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On the Monotonicity of the Time-Map

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1. INTRODUCTION

The equation $\Delta u(y) + \lambda f(u(y)) = 0$ on the unit ball, with homogeneous linear boundary conditions has been much studied (cf. [1, 2 and the references therein]). The change of variable $x = \sqrt{\lambda} y$ transforms this equation to

$$\Delta u(x) + f(u(x)) = 0, \qquad |x| < R, \tag{1.1}$$

where $R = \sqrt{\lambda}$. For radial solutions of this equation, lying in, say, the kth nodal class, R depends only on $p = u(0)$, $R = T(p)$. This function, which describes how the "size of the balls" on which radial solutions exist, vary with the quantity $u(0)$, turns out to be an important function, determining much of the behavior of the solutions.

Thus, if we consider (1.1) together with the boundary conditions

$$\alpha u(x) - \beta\, du(x)/dn = 0, \qquad |x| = R, \tag{1.2}$$

then radial solutions (lying in a prescribed nodal class) can be parametrized by $u(0) = p$; say $u = u(|x|, p)$. These solutions lie on balls whose radii R vary with p, $R = T(p)$, and the function $p \mapsto T(p)$ is called the "time-map." Notice that if T is monotone near a point $\bar{p}$, then for p near $\bar{p}$, we have local uniqueness; i.e., if $u = u(r)$ is any radial solution, and $T(u(0)) = T(\bar{p})$, then $u(0) = \bar{p}$, and $u \equiv u(\cdot, \bar{p})$. In this paper we shall show, in fact, that this is what happens if $u(0)$ is near a hyperbolic zero of f (i.e., $f(\gamma) = 0$, and $f'(\gamma) < 0$).

But the monotonicity of T also plays an important role in symmetry-breaking; that is, in the bifurcation of radial solutions into asymmetric ones. Thus, if we know, say from a linear analysis, that $u(\cdot, p)$ is a bifurcation point, and that the kernel of the linearized operator contains asym-

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metric elements, it does not follow that the symmetry breaks, unless we can rule out radial bifurcation. Indeed, the bifurcation might occur only in the space of radial functions. This occurrence can be ruled out if the kernel does not contain a purely radial element. Now as was shown in [4] under hypothesis (H)(v) below, for p near γ there is a radial component in the kernel if and only if $T'(p)=0$. Thus we can be assured of symmetry-breaking at $u(\cdot, p)$ provided that $T'(p)\neq 0$. It is the purpose of this paper to show that for a wide class of functions f (see hypotheses (H)), $T'(p)\neq 0$ for p near a hyperbolic zero of f. In particular, the result proved here implies that for this class of f's, the symmetry must break on infinitely many radial solutions.

Our method of proof uses a technique we introduced in [2], whereby we construct a system of ordinary differential equations, the analysis of whose solutions yields the desired information on the derivative T'. We point out that for the special case of positive solutions of the Dirichlet problem, Clément and Sweers [1] have obtained a similar result by entirely different methods.

2. The General Framework

Radial solutions of (1.1), (1.2) satisfy the equation

$$u''(r)+\frac{n-1}{r}\,u'(r)+f(u(r))=0, \qquad 0<r<R \tag{2.1}$$

together with the boundary conditions

$$u'(0)=0=\alpha u(R)-\beta u'(R), \tag{2.2}$$

where $r=|x|$ and prime denotes differentiation with respect to r. It is convenient to rewrite (2.1) as the first-order system

$$u'=v, \qquad v'=-\frac{n-1}{r}\,v-f(u), \tag{2.3}$$

together with the boundary conditions

$$v(0)=0=\alpha u(R)-\beta v(R), \tag{2.4}$$

and to consider orbits of (2.3) which satisfy $u'(0)=0$, $u(0)=p>0$; such solutions will be denoted by $u(r,p)$, and p will be considered as a parameter in this paper (cf. [2–5]).

Let

$$\theta_0 = \tan^{-1}(\alpha/\beta), \qquad -\pi/2 \leqslant \theta_0 < \pi/2.$$

If k is a given non-negative integer, and $f(p) > 0$, we define the function $p \mapsto T_k(p)$ (whenever it exists; see Theorem 2.1) by the following two conditions (cf. [5]):

(i) $\alpha u(T_k(p), p) - \beta u'(T_k(p), p) = 0$, and

(ii) if $\theta(r, p) = \tan^{-1}(v(r, p)/u(r, p))$, then $\theta(T_k(p), p) = \theta_0 - k\pi$.

(2.5)

We see then that $T_k(p)$ plays the role of R, and R varies with p (see [2–5]). (If k is fixed, then we write $T(p) = T_k(p)$.) A solution of (2.3), (2.4) which satisfies (2.5) will be said to belong to the "kth-nodal class" of the function f, relative to the given boundary conditions.

We now list the hypotheses on $f \in C^1$ which we shall need. Thus, we assume that there exist points $b < 0 < \gamma$ such that the following hold (cf. [5]):

(i) $f(\gamma) = 0$, $f'(\gamma) < 0$,

(ii) $F(\gamma) > F(u)$ if $b < u < \gamma$; here $F' = f$, and $F(0) = 0$,

(iii) $F(b) = F(\gamma)$, (H)

(iv) if $f(b) = 0$, then $f'(b) < 0$,

(v) $uf(u) + 2(F(\gamma) - F(u)) > 0$ *if* $b < u < \gamma$.

We remark that (ii) is a necessary condition for the existence of solutions of (2.3), (2.4), as is (iii), if b is finite. Condition (i) allows us to prove the existence of radial solutions in the kth nodal class for p near the hyperbolic zero γ of f. (iv) and (v) are mild technical assumptions, which perhaps may be unnecessary.

Under these assumptions, one has the following theorem (see [5] for a proof).

THEOREM 2.1. *Suppose that f satisfies hypotheses* (H)(i), (ii), (iii), *and that $k \in \mathbb{Z}_+$. Then there exists an $\varepsilon > 0$ ($\varepsilon < \gamma$) such that if $\gamma - \varepsilon < p < \gamma$, then $p \in \operatorname{dom}(T_k)$. Moreover, $T_k(p) \to \infty$ as $p \to \gamma$.*

In other words, if $\gamma - \varepsilon < p < \gamma$, then $u(\cdot, p)$ is a radial solution of (1.1), (1.2) lying in the kth nodal class. The main result in this paper is to prove that $T'(p) > 0$ if p is close enough to γ; this will be demontrated in the following sections.

3. Construction of the Coordinate System

In order to show that T is monotone near γ we have to find an expression for $T'(p)$. Thus, we denote by $\sigma_r(q)$ the flow on $\mathbb{R}^3$ generated by the equations

$$u' = v, \qquad v' = -\frac{n-1}{r} v - f(u), \qquad r' = 1,$$

in $r \geqslant 0$, with $u(0) > 0$, and $v(0) = 0$, where $q = (u, v, r)$, $r \geqslant 0$. Hence, if $X = X_q$ is the field

$$X = \left(v, -\frac{n-1}{r} v - f(u), 1 \right),$$

then $\sigma'_r(q) = X_{\sigma_r(q)}$, $\sigma_0(q) = q$, and $\sigma_r(\sigma_s(q)) = \sigma_{r+s}(q)$, for all $r, s \geqslant 0$. Let π denote the projection defined by $\pi(u, v, r) = (u, v, 0)$. The next simple result gives conditions under which we have a good set of coordinates (the proof is straightforward; see [2]).

Proposition 3.1. *Suppose that*

$$v^2(r) + \frac{n-1}{r} u(r)\, v(r) + u(r)\, f(u(r)) > 0 \tag{3.1}$$

along an orbit $\{\sigma_r(q): r \geqslant 0\}$ *of* X, *where* $\sigma_r(q) = \{u(r), v(r), r)$. *Then the vectors* πq, X *and* $\partial/\partial r \equiv (0, 0, 1)$ *form a basis at each point on the orbit.*

Now for $\gamma - \varepsilon < p < \gamma$, let $\bar{p} = (p, 0, 0)$ and $\sigma_r(\bar{p}) = q \equiv (u(r, p), r(r, p), r)$, $r \geqslant 0$, where u and v are C^2-functions of r and p (see [2]). Then assuming (3.1), we can write

$$\frac{\partial}{\partial p} \sigma_r(\bar{p}) = a\, \pi q + b\, X + c\, \partial/\partial r, \tag{3.2}$$

where $a = a(r, p)$, $b = b(r, p)$, and $c = c(r, p)$. Also, from the chain rule,

$$\begin{aligned} \frac{\partial}{\partial p} \sigma_{T(p)}(p) &= \left.\frac{\partial \sigma_r(p)}{\partial p}\right|_{r = T(p)} + \left.\frac{\partial \sigma_r(p)}{\partial r}\right|_{r = T(p)} T'(p) \\ &= a\pi q + (b + T'(p))X + c\, \partial/\partial r, \end{aligned} \tag{3.3}$$

where a, b, and c are all evaluated at $r = T(p)$. The following lemma gives an expression for $T'(p)$ which will be used throughout the paper.

LEMMA 3.2. *If* $\gamma-\varepsilon<p<\gamma$, *and* (3.1) *holds, then*

$$b(T(p),p)=-T'(p).$$

Proof. For the case of Dirichlet boundary conditions (i.e., $\beta=0$ in (1.2)), this result was proved in [2]. We may thus assume that $\beta\neq 0$. Then since

$$\begin{aligned}\sigma_{T(p)}&=[u(T(p),p),v(T(p),p),T(p)]\\&=\left[u(T(p),p),\frac{\alpha}{\beta}u(T(p),p),T(p)\right],\end{aligned}$$

we have

$$\frac{\partial}{\partial p}\sigma_{T(p)}(\bar{p})=\begin{pmatrix}v(T(p),p)\,T'(p)+u_p(T(p),p)\\ \frac{\alpha}{\beta}v(T(p),p)\,T'(p)+\frac{\alpha}{\beta}u_p(T(p),p)\\ T'(p)\end{pmatrix}^t. \tag{3.4}$$

Also, at $r=T(p)$, we have

$$\begin{aligned}&a(\pi q)+(b+T')X+c\,\partial/\partial r\\&=a\begin{pmatrix}u\\ v\\ 0\end{pmatrix}^t+(b+T')\begin{pmatrix}v\\ -\frac{n-1}{r}v-f(u)\\ 1\end{pmatrix}^t+c\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}^t.\end{aligned}$$

Thus from (3.3) and (3.4), we find, equating third components, $T'(p)=(b+T')+c$, so at $r=T(p)$,

$$c=-b. \tag{3.5}$$

Equating first components gives, at $r=T(p)$,

$$ua+v(b+T')=vT'+u_p, \tag{3.6}$$

and finally, from the second components, we have at $r=T(p)$,

$$va+\left(-\frac{n-1}{T(p)}-f(u)\right)(b+T')=\frac{\alpha}{\beta}(vT'+u_p). \tag{3.7}$$

If we set $J = uf(u) + ((n-1)/T(p))\, uv + v^2$, then $J \not\equiv 0$, by hypothesis. Thus (3.6) and (3.7) give, at $r = T(p)$,

$$J(b+T') = \begin{vmatrix} u & vT' + u_p \\ v & \frac{\alpha}{\beta}(vT' + u_p) \end{vmatrix} = 0,$$

since $v = \alpha u/\beta$ at $r = T(p)$. Thus $b(T(p), p) = -T'(p)$, and this completes the proof.

With the aid of this lemma, we shall be able to prove the monotonicity of T for p near γ. Thus, we shall find a pair of differential equations in the unknown functions $a(\cdot, p)$ and $b(\cdot, p)$, and we shall prove that $b(T(p), p) < 0$ if p is near γ. This will be done in the next section. Before doing this, however, we must prove that (3.1) holds, at least if p is near γ.

PROPOSITION 3.3. *Let f satisfy hypotheses* (H). *Then* (3.1) *holds for p near γ.*

The proof of this proposition will follow from a few lemmas. First, using hypothesis (H)(i), we choose $A > 0$ so close to γ that

$$f'(u) < \tfrac{1}{2} f'(\gamma) < 0, \qquad \text{if } A \leqslant u \leqslant \gamma. \tag{3.8}$$

Then from Theorem 2.1, we may take ε so small that $A < \gamma - \varepsilon$, and $p \in \mathrm{dom}(T)$ if $\gamma - \varepsilon < p < \gamma$. For such p we define $T_1^A(p)$ by $u(T_1^A(p), p) = A$, where $T_1^A(p)$ is minimal with respect to this property. We may also assume that A has been taken so close to γ that $v(r, p) = u'(r, p) < 0$ if $0 < r < T_1^A(p)$.

In the following lemmas in this section we assume that f satisfies hypotheses (H), *and that p satisfies $\gamma - \varepsilon < p < \gamma$.*

LEMMA 3.4. *Let $(u(\cdot, p), v(\cdot, p))$ be an orbit of* (2.3); *then* (3.1) *is valid if $0 < r \leqslant T_1^A(p)$.*

Proof. Since $u(r) \geqslant 0$ on $0 \leqslant r \leqslant T_1^A(p)$, it suffices to show that

$$-v'(r) = \frac{n-1}{r}\, v(r) + f(u(r)) > 0 \tag{3.9}$$

on this interval. Now at $r = 0$, $-v'(0) = f(p)/n > 0$. Suppose there were a first point r_1, $0 < r_1 \leqslant T_1^A(p)$, such that $v'(r_1) = 0$. Then if

$$\psi(r) = rf(u(r)) + (n-1)\, v(r), \tag{3.10}$$

we have $\psi(r_1)=0$, and $\psi(r)>0$ if $0<r<r_1$. Also, $\psi'(r)=rf'(u(r))\,v(r)+f(u(r))+(n-1)\,v'(r)$, so

$$\psi'(r_1)=r_1 f'(u(r_1))\,v(r_1)+f(u(r_1))>0,$$

in view of (3.8). This shows that no such r_1 can exist and the proof of the lemma is complete.

LEMMA 3.5. *Suppose $f(b)=0$, so $f'(b)<0$ by* (H)(iv). *Choose B near b such that $b<B<0$, and for $b\leqslant u\leqslant B$,*

$$f'(u)<0, \text{ and so } f(u)<0.$$

Then for p near γ, (3.1) *is valid on $b<u\leqslant B$.*

Proof. Let $T_1^B(p)$ and $T_2^B(p)$ is defined by $u(T_i^B(p),p)=B$, $T_1^B(p)<T_2^B(p)$, $T_1^B(p), T_2^B(p)$ being minimal with respect to these two conditions (cf. Fig. 1). Let $T_1^N(p)$ be defined by $T_1^B(p)\leqslant T_1^N(p)<T_2^B(p)$, and $v(T_1^N(p),p)=0$. Now if $T_1^B(p)\leqslant r\leqslant T_1^N(p)$, then $v(r,p)\leqslant 0$, and $f(u(r,p))<0$. Hence $v'=-((n-1)/r)v-f(u)>0$ on this range. Suppose there was a (smallest) r_2, $T_1^N(p)<r_2\leqslant T_2^B(p)$ with $v'(r_2)=0$ and $v'(r)>0$ for $T_1^B(p)\leqslant r<r_2$; then $rv'(r,p)>0$ on this range. But as $rv'=-(n-1)v-rf(u)$,

$$\begin{aligned}(rv')'\,(r_2)&=-(n-1)\,v'(r_2)-r_2 f'(u(r_2))\,v(r_2)-f(u(r_2))\\&=-r_2 f'(u(r_2))\,v(r_2)-f(u(r_2))\\&>0,\end{aligned}$$

since $v(r_2)>0$. This is impossible and so $(n-1)\,v/r+f(u)<0$ on this range. But since $u<0$ here, it follows that (3.1) holds in this range, and the proof of the lemma is complete.

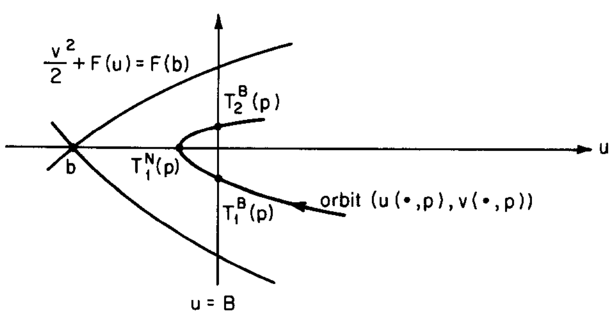

FIGURE 1

Now define $T_2^A(p)$ by $u(T_2^A(p), p) = A$, $T_2^A(p) > T_1^A(p)$, and $T_2^A(p)$ is minimal with respect to these properties; i.e., $T_2^A(p)$ is the "second time" the orbit $(u(\cdot, p), v(\cdot, p))$ meets the line $u = A$ (see Fig. 2). Similarly, let $T_2^N(p)$ be defined by $v(T_2^N(p), p) = u'(T_2^N(p), p) = 0$, $T_2^N(p) > T_2^A(p)$, $T_2^N(p)$ being minimal with respect to these properties (see Fig. 2). Then as both $v(r, p) > 0$ and $f(u(r, p)) > 0$ if $T_2^A(p) \leqslant r \leqslant T_2^N(p)$, we see that (3.1) holds in this region.

Let us remark that we have proved that (3.1) holds in the two wedges W_A and W_B defined by $(v^2/2) + F(u) = F(\gamma)$, $u = A$, $(v^2/2) + F(u) = F(b)$, $u = B$.

We can now complete the proof of Proposition 3.3. Thus, for p near γ, it was shown in [5, Proposition 2.2(iii)], that the orbit segment $(u(r, p), v(r, p))$, $0 \leqslant r \leqslant T(p)$, stays near the corresponding level curve $F(u) + (v^2/2) = F(p)$. That is, given any $\delta > 0$, we can take p so near to γ that all along the orbit segment, $v^2 = 2(F(\gamma) - F(u)) \pm \delta$. Hence

$$v^2 + \frac{n-1}{r} uv + uf(u) = 2(F(\gamma) - F(u)) + \frac{n-1}{r} uv + uf(u) \pm \delta. \tag{3.11}$$

Now by hypothesis (H)(v), $2(F(\gamma) - F(u)) + uf(u)$ is positive along the orbit segments, so that on $B \leqslant u \leqslant A$, it is uniformly bounded away from zero. Since the term $(n-1)\, uv/r$ can be made arbitrarily small on $B \leqslant u \leqslant A$, by taking p near γ (as u and v are uniformly bounded in p; see [5, Proposition 2.1]), and since $\delta \to 0$ as $p \to \gamma$, we see from (3.11) that (3.1) holds outside the wedges. Hence, in view of what we already know, we conclude that (3.1) holds now for $0 < r \leqslant T_N^2(p)$; (cf. Fig. 2). The proof of the theorem is completed by repeating this argument a finite number of times, depending on k, the given nodal class. ∎

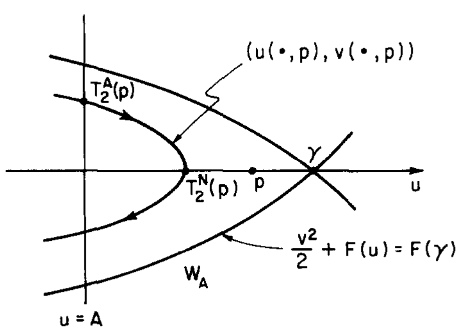

FIGURE 2

4. The Main Theorem

In this section we shall prove the following theorem.

THEOREM 4.1. *Suppose that f satisfies hypotheses* (H). *Then for p near γ*, $(p<\gamma)$, $T'(p)>0$.

As explained in Section 3, we shall show that $b(T(p), p))<0$, and the result will follow from Lemma 3.2, together with Proposition 3.3.

In order to study $b(r, p)$, we shall derive a system of ordinary differential equations satisfied by $a(r, p)$ and $b(r, p)$ (cf. [2]). Thus, if we differentiate the left-hand side of (3.2) with respect to r, we get

$$\frac{\partial^2 \sigma_r(p)}{\partial r\, \partial p}=\frac{\partial^2 \sigma_r(p)}{\partial p\, \partial r}=\frac{\partial}{\partial p} X_{\sigma_r(p)}=dX_{\sigma_r(p)}\frac{\partial \sigma_r(p)}{\partial p},$$

while differentiating the right-hand side gives

$$\begin{aligned} &a'(\pi q)+a(\pi q)'+b'X+bX'+c'\,\partial/\partial r\\ &\quad =a'(\pi q)+a(X-\partial/\partial r)+b'X+b\,dX(X)-b'\,\partial/\partial r. \end{aligned}$$

Then equating both these expressions, we find

$$a\,dX(\pi q)-b\,dX(\partial/\partial r)=a'(\pi q)+a(X-\partial/\partial r)+b'(X-\partial/\partial r). \tag{4.1}$$

Taking inner products of (4.1) with the quantities $(-v, u, 0)$, and $(((n-1)/r)v+f(u), v, 0)$ gives successively the equations

$$\begin{aligned} Ja' &= v\phi a-\frac{n-1}{r^2}\,v^2 b,\\ Jb' &= -u\phi a+\frac{n-1}{r^2}\,uvb, \end{aligned} \tag{4.2}$$

together with the initial conditions

$$a(0)=1/p, \qquad b(0)=0. \tag{4.3}$$

Here as before, $J=uf(u)+(n-1)\,uv/r+v^2$, $\phi=f-uf'$. In view of Proposition 3.3, we know that $J>0$ along an orbit if p is near γ. Furthermore, (3.8) implies that $\phi(u)>0$ if u is near γ. *In the remainder of this section, we assume that p is so close to γ that (3.1) is valid.*

Using (4.3), and the second equation in (4.2), we see that $b'(0)<0$; thus $b(r)<0$ on some interval $0<r\leqslant\sigma$. Since $a(0)=1/p$, we may assume that

$a(r)>0$ on this interval. Note that since $(0, 0)$ is a rest point of (4.2), there is no $\bar{r}>0$ such that $a(\bar{r})=0=b(\bar{r})$. Now we define

$$z(r)=-b(r)/a(r), \qquad r\geqslant 0,$$

where we have suppressed the dependence on p. In view of our above remarks, we see that

$$z(r)>0, \qquad 0<r\leqslant\sigma. \tag{4.4}$$

Next, we shall show that a cannot go through zero unless b has previously gone through zero. Indeed, if $b(r)<0$ on $0<r\leqslant\bar{r}$, and $a(\bar{r})=0$ ($\bar{r}$ being minimal), then from (4.2), $(Ja')(\bar{r})=-(n-1)\, v^2(\bar{r})\, b(\bar{r})/\bar{r}^2>0$ if $v(\bar{r})\neq 0$, which is impossible. If $v(\bar{r})=0$, then $a'(\bar{r})=0=a''(\bar{r})$, and $(Ja''')(\bar{r})=-2(n-1)\, v'(\bar{r})^2\, b(\bar{r})/\bar{r}^2>0$ since $v'(\bar{r})=-f(u(\bar{r}))\neq 0$, because $u(\bar{r})\in(b, B)\cup(A, \gamma)$. Thus $a'''(\bar{r})>0$, which is again impossible. It follows from this that z is differentiable as long as b stays negative. We have thus proved the following lemma.

LEMMA 4.2. *If $z(r)>0$, on $0<r\leqslant T(p)$, then $b(r)<0$ on $0<r\leqslant T(p)$, and z is differentiable on this interval.*

With the aid of this lemma we have a strategy for proving Theorem 4.1; namely, we shall show that $z(r)>0$ on $0<r\leqslant T(p)$. In order to carry out this program, we shall use Eqs. (4.2) in order to derive the differential equation satisfied by the function z. Thus, from the definition of z, we have

$$z'=\frac{ab'-ba'}{-a^2},$$

so from (3.1),

$$-a^2z'=\frac{1}{J}\left[-u\phi a^2+\frac{n-1}{r^2}\,uvab\right]-\frac{1}{J}\left[v\phi ab+\frac{n-1}{r^2}\,v^2b^2\right].$$

It follows that z satisfies the equation

$$z'=-\frac{n-1}{J}\frac{v^2}{r^2}\,z^2+\left[\frac{n-1}{Jr^2}\,uv-\frac{v\phi}{J}\right]z+\frac{u\phi}{J}. \tag{4.5}$$

Our goal is to prove that for p near γ, $z(r)>0$ on $0<r\leqslant T(p)$. Recall that (4.4) shows that z is positive near $r=0$. Our method of proof will be to compare solutions of (4.5) with an equation of the form $z'=-k_1z^2+k_2z+k_3$, for various choices of the constants k_1, k_2, k_3.

Now the proof of the theorem will follow from some lemmas. The basic idea is to show that for p near γ, z can be made large near the rest point γ (and b if $f(b)=0$), and since z spends a finite "time" outside of any (fixed) neighborhood of the rest points, we can maintain control over the size of z all along the orbit. First we define

$$H(z, r)=\frac{1}{J}\left[-\frac{(n-1)\,v^2}{r^2}\,z^2+\left(\frac{n-1}{r^2}\,uv-v\phi\right)z+u\phi\right], \tag{4.6}$$

where $(u, v)=(u(r, p), v(r, p))$, and the p-dependence in H is being suppressed. Note that v/r is continuous at $r=0$, so v/r is bounded (cf. [5, Proposition 2.1]). Next, we define the quantities $\bar{c}_0(p)$ and $\bar{c}_2(p)$ by

$$\bar{c}_0(p)=\min\left\{\frac{u\phi}{J}: \frac{1}{2}\,T_1^A(p)\leqslant r\leqslant T_1^A(p)\right\},$$

$$\bar{c}_2(p)=\frac{\phi(n-1)}{T_1^A(p)^2}\,\frac{\max\{2(F(\gamma)-F(u)): b\leqslant u\leqslant\gamma\}}{\min\{J: \frac{1}{2}T_1^A(p)\leqslant r\leqslant T_1^A(p)\}}.$$

Now if $\frac{1}{2}T_1^A(p)\leqslant r\leqslant T_1^A(p)$, and p is near γ,

$$J=uf(u)+\frac{n-1}{r}\,uv+v^2\leqslant M+\frac{M_1}{r}\leqslant M+\frac{M_1}{T_1^A(p)}\leqslant M_2,$$

where M, M_1, and M_2 are independent of p. Hence on this interval (cf. (3.8))

$$\frac{u\phi}{J}=\frac{uf(u)-u^2f'(u)}{J}\geqslant\bar{c}_0(p)\geqslant\frac{-\frac{1}{2}A^2f'(\gamma)}{M_2}\equiv c_0,$$

c_0 being independent of p. Also, on this same interval

$$J\geqslant(A/2)\,f(A/2)+\frac{n-1}{r}\,uv\geqslant(A/2)\,f(A/2)+\frac{(n-1)\,uv}{\frac{1}{2}T_1^A(p)},$$
$$\geqslant M_3>0,$$

where M_3 is independent of p. Now define

$$c_2=\frac{1}{T_1^A(p)^2}\,\frac{4(n-1)\max\{2(F(\gamma)-F(u): b\leqslant u\leqslant\gamma\}}{M_3}\equiv\frac{\tilde{c}}{T_1^A(p)^2};$$

then $c_2\geqslant\bar{c}_2(p)$, and $\tilde{c}$ is independent of p. Finally, we set

$$H_1(z)=c_0-c_2z^2.$$

Now $T_1^A(p) \to \infty$ as $p \to \gamma$, so if p is near γ, $T_1^A(p) \gg 1$. Thus for p near γ, if $T_1^A(p) \geqslant r \geqslant \frac{1}{2} T_1^A(p)$, we have

$$\frac{1}{J}\left[\frac{n-1}{r^2}\, uv - v\phi\right] > 0.$$

Moreover, we assert that $z(r) > 0$ if $0 < r \leqslant T_1^A(p)$. Indeed, suppose $b(\tilde{r}) = 0$, $0 < \tilde{r} \leqslant T_1^A(p)$, $\tilde{r}$ being minimal. As we have noted earlier, $a(r) > 0$, $0 \leqslant r \leqslant \tilde{r}$. So from (4.2) $(Jb')(\tilde{r}) = -(u\phi a)(\tilde{r}) < 0$, and this is impossible. It follows that $b(r) < 0$, so $a(r) > 0$ on $0 < r \leqslant T_1^A(p)$, and thus $z(r) > 0$ on this interval. Now as the function $F(u) + (v^2/2)$ decreases on orbits of (2.3), we have $F(u) + (v^2/2) \leqslant F(p) < F(\gamma)$. So $v^2 \leqslant 2(F(\gamma) - F(u))$. Thus if $\frac{1}{2} T_1^A(p) \leqslant r \leqslant T_1^A(p)$, then $-(n-1)\, v^2/Jr^2 \geqslant -c_2$. These remarks imply that for p near γ,

$$H(z, r) > H_1(z), \qquad \text{if } T_1^A(p) \geqslant r \geqslant \tfrac{1}{2} T_1^A(p). \tag{4.7}$$

Now we denote by $T_1^A(p) < T_2^A(p) < \cdots$ (resp. $T_1^B(p) < T_2^B(p) < \cdots$) those r-values for which the orbit $(u(\cdot, p), v(\cdot, p))$ meets the line $u = A$ (resp. $u = B$) (see Fig. 3). In these terms, we have the following result.

LEMMA 4.3. (a) *$z(T_1^A(p)) \to \infty$ as $p \to \gamma$.*

(b) *If $z(r) > 0$ on $T_{2s+1}^B(p) \leqslant r \leqslant T_{2s+2}^B(p)$, then $z(T_{2s+2}^B(p)) \to \infty$ as $p \to \gamma$.*

(c) *If $z(r) > 0$ on $T_{2s+2}^A(p) \leqslant r \leqslant T_{2s+3}^A(p)$, then $z[T_{2s+3}^A(p)] \to \infty$ as $p \to \gamma$.*

Proof. Since p will be fixed throughout the argument, we shall suppress this p-dependence.

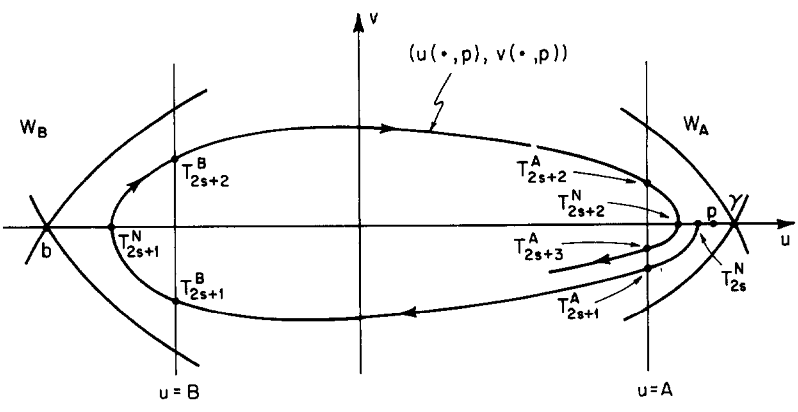

FIGURE 3

We begin by proving (a). Define $z_1(r)$ by

$$z_1' = H_1(z_1), \qquad z_1(\tfrac{1}{2}T_1^A(p)) = z(\tfrac{1}{2}T_1^A(p)). \tag{4.8}$$

Note that $(z - z_1)(\tfrac{1}{2})\, T_1^A(p)) = 0$, and

$$\begin{aligned}(z-z_1)'\,(\tfrac{1}{2}T_1^A(p)) &= H(z(\tfrac{1}{2}T_1^A(p)), \tfrac{1}{2}T_1^A(p)) - H_1(z_1(\tfrac{1}{2}T_1^A(p)))\\ &= H(z(\tfrac{1}{2})T_1^A(p)), \tfrac{1}{2}T_1^A(p)) - H_1(z(\tfrac{1}{2}T_1^A(p)))\\ &> 0,\end{aligned}$$

in view of (4.7), since we have already noted that $z(r) > 0$ if $0 < r \leqslant T_1^A(p)$. Suppose next that for some $\bar{r}$, $\tfrac{1}{2}T_1^A(p) \leqslant \bar{r} \leqslant T_1^A(p)$, that $(z - z_1)(\bar{r}) = 0$, $\bar{r}$ being minimal. Then as above

$$(z - z_1)'\,(\bar{r}) = H(z(\bar{r}), \bar{r}) - H_1(z(\bar{r})) > 0,$$

so that no such $\bar{r}$ can exist. It follows that $z(r) > z_1(r)$ if $T_2^A(p)/2 \leqslant r \leqslant T_1^A(p)$. Next, if $z_1(T_1^A(p)) > \sqrt{c_0/c_2} = T_1^A(p)$, we see that $z_1(T_1^A(p))$ tends to infinity as $p \to \gamma$. If $z_1(T_1^A(p)) < \sqrt{c_0/c_2}$, then we must have $z_1(T_1^A(p)/2) < \sqrt{c_0/c_2}$. Hence since $z_1' > 0$ on $[0, \sqrt{c_0/c_2})$, we have

$$z(T_1^A) > z_1(T_1^A) \geqslant z_1(T_1^A) - z_1(T_1^A/2) = T_1^A/2H_1(\xi),$$

where $z_1(T_1^A) > \xi > z_1(T_1^A/2)$. Now suppose that $H_1(\xi) > c_0/2$; then

$$z(T_1^A) \geqslant z_1(T_1^A) \geqslant \frac{c_0}{2}\,\frac{T_1^A(p)}{2} \to \infty$$

as $p \to \gamma$. If on the other hand $H_1(\xi) \leqslant c_0/2$, then $c_0/2 \geqslant c_0 - c_2\xi^2$ so $\xi > (1/\sqrt{2})\ \sqrt{c_0/c_2} = T_1^A(p)$. But as $z(T_1^A) \geqslant z_1(T_1^A) \geqslant \xi$, we see that we again have $z(T_1^A(p)) \to \infty$ as $p \to \gamma$. This proves (a).

To prove (b), we define, for p near γ, the quantities $\tilde{c}_0(p)$ and $\tilde{c}_2(p)$ by

$$\tilde{c}_0(p) = \min\left\{\frac{u\phi}{J} \colon \frac{1}{2}\,T_{2s+2}^B(p) \leqslant r \leqslant T_{2s+2}^B(p)\right\},$$

and

$$\tilde{c}_2(p) = \frac{4(n-1)}{T_1^A(p)^2}\,\frac{\max\{2(F(\gamma) - F(u) \colon b \leqslant u \leqslant \gamma\}}{\min\{J \colon \tfrac{1}{2}\,T_{2s+2}^B(p) \leqslant r \leqslant T_{2s+2}^B(p)\}}.$$

As before, we can show that $\tilde{c}_0(p) \geqslant \tilde{c}_0 > 0$, and $\tilde{c}_2(p) \geqslant \tilde{c}_2 > 0$, where $\tilde{c}_0$ and $\tilde{c}_2$ are constants independent of p. Define $z_2(r)$ by

$$z_2' = H_2(z_2), \qquad z_2(\tfrac{1}{2}T_{2s+2}^B(p)) = z(\tfrac{1}{2}T_{2s+2}^B(p)),$$

where $H_2(z)=\tilde{c}_0-\tilde{c}_2z^2$. As in part (a), $z(r)\geqslant z_2(r)$ if $\frac{1}{2}T^B_{2s+2}(p)\leqslant r\leqslant T^B_{2s+2}(p)$, and $z(\frac{1}{2}T^B_{2s+2}(p))\to\infty$ as $p\to\gamma$. Since the proof of (c) is similar, it will be omitted, and the proof of the lemma is considered complete.

Now Lemma 4.3 will be used to show that $z(r)>0$ if the orbit $(u(r,p), v(r,p))$ lies in the wedge W_A or W_B (cf. Fig. 3). In order to control z outside of these wedges, we need the following lemma.

LEMMA 4.4. *Given $M>0$, there exists a $\sigma>0$, $0<\sigma<\gamma$, and there exists an $N>0$, such that for any $p\in[\gamma-\sigma,\gamma)$, and any $T\geqslant T^A_1(p)$, if $z(T)>N$, then $z(T+r)>0$ for all r, $0\leqslant r\leqslant M$.*

Thus this lemma shows that if z is sufficiently large at $r=T$, then z will remain positive in the interval $T\leqslant r\leqslant T+M$. In applying this lemma, M will successively play the role of $T^B_1(p)-T^A_1(p)$, $T^A_2(p)-T^B_2(p)$, ..., etc., all of which are uniformly bounded in p (see [5, Lemma 3.12]).

Proof of Lemma 4.4. For $\gamma-\varepsilon<p<\gamma$, fix $\alpha>\sup\{(n-1)v^2: 0\leqslant r\leqslant T(p)\}$, and define k_1 and k_2 by

$$k_1=-\alpha+\min\{u\phi/J: r\geqslant T^A_1(p)\}$$

and

$$k_2=\min\left\{J^{-1}\left(\frac{n-1}{r^2}uv-v\phi\right): r\geqslant T^A_1(p)\right\}.$$

We consider three cases: (i) $k_2\neq 0$; (ii) $k_2=0$, $k_1\leqslant 0$; (iii) $k_2=0$, $k_1>0$.

Case (i). $k_2\neq 0$. Let

$$N>\max\left[\frac{k_1}{k_2}(e^{k_2M-1}), \frac{-k_2}{k_1}, 0\right], \tag{4.9}$$

and choose $\sigma>0$ so small that $T^A_1(p)>N$ if $\gamma-\sigma<p<\gamma$. We will prove that this N "works." Thus let $T>T^A_1(p)$, and $z(T)>N$. Suppose $\exists\bar{r}$, $T<\bar{r}\leqslant T+M$ such that $z(\bar{r})=0$; we shall obtain a contradiction. Choose r_1, $T\leqslant r_1<\bar{r}$ such that $z(r_1)=N$ and $z(r)<N$, $r_1<r<\bar{r}$. Define w by $w'(r)=k_1+k_2w$, $w(r_1)=N$. Then $(z-w)(r_1)=0$ and since $z(r_1)>0$,

$$\begin{aligned}(z-w)'(r_1)=&-\frac{n-1}{Jr_1^2}v^2z(r_1)^2\\ &+\frac{1}{J}\left(\frac{n-1}{r_1^2}uv-v\phi\right)z(r_1)+u\phi/J-k_1-k_2w(r_1),\end{aligned}$$

$$> -\frac{n-1}{r_1^2} v^2 z(r_1)^2 + k_1 + \alpha - k_1,$$

$$= \alpha - \frac{n-1}{r_1^2} v^2 N^2,$$

$$> \alpha - \frac{n-1}{(T_1^A)^2} v^2 N^2,$$

$$> \alpha - (n-1)\, v^2 > 0.$$

Hence $(z-w)'(r_1) > 0$. Now suppose that for some $\tilde{r}$, $(z-w)(\tilde{r}) = 0$, $r_1 < \tilde{r} \leqslant T + M$, $\tilde{r}$ being minimal with respect to these properties (see Fig. 4).

We consider two cases: $\tilde{r} \leqslant \bar{r}$ and $\tilde{r} > \bar{r}$. Thus suppose $\tilde{r} \leqslant \bar{r}$; then $z(\tilde{r}) > 0$ and as before $(z-w)' > 0$, which is impossible. If now $\tilde{r} > \bar{r}$, then

$$0 = z(\bar{r}) > w(\bar{r}) = Ne^{-k_2(r_1 - \bar{r})} + \frac{k_1}{k_2}\left(e^{-k_2(r_1 - \bar{r})} - 1\right). \tag{4.10}$$

Again we consider several cases. Thus, suppose first that $k_2 < 0$. Then since

$$w(\bar{r}) = e^{-k_2(r_1 - \bar{r})}\left(N + \frac{k_1}{k_2}\right) - \frac{k_1}{k_2}, \tag{4.11}$$

we see that $w(\bar{r}) > N > 0$ in view of (4.9); this is a contradiction. If $k_2 > 0$, then if $k_1 \leqslant 0$, then $w(\bar{r}) > e^{-k_2(r_1 - \bar{r})}(N + (k_1)/(k_2)) > 0$, again a contradiction. If $k_1 > 0$, then (4.11) and (4.9) imply

$$w(\bar{r}) > e^{-Mk_2}\left(N + \frac{k_1}{k_2}\right) - \frac{k_1}{k_2} > 0,$$

again a contradiction.

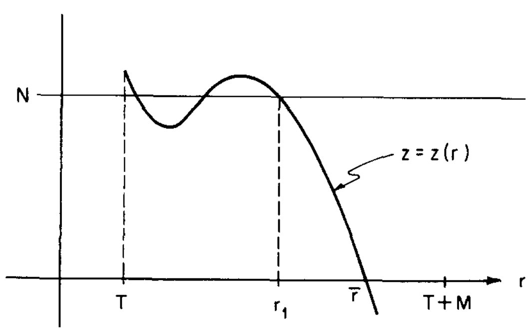

FIGURE 4

This argument shows that no such $\tilde{r}$ can exist. Hence $z(r) > w(r)$ if $r_1 < r \leqslant T + M$. Now again we get the contradiction $0 = z(\bar{r}) > w(\bar{r}) > 0$; this completes the proof in Case (i).

Case (ii). $k_2 = 0$, $k_1 \leqslant 0$. Here we choose $N > -k_1 M$, and let σ be chosen as in Case (i). Then define w to be the solution of $w'(r) = k_1$, $w(r_1) = N$. Now as in Case (i), $(z - w)'\ (r_1) > 0$, and if $(z - w)(\tilde{r}) = 0$, $r_1 < \tilde{r} \leqslant T + M$, $\tilde{r}$ being minimal, $\tilde{r} \leqslant \bar{r}$ leads to a contradiction as before, so $\tilde{r} > \bar{r}$ and

$$\begin{aligned} 0 = z(\bar{r}) > w(\bar{r}) &= k_1 \bar{r} - k_1 r_1 + N \\ &= N + k_1(\bar{r} - r_1) \\ &\geqslant N + k_1 M \\ &> 0. \end{aligned}$$

This contradiction completes the proof in this case.

Case (iii). $k_2 = 0$, $k_1 > 0$. Here we choose $N = 1$, and as in the other case, if $w'(r) = k_1$, $w(r_1) = 1$, then

$$0 = z(\bar{r}) > w(\bar{r}) = 1 + k_1(\bar{r} - r_1) > 0,$$

which is a contradiction. Since we have exhausted all cases, we see that the proof of Lemma 4.4 is complete.

We can now complete the proof of Theorem 4.1. We recall from Lemma 4.2 that it suffices to show $z(r) > 0$ on $0 < r \leqslant T(p)$. Now for p near γ,

$$z(r) > 0 \qquad \text{if } 0 < r \leqslant T_1^A(p);$$

(cf. the discussion preceding Eq. (4.7)).

Next from Lemma 4.3(a), we have $z(T_1^A(p)) \to \infty$ as $p \to \gamma$. Also, the quantity $T_1^B(p) - T_1^A(p)$ is bounded independently of p (see [5, Lemma 3.12]; *the reader should consult Fig.* 3 *at this point*). Hence, using Lemma 4.4, $\exists N_1 > 0$ such that if p is near γ and $z(T_1^A(p)) > N_1$, then $z(r) > 0$ if $T_1^A(p) \leqslant r \leqslant T_1^B(p)$. Since $z(T_1^B(p)) > 0$, we see that $b(T_1^B(p)) < 0$; (cf. the proof of Lemma 4.2). Now suppose $b(\bar{r}) = 0$ for some $\bar{r}$, $T_1^B(p) \leqslant r \leqslant T_2^B(p)$. Then from (4.2), $(kb')(\bar{r}) < 0$ since $\phi < 0$, and $a > 0$ at $\bar{r}$. It follows that no such $\bar{r}$ exists, and so $b(r) < 0$ if $T_1^B(p) \leqslant r \leqslant T_2^B(p)$. This implies that $z(r) > 0$, on this interval. We can now apply Lemma 4.3(b) to conclude that $z(T_2^B(p)) \to \infty$ as $p \to \gamma$. Since $T_2^A(p) - T_2^B(p)$ is bounded independently of p, we conclude, as before, from Lemma 3.4, that $z(r) > 0$ if $T_2^B(p) \leqslant r \leqslant T_2^A(p)$. Again, $b(r) > 0$ on this interval, and by using Lemma 4.3(c) and repeating this argument finitely many times, we conclude that $z(r) > 0$ if $0 < r \leqslant T(p)$, and this completes the proof of the theorem.

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