ELASTO-PLASTIC PLATE BENDING ANALYSIS BY A BOUNDARY ELEMENT METHOD WITH INITIAL PLASTIC MOMENTS

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Abstract—The analysis of elasto-plastic plate bending using a boundary integral equation formulation is described. The integral equations are solved using standard boundary element techniques. The plasticity, as well as the external lateral load, appears in a domain integral. The solution is obtained by an incremental loading procedure with the initial incremental plastic moments calculated by an iterative method. Several study cases are examined and good agreement is shown with published results obtained by the finite element method.

NOMENCLATURE

a	plate half thickness
$[A], [A'], [A^B], [A^D]$	matrices
A_m	symbol for the mth corner
$[B], [B'], [B^B], [B^D]$	matrices
$[C], [C']$	matrices
D	plate stiffness
$[D], [D']$	matrices
$[D]^e, [D]^{ep}$	matrices for stress–strain relations
$[D]^*$	matrix relating plastic and total strain
E	Young's modulus
F	yield surface
$[F^B], [F^D]$	matrices
G	shear modulus
$[G], [G']$	matrices
H'	slope of the uniaxial stress–strain curve
$[H], [H']$	matrices
J	number of boundary elements
k	hardening parameter
K	number of domain elements
L	number of corners
$M_n, M_{nt}, M'_n, M'_{nt}$	moments
M_n^p, M_{nt}^p	plastic moments
M_x^p, M_{xy}^p, M_y^p	plastic moments
$[M]^p$	matrix of plastic moments
$\bar{n}$	outward unit normal
p	lateral load
P	a point
P_0	point on the boundary
$\{P^D\}$	a load vector
Q	a point
$[Q], [Q']$	matrices
r	radius vector
$[R]$	a matrix
S	plate area
$[S]$	a matrix
t	tangential unit vector
$[U], [U']$	matrices
v, v_ξ	fundamental solutions

V_n, V_n^p, V_n'	equivalent shears
w	deflection
$\{W\}, \{W_{,n}\}$	boundary displacement and slope vectors
$\{W^D\}$	vector of domain unknowns
x, y, z	coordinate system
$\{X^H\}$	vector of boundary unknowns
$\{Y^H\}$	vector of boundary knowns
Γ	plate boundary
δ	increment symbol
$\delta(P, Q)$	delta function
Δ	Laplacian operator
$\{\varepsilon\}, \{\varepsilon\}^e, \{\varepsilon\}^p$	total, elastic and plastic strain vectors
λ	proportionality coefficient
ν	Poisson's ratio
$\bar{\sigma}$	yield stress
$\{\sigma\}$	stress vector
ϕ	angle
$\xi\nu$	coordinate system

INTRODUCTION

The Green's function, or boundary integral equation method, is well known as a competitive method of solving linear elastic plate bending problems. Several authors have suggested different ways of formulating the boundary integrals for plate bending (Massonnet[1], Jawson and Maiti[2], Maiti and Chakrabarty[3], Hansen[4], Altiero and Sikarskie[5]). The formulation suggested by Bezine, and independently by Stern, has the advantage of being suitable for solving cases of general boundary conditions while the other methods are restricted to more special configurations (Bezine[6], Stern[7]). The current work extends Stern's formulation to include plasticity.

Through the use of the generalized Rayleigh–Green identity, Bezine and Stern have shown that the boundary integral equations can be obtained in terms of the relevant physical conditions existing along the boundaries. According to this direct approach, a pair of integral equations involving displacement, normal slope, bending moment and equivalent shear on the boundary are defined. Usually two of the above quantities are known for a given boundary. Generally, the suggested pair of equations is reformulated by means of boundary discretization. A discretization involving N elements along the boundary creates a system of equations in $2N$ unknowns.

Although the boundary element method is naturally suitable for linear problems, it is also useful for solving elasto-plastic problems of two and three dimensions by means of initial strain or stress (Banerjee and Butterfield[8]). Usually when using these methods an incremental approach is applied in which the load is divided into increments. An iterative method is then used to evaluate the plastic strain existing at each increment. This basic approach of successive elastic approximations is commonly used in solving elasto-plastic problems by the finite element method, and has been used with the boundary element method. A positive feature is that the matrix involved in the solution does not change during the incrementation and can therefore be inverted in advance and retained.

A boundary element solution for the nonlinear plate bending problem was recently suggested and demonstrated by Morjaria and Mukherjee[9]. The formulation given there has the major drawback that only restricted boundary conditions can be used. The current paper extends the general elastic plate bending formulation of Stern to the elasto-plastic case (Stern[7]). It is shown that this extension retains the advantage of dealing with the physical quantities of displacement, normal slope, normal moment and equivalent shear along the boundary.

Morjaria and Mukherjee used numerical second derivatives of the nonlinear strains in their formulation. In this paper, the problem is formulated such that no derivatives of the plastic strains are needed.

Three study cases are described and a comparison with published results obtained by the finite element method are presented (Popov et al.[10, 11], Owen and Figueiras[12]).

GOVERNING EQUATION

Following Kirchoff's assumptions based on the small deflections theory of thin plates, the biharmonic equation for the elastic plate bending can be reformulated in incremental form to include the plastic strain increments. It is assumed that any in-plane effects are neglected and that the midplane of the plate is a plane of symmetry even in the presence of plastic strains.

The increment of the total strain can be written as a superposition of increments of the elastic strain and the plastic strain as follows:

$$\delta\{\varepsilon\} = \delta\{\varepsilon\}^e + \delta\{\varepsilon\}^p. \tag{1}$$

For the elastic component under plane stress:

$$\delta\sigma_{xx} = \frac{E}{1-v^2}(\delta\varepsilon_{xx}^e + v\delta\varepsilon_{yy}^e),$$

$$\delta\sigma_{yy} = \frac{E}{1-v^2}(\delta\varepsilon_{yy}^e + v\delta\varepsilon_{xx}^e), \tag{2}$$

$$\delta\sigma_{xy} = 2G\delta\varepsilon_{xy}^e.$$

According to the kinematic assumptions the increments of the total strain are given in terms of the partial derivatives of the deflection increment as follows:

$$\delta\varepsilon_{xx} = -z\delta w_{,xx},$$

$$\delta\varepsilon_{yy} = -z\delta w_{,yy}, \tag{3}$$

$$\delta\varepsilon_{xy} = -z\delta w_{,xy}.$$

Following the derivation procedure of the elastic plate bending equation, the incremental form of the equilibrium equation gives

$$\frac{\partial^2 \delta M_x}{\partial x^2} + 2\frac{\partial^2 \delta M_{xy}}{\partial x \, \partial y} + \frac{\partial^2 \delta M_y}{\partial y^2} + \delta p = 0, \tag{4}$$

where the increments of the moments are defined as

$$\delta M_x = \int_{-a}^{a} \delta\sigma_{xx} z \, dz,$$

$$\delta M_y = \int_{-a}^{a} \delta\sigma_{yy} z \, dz, \tag{5}$$

$$\delta M_{xy} = \int_{-a}^{a} \delta\sigma_{xy} z \, dz.$$

Using eqns (1) and (2) the stresses can be expressed as follows:

$$\delta\sigma_{xx} = \frac{E}{1-v^2}(\delta\varepsilon_{xx} + v\delta\varepsilon_{yy}) - \frac{E}{1-v^2}(\delta\varepsilon_{xx}^p + v\delta\varepsilon_{yy}^p),$$

$$\delta\sigma_{yy} = \frac{E}{1-v^2}(\delta\varepsilon_{yy} + v\delta\varepsilon_{xx}) - \frac{E}{1-v^2}(\delta\varepsilon_{yy}^p + v\delta\varepsilon_{xx}^p), \tag{6}$$

$$\delta\sigma_{xy} = 2G\delta\varepsilon_{xy} - 2G\delta\varepsilon_{xy}^p.$$

Substituting eqn (6) into eqn (5), the increments of the plastic moments become defined as

$$\delta M_x^p = \int_{-a}^{a} \frac{E}{1-v^2} (\delta\varepsilon_{xx}^p + v\delta\varepsilon_{yy}^p)z \; dz,$$

$$\delta M_y^p = \int_{-a}^{a} \frac{E}{1-v^2} (\delta\varepsilon_{yy}^p + v\delta\varepsilon_{xx}^p)z \; dz, \tag{7}$$

$$\delta M_{xy}^p = \int_{-a}^{a} 2G\delta\varepsilon_{xy}^p z \; dz.$$

Substituting the plastic and elastic moment increments into the equilibrium eqn (4) and using eqns (3), (5), (6) and (7), the governing equation for the elasto-plastic plate bending in an incremental form is obtained:

$$D\Delta\Delta\delta w = \delta p - \nabla\nabla\delta[M]^p, \tag{8}$$

where Δ is the Laplacian operator and ∇ is defined as

$$\nabla = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right], \tag{9}$$

also

$$\delta[M]^p = \begin{bmatrix} \delta M_x^p & \delta M_{xy}^p \\ \delta M_{xy}^p & \delta M_y^p \end{bmatrix}. \tag{10}$$

The plate bending equation as obtained in eqn (8) includes lateral loading and plastic effects. The plastic moment tensor, which must be unknown at any increment, simply appears in the equation as an additional effective lateral load.

Next, following a short description of the constitutive relations, the governing equation, (8), is transformed into a boundary integral form, with special treatment required for the plasticity terms.

CONSTITUTIVE RELATIONS

The elasto-plastic analysis requires appropriate constitutive relations. The Prandtl–Reuss stress strain relations based on Von Mises' yield condition is used. The formulation given by Zienkiewicz et al.[13] is adopted.

A general yield surface F is given in terms of the stress space and a hardening parameter k,

$$F(\{\sigma\}, k) = 0. \tag{11}$$

Using the normality principle, require:

$$\delta\{\varepsilon\}^p = \lambda \frac{\partial F}{\partial\{\sigma\}}. \tag{12}$$

Rewriting eqn (2) in matrix form using $[D]^e$ as the elasticity matrix and combining the increments of the elastic and the plastic strain results in the increment of the total strain as follows:

$$\delta\{\varepsilon\} = [D]^{e^{-1}}\delta\{\sigma\} + \lambda \frac{\partial F}{\partial\{\sigma\}}. \tag{13}$$

Plasticity occurs on the yield surface F of the stress space. This requires:

$$0 = \left\{\frac{\partial F}{\partial\{\sigma\}}\right\}^T \delta\{\sigma\} + \frac{\partial F}{\partial k} \, dk. \tag{14}$$

Using eqns (13) and (14), the following stress–strain relationship is obtained,

$$\delta\{\sigma\} = [D]^{ep}\delta\{\varepsilon\}, \tag{15}$$

where

$$[D]^{ep} = [D]^e - [D]^e\left\{\frac{\partial F}{\partial\{\sigma\}}\right\}\left\{\frac{\partial F}{\partial\{\sigma\}}\right\}^T [D]^e\left(H' + \left\{\frac{\partial F}{\partial\{\sigma\}}\right\}^T [D]^e\left\{\frac{\partial F}{\partial\{\sigma\}}\right\}\right)^{-1}. \tag{16}$$

It can be shown that H', in eqn (16), is the slope of the uniaxial stress $\bar{\sigma} = \bar{\sigma}(k)$ versus the uniaxial plastic strain as obtained in a uniaxial yield test. The Von Mises' yield surface for the plane stress case is given as

$$F = [\sigma_{xx}^2 + \sigma_{yy}^2 - \sigma_{xx}\sigma_{yy} + 3\sigma_{xy}^2]^{1/2} - \bar{\sigma}. \tag{17}$$

Equations (1), (2) and (15) are used to arrive at the equation

$$\delta\{\varepsilon\}^p = [D]^*\delta\{\varepsilon\}, \tag{18}$$

where

$$[D]^* = [I] - [D]^{e^{-1}}[D]^{ep},$$

$$[D]^* = \left\{\frac{\partial F}{\partial\{\sigma\}}\right\}\left\{\frac{\partial F}{\partial\{\sigma\}}\right\}^T [D]^e\left(H' + \left\{\frac{\partial F}{\partial\{\sigma\}}\right\}^T [D]^e\left\{\frac{\partial F}{\partial\{\sigma\}}\right\}\right)^{-1}. \tag{19}$$

Equation (18) is needed for the iterative procedure. It relates the increments of the plastic strain to the increments of the total strain. The plastic strains are required in eqn (7), and subsequently in the integral equation solution for $w(x, y)$ as presented in the following sections.

THE BOUNDARY INTEGRALS

The generalized Rayleigh–Green identity has been used in the past to obtain boundary integral equations for the biharmonic equation governing elastic plate bending (Bezine[6], Stern[7]). The identity can be written in incremental form as follows:

$$\int_S\int (v\Delta\Delta\delta w - \delta w\Delta\Delta v) \, dS = -D^{-1}\int_\Gamma \{v\delta V_n'(\delta w) - \delta w V_n(v)\} \, ds$$

$$-D^{-1}\int_\Gamma \left\{\frac{\partial}{\partial n}(\delta w)M_n(v) - \frac{\partial v}{\partial n}\delta M_n'(\delta w)\right\} \, ds \tag{20}$$

$$+D^{-1}\sum_{m=1}^L \{[\delta w M_{nt}(v)]_{A_m} - [v\delta M_{nt}'(\delta w)]_{A_m}\}.$$

The above relation holds for all functions v and δw which are four times continuously differentiable. Let S be the domain occupied by the plate and Γ be its boundary. Also let s be the arc length along the boundary, $\bar{n}$ the outward unit normal vector and $\bar{t}$ the unit

vector in the tangential direction of the boundary (see Fig. 1 for notation). On the boundary Γ define M_n, M_{nt} and V_n as the actual distributions of bending moment, twisting moment and equivalent transverse shear force, respectively. The corresponding components M'_n, M'_{nt} and V'_n are defined as functions of the deflection field δw as obtained in an elastic case. In other words, the above functions can be obtained from eqns (3), (2) and (5) when the total strain is taken equal to the elastic strain. Also $M_n(v)$, $M_{nt}(v)$ and $V_n(v)$ are defined as in the elastic case corresponding to a deflection field v. The last summation term in the above equation is a result of possible discontinuities in L corners. The jump at a corner A_m is defined by

$$[\cdot]_{A_m} = (\cdot)_{A_m^+} - (\cdot)_{A_m^-}. \tag{21}$$

Consider two points $P(x_P, y_P)$ and $Q(x_Q, y_Q)$ of the region S and let

$$v(P, Q) = \frac{1}{8\pi} r^2 \log r, \tag{22}$$

where

$$r = \{(x_Q - x_P)^2 + (y_Q - y_P)^2\}^{-1/2}. \tag{23}$$

The Green's function v is a fundamental solution such that

$$\Delta \Delta v(P, Q) = \delta(P, Q). \tag{24}$$

Substitute the governing eqn (8) and the above eqn (24), into eqn (20), and get for any internal point P,

$$\delta w(P) = D^{-1} \int_\Gamma \left\{ v \delta V'_n(\delta w) - \frac{\partial v}{\partial n} \delta M'_n(\delta w) + \frac{\partial}{\partial n}(\delta w) M_n(v) - \delta w V_n(v) \right\} ds_Q$$

$$+ D^{-1} \int_S \int v \{\delta p - \nabla \nabla \delta[M]^p\} \, dS_Q - D^{-1} \sum_{m=1}^L \left\{ [\delta w M_{nt}(v)]_{A_m} - [v \delta M'_{nt}(\delta w)]_{A_m} \right\}. \tag{25}$$

Here, dS_Q and ds_Q denote integration over S and Γ with respect to the coordinates of point Q. Equation (25) is of an inconvenient form since it requires derivatives of the plastic moments and it does not include the actual physical components along the boundary. A better form can be achieved by the following integration by parts:

$$\int_S \int v \nabla \nabla \delta[M]^p \, dS = \int_S \int \left\{ \frac{\partial^2 v}{\partial x^2} \delta M_x^p + 2 \frac{\partial^2 v}{\partial x \, \partial y} \delta M_{xy}^p + \frac{\partial^2 v}{\partial y^2} \delta M_y^p \right\} dS$$

$$- \int_\Gamma \left(\delta M_n^p \frac{\partial v}{\partial n} - v \delta V_n^p \right) ds + \sum_{m=1}^L [v \delta M_{nt}^p]. \tag{26}$$
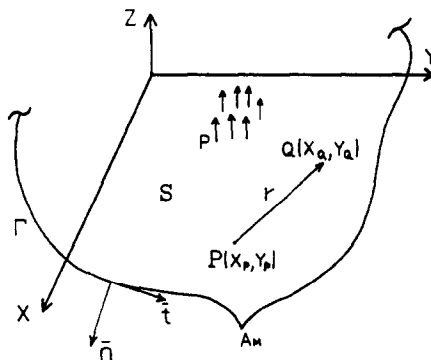

Fig. 1. Notation for the boundary integrals formulation.

Substituting eqn (26) into eqn (25) results in

$$\delta w(P) = D^{-1} \int_{\Gamma} \{v(\delta V_n'(\delta w) - \delta V_n^p) - \delta w V_n(v)\} \, ds_Q$$

$$+ D^{-1} \int_{\Gamma} \left\{ \frac{\partial}{\partial n}(\delta w) M_n(v) - \frac{\partial v}{\partial n}(\delta M_n'(\delta w) - \delta M_n^p) \right\} ds_Q$$

$$- D^{-1} \int_S \int \left\{ \frac{\partial^2 v}{\partial x^2} \delta M_x^p + 2 \frac{\partial^2 v}{\partial x \, \partial y} \delta M_{xy}^p + \frac{\partial^2 v}{\partial y^2} \delta M_y^p - \delta p v \right\} dS_Q$$

$$+ D^{-1} \sum_{m=1}^{L} \{ [\delta w M_{nt}(v)]_{A_m} - [v(\delta M_{nt}'(\delta w) - \delta M_{nt}^p)]_{A_m} \}. \tag{27}$$

Using eqns (5), (6) and (7) the increment of the bending moment δM_n normal to the boundary is

$$\delta M_n = \delta M_n' - \delta M_n^p. \tag{28}$$

Similarly it can be shown that the increment of the effective shear force δV_n on the boundary is

$$\delta V_n = \delta V_n' - \delta V_n^p \tag{29}$$

and the increment of the twisting moment δM_{nt} is

$$\delta M_{nt} = \delta M_{nt}' - \delta M_{nt}^p. \tag{30}$$

Therefore eqn (27) can be rewritten as

$$\delta w(P) = D^{-1} \int_{\Gamma} \{v \delta V_n(\delta w) - \delta w V_n(v)\} \, ds_Q$$

$$+ D^{-1} \int_{\Gamma} \left\{ \frac{\partial}{\partial n}(\delta w) M_n(v) - \frac{\partial v}{\partial n} \delta M_n(\delta w) \right\} ds_Q$$

$$- D^{-1} \int_S \int \left\{ \frac{\partial^2 v}{\partial x^2} \delta M_x^p + 2 \frac{\partial^2 v}{\partial x \, \partial y} \delta M_{xy}^p + \frac{\partial^2 v}{\partial y^2} \delta M_y^p - \delta p v \right\} dS_Q$$

$$+ D^{-1} \sum_{m=1}^{L} \{ [\delta w M_{nt}(v)]_{A_m} - [v \delta M_{nt}(\delta w)]_{A_m} \}. \tag{31}$$

Equation (31) gives the value of δw at any interior point P of S in terms of the physical quantities δw, $(\partial/\partial n)(\delta w)$, δM_n and δV_n on Γ.

Now let point P tend to P_0 on Γ. Taking the limiting values of the integrals, the following equation for point P_0 on Γ is obtained:

$$0.5 \delta w(P_0) = D^{-1} \int_{\Gamma} \{v \delta V_n(\delta w) - \delta w V_n(v)\} \, ds_Q$$

$$+ D^{-1} \int_{\Gamma} \left\{ \frac{\partial}{\partial n}(\delta w) M_n(v) - \frac{\partial v}{\partial n} \delta M_n(\delta w) \right\} ds_Q$$

$$- D^{-1} \int_S \int \left\{ \frac{\partial^2 v}{\partial x^2} \delta M_x^p + 2 \frac{\partial^2 v}{\partial x \, \partial y} \delta M_{xy}^p + \frac{\partial^2 v}{\partial y^2} \delta M_y^p - \delta p v \right\} dS_Q$$

$$+ D^{-1} \sum_{m=1}^{L} \{ [\delta w M_{nt}(v)]_{A_m} - [v \delta M_{nt}(\delta w)]_{A_m} \}, \tag{32}$$

where $v = v(P_0, Q)$.

A second independent equation is needed since two conditions will always be unknown on the boundary. According to Stern a second equation, for the normal derivative of w at point P_0 along the boundary, can be derived as follows (Stern[7]). A local $\xi\eta$-coordinate system is introduced at a point P, as illustrated in Fig. 2. The following fundamental solution is used:

$$V_\xi = \frac{1}{2\pi} r \ln r \cos \phi, \tag{33}$$

where ϕ is the angle of rotation of r with respect to the ξ direction. It is noted that the above fundamental solution gives the following kernel due to the equivalent shear:

$$V(v_\xi) = -D \frac{3-v}{2\pi} \frac{\cos \phi}{r^2}. \tag{34}$$

Due to the $1/r^2$ behavior of this kernel, as Q approaches P_0 the boundary integral does not converge in general. Stern suggested substituting $w - w(P)$ into the generalized Rayleigh–Green identity to achieve convergence (Stern[7]). The second boundary integral equation for a regular point P_0 along the boundary as given by Stern is herewith modified to incremental form with the inclusion of the plastic effects:

$$-\frac{\partial}{\partial n}(\delta w(P_0)) = D^{-1} \int_\Gamma \left\{ v_\xi \delta V_n'(\delta w) - \frac{\partial v_\xi}{\partial n} \delta M_n'(\delta w) \right\} ds_Q$$

$$+ D^{-1} \int_\Gamma \left\{ \frac{\partial}{\partial n}(\delta w) M_n(v_\xi) - (\delta w - \delta w(P_0)) V_n(v_\xi) \right\} ds_Q$$

$$+ D^{-1} \int_S \int v_\xi \{ \delta p - \nabla\nabla \delta[M]^p \} dS_Q$$

$$+ D^{-1} \sum_{m=1}^{L} \{ [(\delta w - \delta w(P_0)) M_{nt}(v_\xi)]_{A_m} - [v_\xi \delta M_{nt}'(\delta w)]_{A_m} \}, \tag{35}$$

where ξ is taken in the normal direction to the boundary at P_0. Replacing v with v_ξ in eqn (26) and substituting into eqn (35), the following equation is obtained:

$$-\frac{\partial}{\partial n}(\delta w(P_0)) = D^{-1} \int_\Gamma \{ v_\xi \delta V_n(\delta w) - (\delta w - \delta w(P_0)) V_n(v_\xi) \} ds_Q$$

$$+ D^{-1} \int_\Gamma \left\{ \frac{\partial}{\partial n}(\delta w) M_n(v_\xi) - \frac{\partial v_\xi}{\partial n} \delta M_n(\delta w) \right\} ds_Q$$

$$- D^{-1} \int_S \int \left\{ \frac{\partial^2 v_\xi}{\partial x^2} \delta M_x^p + 2 \frac{\partial^2 v_\xi}{\partial x \, \partial y} \delta M_{xy}^p + \frac{\partial^2 v_\xi}{\partial y^2} \delta M_y^p - \delta p v_\xi \right\} dS_Q$$

$$+ D^{-1} \sum_{m=1}^{L} \{ [(\delta w - \delta w(P_0)) M_{nt}(v_\xi)]_{A_m} - [v_\xi \delta M_{nt}(\delta w)]_{A_m} \}. \tag{36}$$
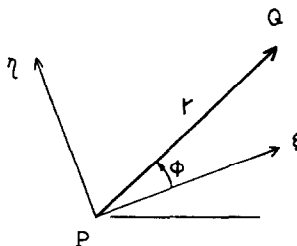

Fig. 2. Notation for ξv-coordinate system.

Using eqn (32) and eqn (36), the unknowns along the boundary can be expressed in terms of the known boundary conditions, the given increment of lateral load, and the increments of the plastic moments. In general the increments of the plastic moments are unknowns but can be found by an iterative procedure. With respect to the iterative procedure it should be noted that the increments of the plastic moments are related to the increments of the plastic strains which are related to the increments of the total strains [eqn (7) and eqn (18)]. To get the increments of the total strains, equations for the second derivatives of the deflection increment are obtained. In general, the above information is needed for a point P inside the domain. Equation (31) is hence differentiated twice with respect to x_P and y_P at the internal point P giving the following equations:

$$
\frac{\partial^2}{\partial x_P^2}(\delta w) = D^{-1} \int_\Gamma \left\{ \frac{\partial^2 v}{\partial x_P^2} \delta V_n(\delta w) - \delta w \frac{\partial^2}{\partial x_P^2}(V_n(v)) \right\} ds_Q
$$

$$
+ D^{-1} \int_\Gamma \left\{ \frac{\partial}{\partial n}(\delta w) \frac{\partial^2}{\partial x_P^2}(M_n(v)) - \frac{\partial^2}{\partial x_P^2}\left(\frac{\partial v}{\partial n}\right) \delta M_n(\delta w) \right\} ds_Q
$$

$$
- D^{-1} \int_S \int \left\{ \frac{\partial^4 v}{\partial x_P^2 \partial x_Q^2} \delta M_x^p + 2 \frac{\partial^4 v}{\partial x_P^2 \partial x_Q \partial y_Q} \delta M_{xy}^p \right\} dS_Q
$$

$$
- D^{-1} \int_S \int \left\{ \frac{\partial^4 v}{\partial x_P^2 \partial y_Q^2} \delta M_y^p - \delta p \frac{\partial^2 v}{\partial x_P^2} \right\} dS_Q
$$

$$
+ D^{-1} \sum_{m=1}^{L} \left\{ \left[\delta w \frac{\partial^2}{\partial x_P^2}(M_{nt}(v)) \right]_{A_m} - \left[\frac{\partial^2 v}{\partial x_P^2} \delta M_{nt}(\delta w) \right]_{A_m} \right\}, \tag{37}
$$

also

$$
\frac{\partial^2}{\partial y_P^2}(\delta w) = D^{-1} \int_\Gamma \left\{ \frac{\partial^2 v}{\partial y_P^2} \delta V_n(\delta w) - \delta w \frac{\partial^2}{\partial y_P^2}(V_n(v)) \right\} ds_Q
$$

$$
+ D^{-1} \int_\Gamma \left\{ \frac{\partial}{\partial n}(\delta w) \frac{\partial^2}{\partial y_P^2}(M_n(v)) - \frac{\partial^2}{\partial y_P^2}\left(\frac{\partial v}{\partial n}\right) \delta M_n(\delta w) \right\} ds_Q
$$

$$
- D^{-1} \int_S \int \left\{ \frac{\partial^4 v}{\partial y_P^2 \partial x_Q^2} \delta M_x^p + 2 \frac{\partial^4 v}{\partial y_P^2 \partial x_Q \partial y_Q} \delta M_{xy}^p \right\} dS_Q
$$

$$
- D^{-1} \int_S \int \left\{ \frac{\partial^4 v}{\partial y_P^2 \partial y_Q^2} \delta M_y^p - \delta p \frac{\partial^2 v}{\partial y_P^2} \right\} dS_Q
$$

$$
+ D^{-1} \sum_{m=1}^{L} \left\{ \left[\delta w \frac{\partial^2}{\partial y_P^2}(M_{nt}(v)) \right]_{A_m} - \left[\frac{\partial^2 v}{\partial y_P^2} \delta M_{nt}(\delta w) \right]_{A_m} \right\} \tag{38}
$$

and

$$
\frac{\partial^2}{\partial x_P \partial y_P}(\delta w) = D^{-1} \int_S \left\{ \frac{\partial^2 v}{\partial x_P \partial y_P} \delta V_n(\delta w) - \delta w \frac{\partial^2}{\partial x_P \partial y_P}(V_n(v)) \right\} ds_Q
$$

$$
+ D^{-1} \int_\Gamma \left\{ \frac{\partial}{\partial n}(\delta w) \frac{\partial^2}{\partial x_P \partial y_P}(M_n(v)) - \frac{\partial^2 \cdot}{\partial x_P \partial y_P}\left(\frac{\partial v}{\partial n}\right) \delta M_n(\delta w) \right\} ds_Q
$$

$$
- D^{-1} \int_S \int \left\{ \frac{\partial^4 v}{\partial x_P \partial y_P \partial x_Q^2} \delta M_x^p + 2 \frac{\partial^4 v}{\partial x_P \partial y_P \partial x_Q \partial y_Q} \delta M_{xy}^p \right\} dS_Q
$$

$$
- D^{-1} \int_S \int \left\{ \frac{\partial^4 v}{\partial x_P \partial y_P \partial y_Q^2} \delta M_y^p - \delta p \frac{\partial^2 v}{\partial x_P \partial y_P} \right\} dS_Q
$$

$$
+ D^{-1} \sum_{m=1}^{L} \left\{ \left[\delta w \frac{\partial^2}{\partial x_P \partial y_P}(M_{nt}(v)) \right]_{A_m} - \left[\frac{\partial^2}{\partial x_P \partial y_P} \delta M_{nt}(\delta w) \right]_{A_m} \right\}. \tag{39}
$$

The behavior of the kernels of the domain integral as $P \to Q$ is required in the plastic moment evaluations using (37) to (39). Using Green's theorem it can be shown that

$$\lim_{P \to Q} \int\int_S \frac{\partial^4 v}{\partial x_P^2 \, \partial x_Q^2} \, dS_Q = \frac{3}{8},\tag{40}$$

$$\lim_{P \to Q} \int\int_S \frac{\partial^4 v}{\partial y_P^2 \, \partial y_Q^2} \, dS_Q = \frac{3}{8},\tag{41}$$

$$\lim_{P \to Q} \int\int_S \frac{\partial^4 v}{\partial x_P^2 \, \partial y_Q^2} \, dS_Q = \frac{1}{8},\tag{42}$$

$$\lim_{P \to Q} \int\int_S \frac{\partial^4 v}{\partial y_P^2 \, \partial x_Q^2} \, dS_Q = \frac{1}{8},\tag{43}$$

$$\lim_{P \to Q} \int\int_S \frac{\partial^4 v}{\partial x_P \, \partial y_P \, \partial x_Q \, \partial y_Q} \, dS_Q = \frac{1}{8},\tag{44}$$

$$\lim_{P \to Q} \int\int_S \frac{\partial^4 v}{\partial x_P^2 \, \partial x_Q \, \partial y_Q} \, dS_Q = 0,\tag{45}$$

$$\lim_{P \to Q} \int\int_S \frac{\partial^4 v}{\partial y_P^2 \, \partial x_Q \, \partial y_Q} \, dS_Q = 0,\tag{46}$$

$$\lim_{P \to Q} \int\int_S \frac{\partial^4 v}{\partial x_P \, \partial y_P \, \partial y_Q^2} \, dS_Q = 0,\tag{47}$$

$$\lim_{P \to Q} \int\int_S \frac{\partial^4 v}{\partial x_P \, \partial y_P \, \partial x_Q^2} \, dS_Q = 0.\tag{48}$$

NUMERICAL SOLUTION

The boundary integrals are discretized into a linear system of algebraic equations. The boundary is approximated by J straight boundary elements. Along the jth element $\delta V_n(\delta w)$, $\delta M_n(\delta w)$, $(\partial/\partial n)(\delta w)$ and δw are assumed to be constants and are denoted as V_j, M_j, W_{nj} and W_j. It is suggested that for analyzing plates with general boundary conditions higher order interpolation functions should be used (Stern[14]). For the numerical evaluation of the domain integrals appearing in eqns (32), (36), (37), (38) and (39), the plate field is approximated by K domain elements (see Fig. 3). For the kth domain element the values of δM_x^p, δM_y^p, δM_{xy}^p and δp are assumed to be constants and have been denoted as δM_{xk}^p, δM_{yk}^p, δM_{xyk}^p and δp_k. A detailed discussion on the incorporation of the twisting moment jumps at corners can be found in Stern[7]. For the sake of simplicity only cases with no corner effects are discussed in the following sections.

Using the above approximations eqn (32) can be written with respect to a point $P_0 = P_i$ corresponding to the ith element on the boundary as follows:

$$0.5W_i = \sum_{j=1}^{J} (a_{ij}^* W_j + b_{ij} W_{nj} + c_{ij} M_j + d_{ij} V_j)$$

$$+ \sum_{k=1}^{K} (g_{ik} \delta M_{xk}^p + h_{ik} \delta M_{yk}^p + q_{ik} \delta M_{xyk}^p + u_{ik} \delta p_k) \qquad (i = 1, 2, \ldots, J). \tag{49}$$

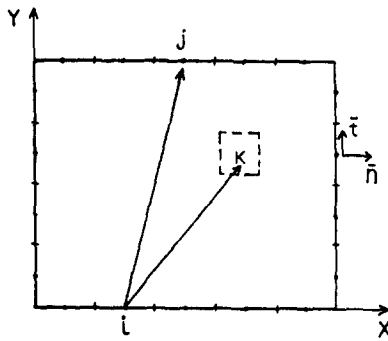Here, the coefficients are the result of integration over the corresponding elements. This

Fig. 3. Illustration of discretization method.

last set of equations, (49), can be written in matrix form as

$$[A]\{W\}+[B]\{W_{,n}\}+[C]\{M\}+[D]\{V\} = [G]\{\delta M_x^p\}+[H]\{\delta M_y^p\}+[Q]\{\delta M_{xy}^p\}+[U]\{\delta p\},$$

$$(50)$$

where

$$a_{ii} = a_{ii}^* - 0.5. \qquad (51)$$

Similarly, discretizing eqn (36) results in

$$-W_{,ni} = \sum_{j=1}^{J} (a_{ij}'^* (W_j - W_i) + b_{ij}' W_{,nj} + c_{ij}' M_j + d_{ij}' V_j)$$

$$+ \sum_{k=1}^{K} (g_{ik}' \delta M_{xk}^p + h_{ik}' \delta M_{yk}^p + q_{ik}' \delta M_{xyk}^p + u_{ik}' \delta p_k) \qquad (i = 1, 2, \ldots, J). \quad (52)$$

This last set of equations can be written in matrix form as

$$[A']\{W\}+[B']\{W_{,n}\}+[C']\{M\}+[D']\{V\}$$

$$= [G']\{\delta M_x^p\}+[H']\{\delta M_y^p\}+[Q']\{\delta M_{xy}^p\}+[U']\{\delta p\}, \qquad (53)$$

where

$$a_{ii}' = - \sum_{j \neq i}^{J} a_{ij}'^* \qquad (54)$$

and

$$b_{ii}' = b_{ii}'^* + 1. \qquad (55)$$

Combine eqns (50) and (53) and rearrange them according to known and unknown boundary displacement and load conditions to obtain

$$[A^B]\{X^B\} = [F^B]\{P^D\}+[B^B]\{Y^B\}. \qquad (56)$$

For example, in a clamped plate case the unknown boundary vector is $\{X^B\} = \{M, V\}^T$ and the known one is $\{Y^B\} = \{W, W_{,n}\}^T$. The load vector is $\{P^D\} = \{\delta M_x^p, \delta M_y^p, \delta M_{xy}^p, \delta p\}^T$. Equation (56) is solved as

$$\{X^B\} = [A^B]^{-1}[F^B]\{P^D\}+[A^B]^{-1}[B^B]\{Y^B\}. \qquad (57)$$

Using a similar approach, a vector of unknown increments of displacements and second partial derivatives inside the domain $\{W^D\}$ can be found using eqns (31), (37), (38) and (39) in a discretized form to be

$$\{W^D\} = [A^D]\{X^B\}+[F^D]\{P^D\}+[B^D]\{Y^B\}. \tag{58}$$

Substituting eqn (57) in eqn (58) results in

$$\{W^D\} = [R]\{P^D\}+[S]\{Y^B\}, \tag{59}$$

where

$$[R] = [A^D][A^B]^{-1}[F^B]+[F^D] \tag{60}$$

and

$$[S] = [A^D][A^B]^{-1}[B^B]+[B^D]. \tag{61}$$

It should be noted that only $\{P^D\}$ in eqn (59) changes during the incremental procedure. The rest of the matrices are found in advance. Gaussian quadrature is used for evaluation of the coefficients. In the case of the domain integrals of eqns (37), (38) and (39), the limiting forms, eqns (40), (41), (42), (43) and (44), are applied in way of the singularities.

The solution procedure is:

1. Apply a load increment δp and calculate the corresponding increments of elastic strains $\{\delta\varepsilon\}^e$ at the chosen integration points using eqns (58) and (3).

2. Determine increments of stress $\{\delta\sigma\}$ corresponding to $\{\delta\varepsilon\}^e$ by eqn (2). Define a multiplication factor r_y which when multiplied by $\{\delta\sigma\}$ will give the elastic limit. Substitute $r_y\{\delta\sigma\}$ into eqn (17) and solve for the multiplication factor r_y. Repeat for all elements. Determine the multiplication factor for the onset of yielding by the smallest r_y obtained.

3. Update p, w, $\{\varepsilon\}^e$ and $\{\sigma\}$ for the onset of yielding, at all domain elements and integration points, using the multiplication factor as obtained in step 3.

4. Apply a load increment δp and assume $\delta[M^p] = 0$ for all domain elements. If returned from step 11, the final value of $\delta[M^p]$ from last increment can be chosen as a first approximation.

5. Evaluate $\{\delta\varepsilon\}$ by eqns (58) and (3) for all integration points using the assumed $\delta[M^p]$ and the load increment.

6. Evaluate the corresponding increments of stress $\{\delta\sigma\}$ by eqn (2).

7. Add $\{\delta\sigma\}$ to $\{\sigma\}$ and check yielding using eqn (17). When yield is checked along the plate thickness it is possible to save computational time by starting with integration points closest to the plate surfaces. Whenever yield does not occur stop checking along the thickness at that point. If the point has yielded at the start of the increment, calculate $\{\delta\varepsilon\}^p$ by eqn (18). If the point yields during the increment at $r\{\delta\varepsilon\}$, use $(1-r)\{\delta\varepsilon\}$ to evaluate $\{\delta\varepsilon\}^p$ by eqn (18) and add $r\{\delta\sigma\}$ to $\{\sigma\}$.

8. Using $\{\delta\varepsilon\}^p$ calculate $\delta[M]^p$ by eqn (7).

9. Check convergence. Convergence occurs if changes in $\delta[M]^p$ are sufficiently small. If convergence was not achieved replace assumed $\delta[M]^p$ by calculated $\delta[M]^p$ and return to step 5.

10. If convergence has been obtained, update $\{\sigma\}$, $\{w\}$ and p. In a case of hardening, the yield stress should be updated too.

11. Return to step 4 unless $p = p_{max}$ or a predetermined maximum number of iterations has been reached.

NUMERICAL STUDIES

Based on the above approach, a Fortran program was prepared and executed on an Apollo DN320 of the Computer Aided Engineering Network (CAEN) of the University

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THICK. = 0.2   G = 11540
E = 30,000   H' = 300
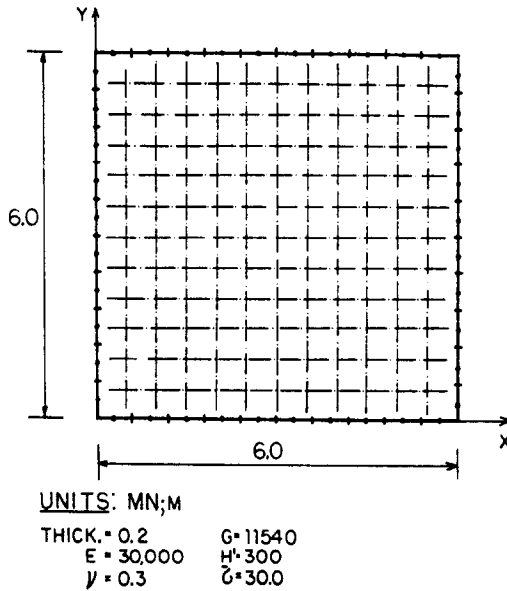γ = 0.3 $\hat{\sigma}$ = 30.0

Fig. 4. Study A—Problem description.

of Michigan. Three typical elasto-plastic plate bending problems were studied to illustrate the effectiveness of the method. The first problem consists of a clamped square plate with linear strain hardening subject to uniform lateral load. The second case is an elastic perfectly plastic simply supported circular plate subject to uniform lateral load. The third case is the bending of a circular clamped plate of hardening material subject to uniform load.

Following the incremental loading procedure described in the preceding section, the deflection curves and plastic zones were found. Due to the asymptotic behavior of the deflection the number of iterations needed for each load increment increases as plasticity develops and therefore the process is generally stopped at some practical load below the limit load. The numerical studies are explained below.

Study A : A clamped square plate of linear hardening material subject to uniform lateral load

A clamped square plate is subject to an increasing uniform load as described in Fig. 4. A linear strain hardening material is assumed. The boundary of the square plate is divided into 4×10 straight boundary elements of equal lengths. The domain of the plate area is divided into 12×12 square elements of equal size. Each domain element has 10 integration points along the thickness for calculating the plastic moments.

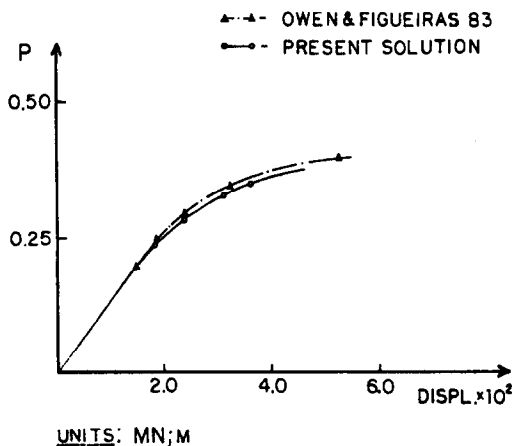The results of the calculations are shown in Figs 5 and 6. In Fig. 5 the deflection of

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Fig. 5. Study A—Central deflection vs load.

P = 0.35

\# = NO. OF PLASTIC INTEGRATION PTS.

10 = TOTAL NO. OF INTEGRATION PTS.
ALONG THE THICKNESS

Fig. 6. Study A—Elastic plastic boundaries.

$E = 10 \times 10^3$ ksi
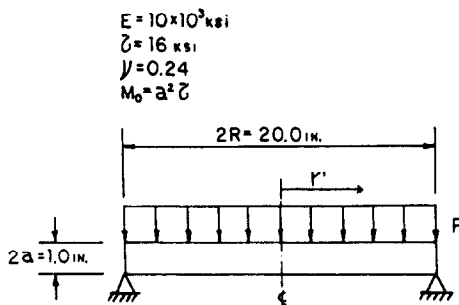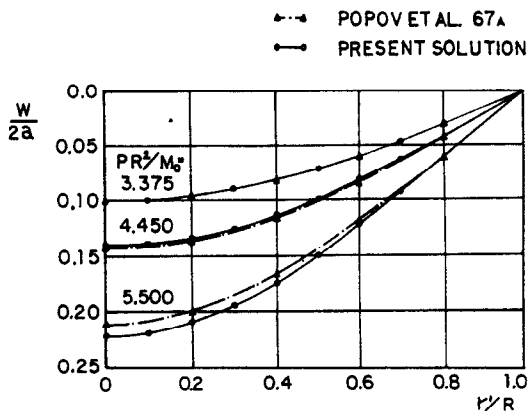$\zeta = 16$ ksi
$\nu = 0.24$
$M_0 = a^2 \zeta$

Fig. 7. Study B—Problem description.

Fig. 8. Study B—Distributions of deflections.

$$-\cdot-\quad \text{POPOV ET AL. 67}_\text{A}$$
$$\rule{0.8cm}{0.4pt}\quad \text{PRESENT SOLUTION}$$

Fig. 9. Study B—Elastic plastic boundaries.

the plate center is given vs the load. The nonlinear behavior, following the linear elastic deflection, is readily observed. The good agreement with finite element results published by Owen and Figueiras[12] is obvious from Fig. 5. Figure 6 indicates the predicted spread of plasticity through the plate volume at a given load close to the limit load. The numbers appearing inside the domain elements indicate the level of plasticity through the thickness, and correspond to the number of integration points at which yielding was reached. Elements that are left blank indicate elastic zones.

Study B : Elastic perfectly plastic simply supported circular plate subject to uniform lateral load

The bending of a simply supported circular plate subject to a uniform lateral load is studied here (see Fig. 7). The plate material is taken to be elastic perfectly plastic. This example of a simply supported circular plate was studied by Popov et al.[10] using a stiffness matrix of ring elements and taking advantage of the problem symmetry. Presently, the use of symmetry is not implemented in the program developed. The plate area was approximated by 144 square and trapezoidal domain elements. It was found that the approximation associated with these element shapes had no strong effects on the results. The boundary of the plate was divided into 48 straight boundary elements of equal length. The results are shown in Figs 8 and 9. In Fig. 8 the distributions of the deflections for three load levels are given. The plasticity starts at the plate center and spreads toward the circular boundary as shown in Fig. 9. The elastic plastic boundary lines in Fig. 9. are due to Popov et al.[10] and are compared to the approximated stepped boundaries that were obtained here. Good agreement with the results obtained by Popov et al. is indicated. The small difference in the results shown in Fig. 8 is increasing with the load, and is believed to be mainly due to the flat asymptotic behavior of the deflections as plasticity spreads.

Study C : Clamped circular plate of hardening material subject to uniform load

This example of a clamped circular plate of hardening material was investigated by Popov et al.[11] and is illustrated in Fig. 10. For simplicity, linear hardening is taken; this approximates the material given in the above reference (see Fig. 11). The boundary and domain elements are taken the same as in Study B described above. The results are summarized in Figs 12 and 13. In Fig. 12 the distributions of the deflections for several
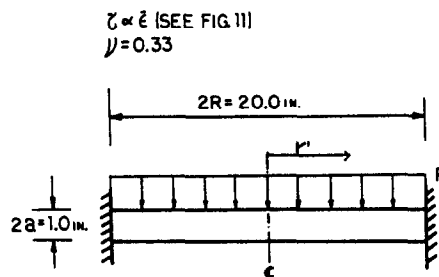
$\zeta \propto \xi$ (SEE FIG 11)
$\nu = 0.33$

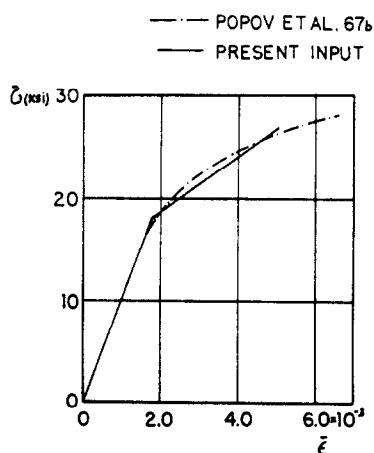Fig. 10. Study C—Problem description.

Fig. 11. Study C—Uniaxial stress–strain curve.

load levels are given. The elastic plastic boundary lines in Fig. 13 are due to Popov *et al.*[11] and are compared to the approximated stepped boundaries that were obtained here. The plasticity starts at the boundary and at the plate center and spreads from the plate surfaces toward the midplane as shown in Fig. 13. A good agreement is shown with the results obtained by Popov *et al.*[11] in spite of the linear hardening approximation used here.

SUMMARY AND CONCLUSIONS

In this work the effectiveness of boundary integrals to solve elasto-plastic plate bending problems using an incremental iterative procedure based on initial plastic moments is studied. The generalized Rayleigh–Green identity is used with the advantage of having the boundary integrals in terms of the actual physical components of deflection, slope, moment and equivalent shear along the boundary. This fact allows different admissible boundary conditions to be imposed. It is shown that the introduction of plastic moments does not change the above property.

Boundary integrals for the second partial derivatives of the deflection are formulated and used for direct calculation of the strain components needed for the evaluation of the initial plastic moments in the iterative procedure. This semianalytical method for the evaluation of the derivatives is believed to give high accuracy in comparison to the methods used for stress evaluation in finite element analysis. Some difficulties which are encountered in plate bending analysis by the finite element method are avoided by the suggested method (Zienkiewicz[15]). Other benefits of the method are the small number of unknowns and the invariance of the inverted matrix with progressive incrementation.
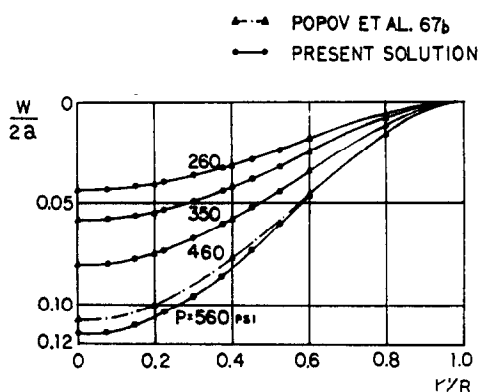

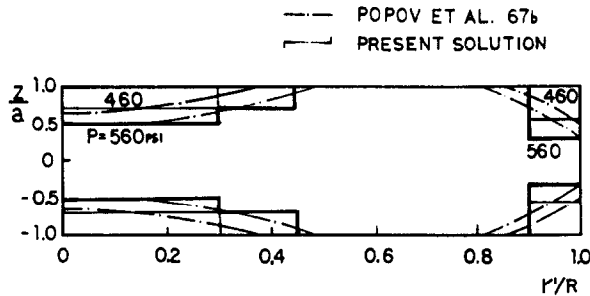
Fig. 12. Study C—Distributions of deflections.

Fig. 13. Study C—Elastic plastic boundaries.

It is demonstrated through the use of several examples that the iterative method for finding the increments of the plastic moments converges. Good agreement with results obtained by the finite element method is shown.

Finally it is concluded that the method developed in this paper is capable of solving a variety of elasto-plastic plate bending problems. It can be easily used with different incremental constitutive relations and different plate shapes and boundary conditions.

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