Nonlinear Analysis, Theory, Methods & Applications, Vol. 9, No. 12, pp. 1487–1493, 1985.
Printed in Great Britain.
0362–546X/85 \$3.00 + .00

CONTROLLABILITY AND FEEDBACK SYSTEMS*

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(*Received* 20 *August* 1984; *received for publication* 9 *April* 1985)

Key words and phrases: Controllability, multifunctions, Carathéodory mappings, Cesari's property Q, fixed points.

1. INTRODUCTION

IN THE present paper we seek sufficient conditions for controllability of the nonlinear feedback control system in Banach spaces of the form

$$(Ex)(t) = f(t, (Mx)(t), u(t)),$$

$$u(t) \in \omega(t, (Mx)(t)), \qquad t \in [0, T]$$

where the function u is the control function and x defined on $[0, T]$ is the trajectory. Many boundary value problems for partial differential equations with controls can be written in this form.

The approach we shall take is to reduce the problem of controllability to the problem of showing a certain multifunction has a fixed point.

Applications of fixed point theorems for multifunctions to problems of similar kinds have been studied by several authors in the past. Lasota and Opial [15, 16] in their papers studied the existence problem of ordinary differential equations with multivalued right-hand sides of the form $\dot{x}(t) \in A(t)x(t) + F(t, x(t))$ with the constraint $Lx = r$. Hermes [9] considered equations of the form $\dot{x} \in R(t, x)$. Tarnove [17] obtained sufficient conditions for A-controllability of the nonlinear system $\dot{x} = f(t, x, u)$, $u(t) \in Q$. Subsequently Dauer [4] treated the question of controllability for systems of the form $\dot{x} = f(t, x) + g(t, u)$, $x(t_0) = x_0$, $x(t_1) = x_1$, $x_0, x_1 \in \mathbb{R}^n$, $u(t) \in \Omega(t)$.

All these authors used various conditions which may be shown (cf. Hou [10]) to imply Cesari's property (Q). We shall see below that Cesari's property (Q) plays an important role in the present work.

Concerning the fixed point theorem for multifunctions used by all these authors, we remark that their statements are all different forms of the following fixed point theorem.

THEOREM 1.1. Let K be a closed convex set in a locally convex Hausdorff space. Let $\Gamma : K \to 2^K$ be a multifunction such that $\Gamma(x)$ is closed and convex for each $x \in K$, Γ has a closed graph and $\cup\{\Gamma(x) : x \in K\}$ is contained in some compact set. Then Γ has a fixed point.

* This research was partially supported by AFOSR Research Project 77-3211 at the University of Michigan, Ann Arbor.
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This theorem is an extension of Kakutani's theorem (cf. Bohnenblust and Karlin [1], Fan [6], Glicksberg [7]).

2. FORMULATION OF CONTROLLABILITY PROBLEM

Let J be the fixed interval $[0, T]$ and let V be a separable reflexive Banach space. For $1 < p < +\infty$, the space $X \equiv L_p(J, V)$ of all pth Bochner integrable functions from J to V is a reflexive Banach space with dual $X^* \equiv L_q(J, V^*)$, where $1/p + 1/q = 1$. Let $\| \|$, $\| \|_*$, $\| \|_X$ denotes the norms in V, V^*, X, respectively.

We shall assume $E: X \to X^*$ to be a closed, linear operator with domain $D(E)$ and $M: D(E) \to L_p(J, Y)$ an operator which maps weakly convergent sequences into sequences strongly convergent in $L_p(J, Y)$ where Y is a Banach space. We shall also assume that ω is a measurable multifunction from $J \times Y$ to 2^U with closed values in a complete separable metric space U, and that f is a Carathéodory mapping from $J \times Y \times U$ to V^*. That is, $f(\cdot, y, u)$ is strongly measurable for fixed (y, u) and $f(t, \cdot, \cdot)$ is separately demicontinuous† in y, u for almost every t in J. Associated with f and ω, we may define a multifunction $\mathfrak{Q}: J \times Y \to 2^{V^*}$ by

$$\mathfrak{Q}(t, y) \equiv \{z \in V^* : z = f(t, y, u), u \in \omega(t, y)\}. \tag{2.1}$$

In what follows, we shall consider pairs of functions (x, u), each pair consisting of a measurable function $u: J \to U$ and a corresponding function $x \in X$, which satifies, almost everywhere, the nonlinear operator equation

$$(Ex)(t) = f(t, (Mx)(t), u(t)), \qquad 0 \leq t \leq T, \tag{2.2a}$$

subject to the constraint

$$u(t) \in \omega(t, (Mx)(t)) \text{ for almost all } t \text{ in } [0, T]. \tag{2.2b}$$

Such pairs of functions are called admissible, and for such a pair, the function x is called a trajectory of the control system while the function u is called the control generating the trajectory x. Mx may be considered as the observed state of the system.

We may now define a very general form of controllability. Let Ψ be a nonempty subset of X.

Definition 2.1. The control system (2.2ab) is said to be Ψ-controllable if there exists an admissible pair (x, u) such that $x \in \Psi$.

Let K be a nonempty subset of Ψ and let Γ denote the multifunction $\Gamma: K \to 2^K$ defined by

$$\Gamma(x) \equiv \{z \in K : (Ez)(t) \in \mathfrak{Q}(t, (Mx)(t)) \quad \text{almost everywhere in } [0, T]\}. \tag{2.3}$$

By a fixed point of Γ we mean an element $x_0 \in K$ such that $x_0 \in \Gamma(x_0)$. Then by the definition of Γ we have $(Ex_0)(t) \in \mathfrak{Q}(t, (Mx_0)(t))$ a.e. in $[0, T]$. With the application of an implicit function theorem we may conclude that there exists a control u_0 for which the pair (x_0, u_0) is admissible, whence the system (2.2ab) is Ψ-controllable.

† A function h from a topological space B_1 to a Banach space B_2 is said to be demicontinuous at a point $x \in B_1$ if $h(x_n) \rightharpoonup h(x)$ weakly for every convergent sequence $x_n \to x$ in B_1 (cf. Kato [12]).

In this fashion, the question of controllability of the system (2.2ab) has been reduced to the problem of finding a fixed point for the multifunction Γ.

There are a number of implicit function theorems guaranteeing, under suitable hypotheses, the existence of an admissible control. The interested reader may consult, e.g. Castaing and Valadier [3], Wagner [18]. Here we will use the following Filippov type implicit function theorem (cf. Hou [11]) which is most appropriate in this setting.

THEOREM 2.1. Suppose that T is a measure space with a complete, σ-finite, nonnegative measure, Z is a Banach space whose dual Z^* has a countable subset that separates points of Z, and U is a separable complete metric space. Let $S: T \to 2^U$ be a measurable multifunction with closed values, let $h: T \times U \to Z$ be a Carathéodory mapping and $g: T \to Z$ a strongly measurable function satisfying the relation $g(t) \in h(t, S(t))$ almost everywhere in T. Then there exists a measurable selection u for S such that $g(t) = h(t, u(t))$ almost everywhere in T.

3. THE CONTROLLABILITY THEOREM

We turn now to a consideration of properties of the control system under which the multifunction Γ defined by (2.3) has a fixed point.

We will assume that, for all $x \in K$, $\Gamma(x) \neq \varnothing$. We shall make use of the following assumptions.

(H1) K is a nonempty closed convex bounded subset of $\Psi \subseteq X$.

(H2) The set valued map $\mathfrak{Q}: J \times Y \to V^*$ satisfies Cesari's property (Q) with respect to y in Y. That is, for almost every t in J and every y in Y, the relation

$$\bigcap_{n=1}^{\infty} \operatorname{clco} \bigcup_{n=k}^{\infty} \mathfrak{Q}(t, y_k) \subseteq \mathfrak{Q}(t, y)$$

holds for every sequence y_n converging to y in Y.

(H3) The sequence $\{Ev_n\}$ is bounded in X^* for every pair of bounded sequences $\{v_n\}$ in X, $\{b_n\}$ in $L_p(J, Y)$ satisfying the relation $(Ev_n)(t) \in \mathfrak{Q}(t, b_n(t))$ almost everywhere in J.

Remark. The assumption (H3) is satisfied, for example, if the function f satisfies the following growth condition:

$$\|f(t, y, u)\|_* \leq \beta(t) + \lambda \|y\|_Y^{p/q} \qquad \text{for} \qquad \beta \in L_q(J, \mathbb{R}),$$

$\lambda > 0$, and $(y, u) \in Y \times U$.

Let us assume that the assumptions (H1), (H2), (H3) are satisfied. We establish first a series of lemmas.

LEMMA 3.1. For every $x \in K$, the set $\Gamma(x)$ is convex.

Proof. Let $z_1, z_2 \in \Gamma(x)$ for some $x \in K$ and let $z = \lambda z_1 + (1 - \lambda) z_2$ for fixed λ, $0 \leq \lambda \leq 1$. Since $(Ez_i)(t) = f(t, (Mx)(t), u_i(t))$ for some $u_i(t) \in \mathfrak{Q}(t, (Mx)(t))$, $i = 1, 2$, we have by linearity of E that

$$\begin{aligned}(Ez)(t) &= \lambda(Ez_1)(t) + (1 - \lambda)(Ez_2)(t) \\ &= \lambda f(t, (Mx)(t), u_1(t)) + (1 - \lambda) f(t, (Mx)(t), u_2(t))\end{aligned} \tag{3.1}$$

for almost every t in J. But the sets $\mathfrak{Q}(t, (Mx)(t))$ are assumed to be convex by (H2), and so the right-hand side of (3.1) belongs to the set $\mathfrak{Q}(t, (Mx)(t))$. We may now conclude that $z \in \Gamma(x)$, this shows that $\Gamma(x)$ is convex. ∎

LEMMA 3.2. The weak topology $\sigma(X, X^*)$ on the set K is metrizable.

Proof. Since $X \equiv L_p(J, V)$ and V is separable, therefore X is also separable (Warga [19, theorem I.5.18]). Then by (H1) and the reflexivity of X, K is weakly compact and now the assertion follows from Dunford and Schwartz [5, theorem V.6.3]. ∎

The following is a Banach space version of Cesari's closure theorem (Hou [10]) which will be used to show that the graph of the multifunction Γ is closed.

THEOREM 3.3. Suppose that G is a finite measure space, Y is a topological space and B is a Banach space. Let $\mathfrak{Q}: G \times Y \to 2^B$ be a multifunction that satisfies Cesari's property (Q) with respect to y in Y. Let ξ, ξ_n, $n = 1, 2, \ldots$, be integrable functions in $L_1(G, B)$ and z, z_n, $n = 1, 2, \ldots$, functions from G to Y, satisfying the relation $\xi_n(t) \in \mathfrak{Q}(t, z_n(t))$ almost everywhere in G, and such that $\xi_n \rightharpoonup \xi$ weakly in $L_1(G, B)$, $z_n(t) \to z(t)$ in Y for almost every t in G as $n \to \infty$. Then $\xi(t) \in \mathfrak{Q}(t, z(t))$ almost everywhere in G.

LEMMA 3.4. The multifunction $\Gamma: K \to 2^K$, has a closed graph in $X_w \times X_w$, X_w being the space X endowed with the weak topology $\sigma(X, X^*)$.

Proof. Since the weak topology $\sigma(X, X^*)$ on the set K is metrizable by lemma 3.2, it is enough to show that $v \in \Gamma(x)$ for any weakly convergent sequences $x_n \rightharpoonup x$, $v_n \rightharpoonup v$ with $v_n \in \Gamma(x_n)$. By definition of Γ, we have $(Ev_n)(t) \in \mathfrak{Q}(t, (Mx_n)(t))$ almost everywhere in J. Since $\{x_n\} \subset K$ is bounded, so does $\{Mx_n\}$, and hence by (H3) $\{Ev_n\}$ is bounded in X^*, and therefore we may extract a subsequence, again denoted by $\{Ev_n\}$, that converges weakly to some point z in X^*. As E is linear and closed, we get $Ev = z$. So we may assume that $Ev_n \rightharpoonup Ev$ weakly in X^*, $(Ev_n)(t) \in \mathfrak{Q}(t, (Mx_n)(t))$ almost everywhere in J, $(Mx_n)(t) \to (Mx)(t)$ strongly in Y for almost every t in J, by choice of appropriate subsequences if necessary. According to (H2), the multifunction $\mathfrak{Q}(t, y)$ has property (Q) with respect to y, we may then conclude from the closure theorem 3.3 that $(Ev)(t) \in \mathfrak{Q}(t, (Mx)(t))$ almost everywhere in J. This implies that $v \in \Gamma(x)$ and the proof is complete. ∎

These three lemmas enable us to establish the following controllability theorem.

THEOREM 3.5. Let the assumptions (H1), (H2) and (H3) hold. Then a sufficient condition for the system (2.2ab) to be Ψ-controllable is that $\Gamma(z) \neq \varnothing$ for every $z \in K$.

Proof. By the lemma above, the multifunction Γ has a closed graph in $X_w \times X_w$ and $\Gamma(z)$ is convex for each $z \in K$. Since K is weakly compact and convex in X_w, the fixed point theorem 1.1 assures us that Γ has a fixed point x in K. It follows from $x \in \Gamma(x)$ that $(Ex)(t) \in h(t, S(t))$ almost everywhere in J, where h is the Carathéodory mapping $(t, u) \mapsto f(t, (Mx)(t), u)$ and $S: J \to 2^U$ is the multifunction defined by $S(t) \equiv \omega(t, (Mx)(t))$. Since V is reflexive and separable, there exists a countable subset in V^{**} that separates points of V^* (cf Hille and Phillips [8]).

We may now invoke the implicit function theorem 2.1 to produce a measurable control u such that the pair (x, u) is admissible, and whence the system (2.2ab) is Ψ-controllable. This argument is valid provided we can show that the multifunction S is measurable from J to U. To this end, let A be a closed subset of U. Then $t \in S^{-}(A)$ if and only if $A \cap S(t) \neq \emptyset$. That is, $(t, (Mx)(t)) \in \omega^{-}(A)$ which is the same as $t \in l^{-1}(\omega^{-}(A))$, where $l(t) \equiv (t, (Mx)(t))$. Thus $S^{-}(A) = l^{-1}(\omega^{-}(A))$ which is measurable since both l and ω are. Therefore S is measurable. ∎

4. AN APPLICATION

In this section we use the controllability theorem 3.5 to prove the existence of admissible pairs for the following control system

$$(\mathcal{R}x)(t) + (\Lambda x)(t) = f(t, (Mx)(t), u(t)), \tag{4.1a}$$

$$u(t) \in \omega(t, (Mx)(t)) \qquad \text{almost everywhere in } [0, T]. \tag{4.1b}$$

Let f, M and ω be given as in the preceeding section. We shall make the following additional assumption on f:

(F) $\|f(t, y, u)\|_* \leq \alpha(t)$ for $\alpha \in L_q(J, R)$ and (y, u) in $Y \times U$.

Furthermore we assume that $\mathcal{R}$ is a linear, maximal monotone operator from X to X^* with domain $D(\mathcal{R})$, Λ is a linear continuous operator from $D(\Lambda) = X$ into X^*, and that Λ is also coercive in the sense that there exists a nonnegative real valued function $h: \mathbb{R} \to \mathbb{R}$ with $h(s) \to +\infty$ as $s \to +\infty$ such that

$$\langle \Lambda x, x \rangle \geq h(\|x\|_X)\|x\|_X \qquad \text{for all } x \in X, \tag{4.2}$$

where $\langle \; , \; \rangle$ denotes the pairing between X and X^*.

We first observe that $\mathcal{R}$ is a closed operator (cf. Brézis [2]), and (H3) is clearly satisfied because of (F). Suppose that assumption (H2) holds. To apply the controllability theorem 3.5, we must choose a nonempty closed convex bounded subset K in X such that the set

$$\Gamma(z) \equiv \{x \in K : (\mathcal{R}x)(t) + (\Lambda x)(t) \in \mathcal{Q}(t, (Mz(t)) \text{ a.e. in } J\}$$

is nonempty for every $z \in K$.

Because of (F), we note that for any $z \in X$ and any measurable function $u: J \to U$ satisfying $u(t) \in \omega(t, (Mz)(t))$ a.e., $\|f(t, (Mz)(t), u(t))\|_* \leq \alpha(t)$ a.e. for some function $\alpha \in L_q(J, R)$. If we define $(N_u z)(t) \equiv -f(t, (Mz)(t), u(t))$, we obtain by (4.2)

$$\begin{aligned}\langle \Lambda x + N_u z, x \rangle &\geq \langle \Lambda x, x \rangle - \|\alpha\|_{L_q(J,\mathbb{R})}\|x\|_X \\ &\geq [h(\|x\|_X) - \|\alpha\|_{L_q(J,\mathbb{R})}]\|x\|_X.\end{aligned}$$

Since $h(s) \to +\infty$, as $s \to +\infty$, thus for sufficiently large $\|x\|_X$ we have $\langle \Lambda x + N_u z, x \rangle \geq 0$. We choose $K \equiv \{x \in X: \|x\|_X \leq C\}$ where C is large enough to make $\langle \Lambda x + N_u z, x \rangle \geq 0$ for all x in the boundary ∂K of K. The set K is independent of z and u.

As in the proof of theorem 3.5, the multifunction $S_z: J \to 2^U$ defined by $S_z(t) \equiv \omega(t, (Mz)(t))$ is measurable with closed values for each z in K. Thus by the Kuratowski and Ryll-Nardzewski's measurable selection theorem [14], S_z has a measurable selection u_z.

If we can show that, for any $z \in K$, the equation

$$\mathcal{R}x + \Lambda x + N_{u_z} z = 0, \tag{4.3}$$

where $(N_{u_z} z)(t) = -f(t, (Mz)(t), u_z(t))$, has a solution $x \in K$, then the set $\Gamma(z)$ is nonempty. That (4.3) has a solution in K follows immediately from an existence theorem for nonlinear equations due to Kemochi [13, theorem 2], since $\langle \Lambda x + N_{u_z} z, x \rangle \geq 0$ for all $x \in \partial K$. Thus we have just proved, by means of theorem 3.5, the following result.

THEOREM 4.1. With the hypotheses (H2), (F) and $\mathcal{R}$, Λ as given above, the system (4.1a, b) admits an admissible pair.

COROLLARY 4.2. With the hypotheses of theorem 4.1, the set Φ of all admissible trajectories in K of the system (4.1ab) is sequentially weakly compact in X.

Proof. Clearly Φ is relatively sequentially weakly compact in X as it is contained in the bounded subset K of the reflexive space X. Let $\{x_n\} \subset \Phi$ and suppose that $x_n \rightharpoonup x$. The proof will be complete if we show that $x \in \Phi$. Clearly $x \in K$. Since $x_n \in \Phi$, x_n is then a fixed point of Γ. That is, $x_n \in \Gamma(x_n)$. Since Γ has closed graph in $X_w \times X_w$, we have $x \in \Gamma(x)$, or equivalently

$$(\mathcal{R}x)(t) + (\Lambda x)(t) \in \mathcal{Q}(t, (Mx)(t))$$

almost everywhere in J. Using the same argument as in the proof of theorem 3.5, we may find a measurable control u for which (x, u) is admissible. Therefore $x \in \Phi$, proving Φ is sequentially weakly compact. ■

Using corollary 4.2, existence theorems for Mayer and Lagrange type optimization problems may be stated. We shall not go into detail here.

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