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HENKIN QUANTIFIERS AND COMPLETE PROBLEMS

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We analyze computational aspects of partially ordered quantification in first-order logic. show that almost any non-linear quantifier, applied to quantifier-free first-order formu suffices to express an $\mathcal{N}\mathcal{P}$-complete predicate. The remaining non-linear quantifiers exp exactly co-$\mathcal{N}\mathcal{L}$ predicates.

Introduction

Certain applications of logic to computer science, for example those discus in [4], have the following features: only finite models are of interest, first-order logic is not sufficiently expressive. Accordingly, various extensions first-order logic have been proposed and analyzed for finite structures. analysis often reveals connections with computational complexity, and ind some of the extensions were proposed just for the purpose of obtaining s connections. (See [4], [7] and the works cited there.) It is natural to consider those extensions of first-order logic which appear in the logical literature whose introduction was motivated by considerations quite distant from comp science.

One such extension, perhaps the oldest that can be meaningfully applie finite models, is the logic with partially ordered quantification introduce Henkin [6]. We show in this paper that formulas obtained from first-order by application of these so-called Henkin quantifiers are able to express complete properties of finite structures, for example 3-colorability of f graphs. This expressive power persists even if rather stringent restrictions imposed on the Henkin quantifiers. However, Henkin quantifiers of extremely restricted sort, which we call narrow Henkin quantifiers, express, applied to first-order formulas, exactly the co-$\mathcal{N}\mathcal{L}$ properties of finite structu

We use the customary notations and conventions of first-order logic equality, except that we also allow Boolean variables, $\alpha, \beta, \ldots,$ which over {**true, false**}. These can easily be eliminated by coding, but the coding

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make some formulas unnecessarily complex. Formally, we work with a two-sorted language whose second sort is to be interpreted as {**true, false**} in all structures and therefore need never be specified when we define a structure. We can view predicates as functions whose values are of this second sort.

We frequently consider algorithms taking finite structures S (for a finite vocabulary σ) as inputs. In such situations, we always assume that the universe of S is $\{0, 1, \dots, n-1\}$, for the appropriate natural number n, and that the structure is presented in some standard format whose length and n are bounded by polynomials of each other.

1. What are Henkin quantifiers?

In [6], Henkin introduced several new sorts of quantifiers, including partially ordered quantifiers as in the expression

$$\begin{pmatrix} \forall x_1 & \exists y_1 \\ \forall x_2 & \exists y_2 \end{pmatrix} \phi(x_1, x_2, y_1, y_2) \tag{1}$$

This expression means that for every x_1 there is a y_1 and for every x_2 there is a y_2 depending only on x_2 such that $\phi(x_1, x_2, y_1, y_2)$. More formally, it means that there are functions $Y_1(x_1)$ and $Y_2(x_2)$ such that

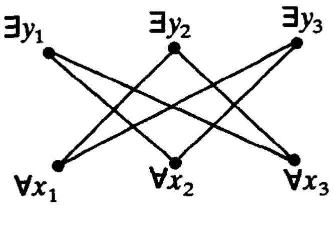
$$\forall x_1 \, \forall x_2 \, \phi(x_1, x_2, Y_1(x_1), Y_2(x_2)).$$

Henkin asked whether (1) is expressible in first-order logic (if ϕ is) and, if not, whether first-order logic enriched by

$$\begin{pmatrix} \forall x_1 & \exists y_1 \\ \forall x_2 & \exists y_2 \end{pmatrix}$$

is recursively axiomatizable. Ehrenfeucht answered both questions negatively by showing how to express the quantifier "for infinitely many x" in this enriched logic; his solution is given in [6].

Henkin quantifiers were extensively studied by Walkoe [10], who considered them to be strict partial orders $Q = \langle A_Q, <_Q \rangle$ where A_Q is a finite set of ordinary first-order quantifiers and $<_Q$ describes the dependence relationships between the existential and universal quantifiers in A_Q. The meaning of such a quantifier Q is defined, as for (1), by Skolemization. For example, if $A_Q = \{\forall x_1, \forall x_2, \forall x_3, \exists y_1, \exists y_2, \exists y_3\}$ and $<_Q$ is given by $\forall x_i <_Q \exists y_j$ when $i \neq j$ as in the figure



then $Q\ \phi(x_1, x_2, x_3, y_1, y_2, y_3)$ means that there are binary functions Y_1, Y_2, Y_3 such that

$$\forall x_1 \forall x_2 \forall x_3\ \phi(x_1, x_2, x_3, Y_1(x_2, x_3), Y_2(x_1, x_3), Y_3(x_1, x_2)).$$

Notice that the only relevant information in $<_Q$ is what universal quantifiers precede what existential quantifiers. Walkoe calls this the essential order of Q, while Henkin considers, in effect, only orders where all of $<_Q$ is essential. The linearly ordered quantifier strings of ordinary first-order logic correspond to the linear Henkin quantifiers, in which $<_Q$ is linear or (since we can ignore the inessential part of $<_Q$) the existential quantifiers are linearly pre-ordered by the inclusion relation on their sets of universal predecessors.

Following Walkoe, we call a Henkin quantifier standard if A_Q is the disjoint union of chains (with respect to $<_Q$) in each of which the last member is existential and all the others are universal. Such a quantifier, like that in (1), can be written as a sequence of rows each of which consists of universal quantifiers followed by a single existential quantifier. It is not difficult to express an arbitrary Henkin quantifier by means of a standard one. For example, if Q is the six-element quantifier above, then $Q\ \phi(x_1, x_2, x_3, y_1, y_2, y_3)$ is equivalent to

$$\begin{pmatrix} \forall x_2\ \forall x_3\ \exists y_1 \\ \forall x_1\ \forall x_3'\ \exists y_2 \\ \forall x_1'\ \forall x_2'\ \exists y_3 \end{pmatrix} [x_1 = x_1' \wedge x_2 = x_2' \wedge x_3 = x_3' \rightarrow \phi(x_1, x_2, x_3, y_1, y_2, y_3)].$$

The concatenation of two Henkin quantifiers is equivalent to a single Henkin quantifier. For example,

$$\begin{pmatrix} \forall x_1\ \exists y_1 \\ \forall x_2\ \exists y_2 \end{pmatrix} \begin{pmatrix} \forall x_3\ \exists y_3 \\ \forall x_4\ \exists y_4 \end{pmatrix}$$

is equivalent to the quantifier Q with domain $\{\forall x_1, \forall x_2, \forall x_3, \forall x_4, \exists y_1, \exists y_2, \exists y_3, \exists y_4\}$ and

$\exists y_1$ preceded by $\forall x_1$,
$\exists y_2$ preceded by $\forall x_2$,
$\exists y_3$ preceded by $\forall x_1$, $\forall x_2$, and $\forall x_3$, and
$\exists y_4$ preceded by $\forall x_1$, $\forall x_2$, and $\forall x_4$.

It follows easily that any formula built according to the formation rules of first-order logic but allowing positive occurrences of Henkin quantifiers can be rewritten as a single standard Henkin quantifier followed by a quantifier-free first-order formula. Walkoe showed [10, Theorem 4.3] that the expressive power of such formulas is exactly the same as the expressive power of existential second-order formulas. For example, the sentence

$$\exists R\ \forall xyz\ [(R(x, y) \rightarrow R(x \cdot z, y \cdot z)) \wedge R(a, b) \wedge \neg R(b, a)]$$

can be rewritten to make the arguments of R simply variables,

$$\begin{aligned}\exists R\ \forall xyzuv[(R(x,y) \wedge u = x\cdot z \wedge v = y\cdot z \rightarrow R(u,v))\\ \wedge (x = a \wedge y = b \rightarrow R(x,y))\\ \wedge (x = b \wedge y = a \rightarrow \neg R(x,y))],\end{aligned}$$

and can then be written in Henkin quantifier form as

$$\begin{pmatrix}\forall x\ \forall y\ \exists\alpha\\ \forall u\ \forall v\ \exists\beta\\ \forall z\end{pmatrix}\begin{aligned}&[(x = u \wedge y = v \rightarrow \alpha = \beta)\\ &\wedge (\alpha \wedge u = x\cdot z \wedge v = y\cdot z \rightarrow \beta)\\ &\wedge (x = a \wedge y = b \rightarrow \alpha) \wedge (x = b \wedge y = a \rightarrow \neg\alpha)].\end{aligned}$$

Here α and β are Boolean variables. So the associated Skolem functions are actually binary predicates, and the conjunct $(x = u \wedge y = v \rightarrow \alpha = \beta)$ forces them to be the same predicate R. The remaining conjuncts are essentially copied from the previous version of the formula, and the last row of the quantifier should, if we insist on standard form, have a dummy existential quantifier added. The same rewriting procedure can be applied to any second-order existential formula once it has been put into Skolem form so that all its first-order quantifiers are universal.

In Theorem 1 below, we apply this result of Walkoe to give the simplest connection between Henkin quantifiers and complexity theory. Before stating the theorem, we introduce some useful terminology.

By an l-ary *global predicate* for a vocabulary σ, we mean a function π assigning to each finite σ-structure S an l-ary predicate $\pi^S : S^l \rightarrow \{\mathbf{true}, \mathbf{false}\}$. The decision problem for π is the problem specified by:

Instance. A finite σ-structure S and $\bar{a} \in S^l$.

Question. Does $\pi^S(\bar{a})$ hold?

In particular, any formula ϕ with l free variables defines an l-ary global predicate. When $l = 0$, the decision problem for this global predicate is the problem of deciding, for any given structure, S, whether the sentence ϕ is true in S.

Theorem 1. *For any global predicate π, the following are equivalent.*

(1) *The decision problem for π is in $\mathcal{NP}$.*

(2) *π is expressible by an existential second-order formula.*

(3) *π is expressible by a formula $Q\phi$ where Q is a Henkin quantifier and ϕ is a quantifier-free first-order formula.*

Proof. The equivalence (1) ↔ (2) is a theorem of Fagin [2], and the equivalence (2) ↔ (3) is the theorem of Walkoe cited above. □

2. Henkin quantifiers and $\mathcal{NP}$-complete problems

Theorem 1 implies in particular that the global predicate defined by a formula $Q\phi$, with Q a Henkin quantifier and ϕ a quantifier-free first-order formula, can be $\mathcal{NP}$-complete. In this section, we investigate how complicated Q must be in order for this phenomenon to occur. Useful measures of the complexity of Q, for this purpose, are the number of rows (if Q is standard) and the type — individual or Boolean — of the existentially quantified variables. For the sake of brevity, we call a Henkin quantifier *mighty* if, by applying it to a quantifier-free first-order formula, one can define an $\mathcal{NP}$-complete global predicate. Linear Henkin quantifiers are, of course, not mightly unless $\mathcal{L} = \mathcal{P} = \mathcal{NP}$, since first-order definable predicates are in $\mathcal{L}$ (logspace computable). We shall show in this section that almost any non-linear Henkin quantifier is mighty.

In the first place, as long as the existentially quantified variables are individual ones ranging over the universe of the structure, rather than Boolean ones, all non-linear Henkin quantifiers are mighty.

Theorem 2. *The quantifier* $\begin{pmatrix} \forall x_1 & \exists y_1 \\ \forall x_2 & \exists y_2 \end{pmatrix}$ *is mighty.*

Proof. In a vocabulary σ with a binary predicate symbol E and three constant symbols 0, 1, 2, let ϕ be the conjunction of

$$x_1 = x_2 \rightarrow y_1 = y_2,$$
$$y_1 = 0 \vee y_1 = 1 \vee y_1 = 2, \quad \text{and}$$
$$Ex_1x_2 \rightarrow y_1 \neq y_2.$$

Then, in any σ-structure where 0, 1, 2 denote distinct elements,

$$\begin{pmatrix} \forall x_1 & \exists y_1 \\ \forall x_2 & \exists y_2 \end{pmatrix} \phi$$

holds if and only if the graph defined by E is 3-colorable. It is well-known [3] that 3-colorability is an $\mathcal{NP}$-complete problem. □

Observe that, in the proof of Theorem 2, the existentially quantified variables ranged, in effect, over only a three-element set, because of the second conjunct in ϕ. If we introduce 'almost Boolean' variables μ, ν to range over $\{0, 1, 2\}$, then the same proof shows:

Corollary. $\begin{pmatrix} \forall x_1 & \exists \mu \\ \forall x_2 & \exists \nu \end{pmatrix}$ *is mighty.* □

It is natural to try to obtain a similar result with Boolean variables α, β in place of the almost Boolean ones. We shall see later that this cannot be done without proving $\mathcal{NL} = \mathcal{NP}$. In the next two theorems, however, we almost achieve this replacement.

Theorem 3. $\begin{pmatrix} \forall x & \exists\alpha \\ \forall y & \exists\beta \\ \forall z & \exists\gamma \end{pmatrix}$ *is mighty.*

Proof. Think of a Boolean circuit built from binary NAND-gates (where NAND means negated conjunction) as being a structure for the vocabulary σ that consists of a constant symbol c (designating the output node) and a binary relation symbol I (where $I(x, y)$ means that node x produces one of the two inputs to node y). We assume, without loss of generality, that the two inputs two each NAND-gate come from distinct nodes; at worst we just make two copies of every node. Then the $\mathcal{NP}$-complete satisfiability problem — decide whether such a circuit has output **true** for some inputs — is given by the formula

$$\begin{pmatrix} \forall x & \exists\alpha \\ \forall y & \exists\beta \\ \forall z & \exists\gamma \end{pmatrix} \phi$$

where ϕ is the conjunction of

$$x = y \rightarrow \alpha = \beta, \quad y = z \rightarrow \beta = \gamma, \quad z = x \rightarrow \gamma = \alpha,$$

$$I(x, z) \wedge I(y, z) \wedge x \neq y \rightarrow \gamma = \neg(\alpha \wedge \beta), \quad \text{and} \quad x = c \rightarrow \alpha.$$

(The third of the five conjuncts is redundant but was included for symmetry.) □

Theorem 4. $\begin{pmatrix} \forall x & \exists\alpha \\ \forall y & \exists\mu \end{pmatrix}$ *is mighty.*

Proof. Think of Boolean circuits as in the previous proof except that, for each NAND-gate, we distinguish a left and a right input. Thus, σ now contains c as before and two binary predicates L and R whose disjunction would be the previous I. The satisfiability problem is described by

$$\begin{pmatrix} \forall x & \exists\alpha \\ \forall y & \exists\mu \end{pmatrix} \phi$$

where ϕ is the conjunction of

$$x = y \rightarrow (\alpha \leftrightarrow (\mu \neq 0)),$$

$$x = c \rightarrow \alpha$$

$$(\neg\alpha) \wedge (L(x, y) \vee R(x, y)) \rightarrow \mu \neq 0,$$

$$\alpha \wedge L(x, y) \rightarrow \mu \neq 1, \quad \text{and}$$
$$\alpha \wedge R(x, y) \rightarrow \mu \neq 2.$$

To see this, note that our formula asserts the existence of a function $\alpha = f(x)$ from the set S of nodes to {**true, false**} (which is to describe the truth value computed at each node) and an auxiliary function $\mu = g(y)$ from S to $\{0, 1, 2\}$ with the following properties. First f is **false** wherever g is 0 and f is **true** wherever g is 1 or 2, so g completely determines f. Second, the value of f at the output node c is **true**. And finally certain values of $g(y)$ are prohibited if f had certain values at the two inputs to y. The prohibitions resulting from each of the last three conjuncts of ϕ are shown in Table 1, along with the permitted values of $g(y)$ and $f(y)$.

Table 1

		prohibited by conjunct				
f (left input)	f (right input)	(3)	(4)	(5)	$g(y)$	$f(y)$
true	**true**		1	2	0	**false**
true	**false**	0	1		2	**true**
false	**true**	0		2	1	**true**
false	**false**	0			1 or 2	**true**

So our formula requires the NAND-gates to produce the correct outputs. □

3. Narrow Henkin quantifiers and nondeterministic log space computability

In this section we study 'narrow' Henkin quantifiers of the form

$$\begin{pmatrix} \forall \boldsymbol{x} & \exists \alpha \\ \forall \boldsymbol{y} & \exists \beta \end{pmatrix}$$

where α, β are, as before, Boolean variables while $\boldsymbol{x}$ and $\boldsymbol{y}$ are tuples of individual or Boolean variables. We say that $\boldsymbol{x}$ and $\boldsymbol{y}$ are compatible if they have the same length and have variables of the same type at corresponding positions. We write $\boldsymbol{x} = \boldsymbol{y}$ as an abbreviation for the conjunction of the equations $x_i = y_i$ between corresponding components of $\boldsymbol{x}$ and $\boldsymbol{y}$ provided $\boldsymbol{x}$ and $\boldsymbol{y}$ are compatible; if they are incompatible, then $\boldsymbol{x} = \boldsymbol{y}$ means **false**. For future reference, we exhibit the definition, i.e. the Skolem form, of

$$\begin{pmatrix} \forall \boldsymbol{x} & \exists \alpha \\ \forall \boldsymbol{y} & \exists \beta \end{pmatrix} \phi(\boldsymbol{x}, \boldsymbol{y}, \alpha, \beta);$$

it is

$$\exists A \, \exists B \, \forall \boldsymbol{x} \, \forall \boldsymbol{y} \, \phi(\boldsymbol{x}, \boldsymbol{y}, A(\boldsymbol{x}), B(\boldsymbol{y})).$$

If $\boldsymbol{x}$ and $\boldsymbol{y}$ are compatible and ϕ has the 'equality bound' form $(\boldsymbol{x}=\boldsymbol{y}\rightarrow \alpha=\beta)\wedge \psi(\boldsymbol{x}, \boldsymbol{y}, \alpha, \beta)$, then this definition reduces to

$$\exists A\, \forall \boldsymbol{x}\, \forall \boldsymbol{y}\, \psi(\boldsymbol{x}, \boldsymbol{y}, A(\boldsymbol{x}), A(\boldsymbol{y})).$$

It will be convenient for us to view 2-SAT, the satisfiability problem for propositional formulas in conjunctive normal form with only two disjuncts in each conjunct, as a quantifier. Given propositional variables $p_0, \ldots, p_{n-1}$, consider a conjunction C of 2-disjunctions of the forms $p_i \vee p_j$, $p_i \vee \neg p_j$, $\neg p_i \vee p_j$, and $\neg p_i \vee \neg p_j$. If we write p and $\neg p$ in the equivalent forms

$$(\textbf{true} \leftrightarrow p) \quad \text{and} \quad (\textbf{false} \leftrightarrow p),$$

then C can be described by a predicate R on $\{0,1,\ldots,n-1\}$, namely $R(x, y, \alpha, \beta) \leftrightarrow C$ has a conjunct $(\alpha \leftrightarrow p_x) \vee (\beta \leftrightarrow p_y)$. Conversely, with each such quaternary predicate R we associate the Boolean formula

$$\bigwedge \{(\alpha \leftrightarrow p_x) \vee (\beta \leftrightarrow p_y) : R(x, y, \alpha, \beta)\}.$$

Let $(\text{2-SAT}\, x, y, \alpha, \beta)\, R(x, y, \alpha, \beta)$ mean that this Boolean formula is satisfiable. More generally, let

$$(\text{2-SAT}\, \boldsymbol{x}, \boldsymbol{y}, \alpha, \beta)\, \phi(\boldsymbol{x}, \boldsymbol{y}, \alpha, \beta)$$

be interpreted as true in a structure S if and only if the Boolean formula

$$\bigwedge \{(\alpha \leftrightarrow p_{\boldsymbol{x}}) \vee (\beta \leftrightarrow p_{\boldsymbol{y}}) : \phi(\boldsymbol{x}, \boldsymbol{y}, \alpha, \beta) \text{ holds in } S\}$$

is satisfiable.

Satisfiability of such a Boolean formula means that there exist suitable truth values for all the propositional variables $p_{\boldsymbol{x}}$ and $p_{\boldsymbol{y}}$. These truth values define predicates A and B on S, where $A(\boldsymbol{x})$ holds if and only if $p_{\boldsymbol{x}}$ was assigned the value **true**, and similarly for B. (If $\boldsymbol{x}$ and $\boldsymbol{y}$ are compatible then A and B must coincide.) Thus, the formula

$$(\text{2-SAT}\, \boldsymbol{x}, \boldsymbol{y}, \alpha, \beta)\, \phi(\boldsymbol{x}, \boldsymbol{y}, \alpha, \beta)$$

is equivalent to

$$\exists A\, \exists B\, \forall \boldsymbol{x}\, \forall \boldsymbol{y}\, [(\boldsymbol{x}=\boldsymbol{y}\rightarrow (A(\boldsymbol{x}) \leftrightarrow B(\boldsymbol{y}))) \wedge \neg\phi(\boldsymbol{x}, \boldsymbol{y}, \neg A(\boldsymbol{x}), \neg B(\boldsymbol{y}))].$$

If $\boldsymbol{x}$ and $\boldsymbol{y}$ are compatible, this reduces to

$$\exists A\, \forall \boldsymbol{x}\, \forall \boldsymbol{y}\, \neg\phi(\boldsymbol{x}, \boldsymbol{y}, \neg A(\boldsymbol{x}), \neg A(\boldsymbol{y})).$$

The main result of this section is that narrow Henkin quantifiers are equivalent, in a suitable sense, to the quantifiers 2-SAT.

Theorem 5. *Narrow Henkin quantifiers and 2-satisfiability quantifiers are each* *expressible by means of the others. More precisely:*

(i) $\begin{pmatrix}\forall x\ \exists\alpha\\ \forall y\ \exists\beta\end{pmatrix}\phi(x,y,\alpha,\beta)$ *is equivalent to*

$$(\text{2-SAT}\,x\gamma, y\delta, \alpha, \beta)\,(\gamma\wedge\neg\delta\wedge\neg\phi(x,y,\neg\alpha,\neg\beta))$$

(ii) $(\text{2-SAT}\,x, y, \alpha, \beta)\,\phi(x,y,\alpha,\beta)$ *is equivalent to*

$$\begin{pmatrix}\forall x\ \exists\alpha\\ \forall y\ \exists\beta\end{pmatrix}[(x=y\rightarrow\alpha=\beta)\wedge\neg\phi(x,y,\neg\alpha,\neg\beta)].$$

Proof. Part (ii) is immediate, by comparison of the existential second-order forms given above for 2-SAT and for narrow Henkin quantifiers. For part (i), we have the following list of formulas in which the equivalence of each consecutive pair is clear either from the discussion above or from pure logic.

$$\begin{aligned}
&(\text{2-SAT}\,x\gamma, y\delta, \alpha, \beta)\,(\gamma\wedge\neg\delta\wedge\neg\phi(x,y,\neg\alpha,\neg\beta)),\\
&\exists A\,\exists B\,\forall x\gamma\,\forall y\delta\,[(x\gamma=y\delta\rightarrow(A(x\gamma)\leftrightarrow B(y\delta)))\\
&\qquad\qquad\wedge\neg(\gamma\wedge\neg\delta\wedge\neg\phi(x,y,A(x\gamma),B(y\delta)))],\\
&\exists A\,\exists B\,\forall x\gamma\,\forall y\delta\,[(x\gamma=y\delta\rightarrow(A(x\gamma)\leftrightarrow B(y\delta)))\\
&\qquad\qquad\wedge(\gamma\wedge\neg\delta\rightarrow\phi(x,y,A(x\gamma),B(y\delta)))],\\
&\exists A\,\exists B\,\forall x\,\forall y\,\phi(x,y,A(x\ \mathbf{true}),B(y\ \mathbf{false})),\\
&\exists A'\,\exists B'\,\forall x\,\forall y\,\phi(x,y,A'(x),B'(y)),\\
&\begin{pmatrix}\forall x\ \exists\alpha\\ \forall y\ \exists\beta\end{pmatrix}\phi(x,y,\alpha,\beta).\qquad\square
\end{aligned}$$

The following corollary is obtained by combining Theorem 5 with the well-known result [8] that the *unsatisfiability* problem for conjunctive normal forms with two literals per clause is complete, with respect to log space computable reductions, for the class $\mathcal{NL}$ of languages nondeterministically recognizable in log space.

Corollary. *All global predicates defined by formulas of the form*

$$\begin{pmatrix}\forall x\ \exists\alpha\\ \forall y\ \exists\beta\end{pmatrix}\phi(x,y,\alpha,\beta),$$

with first-order ϕ, *are in* co-$\mathcal{NL}$, *and some of them, even with quantifier-free* ϕ, *are complete for* co-$\mathcal{NL}$. □

In particular, the narrow Henkin quantifier is not mighty unless co-$\mathcal{NL}=\mathcal{NP}$ (which would imply $\mathcal{NL}=\mathcal{P}=\mathcal{NP}$).

In part (ii) of Theorem 5, we showed how to express 2-SAT by means of an *equality bound* narrow Henkin quantifier, i.e., the first conjunct after the quantifier requires the two Skolem functions to coincide. Combining this observation with part (i), we find that every narrow Henkin quantifier can be expressed by an equality bound one. Indeed, we find that

$$\begin{pmatrix}\forall x & \exists\alpha\\ \forall y & \exists\beta\end{pmatrix}\phi(\boldsymbol{x},\boldsymbol{y},\alpha,\beta)$$

is equivalent to

$$\begin{pmatrix}\forall \boldsymbol{x}\gamma & \exists\alpha\\ \forall \boldsymbol{y}\delta & \exists\beta\end{pmatrix}[(\boldsymbol{x}\gamma=\boldsymbol{y}\delta\rightarrow\alpha=\beta)\wedge(\gamma\wedge\neg\delta\rightarrow\phi(\boldsymbol{x},\boldsymbol{y},\alpha,\beta))].$$

This is easily verified directly (without 2-SAT), and the same method can be used to change any Henkin quantifier into an equality bound one, at the cost of introducing additional universally quantified Boolean variables.

Immerman [7] extended first-order logic by adding various operators, including the transitive closure operator TC, and obtained connections between these extended logics and various complexity classes. In the next theorem, we exhibit a close connection between TC and narrow Henkin quantifiers. Combining this connection with Immerman's results, we obtain a corollary characterizing, more precisely than Theorem 5 does, the expressive power of narrow Henkin quantifiers in logic with linear order.

Following Immerman, we write TC $\psi(\boldsymbol{x},\boldsymbol{y})$, with compatible k-tuples $\boldsymbol{x}$ and $\boldsymbol{y}$, for the reflexive transitive closure of the binary relation on k-tuples defined by the formula ψ. That is, TC $\psi(\boldsymbol{x},\boldsymbol{y})$ is equivalent to the disjunction, over all $n\geqslant 0$, of the formulas

$$\exists z_0\cdots z_n(\boldsymbol{x}=z_0\wedge\psi(z_0,z_1)\wedge\psi(z_1,z_2)\wedge\cdots\wedge\psi(z_{n-1},z_n)\wedge z_n=\boldsymbol{y});$$

it is also equivalent to the second-order formula

$$\neg\exists A\,[A(\boldsymbol{x})\wedge\neg A(\boldsymbol{y})\wedge\forall\boldsymbol{u}\,\forall\boldsymbol{v}\,(A(\boldsymbol{u})\wedge\psi(\boldsymbol{u},\boldsymbol{v})\rightarrow A(\boldsymbol{v}))].$$

(Think of A as the set accessible from $\boldsymbol{x}$.) The notation TC $\psi(\boldsymbol{x},\boldsymbol{y})$ is poorly adapted to situations where ψ contains additional variables as parameters or where terms are to be substituted for $\boldsymbol{x}$ and $\boldsymbol{y}$. For example, if E is the edge relation of a graph, then TC $E(x,y)$ means x is connected to y, so, to say that $F(x)$ is connected to $F(y)$, where F is a function from vertices to vertices, one would write TC $E(F(x),F(y))$; but this last formula also represents the transitive closure of the relation "$F(x)$ is adjacent to $F(y)$", and this transitive closure amounts to "$F(x)$ is connected to $F(y)$ by a path all of whose vertices are in the range of F". Ambiguities like this can be avoided either by adopting a more accurate but longer notation, as in [4], or by *ad hoc* conventions. For our present purposes, it will suffice to adopt the convention that, if $\boldsymbol{x}$ and $\boldsymbol{y}$ are the specified k-tuples of free variables of ψ, then TC $\psi(\boldsymbol{p},\boldsymbol{q})$, for terms $\boldsymbol{p}$, $\boldsymbol{q}$, means the result of substituting $\boldsymbol{p}$ for $\boldsymbol{x}$ and $\boldsymbol{q}$ for $\boldsymbol{y}$ in TC $\psi(\boldsymbol{x},\boldsymbol{y})$, not the transitive closure of $\psi(\boldsymbol{p},\boldsymbol{q})$.

Theorem 6. *Narrow Henkin quantifiers and transitive closures can be expressed in terms of each other; positive occurrences of either can be expressed by negative occurrences of the other.*

Proof. To express TC in terms of a narrow Henkin quantifier, it suffices to rewrite the second-order form given above of the definition of TC as

$$\neg\begin{pmatrix}\forall u & \exists\alpha\\ \forall v & \exists\beta\end{pmatrix}\begin{array}{l}[(u=v\to\alpha=\beta)\wedge(u=x\to\alpha)\wedge(u=y\to\neg\alpha)\\ \quad\wedge(\alpha\wedge\psi(u,v)\to\beta)].\end{array}$$

For the other direction, we note first that any narrow Henkin quantifier can be expressed by one whose tuples of universally quantified variables are compatible:

$$\begin{pmatrix}\forall x & \exists\alpha\\ \forall y & \exists\beta\end{pmatrix}\phi\leftrightarrow\begin{pmatrix}\forall xy' & \exists\alpha\\ \forall x'y & \exists\beta\end{pmatrix}\phi,$$

where x' and y' are tuples of dummy variables compatible with x and y respectively. By a remark above, we can also assume, without loss of generality, that we are dealing with an equality-bound quantifier. We therefore wish to express, in terms of (negatively occurring) TC, a formula

$$\exists A\,\forall x\,\forall y\,\phi(x, A(x), y, A(y)).$$

Think of this formula as asserting the existence of an assignment A of truth values $A(x)$ to all tuples x of the appropriate type, such that all instances of $\phi(x, A(x), y, A(y))$ are true. If, for certain x, α, y, and β, $\phi(x, \alpha, y, \neg\beta)$ is false, and if we assign x the value α, then we must not assign y the value $\neg\beta$, for then we would not satisfy $\phi(x, A(x), y, A(y))$, so the value α for x forces the value β for y. The same holds if $\phi(y, \neg\beta, x, \alpha)$ is false. Let $\psi(x, \alpha, y, \beta)$ be the formula

$$\neg\phi(x, \alpha, y, \neg\beta)\vee\neg\phi(y, \neg\beta, x, \alpha);$$

whenever it holds, the value α for x forces the value β for y. Since this 'forcing' is clearly a reflexive transitive relation on tuples $x\alpha$, we see that $x\alpha$ forces $y\beta$ whenever TC $\psi(x, \alpha, y, \beta)$ holds (where $x\alpha$ and $y\beta$ are the specified tuples of variables for the application of TC). In particular, if TC $\psi(x, \alpha, x, \neg\alpha)$ then x cannot be assigned the value α, and therefore, for an assignment A of the desired sort to exist, it is necessary that

$$\neg\,\exists x\,\exists\alpha\,[\mathrm{TC}\,\psi(x, \alpha, x, \neg\alpha)\wedge\mathrm{TC}\,\psi(x, \neg\alpha, x, \alpha)].$$

We complete the proof of Theorem 6 by showing that this necessary condition is also sufficient.

First, make the preliminary observation that $\psi(x, \alpha, y, \beta)$ and therefore also TC $\psi(x, \alpha, y, \beta)$ are invariant under the operation of interchanging x and y while simultaneously interchanging and negating α and β.

In any structure, TC ψ defines a pre-ordering $\leqslant$ (i.e. a reflexive transitive relation) on the tuples $x\alpha$. Negating the last component of a tuple defines an anti-automorphism of this pre-order (by the preliminary observation) of order two. Furthermore if the structure satisfies the necessary condition above, then we never have $\boldsymbol{x}\alpha$ and $\boldsymbol{x}\neg\alpha$ each $\leqslant$ the other.

Lemma. *Let $(X, \leqslant)$ be a finite pre-ordered set, and let $i: X \rightarrow X$ be an order-reversing function such that every $x \in X$ satisfies $i(i(x)) = x$ and either $x \nleqslant i(x)$ or $i(x) \nleqslant x$ (or both). Then there is a subset T of X, closed upward for $\leqslant$, containing exactly one of x and $i(x)$ for each $x \in X$.*

Granting the lemma for the moment, we apply it to the set of tuples $\boldsymbol{x}\alpha$, pre-ordered according to TC ψ, with $i(\boldsymbol{x}\alpha) = \boldsymbol{x}\neg\alpha$. We obtain a set T as in the conclusion of the lemma, and we define $A(\boldsymbol{x})$ to be **true** if and only if $\boldsymbol{x}$ **true** $\in T$, or equivalently $\boldsymbol{x}$ **false** $\notin T$. Thus, we always have $\boldsymbol{x}A(\boldsymbol{x}) \in T$ and $\boldsymbol{x}\neg A(\boldsymbol{x}) \notin T$. This assignment A has the desired property that $\phi(\boldsymbol{x}, A(\boldsymbol{x}), \boldsymbol{y}, A(\boldsymbol{y}))$ holds for all $\boldsymbol{x}$ and $\boldsymbol{y}$. To see this, suppose $\boldsymbol{x}$ and $\boldsymbol{y}$ were a counter-example, i.e., $\neg\phi(\boldsymbol{x}, A(\boldsymbol{x}), \boldsymbol{y}, A(\boldsymbol{y}))$ holds. Then $\psi(\boldsymbol{x}, A(\boldsymbol{x}), \boldsymbol{y}, \neg A(\boldsymbol{y}))$ would hold and, as ψ implies TC ψ, we would have $\boldsymbol{x}A(\boldsymbol{x}) \leqslant \boldsymbol{y}\neg A(\boldsymbol{y})$. But $\boldsymbol{x}A(\boldsymbol{x}) \in T$, $\boldsymbol{y}\neg A(\boldsymbol{y}) \notin T$, and T is closed upward, a contradiction. This completes the proof of Theorem 6, assuming the lemma.

Proof of Lemma. We proceed by induction on the number of elements of X, the case $X = \emptyset$ being vacuously true. If $X \neq \emptyset$, consider an arbitrary $x \in X$. Replacing x with $i(x)$ if necessary, we may assume $x \nleqslant i(x)$. Then we decide to put into (resp. out of) T all elements $\geqslant x$ (resp. $\leqslant i(x)$); there is no conflict here, as $x \nleqslant i(x)$. Note that an element y has been put into T if and only if $i(y)$ has been put out of T. The set Y of elements whose membership is still in doubt is smaller than X (as $x \notin Y$). Apply the induction hypothesis to Y, with the restrictions of $\leqslant$ and i, to get $T' \subseteq Y$ satisfying the conclusion of the lemma for Y. Finally put all members of T' into T and put all members of $Y - T'$ out of T. It is easy to verify that T has all the required properties. □

Remark. A compactness argument shows that the lemma holds without the hypothesis that X is finite.

By 'logic with linear order', we mean logic with an additional logical symbol $\leqslant$, which functions syntactically as a binary predicate symbol, and which is interepreted in every structure as the usual ordering of the universe $\{0, 1, \ldots, n-1\}$. Immerman [7] proved that the global predicates definable in first-order logic with linear order augmented by TC are exactly those that are in the log-space hierarchy Σ_*^L. He also proved that, if we allow only positive occurrences of TC, then exactly the $\mathcal{NL}$ global predicates are expressible.

The following corollary is obtained by combining these results with Theorem 6.

Proof. If the condition is satisfied, then Q has at least the expressive power of

$$\text{either} \quad \begin{pmatrix} \forall x \ \exists \alpha \\ \forall y \ \exists \beta \\ \forall z \ \exists \gamma \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \forall x \ \exists \alpha \\ \forall y \ \exists \mu \end{pmatrix}$$

and is therefore mighty by the results of Section 2. If the condition is not satisfied, then by eliminating universally quantified Boolean variables as above, from rows without universal quantifiers, we can rewrite any formula $Q\phi$ as $Q'\phi'$ where Q' has, apart from rows consisting just of a single existential quantifier, either just one other row, or two other rows both of whose existentially quantified variables are Boolean. The existential quantifiers that are in rows without universal quantifiers can, without changing the meaning of the formula, be moved to the left, since$(\genfrac{}{}{0pt}{}{\exists x}{R})\phi$ is logically equivalent to $(\exists x)\,R\phi$. Thus, we can rewrite $Q'\phi'$ in the form $\exists x\, Q''\phi'$, where Q'' consists of those rows of Q' that contain universal quantifiers. But, because of the limitation on such rows in Q', we know that Q'' is either linear or narrow, so the global predicate defined by $Q''\phi'$ is at worst co-$\mathcal{NL}$. This complexity estimate is preserved by first-order quantification, so the global predicate defined by $\exists x\, Q''\phi'$ (and by $Q\phi$) is co-$\mathcal{NL}$. Our assumption that $\mathcal{NL} \neq \mathcal{NP}$ prevents any co-$\mathcal{NL}$ predicate from being $\mathcal{NP}$-complete, so Q is not mighty. □

(C) All our theorems have been about formulas in which a Henkin quantifier occurs at the beginning and is applied to a first-order (in fact quantifier-free) matrix. We remarked in Section 1 that formulas built using the formation rules of first-order logic but allowing positive occurrences of Henkin quantifiers can always be rewritten in the 'Henkin prenex' form that we considered. It is natural to ask about the full logic, called H by Walkoe, obtained from first-order logic by adding Henkin quantification as an additional formation rule. This logic contains formulas with negative occurrences of Henkin quantifiers, like

$$\begin{pmatrix} \forall x_1 \ \exists y_1 \\ \forall x_2 \ \exists y_2 \end{pmatrix} \neg \begin{pmatrix} \forall u_1 \ \exists v_1 \\ \forall u_2 \ \exists v_2 \end{pmatrix} \phi(x_1, x_2, y_1, y_2, y_1, u_2, v_1, v_2),$$

which, by definition of Henkin quantifiers, means

$$\exists Y_1, Y_2 \forall x_1, x_2, V_1, V_2 \exists u_1, u_2 \neg \phi(x_1, x_2, Y_1(x_1), Y_2(x_2), u_1, u_2, V_1(u_1), V_2(u_2)).$$

The global predicate defined by this formula is in the class Σ_2^P of the polynomial time hierarchy [9]. Similarly, a formula of H containing n Henkin quantifiers defines a global predicate in the nth level of the polynomial time hierarchy. But, in fact, these global predicates are all in $\Sigma_2^P \cap \Pi_2^P$, by a result of Enderton [1, Theorem 2], which also applies to infinite structures.

Moshe Vardi has pointed out to us that Harel [5] has proposed another semantics for formulas consisting of a quantifier-free matrix preceded by a sequence of Henkin quantifiers and negation symbols. Harel showed that the expressive power of this logic is exactly that of second-order logic. Although his definition looks natural, there is a price for this increased expressive power, namely a strange semantics for nested quantifiers. Whatever the pattern of negations, an existentially quantified variable depends only on universally quantified variables in the same Henkin quantifier and on the existentially quantified variables in previous Henkin quantifiers. On the other hand, a universally quantified variable depends on all variables in previous Henkin quantifiers and on the existentially quantified variables in later Henkin quantifiers. Thus, for example, the Henkin quantifier $(\forall x\ \exists y)$ is not equivalent, in Harel's semantics, to the sequence $(\forall x)(\exists y)$ of Henkin quantifiers; in fact, the latter is equivalent to $(\exists y)(\forall x)$. Note also that, whereas $(\exists y)\,R(x, y)$ and $\neg(\forall y)\neg R(x, y)$ are equivalent, $(\forall x)\,(\exists y)\,R(x, y)$ is not equivalent to $(\forall x)\neg(\forall y)\neg R(x, y)$.

(D) The semantics of Henkin quantifiers treats existential quantifiers quite differently from universal ones. This is reflected in the fact that the dual $\neg Q\neg$ of a Henkin quantifier is usually not a Henkin quantifier; applied to first-order formulas, $\neg Q\neg$ produces co-$\mathcal{NP}$ global predicates, not $\mathcal{NP}$ ones. It is natural to try to restore symmetry by returning to the symmetrical 'games' with which Henkin introduces generalized quantifiers in [6]. If $Q = \langle A_Q, <_Q\rangle$ as in Section 1, then the game for $Q\phi$ is played as follows. One player, called $\exists$, assigns values to the existentially quantified variables; his opponent, $\forall$, assigns values to the universally quantified variables. When assigning a value to a variable x, a player knows the values of only those variables whose quantifiers precede that of x in $<_Q$. A play of the game is won by $\exists$ if and only if the chosen values of the variables make ϕ true. Finally, $Q\phi$ holds if $\exists$ has a winning strategy and fails if $\forall$ has a winning strategy.

This definition can be made precise, and kept symmetrical, as follows. A strategy for $\exists$ consists of functions X_i for all the existential quantifiers $(\exists x_i)$ in Q; the argument places of X_i correspond to the quantifiers $(\forall y_j) <_Q (\exists x_i)$. Strategies for $\forall$ are defined symmetrically. Given strategies for both players, we assign a value to each variable x_i or y_j occurring in Q by applying the corresponding function X_i or Y_j to the values assigned to the variables whose quantifiers occur earlier in $<_Q$. (This recursive definition of the values assigned to variables makes sense because $<_Q$, being a partial order on a finite set, is well-founded.) The outcome, for the two given strategies, is a win for $\exists$ (resp. $\forall$) if these values for the variables make ϕ true (resp. false). Finally, $Q\phi$ is said to hold (resp. fail) if there is a strategy for $\exists$ (resp. $\forall$) which wins against every strategy for $\forall$ (resp. $\exists$).

The price we pay for the symmetry of this definition is that a formula might neither hold nor fail; that is, there might be no winning strategy for either player.

The simplest example of this phenomenon is given by

$$\begin{pmatrix} \forall x \\ \exists y \end{pmatrix} x = y$$

in any structure with at least two elements. Although unpleasant, this lack of determinacy should not be viewed as pathological; it is the usual situation for games of imperfect information.

Miklos Ajtai has suggested applying the von Neumann minimax theorem to these games. The theorem asserts the existence of a number p and a probability distribution $D_\exists$ (resp. $D_\forall$) on the set of $\exists$'s (resp. $\forall$'s) strategies such that, if $\exists$ (resp. $\forall$) chooses a strategy at random according to the distribution $D_\exists$ (resp. $D_\forall$), then he wins with probability at least p (resp. $1-p$), no matter what strategy his opponent uses. Ajtai suggested that the number p be viewed as a truth value for $Q\phi$. This truth value is 1 (resp. 0) if and only if $Q\phi$ holds (resp. fails) in the sense defined above. Formulas which neither hold nor fail have intermediate truth values; the example

$$\begin{pmatrix} \forall x \\ \exists y \end{pmatrix} x = y$$

has truth value $1/n$ in structures of cardinality n.

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