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TOWARD A UNIFIED MODEL FOR ELASTOPLASTIC STRUCTURAL ANALYSIS

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Introduction

This paper deals with a variational model applicable to small-deformation structural analysis expressed for linearly elastic-perfectly plastic (von Mises-Huber-Hencky) material. The model comprises a unification of the *minimum complementary energy principal*, the *Haar-von Karman principle* for deformation elastoplasticity, and (in a sense) the lower bound theorem of *Limit Analysis*. Thus it corresponds directly in terms of meaning and significance to these classical forms for structural analysis [see e.g. 1-4]. The present purpose is mainly to point out the availability of such a unified model, as it appears for structural analysis expressed 'in terms of stresses alone.'

While in terms of solid mechanics analysis the unified formulation comprises in effect just a restatement of the cited classical variational theories, it is distinctive by virtue of features identified with its mathematical modelling form. It is especially convenient that the *entire evolution* of structural response is monotone w.r.t. a single parameter, the 'complementary strain energy bound' value, in the unified problem statement. With an eye toward treatment for numerical solution, this makes it possible to avoid many of the cumbersome features inherent in standard methods for solving problems in elastoplasticity. In fact, the unified model provides the basis for computational solution via a *direct method*. Also there is potential through the application of contemporary results from the field of nonlinear, nonsmooth optimization to gain advantage both in analysis and for computational modelling.

2. The Variational Model

As a first step toward recognition of the unified model, the variational statement associated with the *lower bound theorem* of limit analysis [1,2] is recalled. In terms of scalar load factor α, this statement is written as (the developments are expressed here for a general form of discrete structure):

$$\max_{q \, \varepsilon \, R^n} \alpha$$

$$\text{subject to:} \qquad \qquad (1)$$

$$\text{(equilibrium eqn)} \qquad C^T q + \alpha f = 0$$

$$\text{(yield constraints)} \qquad s_i(D^{-1}q) - \bar{s}_i \le 0$$

Here f and q represent the *load* and *element force* vectors, C is the 'equilibrium matrix,' D transforms 'element stress' to 'element force', and the *yield function* $s_i(\bullet)$ is convex. Values of the yield limit $\bar{s}_i$ and the load vector f are specified, i.e., they represent data. In the case of a structure made up of axially loaded structural elements all made of the same material (e.g., truss members), for example, the yield constraints have the specific form $|D^{-1}q| - \bar{s}_i \le 0$ (see e.g. J. Martin [2] on 'yield conditions').

In order to appreciate how 'elasticity' might be introduced into the model, note that the complementary energy principal for elastic structural response [3], expressed here for systems with stiff supports, can be represented in the following form that is compatible with (1):

$$\max_{q \, \varepsilon \, R^n} \alpha$$

subject to: (2)

$$C^T q + \alpha f = 0$$

(complementary strain energy constraint)

$$\frac{1}{2} q^T Qq - E \leq 0$$

Q symbolizes the compliance matrix, so that $\frac{1}{2} q^T Qq$ is the usual measure of the 'total complementary strain energy.' For this problem statement, load vector f and bound E on the complementary energy represent data. The equivalence between problem (2) and the conventional versions of statement for the complementary potential energy principal may be realized as follows. Considering the load vector f to be specified, there exists a value for data E in problem (2) such that the solution of (2) is identical to the solution obtained for the conventional statement, the latter expressed for the same structure subject to a load that is proportional to f. Note as a property of the relationship that 'minimization of complementary potential energy' associated with the classical form corresponds to 'maximization of the load factor' represented in problem (2).

The two problem statements (1,2) are immediately suggestive of the form for a variational statement of the unified problem. In fact, the statement covering in combined form the several principals named earlier is simply:†

$$\max_{q \,\varepsilon\, R^n} \alpha$$

subject to:

$$C^T q + \alpha f = 0 \qquad (3)$$

$$s_i(D^{-1}q) - \bar{s}_i \leq 0$$

$$\frac{1}{2} q^T Qq - E \leq 0$$

Here load f, yield limits $\bar{s}_i$ and the energy bound E represent data. Suppose that f and $\bar{s}_i$ are specified; then the nature of the structural analysis problem represented — strictly elastic, partially plastic, or plastic limit — depends on the value chosen for energy bound E (within $E \in R+$). Expressed in another way, structural analysis is modelled via problem statement (3) in the form of an evolution from initially elastic through to the limit response, where the evolution is monotone w.r.t. total complementary strain energy.

To elaborate on the latter point, note that the separation between the intervals of purely elastic problems and of response covered under the Haar-von Karman model [1,2] is identified with a value, say E_1, of the energy bound E. Specifically, for given load f and yield limits $\bar{s}_i$ there exists a value E_1 such that for all values E within $0 \lesssim E < E_1$ the yield constraint is inactive, i.e., $s_i(\bullet) < \bar{s}_i$ for all elements of the structure. Furthermore, the value α_1 corresponding to the specification $E = E_1$ is the load associated with the inception of yielding, i.e., α_1 equals the value, within monotone increasing solution values of α for problem (3), at which

$$\max_i s_i(\bullet) = \bar{s}_i$$

first occurs. (Clearly, solution values (say α^*) vary monotonically with E within the range, $0 \leq E < E_1$; this follows from the equilibrium and energy constraints of (3)).

A second characteristic value of E serves to bound from above the interval of elastoplastic solutions to problem (3). The solution associated with this value is of course the limit load, say α_L, and the value itself is symbolized E_L. As E is increased monotonically in the elastoplastic range $E_1 \leq E \lesssim E_L$, a non-decreasing (generally increasing) set of structural elements will have stress at the yield limit $\bar{s}_i$. The limit value

† The author has become aware during the preparation of this material that a unified model expressed in substantially different form was presented earlier by Cyras [5].

E_L of the energy bound is finite for systems of finite extent. A typical form for α^* vs. E in the finite case is shown in the figure. Note that it is possible in general to identify corners between the 'first yield' and 'limit' points as well.

Properties of the solution to problem (3) are observed for the most part directly in the form of the Kurash-Kuhn-Tucker conditions for the problem. Only certain of the features are discussed in what follows. Considering the second and third constraints of (3) in particular, the solution q* must satisfy

$$\lambda_i^* [s_i(q^*) - \bar{s}_i] = 0 \qquad \lambda_i^* > 0 \qquad s_i(q_i^*) - \bar{s}_i < 0 \qquad (4)$$

$$\forall\, i \,\varepsilon\, I$$

$$\lambda_E^* (\tfrac{1}{2} q^{*T} Q q^* - E) = 0 \qquad \lambda_E \geq 0 \qquad \tfrac{1}{2} q^{*T} Q q^* - E \leq 0 \qquad (5)$$

where I stands for the set of all elements of the structure. Stationary of the solution w.r.t. α and q requires additionally that

$$1 - \lambda^{*\,T}_{e} f = 0 \qquad (6)$$

and

$$- C\lambda_e^* + \lambda_E^* Q q^* + \sum_i \lambda_i \partial_\gamma s_i = 0 \qquad (7)$$

Vectors λ_e^*, λ_i and scalar λ_E^* appearing in equations (4 - 7) represent solution values of multipliers on the equilibrium, yield, and complementary energy constraints. $\partial\gamma$ symbolizes the partial derivative w.r.t. element q_γ of the 'member force' vector.

Equations (6) and (7) exclude the (trivial) solution $(q^*, \alpha^*, \ldots) = 0$, and assure $\lambda_E^* \neq 0 \rightarrow (\tfrac{1}{2} q^{*T} Q q^* = E$ from equation (5)). In the particular case where in addition equation (4) is met via $\lambda_i^* = 0$; $s_i^* < \bar{s}_i$ $\forall\, i \,\varepsilon\, I$, these general results together with the original equilibrium constraint serve to identify strictly elastic solutions. Of course the remaining possibilities where $\lambda_i^* > 0 \rightarrow s_i^* = \bar{s}_i \; ; \; i \in I_p \neq \phi$ are associated with 'elastoplastic' solutions (I_p represents the set of members that are stressed to the plastic limit).

Summary

Problem (3) lies essentially within the framework of established results in the field of analysis for nonlinear, nonsmooth optimization problems. Thus both the analysis and the treatment for solution associated with the problem are in general terms well understood. However, it is possible to develop useful special results associated with the particular problem of mechanics. For example, a more convenient version for computational treatment is obtained in the form of a particular dual of problem (3). Also, it is possible for a variety of specific yield laws to produce solutions for the elasto-plasticity problem entirely by analysis. As indicated in the introduction, these details and additional developments are reported elsewhere (Ben-Tal and Taylor [6]).

Along a different line, availability of the unified problem statement characterized in (3) has interesting implications for structural optimization. Specifically, where those constraints in a design problem that reflect the mechanics are expressed via the unified model, the prediction of design may be performed free of narrow restriction on the character of structural response. In other words, one is not necessarily limited to the consideration of *optimal elastic design* separately from design for which inelastic deformation might occur. Thus the unified model might be used to advantage in developments of the kind reported in [7], as an example.

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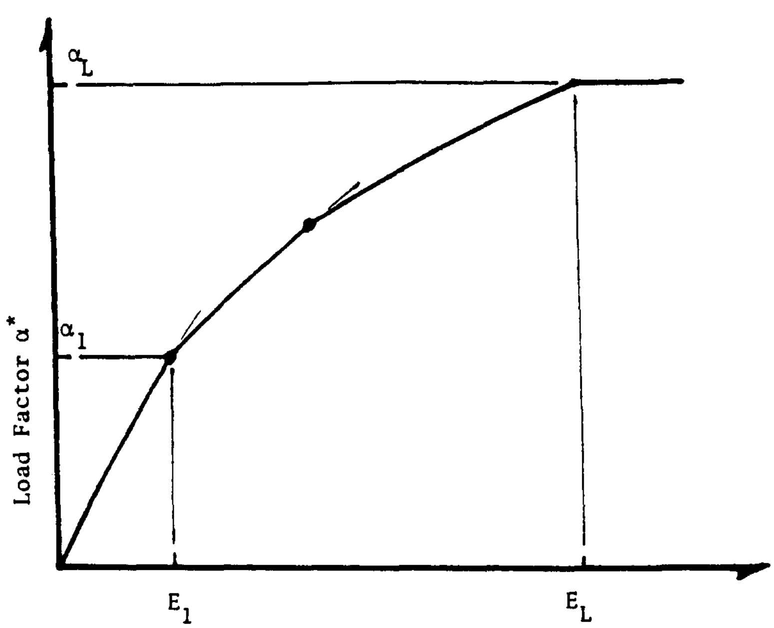

Load Factor α^* vs. Energy Bound E
Typical Evolution through to Plastic Limit State