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EIGENSOLUTION OF PERIODIC ASSEMBLIES OF MULTI-MODE COMPONENT SYSTEMS

1. INTRODUCTION

Many engineering structures feature some kind of repetitiveness, or periodicity; e.g., blade assemblies and multi-span structures such as airframes and truss-type space structures. It is often uneconomical, if not impossible, to calculate the modal characteristics of such large structures by solving the entire eigenvalue problem. However, the periodicity of the structure can often be exploited to simplify greatly the analysis. For instance, several finite-element procedures have been developed [1-3] that take advantage of cyclic symmetry, and they are being used extensively.

This note considers the free vibration eigenvalue problem for a periodic structure made of identical component systems coupled in an identical manner. An analytical solution is obtained for the system's modes of free vibration. Even though the solution is not entirely in closed form, it provides useful information about the effects of the system parameters on the natural frequencies and mode shapes, and it exhibits the sinusoidal variation (in space) of the mode shapes. The procedure presented takes advantage of the repeated geometry of the structure by showing that the eigensolution of the full periodic assembly of N component systems can be extracted by solving N small eigenvalue problems as opposed to a single N times larger eigenvalue problem. The method results in a very significant reduction in computation time (by at least a factor of N).

2. EIGENVALUE PROBLEM FORMULATION

Consider a periodic assembly made of N mono-coupled, identical component systems, as shown in Figure 1. The component systems are coupled through springs of stiffness k_s located at $x = x_c$. For the sake of generality, each component system is described by its modal representation, $(\lambda_i, \phi_i(x))_{i=1,\ldots,M}$, where M is the number of component modes, λ_i the square of the natural frequency, and ϕ_i the associated mode shape. The deflection of the jth component system is expanded as

$$w^j(x, t) = \sum_{i=1}^{M} \phi_i(x)\eta_i^j(t), \qquad j = 1, \ldots, N, \tag{1}$$

where η_i^j is the ith normal co-ordinate for the jth component system. The subscript i refers to the component mode number, while the superscript j designates the component system number. The equations of free motion of the periodic system can be derived by

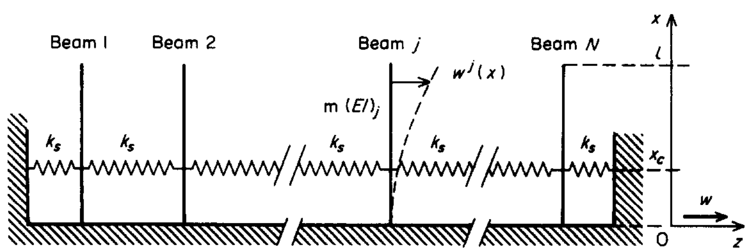

Fig. 1. Assembly of coupled, component systems (beams).

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a component mode analysis [4] and the interested reader will find the detailed derivation in reference [5].

For simple harmonic motion of frequency ω, the free vibration eigenvalue problem of the entire assembly is

$$[A]\bar{\underline{\eta}} = \lambda\bar{\underline{\eta}}, \tag{2}$$

where $\lambda = \omega^2$ is a free vibration eigenvalue and $\bar{\underline{\eta}}$ is an NM-dimensional eigenvector of normal co-ordinate amplitudes

$$\bar{\underline{\eta}} = [\bar{\eta}_1^1, \ldots, \bar{\eta}_i^1, \ldots, \bar{\eta}_M^1 | \ldots | \bar{\eta}_1^N, \ldots, \bar{\eta}_i^N, \ldots, \bar{\eta}_M^N]^{\mathrm{T}}. \tag{3}$$

Here $[A]$ is an NM by NM block tridiagonal matrix given by

$$[A] = \begin{bmatrix} \ddots & \ddots & [0] & \cdots & [0] \\ \ddots & \ddots & \ddots & \ddots & \vdots \\ [0] & -[R]\underline{\phi}\underline{\phi}^{\mathrm{T}} & [\Lambda]+2[R]\underline{\phi}\underline{\phi}^{\mathrm{T}} & -[R]\underline{\phi}\underline{\phi}^{\mathrm{T}} & [0] \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ [0] & \cdots & [0] & \ddots & \ddots \end{bmatrix}, \tag{4}$$

where $[\Lambda] = [\mathrm{diag}\,(\lambda_i)]$ is the M by M diagonal matrix of the eigenvalues of a nominal component system, $[R] = [\mathrm{diag}\,(k_s/M_i)]$ is the M by M diagonal matrix composed of the coupling frequencies, and M_i is the generalized mass for the ith component mode. $\underline{\phi}$ is the M-vector of modal deflections at the constraint location defined by

$$\underline{\phi} = [\phi_1(x_c), \ldots, \phi_i(x_c), \ldots, \phi_M(x_c)]^{\mathrm{T}}, \tag{5}$$

where x_c is the location of the coupling element shown in Figure 1. The submatrices constituting $[A]$ have dimension M by M and correspond to component systems, while the M rows and columns of these submatrices refer to component modes. Once an eigensolution $(\lambda, \bar{\underline{\eta}})$ of $[A]$ has been obtained, the corresponding continuous mode shape for the full assembly is given by equation (1).

3. AN "ALMOST" CLOSED FORM SOLUTION

By exploiting the periodicity of the matrix $[A]$, the eigensolution can be obtained without a global eigenvalue analysis of the entire system. The matrix $[A]$ can be rewritten as

$$[A] = \begin{bmatrix} \ddots & \ddots & [0] & \cdots & [0] \\ \ddots & \ddots & \ddots & \ddots & \vdots \\ [0] & [V] & [U] & [V] & [0] \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ [0] & \cdots & [0] & \ddots & \ddots \end{bmatrix}, \tag{6}$$

where

$$[U] = [\Lambda] + 2[R]\underline{\phi}\underline{\phi}^{\mathrm{T}} \quad \text{and} \quad [V] = -[R]\underline{\phi}\underline{\phi}^{\mathrm{T}}. \tag{7, 8}$$

Abraham and Weiss [6] have shown that the eigensolution of such a block tridiagonal matrix can be obtained by solving the following N eigenvalue problems of order M:

$$([a_j]\underline{\alpha}_j^l = \lambda_j^l \underline{\alpha}_j^l \quad l = 1, \ldots, M), \qquad j = 1, \ldots, N, \tag{9}$$

where $[a_j]$ is an M by M matrix given by

$$[a_j] = [U] + 2[V]\cos\left(\frac{\pi j}{N+1}\right), \qquad j = 1, \ldots, N, \tag{10}$$

and $(\lambda_j^l, \underset{\sim}{\alpha}_j^l)$ constitute the lth eigensolution of $[a_j]$. The $\underset{\sim}{\alpha}_j^l$'s are normalized by using $\underset{\sim}{\alpha}_j^{lT}\underset{\sim}{\alpha}_j^l = 1$. The λ_j^l's, for $j = 1, \ldots, N$ and $l = 1, \ldots, M$, can be shown to be the eigenvalues of the full matrix $[A]$ and a later presentation will show that the $\underset{\sim}{\alpha}_j^l$'s are related to the eigenvectors of $[A]$.

Therefore, for the assembly of component systems considered here, the MM natural frequencies are obtained by solving N eigenvalue problems of reduced size M for the matrices

$$[a_j] = [\Lambda] + 2[R]\underset{-}{\phi}\underset{-}{\phi}^{\mathrm{T}}\left(1 - \cos\frac{\pi j}{N+1}\right), \qquad j = 1, \ldots, N. \tag{11}$$

It is well known that the eigenvalues of the assembly shown in Figure 1 are often distributed in M groups, or clusters, of N eigenvalues each. Hence, in general the eigenvalues of the full matrix $[A]$ are related to those of the $[a_j]$'s by [5]

$$(\lambda_k = \lambda_j^l \qquad k = (l-1)N + j), \qquad l = 1, \ldots, M, \qquad j = 1, \ldots, N, \tag{12}$$

and λ_j^l represents the jth eigenvalue in the lth cluster. Note that equation (12) defines a one-to-one relationship between the indices l and j and the index k. Therefore, for a given j, the eigenvalues of $[a_j]$ are the jth, $(N+j)$th, $\ldots$, $((M-1)N+j)$th eigenvalues of $[A]$; hence the jth eigenvalue of $[A]$ in each cluster. By simply varying j from 1 to N, all eigenvalues of $[A]$ are determined.

Next the eigenvectors of the full matrix $[A]$ are obtained from those of the submatrices $[a_j]$. The eigenvectors of $[A]$, $(\bar{\eta}_k, k = 1, \ldots, NM)$, corresponding to $\lambda_k = \lambda_j^l$ constitute the columns of the NM by NM modal matrix $[P]$ of $[A]$. One can show [6] that if the eigenvectors $\underset{\sim}{\alpha}_j^l$ of the $[a_j]$'s are normalized by using $\underset{\sim}{\alpha}_j^{lT}\underset{\sim}{\alpha}_j^l = 1$, the element P_{lk} of the NM by NM modal matrix $[P]$ is

$$P_{lk} = \sqrt{\frac{2}{N+1}}\sin\left[\frac{\widehat{\mathrm{mod}}(k/N)\,\widehat{\mathrm{int}}(l/M)\pi}{N+1}\right]\alpha_k^{\widehat{\mathrm{mod}}(l/M)}, \tag{13}$$

where

$$\widehat{\mathrm{mod}}(k/N) = \begin{Bmatrix} \mathrm{mod}(k/N) \neq 0 & \\ N & \text{if } \mathrm{mod}(k/N) = 0 \end{Bmatrix},$$

$$\widehat{\mathrm{int}}(l/M) = \begin{Bmatrix} \mathrm{int}(l/M) + 1 & \text{if } l/M \neq 1, 2, \ldots, N \\ \mathrm{int}(l/M) & \text{if } l/M = 1, 2, \ldots, N \end{Bmatrix},$$

and $\alpha_k^{\widehat{\mathrm{mod}}(l/M)}$ is the $\widehat{\mathrm{mod}}(l/M)$th element of the vector $\underset{\sim}{\alpha}_k = \underset{\sim}{\alpha}_j^l$ (the lth eigenvector of $[a_j]$). The quantity $\mathrm{mod}(k/N)$ is the remainder of the integer division of k by N, and $\mathrm{int}(l/M)$ is the integer part of l/M. Again, the relationship between l, j, and k is defined by equation (12).

Upon substituting equation (13) into equation (1), the kth continuous mode shape of the full assembly is given by

$$\bar{w}_k^j(x) = \sqrt{\frac{2}{N+1}}\sin\left[\frac{j\pi\,\widehat{\mathrm{mod}}(k/N)}{N+1}\right]\left(\sum_{i=1}^{M}\alpha_k^i\phi_i(x)\right), \tag{14}$$

where $\bar{w}_k^j(x)$ is the deflection of the j component system in the kth natural mode of frequency ω_k. The coefficients α_k^i can be regarded as modal participation factors. The mode shapes are sinusoidal in terms of j, the component system number. Therefore, for all structural parameter values, the deflection in a mode at a given location x is always sinusoidal throughout the assembly. The corresponding spatial frequency increases with the mode number, as there are N spatial frequencies $l\pi/(N+1)$, $l=1,\ldots,N$. Also, the deflection shape in a given mode is the same for all component systems; only the amplitude varies sinusoidally. Such modes for which the whole structure participates in the motion are characteristic of periodic structures, and are said to be collective, or extended.

Finally, it should be noted that the solution scheme described above is valid for any matrix $[U]$ and $[V]$, regardless of their commutability. When matrices $[U]$ and $[V]$ commute, though, the eigenvalues of $[A]$ can be extracted directly by solving only the eigenvalue problems associated with matrices $[U]$ and $[V]$. From Abraham and Weiss [6], the eigenvalues of $[A]$ for commutable $[U]$ and $[V]$ are given by

$$\lambda_k=\lambda_j^l=\lambda_l(U)+2\lambda_l(V)\cos\frac{\pi j}{N+1},\qquad l=1,\ldots,M,\qquad j=1,\ldots,N, \tag{15}$$

where $\lambda_l(U)$ and $\lambda_l(V)$ are the lth eigenvalues of $[U]$ and $[V]$, respectively. Unfortunately, for the periodic assembly studied here, $[U]$ and $[V]$ do not commute, and one cannot apply the simple formula (15).

4. EXAMPLES

The theory described above can be applied to analyze the assembly of clamped-free Euler-Bernoulli beams shown in Figure 1. The generalized masses are given by $M_i=ml=M$, and the component modes, $\phi_i(x)$, are the modes of a clamped-free beam [5]. Dividing the equations of motion by EI/ml^4 results in the introduction of dimensionless eigenvalues, $\bar{\lambda}=\lambda/(EI/ml^4)$, and of the dimensionless coupling stiffness $\bar{R}=k_s/(EI/l^3)$, yielding

$$\{[\bar{A}]-\bar{\lambda}[I]\}\bar{\underline{\eta}}=\underline{0}, \tag{16}$$

where

$$[\bar{A}]=\begin{bmatrix} \ddots & \ddots & [0] & \cdots & [0] \\ \ddots & \ddots & \ddots & \ddots & \vdots \\ [0] & -\bar{R}\underline{\phi}\underline{\phi}^{\mathrm{T}} & [\bar{\Lambda}]+2\bar{R}\underline{\phi}\underline{\phi}^{\mathrm{T}} & -\bar{R}\underline{\phi}\underline{\phi}^{\mathrm{T}} & [0] \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ [0] & \cdots & [0] & \ddots & \ddots \end{bmatrix}. \tag{17}$$

As an example, a three-beam system has been examined by using both the full structure formulation and the reduced scheme presented in section 3. In the first case, only one component mode is used in the analysis, i.e., $M=1$, and the beam system can be considered as a chain of coupled oscillators. Clearly this is a special case of the results outlined in this letter. The closed form eigensolution for this assembly of single-degree-of-freedom systems has been derived by Chen [7]. Appendix A shows that the formulation outlined above reduces precisely to the results obtained by Chen. In the second case, two-component modes are considered. The analysis of the full structure consists of solving the eigenvalue problem of a six by six matrix whereas, using the above theory, only three two by two eigenvalue problems need to be solved. The results obtained by the two methods are, of course, identical (see Appendix B). Finally, the results were compared in a number of other cases not shown here, over wide ranges of the system parameters. In all cases perfect agreement was observed.

5. CONCLUSIONS

A computationally efficient method for calculating the eigensolution of assemblies of N coupled, identical component systems has been presented, based on the results of reference [6]. The periodicity feature of the global block tridiagonal matrix allows one to extract the full eigensolution by solving N separate eigenproblems each of order M, rather than the complete eigenvalue problem of order NM. When there is a large number of component systems this results in a reduction of computation time by a factor N or N^2 (as the cost of eigenvalue extraction is usually proportional to the square or cube of the system size).

Finally, note that when the periodicity of the full structure is destroyed by imperfections, the reduction procedure described above fails, and the size of the eigenvalue problem which must be solved increases from M to NM. Numerical studies of such mistuned structures can therefore be expensive, and can also face the numerical difficulties associated with very large eigenvalue problems. However, if the magnitude of the imperfections is small, as expected from manufacturing and material tolerances, the eigensolution of mistuned assemblies can still be obtained by applying perturbation theory for the eigenvalue problem to the eigensolution of the tuned assembly obtained above. The interested reader is referred to reference [5] in which perturbation schemes are presented for mistuned assemblies in cases of both strong and weak coupling between component systems.

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APPENDIX A

From the paper of Chen [7], for a tridiagonal matrix whose characteristic determinant is of the form

$$\det\begin{bmatrix} \ddots & \ddots & 0 & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & b & a & b & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & \ddots & \ddots \end{bmatrix} = 0, \qquad \text{(A1)}$$

where $a = \tilde{\omega}^2 - 2b$, the natural frequencies are

$$\tilde{\omega}_i = 2\sqrt{b}\sin\frac{i\pi}{2(1+n)}, \qquad i = 1, \ldots, n, \qquad \text{(A2)}$$

and the corresponding orthonormal mode shapes are

$$\underline{\nu}_i = \sqrt{\frac{2}{1+n}}\left[\sin\frac{i\pi}{1+n}, \sin\frac{2i\pi}{1+n}, \ldots, \sin\frac{ni\pi}{1+n}\right]^{\mathrm{T}}, \qquad i = 1, \ldots, n. \qquad \text{(A3)}$$

When only one mode is used in the analysis of section 3, matrices $[U]$ and $[V]$ become scalars given by $U = \lambda_1 + 2R\phi_1^2$ and $V = -R\phi_1^2$. Shifting the eigenvalues ω^2 of the assembly by λ_1 such that $\tilde{\omega}^2 = \omega^2 - \lambda_1$, and letting $b = R\phi_1^2$, $a = \tilde{\omega}^2 - 2b$, reduces the characteristic determinant for this beam assembly to the form of equation (A1). Substituting the appropriate variables into equation (A2) and simplifying gives the eigenvalues of the beam system as

$$\omega_i^2 = \lambda_1 + 2R\phi_1^2\left(1 - \cos\frac{i\pi}{N+1}\right). \qquad \text{(A4)}$$

For $M = 1$, the diagonal matrix $[\Lambda]$ and the modal deflection vector $\underline{\phi}$ in equation (11) become $[\Lambda] \equiv \lambda_1$, and $\underline{\phi} \equiv \phi_1$, and the eigenvalues given by equation (11) reduce to the expression (A4). Furthermore, for $M = 1$, $\alpha_k^{\widehat{\mathrm{mod}(l/M)}}$ in equation (13) is simply 1, and the normalized modal matrix reduces to

$$P_{lk} = \sqrt{\frac{2}{N+1}}\sin\left(\frac{kl\pi}{N+1}\right), \qquad l = 1, \ldots, N, \qquad k = 1, \ldots, N. \qquad \text{(A5)}$$

For a given mode, equation (A5) simplifies to the expression as given in equation (A3).

APPENDIX B

Consider a three-beam system with two modes for $\bar{R} = 3{\cdot}0$ and $x_c = 1{\cdot}0$; i.e., the spring stiffness equals the static stiffness of the nominal component beam and the constraint location is at the tip. The eigenvalues obtained by solving the three two by two matrices are in perfect agreement with those obtained by solving the full six by six matrix, as follows:

$\bar{\lambda}$	6×6	Three 2×2
1	19·28739	19·28739
2	35·14812	35·14812
3	49·81149	49·81149
4	492·65255	492·65255
5	510·73294	510·73294
6	530·01069	530·01069.

(B1)

The modal matrix, by using equation (13), can be expressed as

$$[P]=\sqrt{\tfrac{2}{4}}\begin{bmatrix} \alpha_1^1\sin\frac{\pi}{4} & \alpha_2^1\sin\frac{2\pi}{4} & \alpha_3^1\sin\frac{3\pi}{4} & \alpha_4^1\sin\frac{\pi}{4} & \alpha_5^1\sin\frac{2\pi}{4} & \alpha_6^1\sin\frac{3\pi}{4} \\ \alpha_1^2\sin\frac{\pi}{4} & \alpha_2^2\sin\frac{2\pi}{4} & \alpha_3^2\sin\frac{3\pi}{4} & \alpha_4^2\sin\frac{\pi}{4} & \alpha_5^2\sin\frac{2\pi}{4} & \alpha_6^2\sin\frac{3\pi}{4} \\ \alpha_1^1\sin\frac{2\pi}{4} & \alpha_2^1\sin\frac{4\pi}{4} & \alpha_3^1\sin\frac{6\pi}{4} & \alpha_4^1\sin\frac{2\pi}{4} & \alpha_5^1\sin\frac{4\pi}{4} & \alpha_6^1\sin\frac{6\pi}{4} \\ \alpha_1^2\sin\frac{2\pi}{4} & \alpha_2^2\sin\frac{4\pi}{4} & \alpha_3^2\sin\frac{6\pi}{4} & \alpha_4^2\sin\frac{2\pi}{4} & \alpha_5^2\sin\frac{4\pi}{4} & \alpha_6^2\sin\frac{6\pi}{4} \\ \alpha_1^1\sin\frac{3\pi}{4} & \alpha_2^1\sin\frac{6\pi}{4} & \alpha_3^1\sin\frac{9\pi}{4} & \alpha_4^1\sin\frac{3\pi}{4} & \alpha_5^1\sin\frac{6\pi}{4} & \alpha_6^1\sin\frac{9\pi}{4} \\ \alpha_1^2\sin\frac{3\pi}{4} & \alpha_2^2\sin\frac{6\pi}{4} & \alpha_3^2\sin\frac{9\pi}{4} & \alpha_4^2\sin\frac{3\pi}{4} & \alpha_5^2\sin\frac{6\pi}{4} & \alpha_6^2\sin\frac{9\pi}{4} \end{bmatrix}. \tag{B2}$$

The values of α_k^i ($i=\widehat{\mathrm{mod}}(l/M)$) are obtained by solving the eigenvectors of three two by two matrices. The matrix $[\alpha]$ made of the α_k^i's, where i and k are the row and column indices, respectively, is given by

$$[\alpha]=\begin{bmatrix} 0{\cdot}9998897 & 0{\cdot}9987226 & 0{\cdot}9963266 & -0{\cdot}0148516 & -0{\cdot}0505287 & -0{\cdot}0856345 \\ 0{\cdot}0148516 & 0{\cdot}0505287 & 0{\cdot}0856345 & 0{\cdot}9998897 & 0{\cdot}9987226 & 0{\cdot}9963266 \end{bmatrix}. \tag{B3}$$

By substituting equation (B3) into equation (B2), the modal matrix was shown to be identical to the one obtained by analyzing the full structure.

The modal participation factors, α_k^i, show that in a given group of modes a single-component mode contributes largely to the motion. For instance, in the lower three modes the beams vibrate primarily in the first-component mode. In general, in the ith group of modes the beams deflect primarily in that component mode.