JOURNAL OF ALGEBRA 110, 13–23 (1987)

On a 2-Generated Infinite 3-Group: The Presentation Problem

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Communicated by Walter Feit

Received June 7, 1984

1. INTRODUCTION

The author and Narain Gupta constructed in [3] a family of residually finite, 2-generated infinite p-groups for every odd prime p, as groups of automorphisms of regular p-trees. This was inspired by Grigorchuk's construction in [2] of a 3-generated infinite 2-group as a permutation group of subintervals of the unit interval. In [6], Merzlyakov relates Grigorchuk's group to p-groups constructed by Aleshin in 1972 [1] by using automatas.

The regularity of our definitions has allowed us to generalize our earlier constructions (see [4]), and gave us ready access to some of their special properties (see [5]).

The prototype of all our constructions has been the infinite 3-group

$$\mathscr{G} = \langle \gamma, x \rangle, \qquad \gamma^3 = x^3 = 1,$$

$$\gamma = (\gamma, x, x^{-1}) \qquad \text{(a recursive definition to be explained later).}$$

We have analyzed its structure in greater detail hoping that the acquired knowledge would clarify ideas concerning finitely generated infinite p-groups which are also residually finite.

This paper addresses the presentation problem for $\mathscr{G}$. We prove that

$\mathscr{G}$ is not finitely presentable,

and provide a presentation for $\mathscr{G}$ with γ and x as generators and recursively defined relators.

Although the definition of $\mathscr{G}$ is simple enough as a group of

0021-8693/87 $3.00

automorphisms of a ternary tree, its presentation requires involved recursion. However, one may identify the starting idea as

$$[\gamma\gamma^{x}, \gamma^{x^{-1}}\gamma] = ([x^{-1}, \gamma^{-1}], 1, 1).$$

The methods developed here are sufficiently general to tackle the presentation problem for the above-mentioned generalizations.

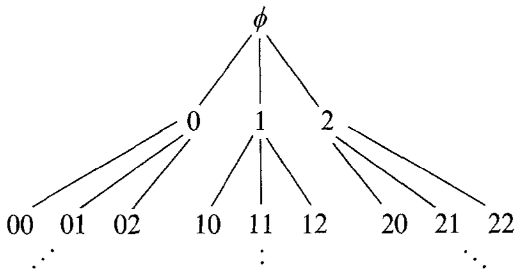
It is to be noted that Grigorchuk gives in [2] a very short, yet unclear, argument for the nonfinite presentability of his group.

In a second paper, we will deal with the subgroup structure of $\mathcal{G}$, and with determining its group of automorphisms.

2. Preliminaries

We recall the construction of $\mathcal{G}$. Let

T:



be the infinite ternary tree having as vertices the finite sequences s in 0, 1, 2, and where these vertices are ordered by $s > s'$ provided s is an initial segment of s'.

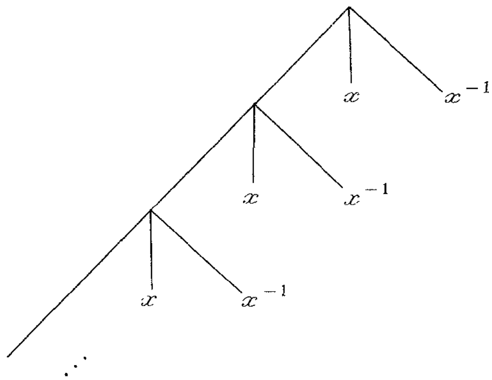
The permutation $x\colon 0 \to 1 \to 2 \to 0$ extends to an automorphism of T by

$$x\colon js \to (j+1)s \qquad (\text{mod } 3)$$

for all finite sequences s.

Given the regularity of T, for any vertex t and any automorphism α of T, we may define an automorphism like α on the subtree headed by t, and extend it to an automorphism of T by fixing the vertices outside the subtree. We denote this new automorphism by the tree T with α attached to its vertex t.

With these comments we introduce the automorphism γ which is denoted by



and note that γ is expressible recursively as

$$\gamma = (\gamma, x, x^{-1}).$$

Our group is simply $\mathscr{G} = \langle \gamma, x \rangle$.

3. A Presentation Diagram

A presentation for $\mathscr{G}$ will be built up in stages. The initial stage is

$$\begin{array}{ccc} & & A_1 = (\langle a_1 \rangle * \langle b_1 \rangle) \wr \langle x_0 \rangle \\ & & \uparrow \nu \\ A_0 = \langle a_0 \rangle * \langle b_0 \rangle & \xrightarrow[\psi_0]{} & G_0 = \langle \gamma_0, x_0 \rangle, \qquad \gamma_0 = (a_1, b_1, b_1^{-1}) \end{array}$$

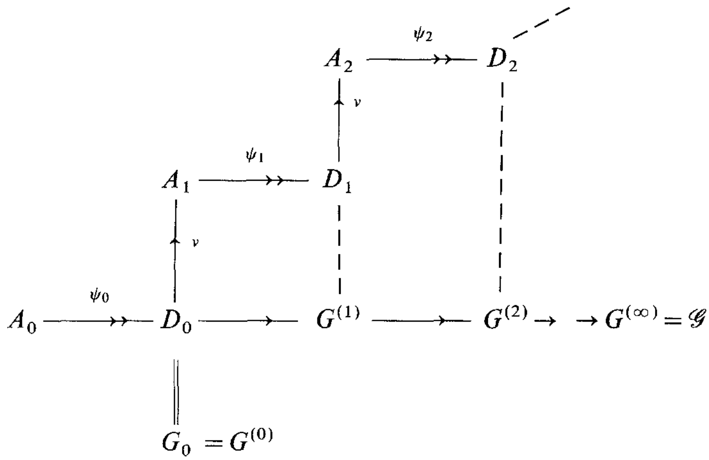
where

$$\langle a_0 \rangle \cong \langle b_0 \rangle \cong \langle a_1 \rangle \cong \langle b_1 \rangle \cong \langle x_0 \rangle \cong C_3,$$

$*$ indicates the free product,
$\wr$ indicates the wreath product,
ψ_0 is an extension of $a_0 \to \gamma_0$, $b_0 \to x_0$,
ν is the identity map.

This is the process of substituting x_0 for b_0, and (a_1, b_1, b_1^{-1}) for a_0 in the free product A_0.

The group $\mathscr{G}$ will be the limit of an infinite iteration of such substitutions which we exhibit in a presentation diagram:



We specify the remaining elements of the diagram by the following list:

$$\langle a_i \rangle \cong \langle b_i \rangle \cong \langle x_i \rangle \cong C_3 \qquad (i \geqslant 0); \tag{1}$$

$$F_i = \langle a_i \rangle * \langle b_i \rangle \qquad (i \geqslant 0); \tag{2}$$

$$G_{i-1} = \langle \gamma_{i-1}, x_{i-1} \rangle \text{ subgroup of } F_i \wr \langle x_{i-1} \rangle \tag{3}$$

where

$$\gamma_{i-1} = (a_i, b_i, b_i^{-1}) \qquad (i \geqslant 1)$$

(note that all the G_i's are isomorphic);

$$\phi_{i-1} : F_{i-1} \twoheadrightarrow G_{i-1} \tag{4}$$

the extension of $a_{i-1} \to \gamma_{i-1}$, $b_{i-1} \to x_{i-1}$,

$$K_{i-1} = \ker(\phi_{i-1}) \qquad (i \geqslant 1)$$

(note that all the K_i's are isomorphic);

$$A_0 = F_0, \qquad A_1 = F_1 \wr \langle x_0 \rangle,$$

$$A_i = ((\ldots (F_i \wr \langle x_{i-1} \rangle) \wr \langle x_{i-2} \rangle) \wr \ldots) \wr \langle x_0 \rangle \qquad (i \geqslant 1); \tag{5}$$

$$D_0 = G_0, \qquad D_1 = G_1 \wr \langle x_0 \rangle,$$

$$D_i = ((\ldots (G_i \wr \langle x_{i-1} \rangle) \wr \langle x_{i-2} \rangle) \wr \ldots) \wr \langle x_0 \rangle \qquad (i \geqslant 1); \tag{6}$$

$$\psi_i\colon A_i \to D_i \text{ is the natural extension of } \phi_i; \tag{7}$$

$$\ker(\psi_i) = \mathsf{X}_{3^i} K_i \qquad (\text{a product of } 3^i \text{ copies of } K_i)\ (i \geq 0);$$

$$G^{(i)} = \psi_i(G^{(i-1)}) \qquad (i \geq 1). \tag{8}$$

Observe that for $i \geq 1$,

$$\gamma^{(i)} = ((...((a_{i+1}, b_{i+1}, b_{i+1}^{-1}), x_i, x_i^{-1})...), x_1, x_1^{-1}),$$

and that

$$G^{(i)} = \langle \gamma^{(i)}, x_0 \rangle.$$

As i increases, the (a, b)-block in $\gamma^{(i)}$ is pushed to the left and in the limit vanishes. Since the x_i's are isolated from each other by nested parentheses, we may rewrite all the x_i's as x, and thus arrive at

$$\gamma^{(\infty)} = (\gamma^{(\infty)}, x, x^{-1}),$$

and

$$G^{(\infty)} = \mathscr{G}.$$

4. Nonfinite Presentability

We will show that

$$\ker \psi_0 = K_0 < F_0' \qquad (\text{the derived group of } F_0),$$

$$\ker \psi_i = \mathsf{X}_{3^i} K_i < \mathsf{X}_{3^i} F_i' < (G^{(i-1)})' \qquad (i \geq 1),$$

$$F_0 \gneqq G^{(0)}, \qquad G^{(i)} \gneqq G^{(i+1)} \qquad (i \geq 0).$$

Given the last result, we would have

$$\mathscr{G} \text{ has no finite presentation.}$$

Lemma 1. *Let H be a group such that $H = \langle a, b \rangle$, $o(a) = o(b) = 3$, and let $\langle x \rangle \cong C_3$. Define $\tilde{H} = H \wr C_3$, and $\gamma = (a, b, b^{-1})$, $\beta = (b, b, b)$ elements of $\tilde{H}$. Also, let*

$$G = \langle \gamma, x \rangle, \qquad \Gamma = \langle \gamma \rangle^G.$$

Then,

(i) $[\gamma\gamma^{x}, \gamma^{x^{-1}}\gamma] = ([b^{-1}, a^{-1}], 1, 1) = [\beta^{-1}, \gamma^{-1}]$,

(ii) $\Gamma' = H' \times H' \times H'$, $\beta \in N_{\bar{H}}(G)$,

(iii) $G/\Gamma' = C_3 \wr C_3$, *if* $H/H' \cong C_3 \times C_3$.

Proof. Part (i) is straightforward. Part (ii) follows from (i) and the fact that Γ is a subdirect product of $H \times H \times H$. Part (iii) is a consequence of

$$\gamma^{i}(\gamma^{x})^{j}(\gamma^{x^{-1}})^{k} \equiv (a^{i}b^{-j+k}, a^{j}b^{i-k}, a^{k}b^{-i+j}) \qquad (\text{mod } \Gamma'),$$

for $i, j, k \in \{0, 1, 2\}$.

Since $G_0/(G_0)' \cong C_3 \times C_3$, clearly $K_0 < F_0'$. By the same token, $K_i < F_i'$ for $i \geq 0$. On applying the previous lemma, we have

$$\ker \psi_1 = \mathop{\times}_{3} K_1 < \mathop{\times}_{3} F_1' < (G^{(0)})'.$$

Similarly, it follows that

$$\ker \psi_i = \mathop{\times}_{3^i} K_i < \mathop{\times}_{3^i} F_i' < (G^{(i-1)})'$$

for $i \geq 1$.

For every $i \geq 1$, F_i' is a free group and $G^{(i-1)}$ contains $\times_{3^i} F_i'$ as a subgroup of finite index. A direct application of the next lemma proves that $G^{(i)} \not\cong G^{(i+1)}$ for $i \geq 0$.

LEMMA 2. *Let X be a finitely generated group with two subgroups Y, Z of finite index in X. Suppose Y and Z decompose as direct sums of k and ℓ free groups, respectively. Then, $k = l$.*

Proof. Let

$$Y = \sum_{i=1}^{k} Y_i, \qquad Z = \sum_{j=1}^{\ell} Z_j,$$

where Y_i's, Z_j's are free groups, and suppose $k \geq \ell$.

Let $M = Y \cap Z$. Then, as $[X: M] < \infty$, it follows that $M \cap Y_i \neq 1$ for all i. Let $M_0 = \sum_{i=1}^{k} M \cap Y_i$, and choose $y_i \in M \cap Y_i$, $y_i \neq 1$, for every i. Then for $y = y_1 y_2 \ldots, y_k$,

$$C_{M_0}(y) \text{ is a direct sum of } k \text{ infinite cyclic groups.}$$

Now, on considering $C_{M_0}(y) \leq C_Z(y)$, we arrive at $k = \ell$.

5. Recursive Presentation

Suppose

$$\{a_0, b_0 \mid a_0^3 = b_0^3 = 1,\ u_1 = u_2 = \cdots = u_k = 1\}$$

is a presentation for $G^{(0)}$, where the u_i's are commutator words. We will show how to extend this presentation recursively to a presentation for $\mathcal{G}$.

The u_i's are words in

$$c_1^{(0)} = [a_0, b_0], \qquad c_2^{(0)} = [a_0, b_0]^{a_0}, \qquad c_3^{(0)} = [a_0, b_0]^{b_0},$$
$$c_4^{(0)} = [a_0, b_0]^{a_0 b_0},$$

the free generators of the derived group F_0'. We will write $u_i = u_i(c_j^{(0)})$.

Note that $K_1 \times 1 \times 1$ is generated by the $F_1 \times 1 \times 1$ conjugates of, and thus by the $\langle \gamma^{(0)} \rangle^{G^{(0)}}$ conjugates of,

$$(u_i(c_j^{(1)}),\ 1,\ 1), \qquad 1 \leqslant i \leqslant k,\ 1 \leqslant j \leqslant 4.$$

Let F be a free group, freely generated by z_1, z_2, z_3, and define

$$\lambda_1 = \lambda_1(z_1, z_2, z_3) = [z_3^{-1} z_1^{-1}, z_1 z_2]$$
$$\lambda_2 = \lambda_1^{z_1}, \qquad \lambda_3 = \lambda_1^{z_3}, \qquad \lambda_4 = \lambda_1^{z_1 z_3}.$$

Then, on substituting $\gamma^{(0)}$, $(\gamma^{(0)})^{t_0}$, $(\gamma^{(0)})^{t_0^{-1}}$ for z_1, z_2, z_3 in the λ_i's we obtain

$$\begin{aligned} \lambda_1^{(1)} = \lambda_1(\gamma^{(0)}, (\gamma^{(0)})^{t_0}, (\gamma^{(0)})^{t_0^{-1}}) &= ([a_1, b_1],\ 1,\ 1) \\ &= (c_1^{(1)},\ 1,\ 1), \end{aligned}$$

$$\lambda_2^{(1)} = (c_2^{(1)},\ 1,\ 1), \qquad \lambda_3^{(1)} = (c_3^{(1)},\ 1,\ 1),$$
$$\lambda_4^{(1)} = (c_4^{(1)},\ 1,\ 1).$$

Thus,

$$\begin{aligned} (u_i(c_j^{(1)}),\ 1,\ 1) &= u_i(\lambda_1^{(1)}, \lambda_2^{(1)}, \lambda_3^{(1)}, \lambda_4^{(1)}) \\ &= u_i(\lambda_j^{(1)}) \qquad (1 \leqslant i \leqslant k). \end{aligned}$$

Let

$$\underline{\lambda_1^{(1)}} = \lambda_1(a_0, a_0^{b_0}, a_0^{b_0^{-1}}),$$
$$\underline{\lambda_2^{(1)}} = (\underline{\lambda_1^{(1)}})^{a_0}, \qquad \underline{\lambda_3^{(1)}} = (\underline{\lambda_1^{(1)}})^{b_0 a_0 b_0^{-1}}, \qquad \underline{\lambda_4^{(1)}} = (\underline{\lambda_1^{(1)}})^{a_0 b_0 a_0 b_0^{-1}}.$$

Then, the following is a presentation for $G^{(1)}$:

$$\{a_0, b_0 \mid a_0^3 = b_0^3 = 1, u_i(c_j^{(0)}) = 1, u_i(\lambda_j^{(1)}) = 1 \qquad (1 \leqslant i \leqslant k, 1 \leqslant j \leqslant 4)\}.$$

This process is clearly inductive and leads to a recursive presentation for $\mathscr{G}$.

6. A Finite Presentation for $G^{(0)}$

We will delete the subscripts from our symbols in $G^{(0)}$ $(= G_0)$. Thus,

$$G = \langle \gamma, x \rangle \leqslant F \wr \langle x \rangle = A,$$
$$F = \langle a \rangle * \langle b \rangle, \qquad \gamma = (a, b, b^{-1}),$$

where

$$\langle a \rangle \cong \langle b \rangle \cong \langle x \rangle \cong \langle \gamma \rangle \cong C_3.$$

Also,

$$G = \Gamma \cdot \langle x \rangle, \qquad \Gamma = \langle \gamma \rangle^G.$$

Furthermore, we have

$$\beta = (b, b, b) \in A$$

induces an automorphism of G.

The main problem in obtaining an explicit presentation for G lies in codifying the "parentheses development." For instance, how does one obtain the element $([a, b], 1, 1)$ from a presentation? This is easily gotten within $\hat{G} = G\langle \beta \rangle$:

$$\gamma^\beta = (a^b, b, b^{-1}) = \gamma([a, b], 1, 1),$$
$$[\gamma, \beta] = ([a, b], 1, 1).$$

We will give a presentation for $\hat{G}$ and from it derive a presentation for G.

PROPOSITION 3. *The following is a presentation for* $\hat{G}$.

$$\underline{\hat{G}} = \{\underline{\beta}, \underline{\gamma}, \underline{x} \mid \text{(i) } \underline{\beta}^3 = 1, \text{ (ii) } \underline{\gamma}^3 = \underline{x}^3 = 1,$$

(iii) $[\underline{x}, \underline{\beta}] = 1$,

(iv) (*Parentheses Development*)

$$[\underline{\gamma}'^i, \underline{\gamma}^j] = [\underline{\beta}^{-i}, \underline{\gamma}^j][\underline{\gamma}', \underline{\beta}^j]^i \qquad (i, j = \pm 1),$$

(v) (*Commutation of Components*) *for* $\mathcal{S} = \{[\underline{\gamma}^i, \underline{\beta}^j] \mid i, j = \pm 1\}$,

$$\mathcal{S} \text{ centralizes } \mathcal{S}^{\underline{x}},$$

(vi) (*Induced Action on Components*)

$$((\underline{\gamma}^{-j})^{\underline{x}^i} s (\underline{\gamma}^j)^{\underline{x}^i}) s^{-\beta^{-i}j} = 1 \qquad (s \in \mathcal{S}, i, j = \pm 1)\}.$$

Proof. Let

$$\phi: \hat{\underline{G}} \to \hat{G}$$

be the extension of $\underline{\gamma} \to \gamma$, $\underline{x} \to x$, $\underline{\beta} \to \beta$. We note that

$$\hat{\underline{G}} = \underline{G} \cdot \langle \underline{\beta} \rangle, \qquad \text{where} \quad \underline{G} = \langle \underline{\gamma}, \underline{x} \rangle,$$

$$\underline{G} = \underline{\Gamma} \cdot \langle \underline{x} \rangle, \qquad \text{where} \quad \underline{\Gamma} = \langle \underline{\gamma} \rangle^{\underline{G}}.$$

Clearly then,

$$|\hat{\underline{G}}/\underline{\Gamma}'| \leqslant 3^5, \qquad \text{and} \qquad \hat{\underline{G}}/\underline{\Gamma}' \cong \hat{G}/\Gamma'.$$

Hence,

$$\ker \phi \leqslant \underline{\Gamma}'.$$

Let ω be a word in γ, β. Then,

$$\omega(\gamma, \beta) = (\omega(a, b), \omega(b, b), \omega(b^{-1}, b)).$$

Clearly,

$$\langle \gamma, \beta \rangle \cong \langle a, b \rangle \cong C_3 * C_3;$$

thus,

$$\phi: \langle \underline{\gamma}, \underline{\beta} \rangle \twoheadrightarrow \langle \gamma, \beta \rangle$$

is an isomorphism.

We will show that $\underline{\Gamma}'$ decomposes as

$$\underline{\Gamma}' = \Delta \Delta^{\underline{x}} \Delta^{\underline{x}^{-1}},$$

where

$$\Delta = \langle \underline{\gamma}, \underline{\beta} \rangle'.$$

By relations (v) in the proposition, the three subgroups

$$\underline{\Delta}, \underline{\Delta}^{\underline{x}}, \underline{\Delta}^{\underline{x}^{-1}}$$

centralize each other. By relations (vi), the product $\underline{\Delta}\underline{\Delta}^{\underline{x}}\underline{\Delta}^{\underline{x}^{-1}}$ is normalized by $\underline{\gamma}$, and thus is a normal subgroup of $\hat{\underline{G}}$. It also contains $\underline{\Gamma}'$, by relations (iv). The desired equality will be a consequence of

$$[\underline{\gamma}^{-1}\underline{\gamma}^{-\underline{x}}, \underline{\gamma}^{-\underline{x}^{-1}}\underline{\gamma}^{-1}] = [\underline{\beta}, \gamma].$$

This is shown by developing the left-hand side to the point where relations (iv) can be used to transform the commutators from $\underline{\Gamma}$ into commutators from $\langle \underline{\gamma}, \underline{\beta} \rangle$, then relations (vi) are used to transform the conjugators from $\underline{\Gamma}$ into conjugators from $\langle \underline{\gamma}, \underline{\beta} \rangle$, then relations (v) are used to collect the $\underline{\Delta}$, $\underline{\Delta}^{\underline{x}}$, $\underline{\Delta}^{\underline{x}^{-1}}$ elements. As

$$\phi\colon [\underline{\gamma}^i, \underline{\beta}^j] \to [\gamma^i, \beta^j] = ([\alpha^i, \ell^j], 1, 1),$$

we have that

$$\phi\colon \underline{\Delta}\underline{\Delta}^{\underline{x}}\underline{\Delta}^{\underline{x}^{-1}} (= \underline{\Gamma}') \twoheadrightarrow F' \times F' \times F'.$$

Now, clearly, as $\underline{\Delta}$ affords a natural F'-presentation, $\phi|_{\underline{\Gamma}'}$ is an isomorphism. But then, by the first paragraph of the proof, $\ker \bar{\phi} = 1$.

At this point one can apply the Schreier–Reidmeister process to obtain a presentation for G. However, we will approach the problem directly by describing the action of β on G. For this purpose, we introduce the following notation:

(i) let

$$\mu(z_1, z_2, z_3) = [z_1 z_2, z_3^{-1} z_1^{-1}]\, z_1,$$

$$\eta(z_1, z_2, z_3) = [z_1 z_2, z_3 z_1] z_1$$

be words in z_1, z_2, z_3 free generators of some free group,

(ii) for ω an element of $\langle \underline{\gamma} \rangle * \langle \underline{x} \rangle$ define the symbols

$$(\omega)^{\beta^0} = \omega, \qquad (\omega)^{\beta^1} = \mu(\omega, \omega^{\underline{x}}, \omega^{\underline{x}^{-1}}),$$
$$(\omega)^{\beta^{-1}} = \eta(\omega, \omega^{\underline{x}}, \omega^{\underline{x}^{-1}})$$

and let

$$[\omega, \beta^{\pm 1}] = \omega^{-1}(\omega)^{\beta^{\pm 1}}, \qquad [\beta^{\pm 1}, \omega] = [\omega, \beta^{\pm 1}]^{-1}.$$

PROPOSITION 4. *Let*

$\underline{G} = \{\underline{\gamma}, \underline{x} \mid$ I. *Relations* (ii)–(iv) *of the previous proposition,*

II. (*Automorphism*)

(i) $((\underline{\gamma})^\beta)^\beta = (\underline{\gamma})^{\beta^{-1}}, ((\underline{\gamma})^{\beta^{-1}})^{\beta^{-1}} = (\underline{\gamma})^\beta, ((\underline{\gamma})^\beta)^{\beta^{-1}} = ((\underline{\gamma})^{\beta^{-1}})^\beta = \underline{\gamma}$,

(ii) *for every relation* $\omega(\underline{\gamma}, \underline{\gamma}^{\underline{x}}, \underline{\gamma}^{\underline{x}^{-1}}) = 1$ *in list* I,

$$\omega((\underline{\gamma})^\beta, (\underline{\gamma})^{\beta \underline{x}}, (\underline{\gamma})^{\beta \underline{x}^{-1}}) = 1,$$

$$\omega((\underline{\gamma})^{\beta^{-1}}, (\underline{\gamma})^{\beta^{-1} \underline{x}}, (\underline{\gamma})^{\beta^{-1} \underline{x}^{-1}}) = 1\} \qquad (116 \textit{ relations}).$$

Then, $\underline{G} \cong G$.

Proof. It is straightforward to check that $\underline{G}$ admits an automorphism $\beta\colon \underline{x} \to \underline{x}, \underline{\gamma} \to (\underline{\gamma})^\beta$, and that

$$\underline{G} \cdot \langle \beta \rangle \cong \hat{G}, \qquad \underline{G} \cong G.$$

ACKNOWLEDGMENTS

This research was done at Universidade de Brasilia and the University of Michigan. The author is grateful to Roger and Margaret Lyndon for their continued encouragement, to CNPq (Brazil) for travel support, and to NSF (U.S.A.) for research support from Grant 82-01601.

REFERENCES

1. S. V. ALESHIN, Finite automata and the Burnside problem for periodic groups, *Mat. Zametki* **11** (1972), 319–328 [Russian]. Engl. transl.: *Math. Notes* **11** (1972), 199–203.
2. R. I. GRIGORCHUK, On the Burnside problem for periodic groups, *Funktsional. Anal. i Prilozhen.* **14** (1980), 53–54, [Russian]. Engl. transl.: *Functional Anal. Appl.* **14** (1980), 41–43.
3. N. GUPTA AND S. SIDKI, On the Burnside problem for periodic groups, *Math. Z.* **182** (1983), 385–388.
4. N. GUPTA AND S. SIDKI, Extensions of groups by free automorphisms, *Contemp. Math.* **33** (1984), 232–246.
5. N. GUPTA AND S. SIDKI, Some infinite *p*-groups, *Algebra i Logika* **22** (1983), 584–589.
6. YU. I. MERZLYAKOV, On infinite finitely generated periodic groups, *Dokl. Akad. Nauk* **268** (1983), 803–805 [Russian]. Engl. transl.: *Soviet Math. Dokl.* **27** (1983), 169–172.