{x}_1(\beta_i) - \bar{x}_1(\alpha_i)| \leq \int_F |\bar{x}_1'(t)|\, dt \leq (\text{meas } F)^{1/2} \left(\int_F |\bar{x}_1'(t)|^2\, dt\right)^{1/2} \leq \delta^{1/2} M_1.$$

The elements $\bar{x}_{01}$ of $T\Omega$ are elements of X_{01} which is finite dimensional and they are equibounded since $|\bar{x}_{01}(t)| \leq R_0$. Thus, $T\Omega$ is a relatively compact subset of the closed convex set Ω in the topology of $X = C[0, a]$. By Schauder's fixed point theorem T has at least one fixed point in Ω. Theorem (4.i) is thereby proved.

5. AN EXAMPLE

Let us consider the problem

$$Ex = x''' = f(t) + g(t, x(t)), \qquad t \in J = [0, a], \tag{16}$$

$$x''(0) = 0, \qquad x''(a) = 0, \qquad 2x(0) + x(a) = 0. \tag{17}$$

The homogeneous problem $x''' = 0$ with the same boundary conditions (17) has the nonzero solution $x(t) = t - a/3$, $0 \leq t \leq a$, and $\ker E = \{c(t - a/3)\}$, $p = 1$. The homogeneous problem is equivalent to the system $x_1' = x_2$, $x_2' = x_3$, $x_3' = 0$ with conditions $x_3(0) = 0$, $x_3(a) = 0$, $2x_1(0)$

$+\ x_1(a) = 0$, or

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1(a) \\ x_2(a) \\ x_3(a) \end{pmatrix} = 0.$$

The dual problem is, therefore,

$$(y_1'\ y_2'\ y_3') = -(y_1\ y_2\ y_3) \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$(y_1(0)\ y_2(0)\ y_3(0)) = (\alpha_1\ \alpha_2\ \alpha_3) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix},$$

$$(y_1(a)\ y_2(a)\ y_3(a)) = -(\alpha_1\ \alpha_2\ \alpha_3) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

with α_1, α_2, α_3 arbitrary, or $y_1' = 0, y_2' = -y_1, y_3' = -y_2$ with conditions $y_1(0) + 2y_1(a) = 0$, $y_2(0) = 0$, $y_2(a) = 0$. Then, $y_3'' = -y_2' = y_1, y_3''' = y_1' = 0$, and by writing y for y_3, we have the dual problem of (16, 17):

$$E^*y = -y''' = 0, \qquad t \in J = [0, a], \qquad y'(0) = 0, \qquad y'(a) = 0, \qquad y''(0) + 2y''(a) = 0.$$

This problem has the nonzero solution $y(t) = 1$, $0 \leqslant t \leqslant a$, and $\ker E^* = \{d\}$, $q = 1$.

By normalization we have

$$\phi(t) = 3a^{-3/2}(t - a/3), \quad \omega(t) = a^{-1/2}, \quad 0 \leqslant t \leqslant a, \quad \ker E = \{c\phi\}, \quad \ker E^* = \{d\omega\}, \quad p = q = 1.$$

Here ϕ and ω do not share regions of positivity and negativity.

The relation $x = Hy$ is now defined by the set of equations

$$x''' = y, \qquad x''(0) = 0, \qquad x''(a) = 0, \qquad 2x(0) + x(a) = 0,$$

$$\int_0^a x(t)(t - a/3)\,dt = 0, \qquad \int_0^a y(t)\,dt = 0. \tag{18}$$

We could express H by an integral operator and estimate H by estimating the kernel. We prefer here to solve problem (18) directly, obtaining the estimates step by step.

First, $x''(t) = \int_0^t y(\alpha)\,d\alpha$ because of $x''(0) = 0$, and then $x''(a) = \int_0^a y(\alpha)\,d\alpha = 0$ because of last relation (18). Moreover, for $0 \leqslant t \leqslant a/2$, $|x''(t)| \leqslant \|y\|_\infty\, a/2$, and by symmetry the same relation holds in $[a/2, a]$. Thus, $\|x''\|_\infty \leqslant \|y\|_\infty\, a/2$. We have now

$$x(t) = A + B(t - a/3) + \int_{a/3}^{t} (t - \beta)x''(\beta)\,d\beta, \qquad A, B \text{ constants}, \tag{19}$$

and the condition $2x(0) + x(a) = 0$ yields

$$0 = 2x(0) + x(a) = 3A + 2\int_{a/3}^{0} (-\beta)x''(\beta)\,d\beta + \int_{a/3}^{a} (a - \beta)x''(\beta)\,d\beta,$$

$$3|A| \leq \|x''\|_\infty \left(2\int_{0}^{a/3} \beta\,d\beta + \int_{a/3}^{a} (a - \beta)\,d\beta\right) = \|x''\|_\infty\, a^2/3,$$

$$|A| \leq (1/9)a^2\|x''\|_\infty. \tag{20}$$

From the relation $\int_0^a x(t)(t - a/3)\,dt = 0$ we derive now

$$A\int_0^a (t - a/3)\,dt + B\int_0^a (t - a/3)^2\,dt + \int_0^a (t - a/3)\,dt \int_{a/3}^{t} (t - \beta)x''(\beta)\,d\beta = 0,$$

$$(1/9)a^3|B| \leq (1/6)a^2|A| + (17a^4/(8.81))\|x''\|_\infty,$$

$$|B| \leq (29/(8.9))a\|x''\|_\infty. \tag{21}$$

By using estimates (20) and (21), and by separating the cases $0 \leq t \leq a/3$ and $a/3 \leq t \leq a$, we obtain from (19),

$$|x(t)| \leq (65/108)a^2\|x''\|_\infty \leq (65/216)a^3\|y\|_\infty.$$

Thus, in the topology of C we have

$$\|H\| \leq (65/216)a^3.$$

To estimate $\|Q\|$ we note that, for $y \in Y$, we have

$$|(Qy)(t)| = \left|\left(\int_0^a \omega(\alpha)y(\alpha)\,d\alpha\right)\omega(t)\right| = \left(\int_0^a a^{-1/2}\,y(\alpha)\,d\alpha\right)a^{-1/2} \leq a^{-1}\,\|y\|_\infty \int_0^a d\alpha = \|y\|_\infty.$$

Thus, $\|Q\| \leq 1$ and, since Q is a projection, also $\|Q\| = 1$.

To estimate S we note that, for $y \in Y_0$, we have $y = d^{\#}\omega = d^{\#}a^{-1/2}$, hence

$$\|y\|_\infty = |d^{\#}|a^{-1/2}, \qquad |d^{\#}| = a^{1/2}\,\|y\|_\infty.$$

On the other hand, we have $q = 1$ and

$$M = \int_0^a \phi\omega\,dt = \int_0^a a^{-1/2}\cdot 3a^{-3/2}(t - a/3)\,dt = 1/2,$$

and by the definition of S, for $x = Sy$, we have $x = d\phi$ with $d = d^{\#}/M$, or

$$\|x\|_\infty = \|Sy\|_\infty = \|d\phi\|_\infty = 2\cdot a^{1/2}\|y\|_\infty\cdot\|3a^{-3/2}(t - a/3)\|_\infty = 4\|y\|_\infty,$$

and thus $\|S\| \leq 4$. We conclude with the list of the constants of interest:

$$\|Q\| = c_3 = 1, \qquad \|I - Q\| \leq c_4 = 2, \qquad \|H\| \leq L = (65/216)a^3, \qquad \|SQ\| \leq c_5 = 4.$$

As a numerical example, let $g: \mathbb{R} \to \mathbb{R}$ be any function satisfying

$$\begin{aligned} &|g(z)| \leq 1, \quad |g(z) - g(z')| \leq (4/3)|z - z'| \quad \text{for} \quad |z|, |z'| \leq R_0 + r, \\ &|z - g(z)| \leq \rho R_0 \quad \text{for} \quad |z| \leq R_0. \end{aligned} \tag{22}$$

Thus $C = 1$, $D = 4/3$. We take $a = 0.1$, $k = 1$, $c = 0.1$, so that $L = (65/216)a^3 = 0.00030092$, and for $r = 0.00067$ we also have

$$Lc_4(c + C) = (0.00030092)(2)(0.1 + 1) = 0.000662024 < 0.00067 = r,$$

$$kc_5(c + Dr) = 4(0.1 + (4/3)(0.00067)) = 0.4035733.$$

For $R_0 = 1.19$, $\rho = 0.165$, we have

$$(1 - \rho c_5)R_0 = (1 - 4(0.165))(1.19) = 0.4046 > 0.4035733.$$

On the other hand, we require that $|z - g(z)| \leq \rho R_0$ for $|z| \leq R_0 = 1.19$. We see that for $1 \leq z \leq 1.19$ we have $g(z) \leq 1$, and the requirement $|z - g(z)| \leq \rho R_0$ can be satisfied since $0.19 = 1.19 - 1 < 0.19635 = \rho R_0$.

Thus, problem (16), (17) has at least one solution $x(t)$ with $|x(t)| \leq R_0 + r = 1.19067$ for $a = 0.1$, $\|f\|_\infty \leq 0.1$, and g satisfying (22) with $R_0 = 1.19$, $r = 0.00067$, $\rho = 0.165$, in other words for any g of Lipschitz constant 4/3, whose graph is within the heavy set box in Fig. 1.

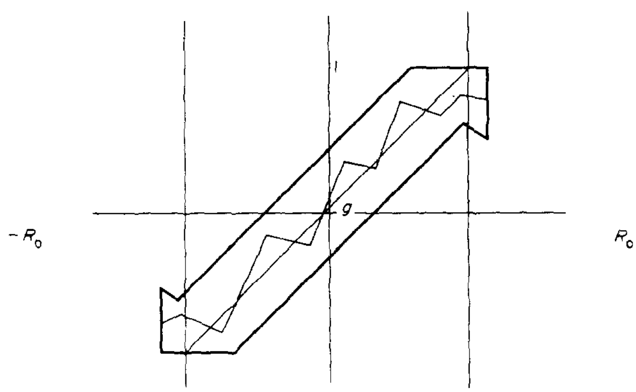

Fig. 1.

6. ANOTHER EXAMPLE

Let us consider the problem

$$Ex = x''' + \lambda^2 x' = f(t) + g(t, x(t)), \qquad t \in J = [0, a], \qquad \lambda = 2\pi/a, \tag{23}$$

$$x(a) + \varepsilon x'(a) = 0, \qquad x''(a) = 0, \qquad x'(0) - x'(a) = 0. \tag{24}$$

The homogeneous problem $x''' + \lambda^2 x' = 0$ with the same boundary conditions (24) has the nonzero solution $x(t) = -\varepsilon\lambda + \sin \lambda t$, $0 \leq t \leq a$, and $\ker E = \{c(-\varepsilon\lambda + \sin \lambda t)\}$, $p = 1$.

The homogeneous problem is equivalent to the system $x_1' = x_2, x_2' = x_3, x_3' = -\lambda^2 x_2$ with conditions $x_1(a) + \varepsilon x_2(a) = 0$, $x_3(a) = 0$, $x_2(0) - x_2(a) = 0$, or

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\lambda^2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} + \begin{pmatrix} 1 & \varepsilon & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1(a) \\ x_2(a) \\ x_3(a) \end{pmatrix} = 0.$$

The dual problem is, therefore,

$$(y_1'\, y_2'\, y_3') = -(y_1\, y_2\, y_3) \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\lambda^2 & 0 \end{pmatrix},$$

$$(y_1(0)\, y_2(0)\, y_3(0)) = (\alpha_1\, \alpha_2\, \alpha_3) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$(y_1(a)\, y_2(a)\, y_3(a)) = -(\alpha_1\, \alpha_2\, \alpha_3) \begin{pmatrix} 1 & \varepsilon & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

with α_1, α_2, α_3 arbitrary, or $y_1' = 0, y_2' = -y_1 + \lambda^2 y_3, y_3' = -y_2$ with conditions $y_3(0) = 0$, $y_1(0) = 0$, $y_2(a) = \varepsilon y_1(a) + y_2(0)$. Then, $y_3'' = -y_2' = y_1 - \lambda^2 y_3$, $y_3''' = y_1' - \lambda^2 y_3' = -\lambda^2 y_3'$, and by writing y for y_3, we have the dual problem of (23), (24):

$$E^* y = -y''' - \lambda^2 y' = 0, \qquad t \in J = [0, a],$$

$$y(0) = 0, \qquad y''(0) = 0, \qquad y'(0) - y'(a) = \varepsilon(y''(a) + \lambda^2 y(a)).$$

This problem has the nonzero solution $y(t) = \sin \lambda t$, $0 \leq t \leq a$, and $\ker E^* = \{d \sin \lambda t\}$, $q = 1$. By normalization we have

$$\phi(t) = 2^{1/2} a^{-1/2} (1 + 2\lambda^2 \varepsilon^2)^{-1/2} (-\varepsilon\lambda + \sin \lambda t), \qquad \omega(t) = 2^{1/2} a^{-1/2} \sin \lambda t, \qquad 0 \leq t \leq a,$$

$$\ker E = \{c\phi(t)\}, \qquad \ker E^* = \{d\omega(t)\}, \qquad p = q = 1.$$

Here ϕ and ω do not share regions of positivity and negativity.

The relation $x = Hy$ is now defined by the set of equations

$$x''' + \lambda^2 x' = y, \qquad x(a) + \varepsilon x'(a) = 0, \qquad x''(a) = 0, \qquad x'(0) - x'(a) = 0,$$

$$\int_0^a x(t)(-\varepsilon\lambda + \sin \lambda t)\, dt = 0, \qquad \int_0^a y(t) \sin \lambda t\, dt = 0.$$

The differential equation yields

$$x(t) = A + B \cos \lambda t + C \sin \lambda t + c_1(t) + c_2(t) \cos \lambda t + c_3 \sin \lambda t,$$

$$c_1(t) = \int_a^t \lambda^{-2} y(\alpha)\,d\alpha, \qquad c_2(t) = -\int_a^t \lambda^{-2} y(\alpha)\cos\lambda\alpha\,d\alpha,$$
$$c_3(t) = -\int_a^t \lambda^{-2} y(\alpha)\sin\lambda\alpha\,d\alpha.$$

The condition $x''(a) = 0$ implies $B = 0$. Since $\lambda = 2\pi/a$, $\sin\lambda t$ and $\cos\lambda t$ are periodic of period a, we have
$$x'(0) - x'(a) = \lambda c_3(0) = \lambda^{-1}\int_a^0 y(\alpha)\sin\lambda\alpha\,d\alpha = 0,$$
and the condition $x'(0) - x'(a) = 0$ is satisfied. The condition $x(a) + \varepsilon x'(a) = 0$ yields $A + \varepsilon\lambda C = 0$; hence $A = -\varepsilon\lambda C$, and
$$x(t) = (-\varepsilon\lambda + \sin\lambda t)C + c_1(t) + c_2(t)\cos\lambda t + c_3(t)\sin\lambda t.$$
Now the condition $\int_0^a x(t)(-\varepsilon\lambda + \sin\lambda t)dt = 0$ yields
$$0 = Ca(2^{-1} + \lambda^2\varepsilon^2) + \int_0^a [-\varepsilon\lambda + \sin\lambda t][c_1(t) + c_2(t)\cos\lambda t + c_3(t)\sin\lambda t]\,dt$$
from which we can derive C. An estimate for C is then
$$|C| \leqslant 2a^{-1}(1 + 2\lambda^2\varepsilon^2)^{-1}(1 + \varepsilon\lambda)\,\lambda^{-2}\int_0^a dt\left|\int_a^t y(\alpha)\,d\alpha + \int_a^t y(\alpha)\cos\lambda(t-\alpha)\,d\alpha\right|$$
$$\leqslant 2a^{-1}(1 + 2\lambda^2\varepsilon^2)^{-1}(1 + \varepsilon\lambda)\,\lambda^{-2}\|y\|_\infty\int_0^a dt\int_t^a 2d\alpha$$
$$= 2a\lambda^{-2}(1 + 2\varepsilon^2\lambda^2)^{-1}(1 + \varepsilon\lambda)\|y\|_\infty.$$
Finally,
$$|x(t)| = |(-\varepsilon\lambda + \sin\lambda t)C + c_1(t) + c_2(t)\cos\lambda t + c_3(t)\sin\lambda t|$$
$$\leqslant (1 + \varepsilon\lambda)|C| + \lambda^{-2}\left|\int_a^t y(\alpha)\,d\alpha + \int_a^t y(\alpha)\cos\lambda(t-\alpha)\,d\alpha\right|$$
$$\leqslant (1 + \varepsilon\lambda)\cdot 2a\lambda^{-2}(1 + 2\varepsilon^2\lambda^2)^{-1}(1 + \varepsilon\lambda)\|y\|_\infty + \lambda^{-2}2a\|y\|_\infty.$$
Thus,
$$\|H\| \leqslant \lambda^{-2}a(2(1 + 2\lambda^2\varepsilon^2)^{-1}(1 + \varepsilon\lambda)^2 + 2).$$
Since $\lambda = 2\pi/a$ we also have
$$\|H\| \leqslant (2\pi)^{-2}a^3(2 + 2^{-1}(1 + 2\lambda^2\varepsilon^2)^{-1}(1 + \varepsilon\lambda)^2).$$

To estimate $\|Q\|$ we note that, for any $y \in Y$, we have
$$|Qy(t)| = \left|\left(\int_0^a \omega(\alpha)y(\alpha)\,d\alpha\right)\omega(t)\right|$$
$$= \left|\left(\int_0^a 2^{1/2}a^{-1/2}y(\alpha)\sin\lambda\alpha\,d\alpha\right)2^{1/2}a^{-1/2}\sin\lambda t\right|$$
$$\leqslant 2a^{-1}\|y\|_\infty\int_0^a|\sin\lambda\alpha|\,d\alpha = 2a^{-1}\|y\|_\infty\cdot 4/\lambda = (4/\pi)\|y\|_\infty,$$

and hence $\|Q\| \leq 4/\pi$. To estimate $\|S\|$ we note that, for $y \in Y_0$, we have $y = d^{\#}\omega = d^{\#}2^{1/2}a^{-1/2}\sin\lambda t$, hence

$$\|y\|_\infty = |d^{\#}|\,2^{1/2}a^{-1/2}, \quad |d^{\#}| = 2^{-1/2}a^{1/2}\|y\|_\infty.$$

On the other hand, we have $q = 1$ and

$$M = \int_0^a \phi\omega\,\mathrm{d}t = \int_0^a 2^{1/2}a^{-1/2}(1 + 2\lambda^2\varepsilon^2)^{-1/2}(-\varepsilon\lambda + \sin\lambda t)\cdot 2^{1/2}a^{-1/2}\sin\lambda t\,\mathrm{d}t$$
$$= (1 + 2\lambda^2\varepsilon^2)^{-1/2},$$

and by the definition of S, for $x = Sy$ we have $x = d\phi$ with $d = d^{\#}/M$, or

$$\|x\|_\infty = \|Sy\|_\infty = \|d\phi\|_\infty = \|2^{1/2}a^{-1/2}(1 + 2\lambda^2\varepsilon^2)^{-1/2}(-\varepsilon\lambda + \sin\lambda t)\cdot 2^{-1/2}a^{1/2}$$
$$\times \|y\|_\infty(1 + 2\lambda^2\varepsilon^2)^{1/2}\|_\infty = \|\varepsilon\lambda + \sin\lambda t\|_\infty = (1 + \lambda\varepsilon)\|y\|_\infty,$$

and thus $\|S\| \leq 1 + \lambda\varepsilon$. We conclude with the list of the constants of interest:

$$\|Q\| \leq c_3 = 4/\pi, \quad \|I - Q\| \leq 1 + 4/\pi, \quad \|SQ\| \leq c_5 = (1 + \lambda\varepsilon)(4/\pi),$$
$$\|H\| \leq L = (2\pi)^{-2}a^3(2 + 2^{-1}(1 + 2\lambda^2\varepsilon^2)^{-1}(1 + \varepsilon\lambda))^2.$$

As a numerical example, let $g\colon \mathbb{R} \to \mathbb{R}$ be any function satisfying

$$|g(z)| < 1, \quad |g(z) - g(z')| \leq (4/3)|z - z'| \quad \text{for } |z|, |z'| \leq R_0 + r,$$
$$|z - g(z)| \leq \rho R_0 \quad \text{for } |z| \leq R_0. \tag{25}$$

Thus, $C = 1$, $D = 4/3$. We take $a = 0.4$, $k = 1$, $\lambda = 2\pi/a = 12.566$, $\varepsilon = 0.01$, $\lambda\varepsilon = 0.12566$, $c_4 = 1 + 4/\pi = 2.27336$, $c_5 = (1 + \lambda\varepsilon)(4/\pi) = 1.4332$, $L = 0.0068148$, $c = 0.1$, $r = 0.018$ we have

$$Lc_4(c + C) = (0.0068148)(2.27324)(0.1 + 1) = 0.017049 < 0.018 = r,$$
$$kc_5(c + Dr) = (1.4332)(0.1 + (4/3)(0.018)) = 0.17772.$$

For $R_0 = 1.5$, $\rho = 0.5$ we have

$$(1 - \rho c_5)R_0 = (1 - (0.5)(1.4332))(1.5) = 0.4251 > 0.17772,$$

while $R_0 - 1 = 0.5 < 0.75 = (0.5)(1.5) = \rho R_0$.

Thus, problem (23), (24) has at least a solution $x(t)$ with $|x(t)| \leq R_0 + r = 1.518$ for $a = 0.5$, $\|f\|_\infty \leq 0.1$ and any g satisfying (25) with $R_0 = 1.5$, $r = 0.018$, $\rho = 0.5$, in other words, for g of Lipschitz constant 4/3 whose graph is within the heavy box in Fig. 2.

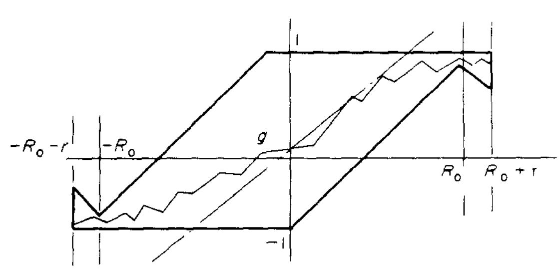

Fig. 2.

APPENDIX

The operator bounds used in this paper are obtained by functional analysis considerations. An elementary analysis of the same operators is contained in previous papers of the authors [1, 2, 6], and we think it relevant for the reader to see briefly the latter in the form which is needed here.

Let us consider the real linear system

$$u' - A(t)u = h(t), \quad 0 \leq t \leq a, \tag{26}$$

$$B_1u(0) + B_2u(a) = 0, \tag{27}$$

where $u(t) = \mathrm{col}(u_1, \ldots, u_n)$, $h(t) = \mathrm{col}(h_1, \ldots, h_n)$, $A = [c_{ij}(t)]$ is an $n \times n$ matrix with bounded measurable entries, and B_1, B_2 are $m \times n$ matrices, $0 \leq m \leq n$. The corresponding homogeneous system is then

$$u' - A(t)u = 0, B_1u(0) + B_2u(a) = 0. \tag{28}$$

The adjoint system, in parametric form is therefore (see, e.g. [6, p. 32] also for references).

$$\mathrm{d}\tilde{v}/\mathrm{d}t + \tilde{v}A(t) = 0, \tilde{v}(0) = \tilde{\alpha}B_1, \quad \tilde{v}(a) = -\tilde{\alpha}B_2, \tag{29}$$

where $v(t) = \mathrm{col}(v_1, \ldots, v_n)$, where $\alpha = \mathrm{col}(\alpha_1, \ldots, \alpha_n)$ is an arbitrary constant matrix, and where $\tilde{M}$ denotes the transpose of any matrix M.

If E denotes the differential operator with boundary conditions defined by relations (27), then ker E is the set of all solutions of problem (28). Analogous definitions have E^* and ker E^* for relations (29).

For any measurable n-vector $z(t) = (z_1, \ldots, z_n)$, $0 \leq t \leq a$, we denote by $\|z\|_\infty$ the usual norm $\|z\|_\infty$ = Ess $\mathrm{Sup}_{0\leq t\leq a}|z(t)|$, where $|z|$ denotes the Euclidean norm of z. If $z \in (L_\nu[0, a])^n$, $\nu \geq 1$, let $\|z\|_\nu$ denote the L_ν-norm $\|z\|_\nu = a^{-1}(\int_0^a |z(t)|^\nu \, \mathrm{d}t)^{1/\nu}$.

Let Y denote the L_ν-space of n-vectors $v(t) = (v_1, \ldots, v_n)$ with $|v(t)|$ L_ν-integrable, and let Y be equipped with the norm $\| \, \|_\nu$.

Let X denote the (Sobolev) space of all n-vectors $u(t) = (u_1, \ldots, u_n)$, $0 \leq t \leq a$, whose elements are AC in $[0, a]$, and whose derivatives $u'(t) = (u_1', \ldots, u_n')$ are L_ν-integrable in $[0, a]$, and let X be equipped with the norm $\|u\|_\nu^1 = \|u\|_\infty + \|u'\|_\nu$.

Let U be the $n \times p$ matrix, whose p columns form a basis for the solutions to the given boundary value problem (28), $0 \leq p \leq n$, and let V be a $q \times n$ matrix whose q rows form a basis for the solutions of the adjoint boundary value problem (29), $0 \leq q \leq n$. Let c and d denote the $p \times p$ and $q \times q$ matrices whose entries are numbers defined by

$$c = \int_0^a \tilde{U}(s)U(s) \, \mathrm{d}s, \qquad d = \int_0^a V(s)\tilde{V}(s) \, \mathrm{d}s.$$

These matrices are nonsingular (cf. [6, p. 35]). Let P: $X \to X$ and Q: $Y \to Y$ be the projection operators defined

$$(Pu)(t) = U(t)\alpha, \qquad \alpha = c^{-1}\int_0^a \tilde{U}(s)u(s) \, \mathrm{d}s, \qquad u \in X,$$

$$(Qv)(t) = \tilde{V}(t)\beta, \qquad \beta = d^{-1}\int_0^a V(s)v(s) \, \mathrm{d}s, \qquad v \in Y,$$

(for $p = 0$ take $P = 0$, for $q = 0$ take $Q = 0$). The operators P, Q are projection operators in the Banach spaces X and Y respectively in the sense that they are linear are

$$PP = P, \qquad \|u\|_\nu^1 \leq \|Pu\|_\nu^1 \leq \lambda_1\|u\|_\nu^1, \qquad u \in X,$$

$$QQ = Q, \qquad \|v\|_\nu \leq \|Qv\|_\nu \leq \lambda_2\|v\|_\nu, \qquad v \in Y,$$

for suitable constants λ_1, λ_2.

Let $X_0 = PX$, $X_1 = (I - P)X$, $Y_0 = QY$, $Y_1 = (I - Q)Y$, so that X and Y have the decompositions $X = X_0 + X_1$, $Y = Y_0 + Y_1$, and thus, for instance, $Px = 0$ if and only if $x \in X_1$, $Qv = 0$ if and only if $v \in Y_1$. Moreover, we have here $X_0 = \ker E$, $Y_0 = \ker E^*$.

The following theorem holds.

THEOREM A.1. If $h \in Y = (L_\nu[0, a])^n$, $\nu \geq 1$, then the boundary value problem (26), (27) has solutions $u \in X$ if and only if $Qh = 0$. If $Qh = 0$, then the boundary value problem (26), (27) has a unique solution Kh with $PKh = 0$, and all other solutions are of the form $Kh + Ua$, a an arbitrary row p-vector. Furthermore the linear map K: $Y_1 \to X_1$ is a bounded linear map in the norms of X_1 and Y_1.

This, which is derived from Fredholm alternative, was proved, for example, in [6] for $\nu = 1$. The proof is analogous for any $\nu \geq 1$. We refer to [6, pp, 32–39]. Thus, for any $h \in Y = (L_\nu[0, a])^n$, with $Qh = 0$ (that is, $h \in Y_1$), the only solution $u(t) = Kh$ of problem (1) with $Pu = 0$ satisfies $\|u\|_\nu^1 \leq \gamma\|h\|_\nu$ for some constant γ independent of h.

As we know, the analogous problem

$$x^{(n)} + p_1(t)x^{(n-1)} + \ldots + p_n(t)x = f(t), \quad 0 \leq t \leq a, \tag{30}$$

$$\sum_{j=0}^{n-1} (a_{ij}x^{(j)}(0) + b_{ij}x^{(j)}(a)) = 0, \quad i = 1, \ldots, m, \tag{31}$$

where a_{ij}, b_{ij} are constants, and $p_1(t), \ldots, p_n(t)$ bounded and measurable functions in $[0, a]$, that is, problem $Ex = f(t)$ with the notations of No. 2, is reduced to problem (26), (27) by the transformation $u_1 = x, \ldots, u_n = x^{(n-1)}$, namely

$$\begin{aligned} &u_1' - u_2 = 0, \ldots, u_{n-1}' - u_n = 0, u_n' + p_n(t)u_1 + \ldots + p_1(t)u_n = f(t), \\ &\sum_{j=1}^{n} (a_{i,j-1}u_j(0) + b_{i,j-1}u_j(a)) = 0, i = 1, \ldots, m. \end{aligned} \tag{32}$$

In other words, $h(t) = \text{col}(0, 0, \ldots, 0, f(t))$, and if $Qh = 0$, then there is one and only one solution $x = Kh$ of (32) with $PKh = 0$. Any other solution is of the form $u = Kh + Uc$, where c is any p column vector. Finally, K: $Y_1 \to X_1$ is a bounded linear map in the norms $\| \|_\nu$ of Y_1 and $\| \|_\nu^1$ of X_1, so that

$$\|Kh\|_\nu^1 \leq \gamma\|h\|_\nu.$$

Thus, for $z = Kh$,

$$\|z\|_\nu^1 = \|(z_1, \ldots, z_n)\|_\infty + \|(z_1', \ldots, z_n')\|_\nu \leq \gamma\|h\|_\nu,$$

and for the particular system (32) we derive

$$\|z_1\|_\infty, \|z_1'\|_\infty, \ldots, \|z_1^{(n-1)}\|_\infty \leq \gamma\|f\|_\nu, \|z_1^{(n)}\|_\nu \leq \gamma\|f\|_\nu.$$

The adjoint system is

$$v_1' = p_n v_n, v_2' + v_1 = p_{n-1}v_n, \ldots, v_n' + v_{n-1} = p_1 v_n \tag{33}$$

which can be reduced to the differential equation

$$v_n^{(n)} - (-1)^{n-1}p_n v_n - (-1)^{n-2}(p_{n-1}v_n)' - \ldots + (p_2 v_n)^{(n-2)} - (p_1 v_n)^{(n-1)} = 0.$$

If $y(t) = \text{col}(y_1, \ldots, y_n)$ is any element of ker E^*, then $y_n = \omega \in \ker E^*$ of Section 2, and the orthogonality of $h(t)$ with y, or $\int_0^a (h_1y_1 + \ldots + h_ny_n)\, dt = 0$, reduces to

$$\int_0^a f(t)\omega(t)\, dt = 0.$$

On the other hand, the orthogonality of $Kh = (z_1, \ldots, z_n)$ with any element $x(t) = \text{col}(x_1, \ldots, x_n)$ of ker E, or $\int_0^a (z_1x_1 + \ldots + z_nx_n)\, dt = 0$, may not be the same as the orthogonality of $z_1(t)$ with $x_1(t) = \phi(t) \in \ker E$ of Section 2, or $\int_0^a z_1(t)\phi(t)\, dt = 0$. Actually, if $\phi_1, \ldots, \phi_p$ denotes the base of ker E we have chosen in Section 2, we may well assume $\phi_1, \ldots, \phi_p$ to be the first row of U, and we have only to determine the solution $x(t) = z_1(t) + \phi_1a_1 + \ldots + \phi_pa_p$ which satisfies $\int_0^a x(t)\phi_i(t)\, dt = 0$, $i = 1, \ldots, p$. We obtain

$$\int_0^a z_1(t)\phi_i(t)\, dt + a_i = 0, i = 1, \ldots, p,$$

or

$$x(t) = z_1(t) - (z_1, \phi_1)\phi_1(t) - \ldots - (z_1, \phi_p)\phi_p(t) = (I - P)z_1(t).$$

Hence

$$x^{(j)}(t) = z_1^{(j)}(t) - (z_1, \phi_1)\phi_1^{(j)}(t) - \ldots - (z_1, \phi_p)\phi_p^{(j)}(t), j = 0, 1, \ldots, n,$$

Here all functions $p_1, \ldots, p_n$ are bounded, say $|p_i(t)| \leq \mu$, all functions $\phi_i^{(j)}(t), j = 0, 1, \ldots, n-1$, are also bounded, say $|\phi^{(j)}(t)| \leq M$, and

$$\|z_i^{(j)}(t)\|_\infty \leq \gamma\|f\|_\nu, \quad j = 0, 1, \ldots, n-1.$$

Thus,

$$\|x^{(j)}(t)\|_\infty \leq \gamma\|f\|_\nu + \gamma\|f\|_\nu \cdot paM^2 = \gamma\|f\|_\nu(1 + paM^2), \quad j = 0, 1, \ldots, n,$$

This proves statement (3) in Section 2.

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