

INTEGRATION FORMULAS TO EVALUATE FUNCTIONS OF RANDOM VARIABLES

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ABSTRACT

Structural reliability analysis often involves extensive integration of a function of multiple basic variables. Special techniques can be used to reduce this numerical effort. This paper develops integration formulas for computing the statistical parameters of a function of a random vector, in particular calculation of the first few moments. The formula is a numerical procedure using selected weights and points to estimate integrals. The points and weights are predetermined in the independent standard normal variable space. The sample points in basic variable space are then obtained by various transformations. The formulas were developed for the basic variables with various probability information known. The procedure is convenient to implement, and allows for a direct use of the available deterministic computer programs.

INTRODUCTION

Recently, there have been extensive new developments in reliability theory and its application to structural engineering. Especially noteworthy has been the work of Ang and Cornell [1], Galambos and Ravindra [2], and Ellingwood et al. [3] in the United States, Lind [4] in Canada, and Ditlevsen [5] and Rackwitz [6] in Europe. A more critical review on structural reliability methods can be found in the book by Madsen et al. [7].

In structural reliability analysis, load and resistance models play a very important role. The models are derived using survey data, tests, analysis and engineering judgement. Often the resistance or load is expressed as a function of multiple basic variables such as material properties, dimensions, geometries, frequency and magnitude of load, and so on. It is then necessary to evaluate the statistical parameters, particularly the first few moments, of these

functions. Various numerical methods have been widely used, such as the Monte Carlo method [8], Latin Hypercube Sampling [9,10], and point estimates [11,12]. However, Monte Carlo simulation or Latin Hypercube Sampling require prohibitively large computational effort, although the number of simulations is independent of the number of basic variables. This paper develops a simple numerical integration procedure for computing the statistical parameters of a function of multiple random variables. The samples of basic variables are obtained by transforming *a priori* the selected points in standard normal space to basic variable space. The method is similar to point estimates, except that the points and weights are predetermined in standard normal space. The procedure is convenient to implement, and allows a direct use of the available deterministic computer programs.

THEORETICAL DEVELOPMENT

For a general function of a random vector $\mathbf{X} = (X_1, X_2, \dots, X_n)$, that is

$$G(\mathbf{X}) = G(X_1, X_2, \dots, X_n) \quad (1)$$

the exact k th moment of G , $E[G^k(\mathbf{X})]$, may be obtained by evaluating the integral:

$$E[G^k(\mathbf{X})] = \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} f_{\mathbf{X}}(x_1, \dots, x_n) G^k(x_1, \dots, x_n) dx_1 \dots dx_n \quad (2)$$

where $f_{\mathbf{X}}(x_1, \dots, x_n)$ is the joint probability density function (PDF) of the random vector \mathbf{X} . In many applications, however, the integration of eqn. (2) may be difficult to perform, and special numerical procedures are needed. Gorman [13] used the quadrature formulas to evaluate eqn. (2) for the special cases of normal and lognormal random variables. In this study numerical integration formulas were formulated for any jointly distributed random vector.

Single standard normal variable

Let Z denote a single standard normal variable with the probability density function defined as

$$f_Z(z) = \phi(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}z^2\right) \quad (3)$$

Substituting eqn. (3) into eqn. (2) and $z = \sqrt{2}u$ ($dz = \sqrt{2} du$) yields

$$E[G^k(Z)] = \int_{-\infty}^{+\infty} \phi(z) G^k(z) dz \quad (4a)$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \exp(-u^2) G^k(\sqrt{2}u) du \quad (4b)$$

The integration of eqn. (4b) can be evaluated using the Gauss-Hermite formula [14]:

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \exp(-u^2) G^k(\sqrt{2}u) du \cong \frac{1}{\sqrt{\pi}} \sum_{j=1}^m \alpha_j G^k(\sqrt{2}u_j) \quad (5)$$

where α_j and u_j are the j th weight factor and zero of the $(2m - 1)$ th order Hermite polynomial, respectively, which are widely available [14], and m is the number of points considered.

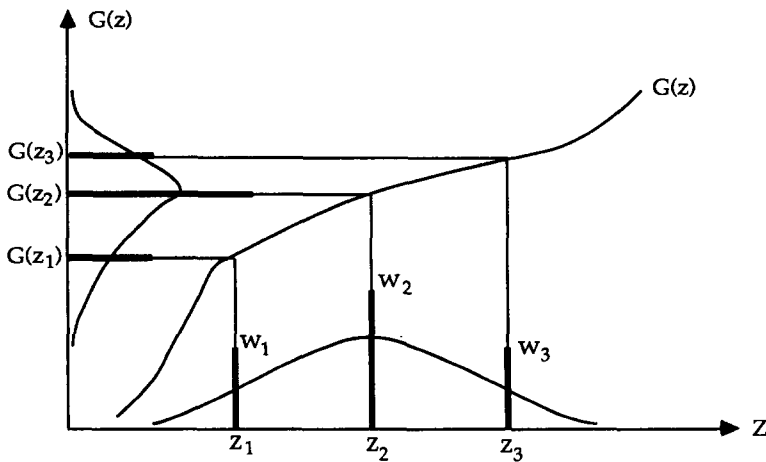


Fig. 1. Illustration of three-point integration.

Using $z_j = \sqrt{2} u_j$ and letting $w_j = \alpha_j / \sqrt{\pi}$, weight factors and points in standard normal space (Z space) are obtained. Thus eqn. (5) becomes

$$\int_{-\infty}^{+\infty} \phi(z) G^k(z) dz \cong \sum_{j=1}^m w_j G^k(z_j) \quad (6)$$

Equation (6) is the integration formula for a single standard normal variable, and it is graphically shown in Fig. 1. When $G^k(z)$ is a polynomial of at most the $(2m - 1)$ th degree, eqn. (6) is exact. Typical normal points z_j and the corresponding weights w_j are given in Table 1.

Single non-normal variable

Let X denote a non-normal variable in eqn. (2) and make a marginal transformation [15]:

$$X = F_X^{-1}(\Phi(z)) \quad (7)$$

where $\Phi(z)$ is the cumulative distribution function (CDF) of standard normal variable Z , and $F_X(x)$ is the cumulative distribution function of non-normal variable X . The transformation is graphically shown in Fig. 2. By this transformation the domain of the integration eqn. (2) is changed from X space to the standard normal space:

$$\int_{-\infty}^{+\infty} f_X(x) G^k(x) dx = \int_{-\infty}^{+\infty} \phi(z) G^k(F_X^{-1}(\Phi(z))) dz \quad (8)$$

The right side of eqn. (8) can be evaluated using eqn. (6)

$$\int_{-\infty}^{+\infty} \phi(z) G^k(F_X^{-1}(\Phi(z))) dz \cong \sum_{j=1}^m w_j G^k(F_X^{-1}(\Phi(z_j))) \quad (9)$$

That is, for a non-normal variable X , eqn. (2) can be calculated using the formula

$$E[G^k(X)] \cong \sum_{j=1}^m w_j G^k(x_j) \quad (10)$$

TABLE 1
Examples of points and weight factors for integration formulas

Number of points m	Points z_j	Weight factors w_j
1	$z_1 = 0$	$w_1 = 1$
2	$z_1 = -1$ $z_2 = +1$	$w_1 = \frac{1}{2}$ $w_2 = \frac{1}{2}$
3	$z_1 = -\sqrt{3}$ $z_2 = 0$ $z_3 = +\sqrt{3}$	$w_1 = \frac{1}{6}$ $w_2 = \frac{4}{6}$ $w_3 = \frac{1}{6}$
4	$z_1 = -\sqrt{3+\sqrt{6}}$ $z_2 = -\sqrt{3-\sqrt{6}}$ $z_3 = +\sqrt{3-\sqrt{6}}$ $z_4 = +\sqrt{3+\sqrt{6}}$	$w_1 = (3-\sqrt{6})/12$ $w_2 = (3+\sqrt{6})/12$ $w_3 = (3+\sqrt{6})/12$ $w_4 = (3-\sqrt{6})/12$
5	$z_1 = -\sqrt{5+\sqrt{10}}$ $z_2 = -\sqrt{5-\sqrt{10}}$ $z_3 = 0$ $z_4 = +\sqrt{5-\sqrt{10}}$ $z_5 = +\sqrt{5+\sqrt{10}}$	$w_1 = (7-2\sqrt{10})/60$ $w_2 = (7+2\sqrt{10})/60$ $w_3 = 8/15$ $w_4 = (7+2\sqrt{10})/60$ $w_5 = (7-2\sqrt{10})/60$

in which

$$x_j = F_X^{-1}(\Phi(z_j)) \tag{11}$$

where z_j and w_j are the points and weight factors associated with standard normal space. Equation (10) is the integration formula for a single non-normal variable.

Multiple standard normal variables

In many practical problems, reliability analysis often involves multiple random variables. In this study, the foregoing method has been generalized to a function of multiple variables. This

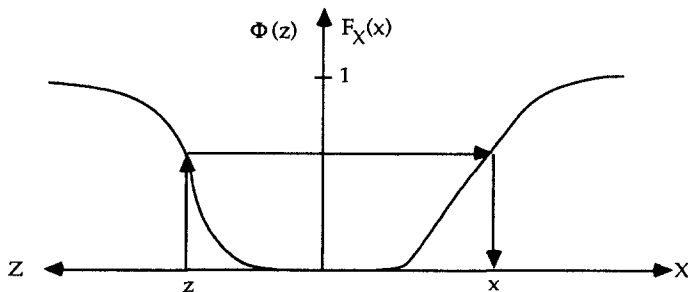


Fig. 2. Transformation of normal variable to non-normal variable.

section describes the development of integration formulas for a function of independent standard normal variables. The following sections describe integration formulas for a function of multiple non-normal variables.

In the case of an independent standard normal vector $\mathbf{Z} = (Z_1, Z_2, \dots, Z_n)$ the joint probability density function is,

$$f_{\mathbf{Z}}(z_1, z_2, \dots, z_n) = \phi(z_1)\phi(z_2)\dots\phi(z_n) \quad (12)$$

Equation (2) becomes

$$E[G^k(\mathbf{Z})] = \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \phi(z_1)\dots\phi(z_n)G^k(z_1, \dots, z_n) dz_1 \dots dz_n \quad (13)$$

Two approaches have been developed to evaluate above integration: the product integration formula and the non-product integration formula.

Product integration formula

The product formula is based on the factorization of G into single-variable functions:

$$G(Z_1, Z_2, \dots, Z_n) = \prod_{i=1}^n G_i(Z_i) \quad (14)$$

Such an operation is difficult since usually the function G is given only implicitly as a function of Z_1, Z_2, \dots, Z_n . Special techniques are required to approximately factorize $G(Z_1, \dots, Z_n)$ into $G_1(Z_1)G_2(Z_2)\dots G_n(Z_n)$. A method suggested by Rosenblueth [11] was modified for the current study.

Based on Rosenblueth's suggestion a function $G = G(Z_1, Z_2, \dots, Z_n)$ is approximated by a function $H = H(Z_1, Z_2, \dots, Z_n)$, which is defined by

$$H = G_0 \prod_{i=1}^n \frac{G_i^*}{G_0} = G_0 \prod_{i=1}^n G_i(Z_i) \quad (15)$$

where $G_0 = G(0, 0, \dots, 0)$, i.e., the function evaluated at the zero means of variables \mathbf{Z} , G_i^* are the functions of G computed as though Z_i were the only random variable, and other variables were set equal to their zero mean values, i.e.,

$$G_i^* = G(0, 0, \dots, Z_i, \dots, 0) \quad (16)$$

$$G_i = \frac{G_i^*}{G_0} \quad (17)$$

At the zero means

$$H = G \quad \text{and} \quad \frac{\partial H}{\partial z_i} = \frac{\partial G}{\partial z_i} \quad (18)$$

Substitution of H for G in eqn. (13) results in the following equation:

$$E[G^k(\mathbf{Z})] = G_0^k \prod_{i=1}^n \int_{-\infty}^{+\infty} \phi(z_i)G_i^k(z_i) dz_i \quad (19)$$

The repeated application of the integration formula, eqn. (6), leads to the approximation of eqn. (19):

$$E[G^k(\mathbf{Z})] \cong G_0^k \prod_{i=1}^n \left(\sum_{j=1}^{m_i} w_j G_i^k(z_{ij}) \right) = G_0^k \prod_{i=1}^n E[G_i^k(Z_i)] \quad (20)$$

where m_i is the number of points considered for variable Z_i . Equation (20) is the product integration formula for multiple independent standard normal variables.

Non-product integration formula

For a multiple independent standard normal vector \mathbf{Z} , the probability density function, eqn. (12), can be further expressed as

$$f_{\mathbf{Z}}(z_1, z_2, \dots, z_n) = \frac{1}{\sqrt{(2\pi)^n}} \exp\left(-\frac{1}{2}z_1^2 - \frac{1}{2}z_2^2 - \dots - \frac{1}{2}z_n^2\right) \quad (21)$$

As in the one-dimensional case, substituting eqn. (21) into eqn. (2) and $z_i = \sqrt{2} u_i$ ($dz_i = \sqrt{2} du_i$), eqn. (2) can be rewritten as

$$E[G^k(\mathbf{Z})] = \frac{1}{\sqrt{\pi^n}} \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \exp(-u_1^2 - u_2^2 - \dots - u_n^2) G^k(\sqrt{2} \mathbf{U}) du_1 \dots du_n \quad (22)$$

The integration of eqn. (22) can be evaluated using various Gauss quadrature rules [16,17]:

$$E[G^k(\mathbf{Z})] \cong \frac{1}{\sqrt{\pi^n}} \sum_{j=1}^m a_j G^k(\sqrt{2} u_{1j}, \dots, \sqrt{2} u_{nj}) \quad (23)$$

where α_j and (u_{1j}, \dots, u_{nj}) are widely available Gauss quadrature weight factors and points [16,17].

Substituting $z_{ij} = \sqrt{2} u_{ij}$ and $w_j = \alpha_j / \sqrt{\pi^n}$ into eqn. (23) the following non-product integration formula is obtained:

$$E[G^k(Z_1, \dots, Z_n)] \cong \sum_{j=1}^m w_j G^k(z_{1j}, z_{2j}, \dots, z_{nj}) \quad (24)$$

Typical weights w_j and points $(z_{1j}, z_{2j}, \dots, z_{nj})$ in independent standard normal space are given in Table 2.

Variables with joint distribution known

The integration formulas for independent standard normal variables can be extended to functions of variables with any type of joint distribution known. The extension is based on the Rosenblatt transformation [15].

Let a set of n random variables $\mathbf{X} = (X_1, \dots, X_n)$ have a joint CDF = $F_{\mathbf{X}}(x_1, x_2, \dots, x_n)$, which can be given in terms of a sequence of conditional distribution functions:

$$F_{\mathbf{X}}(\mathbf{X}) = F_{X_1}(x_1) F_{X_2|X_1}(x_2) \dots F_{X_n|X_1 \dots X_{n-1}}(x_n) \quad (25)$$

The original vector \mathbf{X} can be transformed sequentially and componentwise, using the following equalities;

$$\Phi(z_1) = F_{X_1}(x_1) \quad (26a)$$

$$\Phi(z_2) = F_{X_2|X_1}(x_2) \quad (26b)$$

$$\Phi(z_3) = F_{X_3|X_1 X_2}(x_3) \quad (26c)$$

⋮

$$\Phi(z_n) = F_{X_n|X_1 \dots X_{n-1}}(x_n) \quad (26n)$$

TABLE 2
Examples of points and weights for non-product integration formulas

Formulas m	Points $(z_{1j}, \dots, z_{ij}, \dots, z_{nj})$	Weight factors w_j
$n+1$	$Z_1 = (\sqrt{n}, 0, 0, \dots, 0)$ $Z_2 = \left(-\sqrt{\frac{1}{n}}, \sqrt{\frac{(n+1)(n-1)}{n}}, 0, \dots, 0\right)$ $Z_3 = \left(-\sqrt{\frac{1}{n}}, \sqrt{\frac{(n+1)}{n(n-1)}}, \sqrt{\frac{(n+1)(n-2)}{n-1}}, 0, \dots, 0\right)$... $Z_n = \left(-\sqrt{\frac{1}{n}}, -\sqrt{\frac{(n+1)}{n(n-1)}}, -\sqrt{\frac{(n+1)}{(n-1)(n-2)}}, \dots, \sqrt{\frac{n+1}{2}}\right)$ $Z_{n+1} = \left(-\sqrt{\frac{1}{n}}, -\sqrt{\frac{(n+1)}{n(n-1)}}, -\sqrt{\frac{(n+1)}{(n-1)(n-2)}}, \dots, -\sqrt{\frac{n+1}{2}}\right)$	$w_1 = \frac{1}{n+1}$ $w_2 = \frac{1}{n+1}$ $w_3 = \frac{1}{n+1}$... $w_n = \frac{1}{n+1}$ $w_{n+1} = \frac{1}{n+1}$
$2n$	$Z_1 = -Z_{n+1} = (\sqrt{n}, 0, \dots, 0)$ $Z_2 = -Z_{n+2} = (0, \sqrt{n}, 0, \dots, 0)$... $Z_n = -Z_{n+n} = (0, 0, \dots, \sqrt{n})$	$w_1 = w_{n+1} = \frac{1}{2n}$ $w_2 = w_{n+2} = \frac{1}{2n}$... $w_n = w_{n+n} = \frac{1}{2n}$
$2n^2+1$	$Z = (0, 0, \dots, 0)$ $Z = (\pm\sqrt{n+2}, 0, \dots, 0)^a$ $Z = \left(\pm\sqrt{\frac{n+2}{2}}, \pm\sqrt{\frac{n+2}{2}}, \dots, 0\right)^a$	$w_j = \frac{2}{n+2}$ $w_j = \frac{4-n}{2(n+2)^2}$ $w_j = \frac{1}{(n+2)^2}$

^a Points include all possible permutations of coordinates.

and hence

$$x_1 = F_{X_1}^{-1}[\Phi(z_1)] \quad (27a)$$

$$x_2 = F_{X_2|X_1}^{-1}[\Phi(z_2) | x_1] \quad (27b)$$

$$x_3 = F_{X_3|X_1X_2}^{-1}[\Phi(z_3) | x_1x_2] \quad (27c)$$

⋮

$$x_n = F_{X_n|X_1 \dots X_{n-1}}^{-1}[\Phi(z_n) | x_1 \dots x_{n-1}] \quad (27n)$$

The Jacobian of the transformation is written as:

$$J = \frac{\phi(z_1) \dots \phi(z_n)}{f_X(x_1, \dots, x_n)} \quad (28)$$

By the above transformation, the integration of eqn. (2) over the non-normal space X is performed over the independent standard normal space Z :

$$\begin{aligned} E[G^k(X)] &= \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} f_X(x_1, \dots, x_n) G^k(x_1, \dots, x_n) dx_1 \dots dx_n \\ &= \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} f_X(x_1, \dots, x_n) G^k(x_1, \dots, x_n) J dz_1 \dots dz_n \\ &= \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \phi(z_1) \dots \phi(z_n) G^k(x_1, \dots, x_n) dz_1 \dots dz_n \end{aligned} \quad (29)$$

Equation (29) can be evaluated using either the product integration formula eqn. (20) or the non-product integration formula eqn. (24). Suppose eqn. (24) is used, then eqn. (29) becomes:

$$E[G^k(X_1, \dots, X_n)] \cong \sum_{j=1}^m w_j G^k(x_{1j}, \dots, x_{nj}) \quad (30)$$

with

$$x_{ij} = F_{X_i|X_1, \dots, X_{i-1}}^{-1}[\Phi(z_{ij}) | x_{1j}, \dots, x_{(i-1)j}] \quad (31)$$

where i denotes the i th variable and j denotes j th point; w_j and $(z_{1j}, \dots, z_{ij}, \dots, z_{nj})$ can be obtained from Table 2.

Therefore, arbitrary distributions can also be handled by integration formulas for normal distributions provided that the conditional distribution functions can be produced. In principle, there are $n!$ probabilistically equivalent transformations differing in the ordering of the variables. In general, not all of them have the same favorable numerical properties [18]. Therefore, in some problematic cases it is worthwhile to try one of the alternatives.

Variables with marginal distributions known

In many applications, the joint PDF $f_X(x_1, \dots, x_n)$ is often unknown, as complete statistical information on X is seldom available. In most situations the information available on X consists of only the mean vector \bar{X} , the correlation matrix C , and the set of marginal CDF's, $F_{X_i}(x_i)$, $i = 1, \dots, n$. In this situation the integration formulas derived in the previous sections cannot be directly applied. To use the formulas it is natural to seek a multivariate distribution model which itself is based on the normal distribution. In addition, the model sought must be consistent with the known marginals and the correlation matrix of the random vector X . There are a large number of such models available in the literature [19]. The model used by Der Kiureghian and Liu [20] for reliability analysis under incomplete probability information is chosen for the current study.

First define $Y = (Y_1, Y_2, \dots, Y_n)$ as joint standard normal variables by assigning a joint distribution to X_1, \dots, X_n :

$$f_X(x_1, \dots, x_n) = \phi_n(y_1, \dots, y_n, C_0) \frac{f_{X_1}(x_1) \dots f_{X_n}(x_n)}{\phi(y_1) \dots \phi(y_n)} \quad (32)$$

in which $f_{X_i}(x_i) = dF_{X_i}(x_i)/dx_i$ is the marginal PDF of X_i , and $\phi_n(\mathbf{Y}, \mathbf{C}_0)$ is the joint PDF of n -dimension normal variates with zero means, unit standard deviations, and correlation matrix \mathbf{C}_0 . The correlation coefficients $\rho_{0,ij}$ of \mathbf{C}_0 are expressed in terms of the correlation coefficient ρ_{ij} of X_i and X_j through the integration:

$$\begin{aligned} \rho_{ij} &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\frac{x_i - \bar{X}_i}{\sigma_i} \right) \left(\frac{x_j - \bar{X}_j}{\sigma_j} \right) \phi_2(y_i, y_j, \rho_{0,ij}) \frac{f_{X_i}(x_i) f_{X_j}(x_j)}{\phi(y_i) \phi(y_j)} dx_i dx_j \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\frac{x_i - \bar{X}_i}{\sigma_i} \right) \left(\frac{x_j - \bar{X}_j}{\sigma_j} \right) \phi_2(y_i, y_j, \rho_{0,ij}) dy_i dy_j \end{aligned} \quad (33)$$

Equation (33) can be solved iteratively to find $\rho_{0,ij}$. To avoid such tedious calculations, approximate formulas for the ratio

$$F = \rho_{0,ij} / \rho_{ij} \quad (34)$$

are available in the literature [20,21].

After establishing the multivariable distribution model, eqn. (32), the k th moment of the function $G(X_1, \dots, X_n)$ can be evaluated as the mathematical expectation of $G^k(\mathbf{X})$

$$\begin{aligned} E[G^k(\mathbf{X})] &= \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} f_{\mathbf{X}}(x_1, \dots, x_n) G^k(\mathbf{X}) dx_1 \dots dx_n \\ &= \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \phi_n(\mathbf{Y}, \mathbf{C}_0) \frac{f_{X_1}(x_1) \dots f_{X_n}(x_n)}{\phi(y_1) \dots \phi(y_n)} G^k(\mathbf{X}) d\mathbf{X} \end{aligned} \quad (35)$$

The domain of the above integration can be converted from non-normal space \mathbf{X} to the correlated standard normal space \mathbf{Y} by the marginal transformation:

$$Y_i = \Phi^{-1}[F_{X_i}(x_i)] \quad i = 1, 2, \dots, n \quad (36)$$

To use the integration formulas derived in previous sections the correlated normal space \mathbf{Y} must be mapped onto uncorrelated standard normal space \mathbf{Z} by the linear transformation:

$$\mathbf{Z} = \mathbf{L}_0^{-1} \mathbf{Y} \quad (37)$$

where \mathbf{L}_0 = the lower triangular matrix obtained from Cholesky decomposition of \mathbf{C}_0 .

Since eqns. (36) and (37) are two one-to-one transformations, their inverse transformations exist and therefore \mathbf{Z} can be mapped to \mathbf{Y} , and then \mathbf{Y} to \mathbf{X}

$$\mathbf{Y} = \mathbf{L}_0 \mathbf{Z} \quad (38)$$

$$X_i = F_{X_i}^{-1}[\Phi(y_i)] \quad (39)$$

Combining eqn. (38) with eqn. (39) finally \mathbf{Z} is mapped to \mathbf{X} :

$$\mathbf{X} = \mathbf{F}^{-1}[\Phi(\mathbf{L}_0 \mathbf{Z})] \quad (40)$$

The Jacobian of the combined transformation, eqn. (40), is obtained as:

$$J = \frac{\phi(y_1) \dots \phi(y_n)}{f_{X_1}(x_1) \dots f_{X_n}(x_n)} |\mathbf{L}_0| \quad (41)$$

where

$$|\mathbf{L}_0| = \begin{vmatrix} \frac{\partial y_1}{\partial z_1} & \dots & \frac{\partial y_1}{\partial z_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_n}{\partial z_1} & \dots & \frac{\partial y_n}{\partial z_n} \end{vmatrix} \quad (42)$$

By the transformation, eqn. (40), the integration, eqn. (35), finally is performed over the domain of the uncorrelated standard normal space:

$$\begin{aligned} & \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \phi_n(\mathbf{Y}, \mathbf{C}_0) \frac{f_{X_1}(x_1) \dots f_{X_n}(x_n)}{\phi(y_1) \dots \phi(y_n)} G^k(\mathbf{X}) dx_1 \dots dx_n \\ &= \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \phi_n(\mathbf{Y}, \mathbf{C}_0) \frac{f_{X_1}(x_1) \dots f_{X_n}(x_n)}{\phi(y_1) \dots \phi(y_n)} G^k(\mathbf{X}) J dz_1 \dots dz_n \\ &= \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \phi(z_1) \dots \phi(z_n) G^k(\mathbf{X}) dz_1 \dots dz_n \end{aligned} \quad (43)$$

The integration, eqn. (43), can be evaluated using either product or non-product integration formulas. Suppose the non-product formula eqn. (24) is used, then eqn. (43) is calculated by

$$E[G^k(\mathbf{X})] \cong \sum_{j=1}^m w_j G^k(x_{1j}, x_{2j}, \dots, x_{nj}) \quad (44)$$

in which $(x_{1j}, x_{2j}, \dots, x_{nj})$ is obtained using eqns. (38) and (39):

$$\begin{pmatrix} y_{1j} \\ \vdots \\ y_{nj} \end{pmatrix} = \begin{pmatrix} L_{11} & \dots & 0 \\ \vdots & L_{ii} & \vdots \\ L_{n1} & \dots & L_{nn} \end{pmatrix} \begin{pmatrix} z_{1j} \\ \vdots \\ z_{nj} \end{pmatrix} \quad (45)$$

$$\begin{pmatrix} x_{1j} \\ \vdots \\ x_{nj} \end{pmatrix} = \begin{pmatrix} F_{X_1}^{-1}[\Phi(y_{1j})] \\ \vdots \\ F_{X_n}^{-1}[\Phi(y_{nj})] \end{pmatrix} \quad (46)$$

where the first subscript i denotes the i th variable and the second subscript j denotes the j th point; w_j and (z_{1j}, \dots, z_{nj}) can be obtained from Table 2.

NUMERICAL EXAMPLES

Three examples are given to illustrate the use of the developed integration formulas. The application of the approach to bridge reliability analysis can be found in Ref. [22].

Example 1. Single variable function

A simple example is given here to show the use of integration formula eqn. (6) and eqn. (10). Consider a function of a single variable

$$G(X) = X^2 \quad (47)$$

Case 1: Standard normal variable. Let X be a standard normal variable denoted by Z with zero mean and unit variance. The mean value of G is calculated using three points. From eqn. (6):

$$\begin{aligned} \bar{G} &= E[G(Z)] \cong \sum_{j=1}^3 w_j G(z_j) \\ &= \frac{1}{6}(-\sqrt{3})^2 + \frac{4}{6}(0)^2 + \frac{1}{6}(+\sqrt{3})^2 = 1 \end{aligned} \quad (48)$$

The result is exact since in this case the mean of G is equal to the variance of Z ($\sigma_Z^2 = 1$).

Case 2: Non-normal variable. Assume that X is a lognormally distributed random variable with cumulative distribution function (CDF):

$$F_X(x) = \Phi\left(\frac{\ln x - 2.283}{0.198}\right) \quad (49)$$

In this case a four-point integration formula is used. The corresponding points and weight factors in standard normal space are taken from Table 2:

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} -\sqrt{3 + \sqrt{6}} \\ -\sqrt{3 - \sqrt{6}} \\ +\sqrt{3 - \sqrt{6}} \\ +\sqrt{3 + \sqrt{6}} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{pmatrix} = \frac{1}{12} \begin{pmatrix} 3 - \sqrt{6} \\ 3 + \sqrt{6} \\ 3 + \sqrt{6} \\ 3 - \sqrt{6} \end{pmatrix} \quad (50)$$

The points in non-normal space are obtained using eqn. (11)

$$x_j = F_X^{-1}(\Phi(z_j)) = \exp(2.283 + 0.198z_j) \quad (51)$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 6.18 \\ 8.47 \\ 11.4 \\ 15.6 \end{pmatrix} \quad (52)$$

The mean value of G is calculated by using the integration formula eqn. (10),

$$\bar{G} \cong \sum_{j=1}^4 w_j G(x_j) = 104 \quad (53)$$

The exact value of \bar{G} is also 104.

Calculations are also performed for X with other types of distribution. The results are compared with the exact value. The comparisons are summarized in Table 3. The most accurate results are obtained for the integration formula eqn. (10). The mean value of G is also evaluated using eqn. (10) with a different number of points. The results are shown in Table 4. From the

TABLE 3

Two-point integration vs. types of random variables; $G = X^2$ with $\bar{X} = 10, \sigma_X = 2$

Distribution	x_1	x_2	$\frac{x_1 - \bar{X}}{\sigma_X}$	$\frac{x_2 - \bar{X}}{\sigma_X}$	\bar{G}	Error (%)
Normal	8.00	12.00	-1.000	1.000	104.00	0.00
Gamma	8.01	11.99	-0.995	0.995	103.95	0.05
Lognormal	8.04	11.95	-0.980	0.975	103.82	0.17
Beta	7.96	12.04	-1.020	1.020	104.16	0.16
Weibull	7.98	12.00	-1.010	1.000	104.04	0.04
Extreme Type I	8.15	11.84	-0.925	0.920	103.40	0.57
Extreme Type II	8.35	11.57	-0.825	0.785	102.59	1.36

TABLE 4

Accuracy vs. number of integration points

Distribution	Mean value of function: $G = X^2$				
	2 points	3 points	4 points	5 points	6 points
Normal	104.0000				
Gamma	103.9470	104.0000			
Lognormal	103.8209	103.9968	104.0000		
Beta	104.1649	103.9932	104.0004	104.0000	
Weibull	104.0421	104.0285	103.9944	104.0004	104.0000
Extreme Type I	103.4037	103.9888	104.0032	103.9992	104.0000
Extreme Type II	102.5860	103.6864	103.9470	103.9912	103.9992

table one can see how many points are required in integration to obtain accurate results for different variables.

Example 2. Multivariate function

To illustrate the use of integration formulas for multivariate, consider the wind pressure acting on a window. The pressure is given by the formula

$$G = \frac{1}{2}\gamma (X_1 + X_2)^2 \tag{54}$$

where G is the wind pressure, γ is the density of air, X_1 is the basic wind speed, and X_2 is the gust wind speed. Consider γ to be deterministic with a constant value of 2, X_1 and X_2 to be random variables with $\bar{X}_1 = 10, \sigma_{X_1} = 1, \bar{X}_2 = 20,$ and $\sigma_{X_2} = 1.$

Case 1: X_1 and X_2 with normal distribution. The independent standard normal variables are defined as

$$Z_1 = \frac{X_1 - \bar{X}_1}{\sigma_{X_1}} = X_1 - 10 \tag{55a}$$

$$Z_2 = \frac{X_2 - \bar{X}_2}{\sigma_{X_2}} = X_2 - 20 \tag{55b}$$

The wind pressure G is further expressed in terms of the independent standard normal vector \mathbf{Z} :

$$G = (Z_1 + Z_2 + 30)^2 \quad (56)$$

Both product and non-product formulas are used to estimate the statistics of function G .

Using the product integration formula, eqn. (20), G_0 , G_1^* , and G_2^* are obtained as

$$G_0 = G(0, 0) = 900 \quad (57a)$$

$$G_1^* = G(Z_1, 0) = (Z_1 + 30)^2 \quad (57b)$$

$$G_2^* = G(0, Z_2) = (Z_2 + 30)^2 \quad (57c)$$

From eqns. (15–17) the wind pressure G is approximated by the product of two single-variable functions:

$$G \cong H = 900 \frac{(Z_1 + 30)^2}{900} \frac{(Z_2 + 30)^2}{900} \quad (58)$$

From eqn. (20) the mean of the wind pressure acting on a window can be calculated by

$$\bar{G} \cong 900 \left(\sum_{j=1}^{m_1} w_j \frac{(z_{1j} + 30)^2}{900} \right) \left(\sum_{j=1}^{m_2} w_j \frac{(z_{2j} + 30)^2}{900} \right) \quad (59)$$

If the two-point formula ($m_1 = m_2 = 2$) is used, then from Table 1,

$$\begin{pmatrix} z_{11} \\ z_{12} \end{pmatrix} = \begin{pmatrix} z_{21} \\ z_{22} \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} \quad (60)$$

Thus the mean of G is

$$\bar{G} \cong 900 \left(\frac{29^2 + 31^2}{1800} \right)^2 = 902.0011 \quad (61)$$

The exact mean of G is equal to 902.

The non-product integration formula, eqn. (24), approximates the mean of G by

$$\bar{G} \cong \sum_{j=1}^m w_j (z_{1j} + z_{2j} + 30)^2$$

If a $(n + 1)$ -point formula is used, then from Table 2 the points and the weight factors are

$$\begin{pmatrix} z_{11} & z_{21} \\ z_{12} & z_{22} \\ z_{13} & z_{23} \end{pmatrix} = \begin{pmatrix} \sqrt{2} & 0 \\ -\sqrt{\frac{1}{2}} & \sqrt{\frac{3}{2}} \\ -\sqrt{\frac{1}{2}} & -\sqrt{\frac{3}{2}} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} \quad (62)$$

The mean value of G , \bar{G} , is calculated as

$$\bar{G} = 902 \quad (63)$$

This is the exact mean of G .

Case 2: X_1 and X_2 with joint PDF known.

$$f_X(x_1, x_2) = \frac{1}{\sqrt{2\pi} \sigma_{X_1}} \exp\left(-\frac{1}{2} \left(\frac{x_1 - \bar{X}_1}{\sigma_{X_1}}\right)^2\right) \frac{1}{\sqrt{2\pi} \sigma_{\ln X_2} \sqrt{1 - \rho_{X_1, \ln X_2}^2}} \frac{1}{x_2} \\ \times \exp\left(-\frac{1}{2} \left(\frac{\ln x_2 - \overline{\ln X_2} - \rho_{X_1, \ln X_2} (\sigma_{\ln X_2} / \sigma_{X_1}) (x_1 - \bar{X}_1)}{\sigma_{\ln X_2} \sqrt{1 - \rho_{X_1, \ln X_2}^2}}\right)^2\right) \quad (64)$$

where

$$\sigma_{\ln X_2} = \sqrt{\ln(1 + V_{X_2}^2)} \quad (65a)$$

$$\overline{\ln X_2} = \ln \bar{X}_2 - \frac{1}{2} \sigma_{X_2}^2 \quad (65b)$$

$$\rho_{X_1, \ln X_2} = \frac{V_{X_2} \rho_{X_1, X_2}}{\sqrt{\ln(1 + V_{X_2}^2)}} \quad (65c)$$

and

$$\begin{bmatrix} \rho_{X_1, X_1} & \rho_{X_1, X_2} \\ \rho_{X_2, X_1} & \rho_{X_2, X_2} \end{bmatrix} = \begin{bmatrix} 1 & 0.6 \\ 0.6 & 1 \end{bmatrix} \quad (66)$$

The solution for the first two moments of G proceeds as follows:

(1) Find conditional CDF's from the given joint distribution.

$$F_{X_1}(x_1) = \Phi\left(\frac{x_1 - \bar{X}_1}{\sigma_{X_1}}\right) \quad (67a)$$

$$F_{X_2|X_1}(x_2|x_1) = \Phi\left(\frac{\ln x_2 - \overline{\ln X_2} - \rho_{X_1, \ln X_2} (\sigma_{\ln X_2} / \sigma_{X_1}) (x_1 - \bar{X}_1)}{\sigma_{\ln X_2} \sqrt{1 - \rho_{X_1, \ln X_2}^2}}\right) \quad (67b)$$

(2) Perform the Rosenblatt transformation using eqn. (27). In this example, the transformation can be obtained as follows,

$$X_1 = F_{X_1}^{-1}[\Phi(z_1)] = \bar{X}_1 + \sigma_{X_1} Z_1 \quad (68a)$$

$$X_2 = F_{X_2|X_1}^{-1}[\Phi(z_2)|x_1] = \exp\left(\overline{\ln X_2} + \rho_{X_1, \ln X_2} (\sigma_{\ln X_2} / \sigma_{X_1}) (x_1 - \bar{X}_1) + \sigma_{\ln X_2} \sqrt{1 - \rho_{X_1, \ln X_2}^2} Z_2\right) \quad (68b)$$

Using the given parameters the above two equations can be simplified as

$$X_1 = 10 + Z_1 \quad (69a)$$

$$X_2 = \exp(2.9956 + 0.03Z_1 + 0.04Z_2) \quad (69b)$$

(3) Select the points and weight factors in the standard normal space. If the $(n + 1)$ -point formula is used, then from Table 2 the points and the weight factors are

$$\begin{pmatrix} z_{11} & z_{21} \\ z_{12} & z_{22} \\ z_{13} & z_{23} \end{pmatrix} = \begin{pmatrix} \sqrt{2} & 0 \\ -\sqrt{\frac{1}{2}} & \sqrt{\frac{3}{2}} \\ -\sqrt{\frac{1}{2}} & -\sqrt{\frac{3}{2}} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} \quad (70)$$

(4) Calculate (x_{1j}, x_{2j}) using eqn. (31)

$$\begin{pmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \\ x_{13} & x_{23} \end{pmatrix} = \begin{pmatrix} 11.414 & 20.866 \\ 9.293 & 20.563 \\ 9.293 & 18.644 \end{pmatrix} \quad (71)$$

(5) Calculate the first two moments of G using eqn. (30)

$$E[G(X)] = \sum_j^3 \frac{1}{3} G(x_{1j}, x_{2j}) = 904.484 \quad (72)$$

$$E[G^2(X)] = \sum_j^3 \frac{1}{3} G^2(x_{1j}, x_{2j}) = 829821.3 \quad (73)$$

The exact value of \bar{G} for this case is equal to 902.72.

Case 3: X_1 and X_2 with marginal distributions known

$$F_{X_1}(x_1) = \exp(-\exp(-1.283(x_1 - 9.55))) \quad (74a)$$

$$F_{X_2}(x_2) = \Phi\left(\frac{\ln x_2 - 2.996}{0.05}\right) \quad (74b)$$

and the correlation matrix

$$C = \begin{bmatrix} 1 & 0.6 \\ 0.6 & 1 \end{bmatrix} \quad (75)$$

The solution for the first two moments of G proceeds as follows:

(1) Construct the correlation matrix C_0 by using the available formulas [20,21],

$$C_0 = \begin{bmatrix} 1 & 0.6158 \\ 0.6158 & 1 \end{bmatrix} \quad (76)$$

(2) Determine L_0 by Cholesky decomposition of C_0 :

$$L_0 = \begin{bmatrix} 1 & 0 \\ 0.6158 & 0.7879 \end{bmatrix} \quad (77)$$

(3) Select normal integration formulas. Suppose the $(n + 1)$ non-product formula is used,

$$\begin{bmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \end{bmatrix} = \begin{bmatrix} \sqrt{2} & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} \quad (78)$$

(4) Calculate correlated standard normal vectors Y using eqn. (45)

$$\begin{pmatrix} y_{1j} \\ y_{2j} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0.6158 & 0.7879 \end{pmatrix} \begin{pmatrix} z_{1j} \\ z_{2j} \end{pmatrix} \tag{79}$$

$$\begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \end{bmatrix} = \begin{bmatrix} 1.414 & -0.707 & -0.707 \\ 0.871 & 0.530 & -1.40 \end{bmatrix} \tag{80}$$

(5) Perform marginal transformation using eqn. (46)

$$x_{1j} = F_{X_1}^{-1}(\Phi(y_{1j})) = 9.55 - \frac{1}{1.283} \ln(-\ln(\Phi(y_{1j}))) \tag{81a}$$

$$x_{2j} = F_{X_2}^{-1}(\Phi(y_{2j})) = \exp(2.995 + 0.05y_{2j}) \tag{81b}$$

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix} = \begin{bmatrix} 11.495 & 9.270 & 9.270 \\ 20.887 & 20.534 & 18.645 \end{bmatrix} \tag{82}$$

(6) Calculate the first two moments of G

$$E[G] = \frac{1}{3}G(x_{11}, x_{21}) + \frac{1}{3}G(x_{12}, x_{22}) + \frac{1}{3}G(x_{13}, x_{23}) = 905.4 \tag{83}$$

$$E[G^2] = \frac{1}{3}G^2(x_{11}, x_{21}) + \frac{1}{3}G^2(x_{12}, x_{22}) + \frac{1}{3}G^2(x_{13}, x_{23}) = 831930.5 \tag{84}$$

The exact mean of G is 902.72. The 0.2% error is due to the approximation in calculating the correlation coefficient $\rho_{0,12}$ and the nonlinearity of G expressed in terms of independent standard variables. The error due to the nonlinearity of G can be reduced by using integration formulas with more points.

Example 3. Modeling of reinforced concrete section

In this example the ultimate bending moment of the reinforced section shown in Fig. 3 is considered. This case was also studied by Ditlevsen [23]. The ultimate bending moment is

$$M_U = X_1 X_2 X_3 - X_4 \frac{X_1^2 X_2^2}{X_5 X_6} \tag{86}$$

where X_1 is the area of reinforcement, X_2 the yield stress of the reinforcement, X_3 the effective depth of the reinforcement, X_4 a factor related to the stress-strain relationship of concrete, X_5

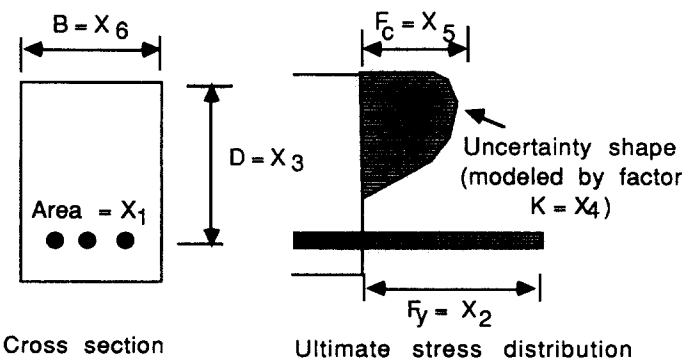


Fig. 3. State of ultimate stresses for reinforced concrete section.

TABLE 5
Statistics of basic variables in Example 3

Variable	Mean value	Standard deviation
X_1	1260 mm ²	63 mm ²
X_2	250 N/mm ²	17.5 N/mm ²
X_3	770 mm	10 mm
X_4	0.55	0.055
X_5	30 N/mm ²	4.5 N/mm ²
X_6	250 mm	5 mm

TABLE 6
Comparison of integration method with Monte Carlo method

Method	\bar{R}_{FM} (10 ⁶) (N mm)	V_{FM}
3-point integration	235	0.085
5-point integration	235	0.085
100 simulations	237	0.074
500 simulations	234	0.086
5000 simulations	235	0.084

the maximum compressive strength of the concrete, and X_6 is the width of the beam. (X_1, X_2, \dots, X_6) are basic variables with lognormal distributions, and mutually uncorrelated. The mean values and standard deviations are given in Table 5.

The mean and coefficient of variation of M_U were calculated using 3- and 5-point product integration formulas, and also evaluated using Monte Carlo simulations. The results given in Table 6 indicate that integration formulas give results with good accuracy.

CONCLUSIONS

A set of numerical integration formulas were developed to compute the statistical parameters of a function of multiple random variables. The formula is a numerical procedure to estimate integrals using selected weights and points. The points and weights are predetermined in the independent standard normal variable space. The sample points in basic variable space are obtained by special transformations. The formulas were developed depending on the available statistical data about the basic variables.

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