Stochastic Processes and their Applications 28 (1988) 301-315
North-Holland

ASYMPTOTIC EXPANSIONS FOR FIRST PASSAGE TIMES

Michael WOODROOFE
Department of Statistics, The University of Michigan, Ann Arbor, MI, USA

Received 4 December 1986
Revised 10 December 1987

Let F be a strongly non-lattice distribution function with a positive mean, a positive variance, and a finite third moment. Let $X_1, X_2, \ldots$ be i.i.d. with common distribution function F; and let $S_n = X_1 + \cdots + X_n$ and $t^a = \inf\{n \geq 1: S_n > a\}$ for $n \geq 1$ and $a > 0$. The main result reported here is a two term asymptotic expansion for $H_a(n, z) = P\{t^a < n, S_n - a \leq z\}$ as $a \to \infty$. Assuming higher moments, a three term expansion for $P\{t^a \leq n\}$ and refined estimates for the probability of ruin in finite time are obtained as simple corollaries. A key tool is an asymptotic expansion in Stone's formulation of the local limit theorem.

AMS (1980) Classifications: Primary 60F05; Secondary 60J15.

Edgeworth expansions * local limit theorem * ruin problems

1. Introduction

The purpose of this article is to develop asymptotic expansions related to first passage times, under weak moment and smoothness conditions. Towards this end, let F denote a (right continuous) distribution function with a positive mean μ, a positive variance σ^2, a finite third moment, and higher moments as needed. Suppose further that F satisfies Cramér's Condition (is strongly non-lattice); that is,

$$\limsup_{s \to \infty} |\psi(s)| < 1,$$

where

$$\psi(s) = \int_{\mathbb{R}} e^{isx} F(dx), \quad s \in \mathbb{R}.$$

Let $X_1, X_2, \ldots$ be i.i.d. random variables with common distribution function F; and denote the random walk and first passage times by $S_0 = 0$,

$$S_n = X_1 + \cdots + X_n, \quad n \geq 1,$$

$$t^a = \inf\{n \geq 1: S_n > a\}, \quad a > 0,$$

Research supported by the National Science Foundation under DMS-8413452.

and

$$t_a = \inf\{n \geq 1 : S_n \leq a\}, \quad a \in \mathbb{R},$$

where the infimum of the empty set is understood to be ∞. Here $t^a < \infty$, w.p. 1 for all $a > 0$, since $\mu > 0$; the probability that $t_a < \infty$ is estimated below.

The main result of this paper is a two term asymptotic expansion, as $a \to \infty$, for the function

$$H_a(n, z) = P\{t^a < n, S_n - a \leq z\},$$

defined for $z \in \mathbb{R}$, $n \geq 1$, and $a > 0$. A three term asymptotic expansion for the distribution of t^a is derived as a simple corollary.

These results have potential applications to sequential analysis, where stopping times of the form t^a and t_a arise naturally. For example, the stopping time of the sequential probability ratio test is the minimum of two such first passage times. After tilting, there are also applications to the ruin problem. These are indicated in Corollary 3.

There are also more speculative potential applications. In principle, the main result allows one to compute higher order expansions for the distributions of randomly stopped sums, as in Woodroofe and Keener (1987); and the main result here is a first step towards its own extension to non-linear boundaries. These possibilities are sufficiently complicated to warrant separate consideration, however.

This paper is similar to Takahashi (1987) and Keener (1987) in that higher order expansions for boundary crossing probabilities are sought. It is also similar to Lalley (1984) in that weak moment and smoothness conditions are sought.

A useful alternative expression for $H_a(n, z)$ is presented in Section 2. An asymptotic expansion in Stone's (1965) formulation of the local limit theorem is developed in Section 3. This is a key tool which may be of independent interest. The main result and its corollaries are presented in Section 4; and the main result is proved in Section 5. Remarks and examples occupy Section 6.

The notations and assumptions listed in the first two paragraphs are used throughout this paper, except in Section 3, where the mean may be non-positive. They are not repeated in the statements of lemmas and theorems.

2. An identity

Let F^k denote the distribution function of S_k, the k-fold convolution of F with itself, for $k \geq 1$; and let M_k denote a regular conditional distribution for $\min\{S_1, \ldots, S_{k-1}\}$ given S_k for $k \geq 2$. In symbols,

$$F^k(x) = P\{S_k \leq x\}$$

and

$$M_k(x; y) = P\{\min[S_1, \ldots, S_{k-1}] \leq y | S_k = x\}$$

for all x, $y \in \mathbb{R}$ and $k \geqslant 2$. In addition, it is convenient to let $M_1(x; y) = 0$ for all x and y. The same symbol is used to denote a distribution function and the induced measure.

If G is a monotone function, then $G(y-)$ denotes the limit of $G(x)$ as $x \to y$ from below. The symbol $\int_a^b$ means $\int_{(a,b]}$; and $\int_{a-}^b = \int_{[a,b]}$, etc.

Lemma 1. *With the notation of the previous two paragraphs,*

$$H_a(n, z) = \sum_{k=1}^{n-1} h_a(n, k, z), \tag{1}$$

where

$$h_a(n, k, z) = \int_{-\infty}^{z} \int_{x}^{z} [1 - M_k(x; y-)] F^{n-k}(\mathrm{d}y + a - x) F^k(\mathrm{d}x)$$

for $k = 1, \ldots, n-1$, $n > 1$, $z \in \mathbb{R}$, $a > 0$.

Proof. It suffices to consider fixed n, z, and a. Since the joint distributions of $S_1, \ldots, S_n$ and $S_n - S_{n-k}$, $k = 1, \ldots, n$, are the same,

$$\begin{aligned} H_a(n, z) &= P\{S_k > a, \exists k < n, S_n - a \leqslant z\} \\ &= P\{S_k < S_n - a, \exists k < n, S_n - a \leqslant z\} \\ &= \sum_{k=1}^{n-1} P\{S_j \geqslant S_n - a, \forall j < k, S_k < S_n - a \leqslant z\}. \end{aligned}$$

Next, for fixed $2 \leqslant k \leqslant n-1$, (a version of) the conditional distribution of $\min\{S_1, \ldots, S_{k-1}\}$ and $S_n - S_k$ given $S_k = x$ is the product of $M_{k-1}(x; \cdot)$ and F^{n-k} for a.e. $x \in \mathbb{R}(F^k)$. So,

$$\begin{aligned} &P\{S_j \geqslant S_n - a, \forall j < k, S_k < S_n - a \leqslant z \mid S_k = x\} \\ &\quad = \int_x^z [1 - M_k(x; y-)] F^{n-k}(\mathrm{d}y + a - x) \end{aligned}$$

for a.e. $x \leqslant z(F^k)$ and $k = 2, \ldots, n-1$ by Fubini's Theorem applied to the conditional distributions; and the last relation is also true if $k = 1$. The lemma then follows by integrating over $x \leqslant z$ and summing over $k = 1, \ldots, n-1$. □

Many of the terms in (1) contribute negligibly. To see why, let

$$q_m = P\{|S_k - k\mu| > k\mu/2, \exists k \geqslant m\}$$

and

$$q(z) = P\{t_z < \infty\} = P\{S_k \leqslant z, \exists k \geqslant 1\}$$

for $m = 1, 2, \ldots$ and $z \in \mathbb{R}$. Then, since F is assumed to have a finite third moment,

$$\sum_{m=1}^{\infty} m q_m < \infty \quad \text{and} \quad \int_{-\infty}^{0} |z| q(z)\, \mathrm{d}z < \infty. \tag{2}$$

See, for example, Baum and Katz (1965) and Chow and Teicher (1978, pp. 362 and 368). Of course, (2) implies that $q_m + q(-m) = o(m^{-2})$, since both are non-increasing.

Next, let

$$H_{a,m}(n, z) = \sum_{k=1}^{m} h_{a,m}(n, k, z),$$

where

$$h_{a,m}(n, k, z) = \int_{-m}^{z} \int_{x}^{z} [1 - M_k(x; y-)] F^{n-k}(dy + a - x) F^k(\mathrm{d}x)$$

for $k = 1, \ldots, n-1$, m, $n \geq 1$, $z \in \mathbb{R}$, and $a > 0$.

Lemma 2. *With the notation of the previous two paragraphs,*

$$|H_a(n, z) - H_{a,m}(n, z)| \leq q_m + q(-m) \tag{3}$$

for $-m < z < m\mu/2$, $n > m$, *and* $a > 0$.

Proof. If $z < m\mu/2$ and $n > m$, then the left side of (3) is at most

$$\sum_{k=m}^{n-1} h_a(n, k, z) + \sum_{k=1}^{m} [h_a(n, k, z) - h_{a,m}(n, k, z)] \leq P\{S_k < z, \exists k \geq m\}$$
$$+ P\{S_k \leq -m, \exists k \geq 1\} \leq q_m + q(-m). \quad \square$$

There is a useful alternative expression for $H_{a,m}$, which may be obtained by integration by parts.

Lemma 3. *For* $z > -m$, $n > m$, *and* $a > 0$,

$$\begin{aligned} H_{a,m}(n, z) = &\sum_{k=1}^{m} \int_{-m}^{z} \int_{x}^{z} [F^{n-k}(a + y - x) - F^{n-k}(a)] M_k(x; \mathrm{d}y) F^k(\mathrm{d}x) \\ &+ \sum_{k=1}^{m} \int_{-m}^{z} [F^{n-k}(a + z - x) - F^{n-k}(a)][1 - M_k(x; z)] F^k(\mathrm{d}x). \end{aligned} \tag{4}$$

Proof. If M is a distribution function and G is non-decreasing and right continuous, then

$$\begin{aligned} \int_{x}^{z} [1 - M(y-)] G(\mathrm{d}y) &= \int_{x}^{z} \left\{ \int_{y-}^{z} + \int_{z}^{\infty} \right\} M(\mathrm{d}w) G(\mathrm{d}y) \\ &= \int_{x}^{z} [G(w) - G(x)] M(\mathrm{d}w) + [1 - M(z)][G(z) - G(x)] \end{aligned}$$

for all $x < z$, by Fubini's theorem. The lemma follows easily. $\square$

3. A local expansion

In this section only, the mean μ may be non-positive.

If F has a finite pth moment, where $p \geq 3$, then F^n has an Edgeworth expansion; that is, uniformly in $s \in \mathbb{R}$,

$$F^n(s) = \Phi_{p,n}\left[\frac{s-n\mu}{\sigma\sqrt{n}}\right] + o[n^{-(p-2)/2}], \tag{5}$$

where

$$\Phi_{p,n} = \Phi + \sum_{j=1}^{p-2} n^{-j/2} Q_j,$$

Φ denotes the standard normal distribution function, and $Q_1, Q_2, \ldots$ are linear combinations of the derivatives of $\phi = \Phi'$, whose coefficients are determined by the cumulants of F. See, for example, Feller (1966, pp. 509 and 515) or Gnedenko and Kolmogorov (1954, Sections 38 and 45). In particular, $Q_1 = -(\rho/6\sigma^3)\phi''$ and $Q_2 = (\kappa/24\sigma^4)\phi''' + (\rho^2/72\sigma^6)\phi^{(\mathrm{iv})}$, where ρ and κ denote the third and fourth cumulants.

Theorem 1. *If F has a finite pth moment, where $p \geq 3$, then there is a $0 < \delta < 1$ for which*

$$F^n(b+c) - F^n(b-c) = \Phi_{p,n}\left[\frac{b+c-n\mu}{\sigma\sqrt{n}}\right] - \Phi_{p,n}\left[\frac{b-c-n\mu}{\sigma\sqrt{n}}\right] + o[n^{-(p-1)/2}]\{c+\delta^n\} \tag{6}$$

uniformly with respect to $b \in \mathbb{R}$ and $c > 0$.

Proof. There is no loss of generality in supposing that $\mu = 0$ and $\sigma = 1$. For fixed $c > 0$ and $m > 1$, let $g = g(\cdot\,; m, c)$ be the symmetric function for which $g(x) = 1$ for $0 \leq x \leq c$, $g(x) = 1 - m(x-c)$ for $c < x \leq c + 1/m$, and $g(x) = 0$ for $x > c + 1/m$. Then g dominates the indicator of the interval for $-c$ to c. So,

$$F^n(b+c) - F^n(b-c) \leq E\{g(S_n - b)\} \tag{7}$$

for all b, c, m, and n. Let $\hat{g}$ denote the Fourier transform of g; that is $\hat{g}(s) = \int_{\mathbb{R}} e^{isx} g(x)\,dx$, $s \in \mathbb{R}$. Then it is easily seen that $\hat{g}(0) = 2c + 1/m$ and $\int_{\mathbb{R}} |\hat{g}(s)|\,ds \leq (4mc+2)\pi$. Now,

$$E\{g(S_n - b)\} = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{g}(s)\, e^{-isb} \psi(s)^n\,ds$$

and

$$\int_{\mathbb{R}} g(x-b)\Phi_{p,n}\left(\frac{dx}{\sqrt{n}}\right) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{g}(s)\, e^{-isb}\psi_{p,n}(s\sqrt{n})\,ds,$$

where $\psi_{p,n}$ denotes the Fourier transform of (the signed measure) $\Phi_{p,n}$. Let $\varepsilon > 0$ be so small that $|\psi(s)| \leq \exp(-s^2/4)$ for all $|s| \leq \varepsilon$. Then

$$\left| E\{g(S_n - b)\} - \int_{\mathbb{R}} g(x-b)\Phi_{p,n}\left(\frac{\mathrm{d}x}{\sqrt{n}}\right)\right| \leq \frac{1}{2\pi}(I+J), \tag{8}$$

where

$$I = \int_{-\varepsilon}^{\varepsilon} |\hat{g}(s)|\,|\psi(s)^n - \psi_{p,n}(s\sqrt{n})|\,\mathrm{d}s$$

and

$$J = 2\int_{\varepsilon}^{\infty} |\hat{g}(s)|[|\psi(s)|^n + |\psi_{p,n}(s\sqrt{n})|]\,\mathrm{d}s.$$

In view of Cramér's Condition, the form of the derivative of $\Phi_{p,n}$, and the bound on the integral of $|\hat{g}|$, it is easily seen that there is a $0 < \delta < 1$, for which

$$J \leq 4\delta^{3n}\int_{\varepsilon}^{\infty} |\hat{g}(s)|\,\mathrm{d}s \leq (8mc+4)\pi\delta^{3n} \tag{9}$$

for all b, c, m, and n. Moreover, since $|\hat{g}(s)| \leq \hat{g}(0) = 2c + 1/m$, for all $s \in \mathbb{R}$,

$$\begin{aligned} I &\leq \frac{2c+1/m}{\sqrt{n}}\int_{-\varepsilon\sqrt{n}}^{\varepsilon\sqrt{n}} \left|\psi\left(\frac{s}{\sqrt{n}}\right)^n - \psi_{p,n}(s)\right|\,\mathrm{d}s \\ &= \frac{2c+1/n}{\sqrt{n}}\,\mathrm{o}[n^{(p-2)/2}], \end{aligned} \tag{10}$$

uniformly in b and c, where the final equality follows from standard arguments. (See, for example, Feller (1966, p. 507) or Gnedenko and Kolmogorov (1954, Section 45).) Finally, it is easily seen that there is a C for which

$$\int_{\mathbb{R}} g(x-b)\Phi_{p,n}\left(\frac{\mathrm{d}x}{\sqrt{n}}\right) - \left[\Phi_{p,n}\left(\frac{b+c}{\sqrt{n}}\right) - \Phi_{p,n}\left(\frac{b-c}{\sqrt{n}}\right)\right] \leq \frac{C}{m\sqrt{n}} \tag{11}$$

for all b, c, m, and n. Letting $m = 1/\delta^{3n/2}$ and combining (7)-(11) shows that the left side of (6) is less than or equal to the right; and a similar lower bound, using $g(\cdot\,; c - 1/m, m)$ in place of $g(\cdot\,; c, m)$ may be obtained to complete the proof. □

4. The main theorem

In the statement of the main theorem and its corollaries, let

$$K_1(z) = \int_{-\infty}^{z} q(y)\,\mathrm{d}y = \int_{-\infty}^{z} P\{t_y < \infty\}\,\mathrm{d}y,$$

and

$$K_2(z) = \int_{-\infty}^{z}\left\{\int_{\{t_y<\infty\}} (y - S_{t_y} + \mu t_y)\,\mathrm{d}P\right\}\mathrm{d}y,$$

The relation between K_1, K_2, and F is explored in Section 6. That K_2 is finite is shown in the proof.

Theorem 2. *Consider values of $n = n(a)$ and $z = z(a)$ for which*

$$a_n := \frac{a - n\mu}{\sigma\sqrt{n}} = \mathrm{o}(\sqrt{n}) \quad \text{and} \quad z = \mathrm{o}(\sqrt{n}). \tag{12}$$

Then

$$H_a(n, z) = \frac{1}{\sigma\sqrt{n}}\,\phi(a_n)K_1(z) + \frac{1}{n}\left\{\frac{1}{\sigma^2}\,\phi'(a_n)K_2(z) + \frac{1}{\sigma}\,Q_1'(z)K_1(z)\right\}$$
$$+\,\mathrm{o}\left(\frac{1}{n}\right)[1 + (z^+)^2], \tag{13}$$

as $a \to \infty$, where $z^+ = \max\{0, z\}$.

It is easy to describe the proof of Theorem 2. For appropriately chosen $m = m(a)$, the expansion (6) may be substituted for F^{n-k} in (4), and the resulting expression may be simplified. The details are presented in the next section.

The following corollary provides an asymptotic expansion for the distributions of t^a and $\max\{S_1, \ldots, S_n\}$.

Corollary 1. *Suppose that F has a finite fourth moment. If $n = n_a \to \infty$ as $a \to \infty$ in such a manner that $a_n = \mathrm{o}(\sqrt{n})$, then*

$$P\{t^a \leq n\} = (1 - \Phi)(a_n) + \frac{1}{\sqrt{n}}\left\{\frac{1}{\sigma}\,\phi(a_n)K_1(0) - Q_1(a_n)\right\}$$
$$+\,\frac{1}{n}\left\{\frac{1}{\sigma^2}\,\phi'(a_n)K_2(0) + \frac{1}{\sigma}\,Q_1'(a_n)K_1(0) - Q_2(a_n)\right\} + \mathrm{o}(1/n).$$

Proof. Since $S_n > a$ implies $t^a \leq n$,

$$P\{t^a \leq n\} = P\{S_n > a\} + P\{t^a < n, S_n \leq a\}$$
$$= (1 - F^n)(a) + H_a(n, 0) \tag{14}$$

for all n and a. The corollary follows by substituting the expansions (5) and (13) into (14). $\square$

The next corollary provides an alternative formulation of the main result. Let

$$H^a(n, z) = P\{t^a \geq n, 0 < S_n - a \leq z\},$$

for all $z > 0$, $n \geq 1$, and $a > 0$. Alternatively, $H^a(n, z) = P\{t^a = n, S_n - a \leq z\}$, since $\{t^a = n\} \subset \{S_n > a\} \subset \{t^a \leq n\}$.

Corollary 2. *Under the conditions of Theorem 2 with $z>0$,*

$$H^a(n,z)=\frac{1}{\sigma\sqrt{n}}\phi(a_n)K^1(z)+\frac{1}{n}\left\{\frac{1}{\sigma^2}\phi'(a_n)K^2(z)+\frac{1}{\sigma}Q_1'(a_n)K^1(z)\right\}$$
$$+\mathrm{o}\left(\frac{1}{n}\right)[1+z^2],$$

where

$$K^1(z)=K_1(0)-K_1(z)+z$$

and

$$K^2(z)=K_2(0)-K_2(z)+z^2/2,\quad z>0.$$

Proof. Corollary 2 follows easily from Theorems 1 and 2 and the relations

$$H^a(n,z)=P\{0<S_n-a\leqslant z\}-P\{t^a<n, 0<S_n-a\leqslant z\}$$

and

$$P\{t^a<n, 0<S_n-a\leqslant z\}=H_a(n,z)-H_a(n,0).\quad \square$$

Corollary 2 may be used to refine Lalley's (1984) estimates for the probability of ruin in finite time. To see how, let G denote a distribution function with a negative mean, a positive variance, and a moment generating function which is finite on some neighborhood Ω of the origin. Suppose also that G satisfies Cramér's Condition. Then G may be embedded in an exponential family; that is, $G=G_0$, where

$$G_\omega(dx)=\exp[\omega x-\gamma(\omega)]G(\mathrm{d}x),\quad x\in\mathbb{R},\quad \omega\in\Omega,\tag{15}$$

and $\exp[\gamma(\omega)]$ is the moment generating function of G at ω. The mean and variance of G_ω are then $\mu(\omega)=\gamma'(\omega)$ and $\sigma^2(\omega)=\gamma''(\omega)$ for all $\omega\in\Omega$.

Let $X_1, X_2,\ldots$ denote the coordinate functions on R^∞, and let P_ω denote the (unique) probability measure which makes $X_1, X_2,\ldots$ independent with common distribution function G_ω for each $\omega\in\Omega$. If $X_1, X_2,\ldots$ are regarded as the monthly losses of an insurance company with initial capital $a>0$, under P_0, then

$$p(a,N)=P_0\{t^a\leqslant N\}$$

is the probability of ruin within N months. This is approximated below when N is of the form $N=a/\mu(\omega)$ for some fixed ω for which $\mu(\omega)>0$ and $\gamma(\omega)>0$.

Corollary 3. *With the assumptions of the previous two paragraphs, let $a\to\infty$ through integer multiples of $\mu(\omega)$. Then*

$$p(a,N)=\left[\frac{C}{\sqrt{N}}+\mathrm{o}\left(\frac{1}{N}\right)\right]\mathrm{e}^{-N\Delta},\tag{16}$$

where

$$\Delta = \omega\mu(\omega) - \gamma(\omega) > 0,$$

$$C = \left\{\frac{\omega}{\sqrt{[2\pi\gamma''(\omega)][1-e^{-\gamma(\omega)}]}}\right\} \int_0^\infty K^1(z;\omega)\, e^{-\omega z}\, dz,$$

and $K^1(\cdot\,;\omega)$ *is as in Corollary* 2 *with* $F = G_\omega$ (*and* $\sqrt{[x]}$ *denotes the square root of* x).

Proof. Since the proof is similar to Lalley's, which yields (16) with $o(1/N)$ replaced by $o(1/\sqrt{N})$, it is sufficient to outline the argument. Writing F for G_ω and P for P_ω, it is easily seen that the conditions imposed in the Introduction are satisfied. (See Lemma 4 of Bahadur and Rao (1960) for Cramér's Condition.) By tilting and integration by parts,

$$p(a, N) = q(a)\, e^{-N\Delta},$$

where

$$q(a) = \sum_{n=1}^{N} \int_0^\infty \omega \exp[-\omega z + (n-N)\gamma(\omega)] H^a(n, z)\, dz.$$

It is easily seen that only values of n and z for which $0 \leqslant N - n \leqslant N^{1/8}$ and $0 \leqslant z \leqslant N^{1/8}$ need be considered. Then the expansion of Corollary 2 may be substituted for H^a; and (16) results. The reason that the coefficient of $1/N$ vanishes is that $a_N = 0$ and, therefore, $\phi'(a_N) = 0 = Q_1'(a_N)$. $\square$

5. Proof of Theorem 2

There is no loss of generality in supposing that $\sigma = 1$, since a may be replaced by a/σ; and it suffices to prove the theorem for fixed functions $n = n(a)$ and $z = z(a)$ which satisfy (12). Thus, n and z are functions of a in the proof, and limits are taken as $a \to \infty$, unless otherwise specified. Observe that $n/a \to 1/\mu$, by (12). Let

$$c_k^2 = k^2[q_k + q(-k)] + \sum_{j=k}^{\infty} j[q_j + q(-j)], \quad k \geqslant 1.$$

Then $c_k \to 0$ as $k \to \infty$. So, there is an integer valued function $m = m(a)$ for which

$$m/\sqrt{n} \to 0, \quad |z|/m \to 0 \quad \text{and} \quad \sqrt{n} \cdot c_m/m \to 0$$

as $a \to \infty$. With this choice of m, the right side of (3) is of smaller order of magnitude than $1/n$; so, it suffices to prove the theorem with H_a replaced by $H_{a,m}$.

The next step is to substitute (6) into (4). Here (6) must be applied with n replaced by $n-k$, where $1 \leqslant k \leqslant m$, $b - c = a$ and $b + c = a + w$, where $0 \leqslant w \leqslant z + m$. Now

$$\frac{a + w - (n-k)\mu}{\sqrt{(n-k)}} = a_n + \frac{b(w)}{\sqrt{(n-k)}},$$

where

$$b(w)=[\sqrt{n}-\sqrt{(n-k)}]a_n+(w+k\mu)$$

for all w (and the dependence of b on n, k, and a has been suppressed in the notation). Observe that $|b(w)|\leq C(k+w)$ for some constant C for all $1\leq k\leq m$ and $0\leq w\leq 2m$, so that $b(w)/\sqrt{(n-k)}\to 0$ uniformly in $1\leq k\leq m$ and $0\leq w\leq 2m$. So, by (6), Taylor's Theorem, and the uniform continuity of ϕ'',

$$\begin{aligned}F^{n-k}(a+w)-F^{n-k}(a)&=\Phi_{3,n-k}\left[a_n+\frac{b(w)}{\sqrt{(n-k)}}\right]-\Phi_{3,n-k}\left[a_n+\frac{b(0)}{\sqrt{(n-k)}}\right]\\&\quad+\mathrm{o}\left(\frac{1}{n-k}\right)[w+\delta^n]\\&=\frac{1}{\sqrt{n}}\phi(a_n)w+\frac{1}{2n}\{\phi'(a_n)w(w+2k\mu)+2Q_1'(a_n)w\}\\&\quad+\mathrm{o}(1/n)(k^2+w^2),\end{aligned}\tag{17}$$

uniformly in $1\leq k\leq m$ and $0\leq w\leq 2m$, since $\sqrt{(n-k)}\leq 2/n$ for $k\leq m$ and large a. When combined with (4), (17) shows that

$$\begin{aligned}H_{a,m}(n,z)&=\frac{1}{\sqrt{n}}\phi(a_n)\,\mathrm{sum}_1+\frac{1}{n}\phi'(a_n)\,\mathrm{sum}_2+\frac{1}{n}Q_1'(a_n)\,\mathrm{sum}_1\\&\quad+\mathrm{o}\left(\frac{1}{n}\right)(R_{a,1}+R_{a,2}),\end{aligned}\tag{18}$$

where

$$\begin{aligned}\mathrm{sum}_1&=\sum_{k=1}^{m}\int_{-m}^{z}\int_{x}^{z}(y-x)M_k(x;\mathrm{d}y)F^k(\mathrm{d}x)\\&\quad+\sum_{k=1}^{m}\int_{-m}^{z}(z-x)[1-M_k(x;z)]F^k(\mathrm{d}x),\\2\,\mathrm{sum}_2&=\sum_{k=1}^{m}\int_{-m}^{z}\int_{x}^{z}(y-x)(y-x+2k\mu)M_k(x;\mathrm{d}y)F^k(\mathrm{d}x)\\&\quad+\sum_{k=1}^{m}\int_{-m}^{z}(z-x)(z-x+2k\mu)[1-M_k(x;z)]F^k(\mathrm{d}x),\\|R_{a,1}|&\leq\sum_{k=1}^{m}\int_{-m}^{z}k^2[1-M_k(x;x)]F^k(\mathrm{d}x),\end{aligned}$$

and

$$\begin{aligned}|R_{a,2}|&\leq\sum_{k=1}^{m}\int_{-m}^{z}\int_{x}^{z}(y-x)^2M_k(x;\mathrm{d}y)F^k(\mathrm{d}x)\\&\quad+\sum_{k=1}^{m}\int_{-m}^{z}(z-x)^2[1-M_k(x;z)]F^k(\mathrm{d}x).\end{aligned}$$

The next step is to estimate the remainder terms. Towards this end, let $T = \inf\{k: S_k = \inf_j S_j\}$. Then

$$\int_{-m}^{z} [1 - M_k(x; x)] F^k(dx) \leqslant P\{S_j > S_k \ \forall j < k\}$$

$$\leqslant CP\{T = k\} \quad \forall k \geqslant 1,$$

where $C = 1/P\{S_j \geqslant 0, \forall j \geqslant 1\}$. Now $E(T^2) < \infty$, since

$$P\{T > k\} \leqslant P\{S_j < X_1, \exists j > k\} \leqslant q_k \quad \text{for all } k \geqslant 1.$$

So,

$$|R_{a,1}| \leqslant CE(T^2) < \infty. \tag{19}$$

Next, a simple integration by parts, as in the proof of Lemma 3, shows that for $-m < x \leqslant z$ and $k \leqslant m$,

$$\int_x^z (y-x)^2 M_k(x; dy) + (z-x)^2[1 - M_k(x; z)] = 2\int_x^z (y-x)[1 - M_k(x; y)]\,\mathrm{d}y.$$

So,

$$|R_{a,2}| \leqslant \sum_{k=1}^{m} \int_{-m}^{z} \int_x^z 2(y-x)[1 - M_k(x; y)]\,\mathrm{d}y F^k(\mathrm{d}x)$$

$$\leqslant 2\int_{-m}^{z} \int_{\{t_y < \infty\}} (y - S_{t_y})\,\mathrm{d}P\,\mathrm{d}y$$

$$\leqslant 2\int_{-m}^{z} \int_0^{\infty} P\{S_k \leqslant y - x, \exists k \geqslant 1\}\,\mathrm{d}x\,\mathrm{d}y$$

$$\leqslant 2\int_{-\infty}^{z} \int_{-\infty}^{y} q(w)\,\mathrm{d}w\,\mathrm{d}y \leqslant C[1 + (z^+)^2] \tag{20}$$

for some constant C.

It remains to simplify the two sums. For the first integration by parts (as above) shows that

$$\text{sum}_1 = \sum_{k=1}^{m} \int_{-m}^{z} \int_{-m}^{y} [1 - M_k(x; y)] F^k(\mathrm{d}x)\,\mathrm{d}y = \int_{-m}^{z} q_m(y)\,\mathrm{d}y$$

where

$$q_m(y) = \sum_{k=1}^{m} P\{S_j > y, \forall j < k, -m < S_k \leqslant y\}$$

$$= P\{t_y \leqslant m, S_{t_y} > -m\}, \quad y > -m.$$

Now, $0 \leq q(y) - q_m(y) \leq q(-m) + q_m$, as in the proof of Lemma 2. So, for all sufficiently large a,

$$\int_{-m}^{z} [q(y) - q_m(y)]\,\mathrm{d}y \leq 2m[q(-m) + q_m] = \mathrm{o}(1/\sqrt{n});$$

and $\int_{-\infty}^{-m} q(y)\,\mathrm{d}y = \mathrm{o}(1/\sqrt{n})$. So,

$$\mathrm{sum}_1 = \int_{-\infty}^{z} q(y)\,\mathrm{d}y + \mathrm{o}(1/\sqrt{n}) = K_1(z) + \mathrm{o}(1/\sqrt{n}). \tag{21}$$

The analysis of sum_2 is similar to that of sum_1 with one exception; it is not a priori clear that $K_2(z)$ is finite. That $\int_{-\infty}^{z} \{\int_{t_y<\infty} (y - S_{t_y})\,\mathrm{d}P\}\,\mathrm{d}y$ is finite is shown in (20). For the term involving t_y, let τ_k, $k \geq 0$, denote the (strict) descending ladder epochs; that is, $\tau_0 = 0$ and $\tau_k = \inf\{n: S_n < S_{\tau_{k-1}}\} \leq \infty$ for $k \geq 1$. Also, let $J = \max\{k: \tau_k < \infty\}$. Then

$$\begin{aligned}\int_{-\infty}^{0} \left\{\int_{t_y<\infty} t_y\,\mathrm{d}P\right\}\mathrm{d}y &= \int_{t_0<\infty} \left\{\sum_{k=1}^{J} \tau_k (S_{\tau_{k-1}} - S_{\tau_k})\right\}\mathrm{d}P \\ &= \int_{t_0<\infty} \left\{\sum_{k=1}^{J} (\tau_k - \tau_{k-1})(S_{\tau_{k-1}} - S_{\tau_J})\right\}\mathrm{d}P \\ &\leq -\int \tau_J S_{\tau_J}\,\mathrm{d}P \leq \sqrt{E(\tau_J^2)}\sqrt{E(S_{\tau_J}^2)};\end{aligned}$$

and the last two terms are finite by (2). That $K_2(z)$ is finite for all z follows easily; and then an argument which is similar to, but lengthier than (21), shows that

$$\mathrm{sum}_2 = K_2(z) + \mathrm{o}(1)[1 + (z^+)^2]. \tag{22}$$

The theorem follows from (18)-(22). □

6. Remarks and examples

The functions K_1 and K_2 may be related to F as follows. First write $K_2(z) = K_{2,1}(z) - K_{2,2}(z)$, where

$$K_{2,1}(z) = \int_{-\infty}^{z} y q(y)\,\mathrm{d}y$$

and

$$K_{2,2}(z) = \int_{-\infty}^{z} \left\{\int_{t_y<\infty} (S_{t_y} - \mu t_y)\,\mathrm{d}P\right\}\mathrm{d}y.$$

Let

$$M = \min\{0, S_1, S_2, \ldots\},$$

so that $q(y)=P\{t_y<\infty\}=P\{M\leq y\}$ for all $y<0$. The characteristic function of M is known to be

$$E\{e^{irM}\}=\exp\left\{\sum_{k=1}^{\infty}\frac{1}{k}\int_{\{S_k\leq 0\}}[e^{irS_k}-1]\,dP\right\}$$

for all $r\in\mathbb{R}$. See, for example, Feller (1966, p. 576). The functions K_1 and $K_{2,1}$ are simply related to q and, therefore, implicitly determined by the latter expression. In particular,

$$K_1(0)=-E(M)=\sum_{k=1}^{\infty}\frac{1}{k}E\{S_k^-\}$$

and

$$-2K_{2,1}(0)=E(M^2)=\sum_{k=1}^{\infty}\frac{1}{k}E\{(S_k^-)^2\}+E(M)^2,$$

where $s^-=\max\{0,-s\}$.

The function $K_{2,2}$ may be computed when F is embedded in an exponential family, as in (15). When $\mu(\omega)>0$, in the notation of Corollary 3, K_1, $K_{2,1}$, and $K_{2,2}$ may be formed with F replaced by G_ω; these are denoted by $K_1(\cdot\,;\omega)$, $K_{2,1}(\cdot\,;\omega)$, and $K_{2,2}(\cdot\,;\omega)$.

Lemma 4. *With the notation of the previous paragraph,*

$$K_{2,2}(z;\omega)=\frac{\partial}{\partial\omega}K_1(z;\omega)$$

for all $z\in\mathbb{R}$ and all $\omega\in\Omega$ for which $\mu(\omega)>0$.

Proof. For $y\in\mathbb{R}$,

$$\begin{aligned}\int_{\{t_y<\infty\}}[S_{t_y}-\mu(\omega)t_y]\,dP_\omega&=\int_{\{t_y<\infty\}}[S_{t_y}-\mu(\omega)t_y]\exp[\omega S_{t_y}-t_y\gamma(\omega)]\,dP_0\\&=\frac{\partial}{\partial\omega}\int_{\{t_y<\infty\}}\exp[\omega S_{t_y}-t\gamma(\omega)]\,dP_0\\&=\frac{\partial}{\partial\omega}P_\omega\{t_y<\infty\}\end{aligned}$$

for all ω for which $\mu(\omega)>0$. The lemma now follows by integrating over $y\leq z$. □

Example 1. If F is the normal distribution with mean μ and variance $\sigma^2=1$, then

$$K_1(0)=\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}[\phi(\mu\sqrt{k})-(\mu\sqrt{k})\Phi(-\mu\sqrt{k})],$$

$$2K_{2,1}(0,0)=\sum_{k=1}^{\infty}[(1+k\mu^2)\Phi(-\mu\sqrt{k})-(\mu\sqrt{k})\phi(-\mu\sqrt{k})\}+K_1(0)^2,$$

and

$$K_{2,2}(0)=\sum_{k=1}^{\infty}\Phi(-\mu\sqrt{k}).$$

Remarks. 1. The proofs of Lemmas 1 and 3 did not require any moment assumptions.

2. If the fourth moment is finite, then $o(1/n)$ may be replaced by $O(n^{-3/2})$ in Theorem 1, and $o(1/n)[1+(z^+)^2]$ by $O(n^{-3/2})[1+(z+)^3]$ in Theorem 2. This is clear for Theorem 1. For Theorem 2, first observe that m and $|z|$ may be replaced by m^2 and z^2 in (2); and if c_k is replaced by c'_k, where

$$c_k'^3=k^3[q_k+q(-k)]+\sum_{j=k}^{\infty}j^2[q_j+q(-j)],$$

in the proof of Theorem 2, then the right side of (3) is $o(n^{-3/2})$. Taking an additional term in the Taylor series expansion in (17) shows that $o(1/n)(k^2+w^2)$ may be replaced by $O(n^{-3/2})(k^3+w^3)$; and an examination of (18)–(22) then shows that the remainder is $O(n^{-3/2})[1+(z^+)^3]$.

3. There is some uniformity with respect to F implicit in the proof of Theorem 2. For a fixed F, as described in the Introduction, let $\mathcal{F}$ be the class of distributions of $\alpha X+\beta$, where X has distribution function F, and (α,β) varies in a bounded set for which $\alpha\mu+\beta$ and $\alpha\sigma$ remain bounded away from 0. Then (6) holds uniformly in this class, since the dependence on α and β may be absorbed into the dependence on b and c; and it is easily seen that the series and integral in (2) converge uniformly with respect to α and β. These are the basic ingredients in the proof of Theorem 2; and an examination of (18)–(22) shows that (13) holds uniformly in any such class. This remark may be useful in extensions of the main result to non-linear problems.

4. The final example shows that (13) need not hold, if Cramér's Condition is replaced by the condition that F be nonlattice.

Example 2. Let $Y_1, Y_2,\ldots$ and $Z_1, Z_2,\ldots$ denote independent random variables which take the values ±1 with probability $\frac{1}{2}$ each; and let $X_k=1+Y_k+\sqrt{2}Z_k$, $k=1,2,\ldots$. Then F is non-lattice. In this case, F^n has a discontinuity of size

$$P\{S_n=n\}=P\{Y_1+\cdots+Y_n=0\}P\{Z_1+\cdots+Z_n=0\}\sim 2/\pi n$$

at n for large, even n (cf. Gnedenko and Kolmogorov (1954, Section 45)). So, if $a=n$ is a large, even integer, then

$$H_a(n,0)-H_a(n,0-)=P\{t^a<n, S_n=n\}$$

$$=P\{S_k<0,\exists k<n|S_n=n\}P\{S_n=n\}.$$

Since $S_n=n$ iff $Y_1+\cdots+Y_n=0$ and $Z_1+\cdots+Z_n=0$, it is not difficult to see that the conditional probability converges to $P\{S_k<0,\exists k\geq 1\}$, as in Woodroofe (1982, Ch. 5). Thus, $H_a(n,z)$ has a discontinuity of order $1/n$, when $a=n$ is a large even integer; and, therefore, (13) cannot hold in this case.

References

R. Bahadur and R. Rao, On deviations of the sample mean, Ann. Math. Statist. 31 (1960) 1015-1030.

L. Baum and M. Katz, Convergence rates in the law of large numbers, Trans. Amer. Math. Soc. 120 (1965) 108-123.

Y.S. Chow and H. Teicher, Probability Theory (Springer, New York, 1978).

W. Feller An Introduction to Probability Theory and its Applications, vol. 2 (Wiley, New York, 1966).

B. Gnedenko and A. Kolmogorov, Limit Theorems for Sums of Independent Random Variables (Addison-Wesley, Cambridge, Mass., 1954).

R. Keener, Asymptotic expansions in non-linear renewal theory, in: M.L. Puri, ed., New Perspectives in Theoretical and Applied Statistics (Wiley, New York, 1986) 479-502.

S. Lalley, Limit theorems for first passage times in linear and nonlinear renewal theory, Adv. Appl. Probab., 16 (1984) 766-803.

C. Stone, A local limit theorem for non-lattice multidimensional distribution functions, Ann. Math. Statist., 36 (1965) 546-551.

H. Takahashi, Asymptotic expansions for Anscombe's theorem in repeated significance tests and estimation after sequential testing, Ann. Statist. 15 (1987) 278-295.

M. Woodroofe, Non-linear Renewal Theory in Sequential Analysis (SIAM, Philadelphia (1982)).

M. Woodroofe and R. Keener, Asymptotic expansions in boundary crossing probabilities, Ann. Probab 15 (1987) 102-114.