Discrete Applied Mathematics 12 (1985) 155-164
North-Holland

ON SIGNED DIGRAPHS WITH ALL CYCLES NEGATIVE

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Received 22 September 1983
Revised 19 June 1984 and 5 February 1985

It is known that signed graphs with all cycles negative are those in which each block is a negative cycle or a single line. We now study the more difficult problem for signed digraphs. In particular we investigate the structure of those digraphs whose arcs can be signed (positive or negative) so that every (directed) cycle is negative. Such digraphs are important because they are associated with qualitatively nonsingular matrices. We identify certain families of such digraphs and characterize those symmetric digraphs which can be signed so that every cycle is negative.

1. Introduction

We shall have occasion below to make use of graphs, digraphs, signed graphs, and signed digraphs. Consequently we begin with a very brief review of these concepts which will also serve to standardize our notation.

A *graph* $G=(V,E)$ consists of a finite nonempty set V of points and a set E of lines, each a 2-subset of V. A *digraph* $D=(V,X)$ has in addition to set V a collection $X\subset V\times V$ of arcs (u,v) where $u\neq v$. A *signed graph* $H=(V,E,\sigma)$ consists of a graph (V,E) together with a sign function $\sigma: E\rightarrow\{1,-1\}$. Similarly a *signed digraph* $S=(V,X,\sigma)$ is a digraph (V,X) whose arcs have been signed positive or negative by σ. Terminology not given here can be found in Harary [3] or Harary, Norman, and Cartwright [4].

The signed graphs in which every cycle is negative were easily characterized in [2] where the following result appears.

Theorem A (Harary). *A signed graph H has all cycles negative if and only if each block of H is either a line or a negative cycle.*

Our object is to study the class $\mathcal{N}$ of all signed digraphs with all (directed) cycles negative. The primary reason why this class is of interest is because of its importance to the sign solvability problem. It was already implicit in the original paper of Bassett, Maybee, and Quirk [1] that such signed digraphs belong to qualitatively invertible matrices. Since that time it has become clear from work by Klee and his associates [5, 8], and the work of Maybee [9] on sign solvable graphs and of Lady [7], that this class of signed digraphs plays a central role in the analysis of sign solvable systems. We do not have a characterization of the set $\mathscr{N}$ similar to that given by Theorem A and it may be that a characterization will be too complicated to be useful. But we shall identify three large classes of signed digraphs in $\mathscr{N}$.

Let $\mathscr{M}$ be the set of all digraphs D for which there exists a sign function σ such that $\sigma D \in \mathscr{N}$. Obviously both sets $\mathscr{N}$ and $\mathscr{M}$ are hereditary, i.e.,

If T is a subgraph of $S \in \mathscr{N}$, then $T \in \mathscr{N}$.

If F is a subgraph of $D \in \mathscr{M}$, then $F \in \mathscr{M}$.

We shall be interested primarily in elements of $\mathscr{M}$ and $\mathscr{N}$ that are strong.

For a digraph (or a signed digraph), a *symmetric cycle* C_n has $n \geq 3$ and consists of a directed cycle of length n and its converse. If D is a digraph we shall denote by $G_0(D)$ *the symmetric part* of D, i.e., the largest symmetric subdigraph contained in D. As the notation implies, we consider $G_0(D)$ to be itself a graph because whenever the arc (u, v) belongs to $G_0(D)$ so does (v, u). If D is a symmetric digraph, then we identify $G_0(D)$ with D itself.

2. Upper digraphs

For a graph G (or a digraph D) the adjacency matrix $A(G)$ (or $A(D)$) is binary (consists of 0 and 1 entries) and has zeros on its principal diagonal. Also $A(G)$ is symmetric but $A(D)$ need not be. In fact, $A(D)$ is symmetric if and only if D is a symmetric digraph. Similarly for a signed graph H or signed digraph S, $A(H)$ has entries 0, 1, or -1.

A matrix $A = [a_{ij}]$ is called *upper Hessenberg* [10, p. 218] if $a_{ij} = 0$ whenever $i - j > 1$. For want of a better term we shall call a digraph *upper* if there is a labelling of V such that the resulting adjacency matrix $A(D)$ is upper Hessenberg.

We will now characterize strongly connected upper digraphs.

Theorem 1. *A digraph D is strong and upper if and only if*

(1) *it has a hamiltonian path, say* $(v_p v_{p-1} \cdots v_2 v_1)$,

(2) *for each* $i \neq p$, *there is a path from* v_i *to* v_p, *and*

(3) *there is no arc* (v_i, v_j) *with* $i - j > 1$.

Proof. If D satisfies (1), (2), and (3), then clearly D is strong and upper. For the converse let D be strong and upper. Then (3) follows at once. On the other hand, there exists a path from v_p to v_1 because D is strong. The truth of (1) follows from this fact and (3). Finally (2) must hold because D is strong. □

Theorem 2. *If D is strong and upper, then $D \in \mathscr{M}$.*

Proof. We have to produce a function σ such that $\sigma D \in \mathscr{N}$. By condition (1) each of the arcs (v_i, v_{i-1}) belongs to D, $2 \le i \le p$. Set $\sigma(v_i, v_{i-1}) = -1$. It remains to define σ on any arc of the form (v_i, v_j), $j \ge i+1$ which we do by setting $\sigma(v_i, v_j) = (-1)^{j-i+1}$ if the arc belongs to D. Now suppose Z is a cycle in the resulting signed digraph S. Then Z must have the form $(v_i v_j v_{j-1} \cdots v_{i+1} v_i)$ for some $j \ge i+1$. Hence Z consists of the arcs $(v_i, v_j)(v_j, v_{j-1}) \cdots (v_{i+1}, v_i)$ and

$$\sigma Z = \sigma(v_i, v_j)\sigma(v_j, v_{j-1}) \cdots \sigma(v_{i+1}, v_i) = (-1)^{j-i+1}(-1) \cdots (-1)$$

where there are $j-i$ arcs of sign (-1). Thus $\sigma Z = (-1)^{j-i+1}(-1)^{j-i} = -1$ so Z is a negative cycle and $\sigma D \in \mathscr{N}$. □

We remark that to test D to see if D is upper is an NP-complete problem.

It is of interest to observe that every arc of the form (v_i, v_j), $j \ge i+1$, $i < p$, can belong to D if D is upper and σD will belong to $\mathscr{N}$. A digraph D is *maximal upper* if D is upper and for each arc x in the complement $\bar{D}$, $D+x$ is no longer in $\mathscr{M}$, i.e., if $D_1 = D+x$, then there is no σ such that $\sigma D_1 \in \mathscr{N}$.

Theorem 3. *If for a strong upper digraph D, its upper Hessenberg adjacency matrix satisfies $a_{ij} = 1$ for $i \ne j$ and $i-j \le 1$, then D is maximal upper.*

Proof. Suppose D is a strong upper digraph and suppose the points are labeled so that $A(D)$ is upper Hessenberg. Then if D_1 results from D by adjoining an arc, the arc must have the form $(v_i v_j)$ where $i-j > 1$. Suppose σ exists such that $\sigma D_1 \in \mathscr{N}$. Then the cycles $Z_1 = (v_j v_{i-1} v_i v_j)$ and $Z_2 = (v_j v_i v_{i-1} \cdots v_{j+1} v_j)$ are both negative, so that $(\sigma Z_1)(\sigma Z_2) - 0$. But then

$$(\sigma Z_1)(\sigma Z_2) = \sigma(v_j v_i)\sigma(v_i v_j)\sigma(v_{i-1} v_i)\sigma(v_i v_{i-1})\sigma(v_j v_{i-1} v_{i-2} \cdots v_{j+1} v_j).$$

The first two pairs in this product correspond to 2-cycles and are both negative and the last term represents the sign of a cycle and must also be negative. The sign of the product must be negative, a contradiction. It follows that σ does not exist such that $\sigma D_1 \in \mathscr{N}$. □

Note that if D is a maximal upper digraph, then $G_0(D)$ is a path of length $p-1$.

Note also that, if D is not maximal, then it may be possible to adjoin an arc to D in such a way that $D+x \in \mathscr{M}$ but $D+x$ is not upper given in Fig. 1. Here we use solid lines for positive arcs, dashed lines for negative arcs.

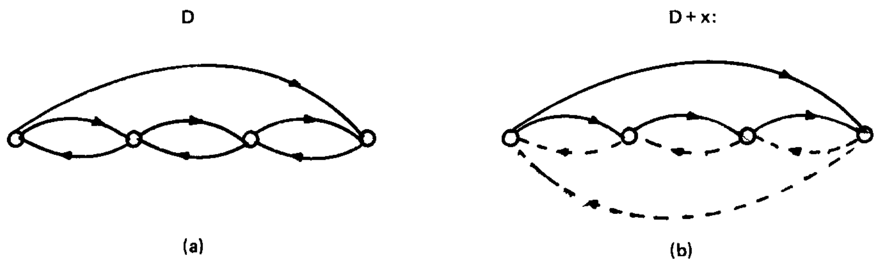

Fig. 1. A non-maximal upper digraph D in (a) and its signed augmentation (b), in $\mathcal{N}$.

3. A generalization of unipathic digraphs

We may arrive at another subclass of $\mathcal{M}$ in the following way. The class of unipathic digraphs was identified in [4, p. 218]. A digraph D is *unipathic* if whenever v is reachable from u, there is exactly one path from u to v. We start with the following basic result.

Theorem 4. *If D is unipathic and x is an arc of D, then x belongs to at most one cycle.*

Proof. Let x be the arc (u, v) and suppose $x \in Z_1$ and $x \in Z_2$. Then we can write $Z_1 = (u, v)p_1(v \to u)$ and $Z_2 = (u, v)p_2(v \to u)$ where $p_1(v \to u)$ and $p_2(v \to u)$ are paths from v to u. Since Z_1 and Z_2 are distinct, p_1 and p_2 are not the same path, contradicting the fact that D is unipathic. □

We can use the result of Theorem 4 to find an interesting generalization of the class of unipathic digraphs. Denote by U the latter class and define the class $\bar{U}$ as follows. The digraph $D \in \bar{U}$ if every cycle Z of D contains at least one arc x which is not in any other cycle of D.

Theorem 5. *If $D \subset \bar{U}$, then $D \in \mathcal{M}$.*

Proof. For each cycle Z of D choose an arc x belonging to no other cycle of D. Set $\sigma x = -1$. For each remaining arc y we set $\sigma y = 1$, and so $\sigma D \in \mathcal{N}$. □

It follows, of course, from Theorem 5 that every unipathic digraph belongs to $\mathcal{M}$.

The class $\bar{U}$ contains many elements which are not unipathic. In Fig. 2 we show two quite different examples. Note that in these examples, there is an arc belonging to both cycles. If this common arc is given a negative sign and all other arcs are given a positive sign, then $\sigma D \in \mathcal{N}$ in each case.

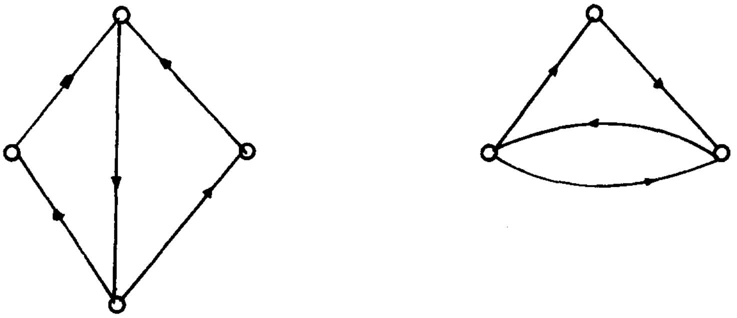

Fig. 2. Simple examples of non-unipathic digraphs in the class $\bar{U}$.

4. Symmetry

We start with a result of fundamental importance to the study of the properties of signed digraphs in $\mathcal{N}$.

Theorem 6. *Let $S \in \mathcal{N}$ and suppose that C_n, $n \geq 3$, is a symmetric cycle of S. Then n is even.*

Proof. We may assume that C_n is composed of the two directed cycles $Z_1 = (v_1 v_2 \cdots v_n v_1)$ and $Z_2 = (v_1 v_n v_{n-1} \cdots v_2 v_1)$ which is the converse Z_1' of Z_1. Since $S \in \mathcal{N}$ we have $\sigma Z_1 = -1$ and $\sigma Z_2 = -1$. Moreover, we have $\sigma(v_1 v_2 v_1) = -1$, $\sigma(v_2 v_3 v_2) = -1, \ldots, \sigma(v_1 v_n v_1) = -1$. It follows that

$$(-1)^n = \sigma(v_1 v_2 v_1)\sigma(v_2 v_3 v_2) \cdots \sigma(v_1 v_n v_1) = (\sigma Z_1)(\sigma Z_2) = 1.$$

Thus n must be even. □

The following corollary of Theorem 6 is also very useful.

Corollary 6a. *Let $S \in \mathcal{N}$ and suppose a symmetric cycle C_{2n} belongs to S. Let u, v be two distinct points of C_{2n} whose distance along the cycle is even. Then, in the signed digraph obtained from S by removing all arcs of C_{2n} and all points of C_{2n} except u and v, the points u and v are not unilaterally connected.*

Proof. Assume, for contradiction, there is a path from u to v which is disjoint from C_{2n} except for the points u and v. Denote this path by $P_0(u \to v)$. Now in C_{2n} there are two paths from v to u, say $P_1(v \to u)$ and $P_2(v \to u)$, and they must have opposite signs because $d(u, v)$ along C_{2n} is even. But then $P_1(v \to u)P_0(u \to v) = Z_1$, and $P_2(v \to u)P_0(u \to v) = Z_2$ are both cycles of S and both must be negative. But this is impossible so $P_0(u \to v)$ cannot exist in S. A similar argument shows that no path outside of C_{2n} can exist from v to u. □

We illustrate in Fig. 3 two signed digraphs S with a symmetric 4-cycle. In Fig. 3a no matter how the signs of the arcs (25) and (54) are chosen there is always a positive cycle while 3b shows that adjacent points can be joined by paths exterior to C_4. We call such paths *exterior paths* and refer to the condition of the corollary as the *exterior path condition.*

Let D be a symmetric digraph and set $G(D) \equiv G_0(D)$. What properties must $G(D)$ have in order to insure that $D \in \mathcal{M}$? We know from Theorem 6 that every cycle of $G(D)$ must have even length, thus G must be bipartite. But this condition is not sufficient by virtue of Corollary 6a. A counterexample is shown in Fig. 4a. Because of the line [14], the cycle [1 2 5 6 1] has the property that its two points 1 and 5 are an even distance apart and are joined by an exterior path. Nevertheless the graph is bipartite.

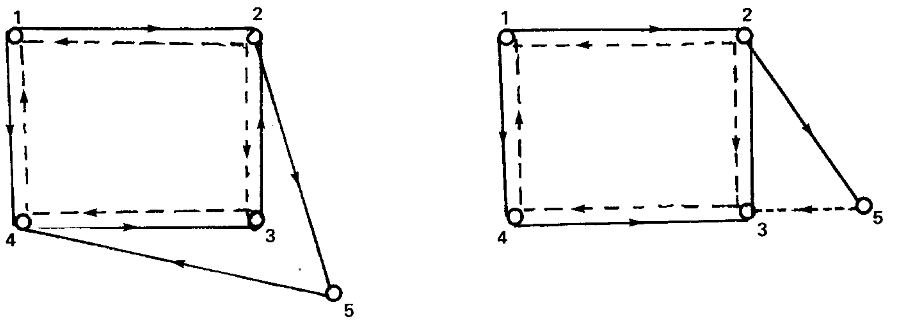

Fig. 3. Exterior paths.

We note that in the example of Fig. 4a the cycles [1 4 5 6 1] and [1 2 5 6 1] have two adjacent common lines, namely [16] and [56]. On the other hand, in Fig. 4b we have three cycles [1 2 3 4 5 6], [1 2 5 6], [2 3 4 5]. Each pair of cycles intersect in a path of odd length and it is easy to verify that $D \in \mathcal{M}$.

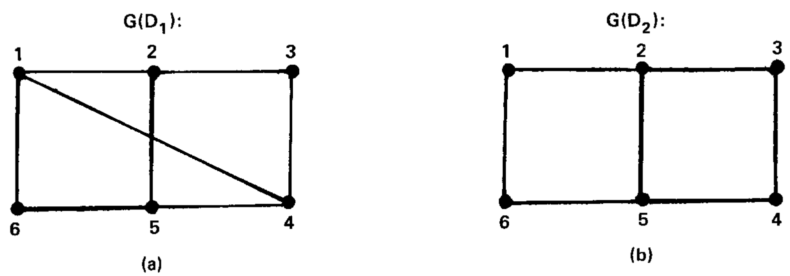

Fig. 4. Both $G(D)$ are bipartite, but (a) $D_1 \notin \mathcal{M}$, (b) $D_2 \in \mathcal{M}$.

5. Symmetric digraphs

We wish to characterize the symmetric digraphs in $\mathcal{M}$. To this end we require the following results.

Lemma 7. *Let D be a symmetric digraph in $\mathcal{M}$. If two cycles C_1 and C_2 of $G(D)$ intersect in a path of length r, then r must be odd..*

Proof. The result follows from Corollary 6a. If r were even, then we would have an exterior path joining points an even distance apart in a cycle. □

Now we come to our main result. We are greatly indebted to a referee for suggesting the elegant proof we will give below for Theorem 8. It is based upon a very nice result of T. Zaslavsky [11] and replaces a much longer and more intricate proof originally presented by the authors.

Let G be a graph and let $\mathcal{B}$ be a set of cycles of G. Zaslavsky calls $\mathcal{B}$ *theta additive* if, whenever C_1 and C_2 are cycles for which C_1+C_2 (where C_1+C_2 is the set of lines in C_1, C_2, or in both cycles.) He has proved in [11] the following key result.

Theorem B (Zaslavsky). *Given any set $\mathcal{B}$ of cycles in G, there exists a signed graph on G whose set of positive cycles is $\mathcal{B}$ if and only if $\mathcal{B}$ is theta additive.*

Theorem 8. *Let D be a symmetric digraph. Then $D\in\mathcal{M}$ if and only if $G(D)$ is bipartite and does not contain any exterior path joining two points an even distance apart in any cycle of G.*

Proof. Assume first that $D\in\mathcal{M}$ is symmetric. Since $D\in\mathcal{M}$, signs can be assigned to the arcs of D so that the resulting signed digraph $S\in\mathcal{N}$. But then Theorem 6 implies that all cycles of D have even length so $G(D)$ is bipartite. Also Corollary 6a applies so that $G(D)$ does not contain any exterior path joining two points an even distance apart in any cycle of G. Thus the only if portion of the theorem is true. To prove the if portion, let D be bipartite and satisfy the exterior path condition and define $\mathcal{B}$ to be the set of even cycles of $G(D)$ of length $2p$ for some odd number $p>1$. Since no two cycles of G can intersect in a path of even length, $\mathcal{B}$ is theta additive. Therefore there exists a sign function σ_1 on G whose set of positive cycles is $\mathcal{B}$. Now suppose that $V(G)=A\cup B$ is a bipartition of $G(D)$. We then define a sign function σ_2 on D as follows. Let each arc (u,v) with $u\in A$ and $v\in B$ have the same sign as the corresponding line of $G(D)$, and each arc (u,v) with $u\in B$ and $v\in A$ the opposite sign. Now suppose z is any cycle in D, and C is the corresponding cycle in $G(D)$. If z has length l, then

$$\sigma_2(z)=(-1)^{l/2}\sigma_1(C).$$

But for $l=2p$ where p is even, $\sigma_1(C)=-1$ and $(-1)^{l/2}=1$ by Theorem B. Similarly, again by Theorem B, if $l=2p$ where p is odd, $\sigma_1(C)=1$ and $(-1)^{l/2}=-1$. It follows that $\sigma_2(z)=-1$ for any $z\in D$. Thus $D\in\mathcal{M}$, as was to be shown. □

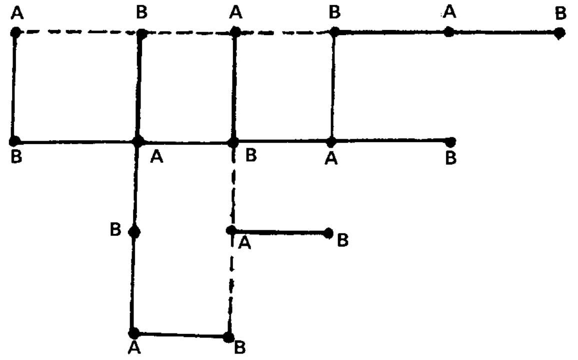

Fig. 5. A graph satisfying the conditions of Theorem 8 with a sign function σ_1 and a bipartition of its points.

We can illustrate the if portion of the above proof using the example shown in Fig. 5. The graph has the negative edges shown by the dotted lines and its points have been labeled A or B to illustrate a bipartition of the points. Note that the sign function σ_1 has the required properties.

The underlying symmetric digraph for the graph of Fig. 5 is shown in Fig. 6. The sign function σ_2 *induced on D* by σ_1 is illustrated using dotted lines for negative arcs. Note that $D\in\mathcal{M}$.

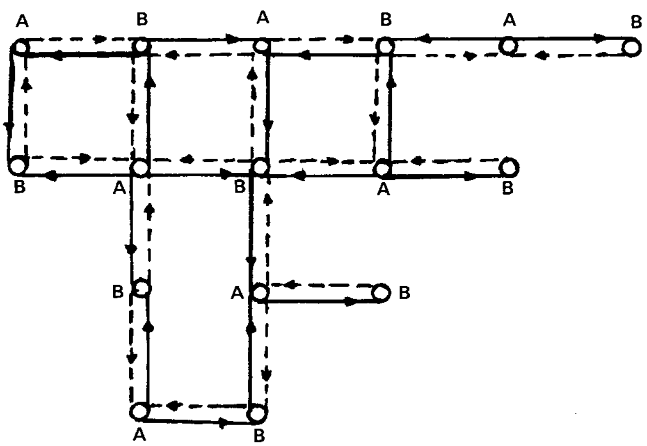

Fig. 6. The signed digraph $S\in\mathcal{M}$ arising from the sign function σ_2 on D induced by σ_1 on $G(D)$ shown in Fig. 5.

6. Unsolved problems

As we mentioned above, we have not presented a characterization of signed digraphs with all negative cycles. However complex such a characterization may be, we feel that it could prove very useful in view of the importance of sign solvable systems in a variety of fields. The classes we have introduced in Sections 2, 3, and 5 do provide us with a large stock of elements in $\mathcal{N}$.

We pointed out in Section 2 that if D is a maximal upper digraph, then $G_0(D)$ is a path. Turning this around, we observe that each maximal upper digraph may be regarded as the result of turning a path G into a digraph and adjoining as many arcs as possible. Now a path is a particular instance of a tree. Thus an unsolved problem arising from Section 2 is the following. If the graph G is a tree which is not a path, can be construct a maximal digraph D from G in some manner similar to that used in constructing a maximal upper digraph from a path? If such a construction cannot be made for all trees, then what is the subset of trees for which it can be done? We know this subset is not empty. In fact, we have recently been able to construct maximal digraphs from all trees which are caterpillars by a method similar to that used here.

In Section 3 we have introduced an interesting generalization of the class of unipathic digraphs, namely the class of digraphs D such that every cycle of D contains at least one arc not in any other cycle of D. Suppose we call such digraphs *free cyclic*. Since each free cyclic digraph belongs to $\mathcal{M}$, it would be of considerable interest to characterize the digraphs in this class.

A general unsolved problem can be formulated in terms of the graph $G_0(D)$ for $D \in \mathcal{M}$. Suppose $D \in \mathcal{M}$, then what can be said about $G_0(D)$? Conversely, would it be useful to attempt to classify the elements $D \in \mathcal{M}$ in terms of their symmetric part $G_0(D)$? It is clear from the results of Sections 4 and 5 that $G_0(D)$ cannot be an arbitrary graph since it must be bipartite and satisfy the exterior path condition. When will $G_0(D)$ be connected? When will it be a spanning subgraph?

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