

# The economic lot and delivery scheduling problem: The single item case\*

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## Abstract

We have studied the problem of determining the frequency of production of a single component and the frequency of delivery of that component to a customer which uses this component at a constant rate. The objective is to minimize the average cost per unit time of production setup costs, inventory holding costs at both the supplier and the customer, and transportation costs. The model allows positive production setup times. We prove that the ratio between the production interval and delivery interval must be an integer in an optimal solution. This provides the basis for a very simple, optimal solution procedure. We use these results to characterize situations in which it is optimal to have synchronized production and delivery, and discuss the ramifications of these conditions on strategies for setup cost and setup time reductions.

## 1. Introduction

The crux of just-in-time as it relates to the relationship between a supplier and its customers is the synchronization of production and shipments. One key question is the frequency with which these activities should occur in the presence of transportation and production economies of scale. We address this issue in the context of the linkage between a major assembly facility, such as an automotive assembly plant, and one of its major suppliers which ships directly to the assembly facility.

In this paper, we are concerned with only a single component produced on a single machine at the supplier and delivered at regular intervals (to be determined) to the assembly facility at a fixed delivery cost per shipment. (Later in the paper we relax the assumption of a fixed delivery cost

per shipment.) The assembly facility uses this component at a constant rate. This assumption is consistent with a just-in-time context wherein demand fluctuations are smoothed (see, for example, Refs. [1,2]), and is especially realistic in automotive applications where the assembly plant uses paced assembly lines.

The supplier produces this component in batches at regular intervals (to be determined) which may differ from the delivery interval. The reasons for batch production in this context may include: (a) production rate greater than the demand rate, (b) administrative convenience, and (c) setup costs and/or setup times incurred because of multiple parts being produced on the same machine. Even though we are dealing with only a single component, we do not preclude the production of other components on the same machine, provided that the component under consideration can be produced at regular intervals. The objective is to minimize the average cost per unit time of setups at the supplier, inventory holding costs at both the supplier and the assembly facility, and transportation while ensuring no backorders at the assembly facility.

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This is admittedly a very simplified version of the problem, but there are many applications for which this model captures the essence and fundamental trade-offs. For example, there are numerous applications where a component, or a family of similar components is supplied principally or exclusively by a single source. Indeed, the adoption of just-in-time policies has made this commonplace. If the components are partially or fully customized, which is also quite common, the supplier needs to make decisions regarding the frequency of production of each customer's components. Representing an entire production process by one "machine" may be a reasonable approximation when there is an identifiable bottleneck, or when production planning decisions are focused on one stage of production, such as a final assembly line (at the supplier). Finally, direct shipments are the norm when product flows between the supplier and the customer are relatively high, and it is in these instances where coordination between transportation and delivery schedules has the greatest potential benefit. Even if deliveries are not direct, modeling them as such will capture the first-order effects of the delivery frequency.

Although our model is simple, it is useful in providing insights into the basic interactions between the production interval and the delivery interval. In addition, the model is simple enough so that we can investigate the role of production setup costs and times in determining the frequency of shipments. In particular, we investigate the question of how small setup costs and times need to be for synchronization of production and shipments to be optimal, given existing transportation costs. We study more general versions of this problem in sequels [3–5], but in these papers, the production sequencing and scheduling issues are complicated enough to obscure the qualitative impact of setup cost and setup time reductions.

In the next Section, we provide a formal statement of the problem assumptions. This is followed by a brief review of the literature. We then present a formulation of the problem which is simplified significantly by a Theorem which says that the ratio between the production interval and delivery interval must be an integer in an opti-

mal solution. This provides the basis for a very simple, optimal solution procedure. We use these results to characterize situations in which it is optimal to have synchronized production and delivery, and discuss the ramifications of these conditions on strategies for setup cost and setup time reductions. We also extend the basic model to consider capacitated shipments.

## 2. Problem assumptions

The assumptions throughout this study are as follows.

(1) The assembly facility is scheduled in such a way that the usage of the component is effectively constant.

(2) Production setup costs and times at the assembly facility are assumed to be negligible, and therefore do not have a significant impact on the component ordering policy. This is reflective of many assembly environments.

(3) The component is produced on one production line or machine.

(4) A fixed cost and/or time is incurred at the supplier for the setup of the component.

(5) Inventory levels are reviewed continuously, i.e., this is a continuous-time model.

(6) The inventory holding cost at the supplier and the assembly facility are identical. Inventory costs are charged per unit time.

(7) Both deliveries and production runs are equally spaced in time, although their frequency may differ. The schedule is repeated indefinitely.

(8) The delivery quantity is exactly equal to the demand at assembly facility during the upcoming delivery interval. Thus, since the demand rate is constant and deliveries are equally spaced in time, the delivery quantity is constant.

(9) The delivery lead time is constant and deterministic and, therefore, without loss of generality, equal to zero. (We do not include the cost of holding in-transit inventory in the model, but this is a straightforward generalization.)

(10) The delivery cost is a constant ( $A$ ) irrespective of the delivery quantity (i.e., each "truck" has infinite capacity). Later in the paper, a volume-sensitive transportation cost (finite truck capacity) will be considered.

Most of the assumptions are self-explanatory or are justified in the previous Section. We assume (6) because only the locations of storage

differ. However, if the inventory holding cost at the assembly facility is higher than that at the supplier, the formulations and the solution procedures for the problem can be adapted with little modification.

The cost of in-transit inventory discussed in assumption (9) is proportional to the product of the delivery lead time and the delivery quantity, the latter of which is proportional to the delivery interval. Consequently, the cost of in-transit inventory is a linear function of the delivery interval, and this term can be incorporated into our solution procedure with a minor modification of the cost parameters.

The objective is the sum of setup costs at the supplier, delivery costs and inventory holding costs (at both the supplier and the assembly facility) per unit time. The decision variables are the lengths of the production (or setup) interval for the component and the delivery interval.

### 3. Literature review

Although a considerable amount of work has been done on multi-stage production systems with known constant demands, little research has been done that considers both the cost of inventory accumulation prior to delivery and the cost of transportation in the determination of jointly optimal production and delivery schedules. One reason why the former has been ignored is that many models assume instantaneous production (e.g., Refs. [6–9]). Other papers that do incorporate capacity constraints ignore some or all of the accumulation inventory (e.g., Refs. [10–12]), or treat transportation costs as fixed [13]. Likewise, there is some research on procurement policies that include transportation costs, either as quantity discounts [14] or a fixed charge per shipment [15,16], but none explicitly considers the impact of the selected delivery schedule on the *supplier's* inventory levels. Only a few papers deal with all of the issues that we consider; they are reviewed at the end of this Section. In the interest of brevity, we will review only those models with *finite production rates*, and we concentrate on *continuous review models* because inventory accumulation and depletion are reflected differently in discrete time models. We first review pure production lot-sizing models, and then turn to

models that explicitly consider transportation costs.

In continuous review lot-sizing models, policies are often assumed to be stationary and nested. A policy is called “stationary” if each stage orders at equally-spaced points in time and in equal amounts. A policy is called “nested” if, each time any stage orders, its immediate successors also order. These two properties together naturally imply that the lot-size at a stage is an integer multiple of the lot-size at successor stages. We refer to this assumption as the “integer multiple” assumption hereafter.

The optimality of an integer multiple policy under the stationarity assumption still remains an open question. Williams [17] shows that, unless a stationary policy is assumed, an integer multiple policy does not necessarily yield an optimal solution. Jensen and Kahn [18], and Sznadrobits [19] show that the integer multiple policy need not be optimal in noninstantaneous production cases.

With finite production rates, the delivery lag must be considered to prevent interstage stock-outs. In the studies of Schussel [20], Taha and Skeith [21], Schwarz and Schrage [22], and Graves and Schwarz [23], it is assumed that a given stage does not begin production until its immediate predecessor completes an entire lot. Schussel [20] studies assembly systems in which a “learning curve” function reflects a decrease in marginal unit production costs with the lot-size. He proposes a heuristic decision rule which is based on the assumption that an integer multiple policy is optimal. Taha and Skeith [21] consider a serial system in which backorders for the finished product are allowed. They assume an integer multiple policy and solve the problem by examining all combinations of such integer values.

Schwarz and Schrage [22] develop a branch and bound approach for lot-sizing in assembly systems which is based on a set of modified inventory holding costs. They also present a “system-myopic” approach which optimizes an objective function with respect to two adjacent stages and ignores other multi-stage interaction effects. This approach is later refined by Graves and Schwarz [23], and Blackburn and Millen [7].

The assumption of no lot-splitting (i.e., transfer batch equal to production batch) in the aforementioned studies may cause unnecessary inventories. To avoid this, Jensen and Kahn [18], and Bigham and Mogg [24] assume that the inventory at a stage is depleted constantly and continuously during the production time of the successive stage (i.e., that units are transferred between stages one at a time).

Jensen and Kahn [18] do not use the integer multiple assumption. Instead they constructed a simulation model which calculates the minimum required delay of the initial startup at each stage and evaluates the average inventory at each stage in a serial system, given the lot-size at that stage and at the successor stage. They formulate a dynamic program in which the simulation model is used to evaluate each functional equation. They note that high average inventories result if integer multiples are not used and discuss a problem for which nonconstant lot-sizes are optimal.

Bigham and Mogg [24] present a heuristic procedure to determine lot-sizes for an assembly system under the integer multiple assumption. Initially, all the multipliers are set equal to one. Then an iterative search procedure is performed in which, at each iteration, the alternatives of increasing or decreasing (when possible) each multiplier are considered and the change resulting in the largest cost decrease is made. The procedure is repeated until no improvement is achieved.

Moily [25] studies an assembly system where the lot-size at a stage is an integer divisor of the lot-size at the immediate successor stage. He presents optimal and heuristic solution procedures and shows that considerable cost savings can occur if this policy is employed under favorable conditions such when the natural (most economical independently determined) production cycle of a stage is shorter than that of its immediate successor stage.

We discuss the integer multiplier issue in the context of our problem in the latter part of the next Section. In our problem, an integer divisor policy would mean that the supplier produces in smaller lots than are ordered by the customer, and delivers them periodically during the production run at the customer. Practically speaking, how-

ever, since most production runs are short (maybe a week, but certainly not months in most instances), the inventory savings from an integer divisor policy would be more than off-set by the increase in transportation costs in most cases. In principle, the model of Moily can be modified (by including the transportation cost in the supplier's setup cost) to handle the integer divisor policy.

We now turn to models that explicitly consider transportation costs. In the context of comparing direct shipment policies with peddling, where one truck makes deliveries to multiple customers on the same trip, Burns et al. [26] develop a single-item model with the objective of minimizing transportation and inventory costs per unit time. Production-related costs are not included. The transportation cost consists of a fixed charge per truck, and the trucks have capacity constraints. While the cost of inventory accumulation prior to delivery is considered, it is assumed that production is *not* synchronized with delivery, so a simple representation for accumulation inventory is used. Under these assumptions, the optimal delivery quantity can be obtained by EOQ-type analysis.

Benjamin [27] analyzes a problem similar to ours, but his model differs from ours in two important respects. First, he does not include the cost of inventory accumulation prior to delivery in his objective function (although it appears that he intended to, based on figures in the paper). Second, he does not account for additional inventories that would accrue when the production interval (or batch) is not an integer multiple of the delivery interval (batch). As a consequence of these two implicit assumptions, he suggests that the problem can be solved optimally by independent EOQ-type formulas for the production and delivery batches. Unfortunately, the independent solutions generally will not have the integer multiple property that is implicit in the formulation of the objective function.

Blumenfeld et al. [28] study a problem in which the supplier uses a single machine to produce several components, each of which is shipped to a unique destination. Setup times are not incorporated. Their model allows each component to be produced more than once in each

production cycle. Unlike Benjamin's formulation, it *does* include accumulation inventories that accrue when production runs are equally spaced in time, and when production batch sizes are integer multiples of the respective delivery batches. However, it does *not* include accumulation inventories that must be held if either of these conditions is not satisfied. Consequently, when their results are specialized to the problem treated by Benjamin, they arrive at similar conclusions. They suggest rounding the ratio of the production batch to the delivery batch to obtain an integer multiple, but do not indicate how the rounding should be accomplished. For the case of  $N$  components with identical cost and demand characteristics, they present results for the special case in which the machine is 100% utilized. Unfortunately, for this case, the results do not specialize to the case of  $N=1$ .

Our model and results differ from earlier research in several ways. First, unlike most of the pure production lot-sizing literature, we explicitly model the finite production rate at the supplier, and thereby also capture the impact of inventory that must be accumulated prior to each delivery. Second, we incorporate transportation costs for the movement of goods between the two facilities, and these movements may occur at times other than the start of production runs. Finally, we prove that for any solution with a rational multiplier, we can construct a better solution with the integer multiple property. Thus, with the possible exception of policies with irrational multipliers (which, incidentally, are impossible to implement), optimal solutions must have the integer multiple property. We use this fact to construct an exact objective function. Moreover, we explicitly constrain the solution to those with integer multipliers. Consequently, unlike the several papers that consider transportation costs, we provide truly optimal results.

It is useful to point out that several of the above authors have used their analyses of single-item systems to form a foundation for studies of more complex systems [29–31]. Thus, since our results are exact, yet easy to interpret and apply, they can be incorporated easily into more realistic models.

#### 4. Formulation

Notation:

- $D$  = demand per unit time,
- $T$  = production interval (time between setups),
- $R$  = delivery interval (time between deliveries),
- $p$  = processing time per unit,
- $A$  = delivery cost,
- $S$  = setup cost,
- $h$  = inventory holding cost per unit per unit time,
- $s$  = setup time.

Without loss of generality, the time period is assumed to be a year, and  $T$ ,  $R$ ,  $s$ , and  $p$  are expressed as a fraction of a year. The two decision variables are  $T$  and  $R$ . Figure 1 shows a plot of inventory levels at the supplier and assembly facility for a case where  $T=3R$ . Even with deterministic demand, the problem becomes very complicated if  $T$  is a non-integer multiple of  $R$ . In addition, a non-integer-multiple policy is difficult to implement. We prove that an optimal policy has  $T=MR$  with  $M$  integer. Let  $\lceil x \rceil$  denote the smallest integer greater than or equal to  $x$ .

**Theorem 1.** Let  $M$  be non-integer but a positive rational number and let  $M' = \lceil M \rceil$ . Then,  $TC(M,R) - TC(M',R) > 0$  for any  $M$  and  $R$ , where  $TC$  is the total cost per unit time.

**Proof.** See Appendix A.

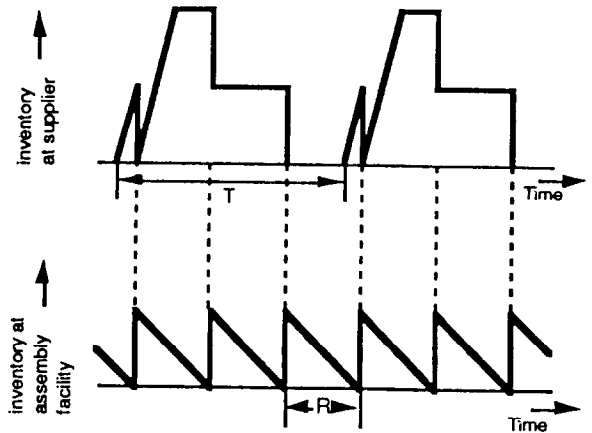


Fig. 1. Plot of the inventory levels.

The theorem states that it is less expensive to round the value of  $M$  to the next larger integer than to use a non-integer value of  $M$ . Therefore, we can formulate the problem in terms of alternate decision variables  $R$  and  $M$ , where  $T=MR$  for some positive integer  $M$ .

Note that, in Fig. 1, the inventory level at the supplier does not follow the usual sawtooth pattern even though the end usage is deterministic and constant over time. Moreover, the inventory depends on  $p$  (processing time per unit), which complicates the calculation of average inventory at the supplier.

The average inventory at the assembly facility is given by  $\frac{1}{2}DR$ . Some algebra will show that the average inventory level at the supplier is given by

$$\frac{1}{2}DT(1-pD) + pD^2R - \frac{1}{2}DR.$$

Although the detailed derivation of the above formula is complicated, it has a simple intuitive explanation. The average total inventory in the two locations clearly must be at least as large as in the standard economic production quantity (EPQ) model, but the only difference in our model is the inventory accumulation prior to each delivery. In our model, the supplier must have the entire  $DR$  units available  $pDR$  time units earlier than if the supplier and assembly facility were at the same site, and the associated cost is incurred in all delivery intervals (each with duration  $R$ ). The resulting incremental inventory per unit time is  $pD^2R$ . (It is instructive to point out that the incremental inventory is not  $\frac{1}{2}pD^2R$  because the *entire* quantity  $DR$ , and not just half of it, must be available  $pDR$  time units earlier than in the EPQ model.) To determine the inventory at the supplier, we must then subtract the inventory at the assembly facility from the total inventory. The three terms in the above expression reflect the average inventory in the EPQ model, the adjustment for accumulation inventory, and the adjustment for inventory at the assembly facility, respectively.

A setup cost  $S$  is incurred every  $T$  time units and the delivery cost  $A$  is incurred every  $R$  time units. Therefore, the corresponding average costs per unit time are  $S/T$  and  $A/R$ , respectively. Therefore, the total controllable cost per unit time is

$$\frac{S}{T} + \frac{1}{2}D(T-pDT-R)h + pD^2Rh + \frac{1}{2}DRh + \frac{A}{R},$$

which simplifies to

$$\frac{S}{T} + \alpha T + \beta R + \frac{A}{R},$$

where  $\alpha = \frac{1}{2}(1-pD)Dh$  and  $\beta = pD^2h$ . Note that  $\alpha$  and  $\beta$  are parameters that depend only upon the problem data.

Note that the objective function is divided into two parts: A part associated with the production interval ( $T$ ), and a part associated with the delivery interval ( $R$ ). The first part represents the cost when deliveries are ignored, while the second part represents costs incurred solely because of the delivery interval:  $\beta R$  (the additional average inventory holding cost incurred because of the delivery interval  $R$ ) and  $A/R$  (the average delivery cost per unit time). If we consider the two parts separately,  $T$  would be determined by a trade-off between the setup cost and a portion of the inventory holding cost and  $R$  would be determined by a similar trade-off between the delivery cost and the remaining inventory holding cost. If the resulting value of  $T/R$  is an integer, these  $T$  and  $R$  values are optimal if a certain condition (which will be discussed later) holds. However,  $T/R$  is unlikely to be an integer. Therefore, we need a systematic approach to adjust the values of  $T$  and  $R$  while keeping the cost as low as possible.

The interpretation of the objective function discussed above leads us to another way to construct the expression for inventory holding costs. Figure 2 shows the total inventory at the two locations as a function of time. The area under the inventory curve during the production interval is

$$\frac{1}{2}(1-pD)DT^2 + pD^2RT.$$

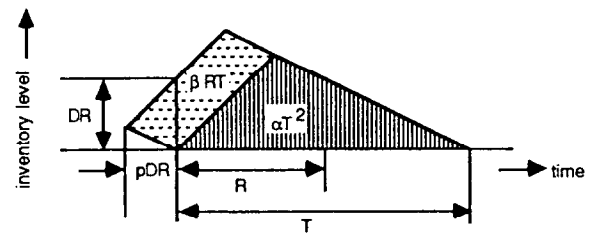


Fig. 2. Alternate representation of the inventory.

If we ignore deliveries, the area under the inventory curve during a duration  $T$  is  $\frac{1}{2}(1-pD)DT^2$ . On the other hand, if we consider deliveries, the issue of “availability” arises. That is, at each delivery time, the delivery amount must be available at the supplier. Therefore, the supplier must start production at least  $pDR$  time before the inventory level reaches zero at the assembly facility. Henceforth, we will refer to the amount of time between the start of production and the zero-inventory point as “earliness”.

When the production interval is  $T$ , a little algebra shows that the increase in the area under the inventory curve caused by earliness is  $eDT$  and  $e$  must be greater than or equal to  $pDR$ . However, in a single component model, there is no reason to set the value of  $e$  greater than  $pDR$ . Therefore, in this model,  $e$  is equal to  $pDR$ , and the increase in the area under the inventory curve in this model is  $pD^2RT$ . As a result, the average inventory caused by earliness is  $pD^2R$ , and the average inventory cost per unit time is  $pD^2Rh$ , which is equivalent to  $\beta R$ .

We assume that just after a setup, production starts immediately and continues without interruption until the scheduled amount is produced. The next setup starts after  $T$  time units has elapsed. Therefore,  $T$  must be larger than sum of the setup time and the production time, i.e.,

$$T \geq s + pDT \quad \text{or} \quad T \geq \frac{s}{1-pD}$$

We now give the formulation.

$$\text{Minimize: } \frac{S}{T} + \alpha T + \beta R + \frac{A}{R}$$

$$\text{Subject to: } T = MR, \quad T$$

$$\geq \tau, \quad M \geq 1, \text{ integer,} \quad (S1)$$

$$\text{where } \alpha = \frac{1}{2}(1-pD)Dh, \quad \beta = pD^2h, \quad \text{and } \tau = s/(1-pD).$$

### 5. Solution approach

The objective function is not guaranteed to be jointly convex in  $M$  and  $R$ , so standard nonlinear programming techniques do not apply. However, it is possible to define a region which con-

tains the optimum values so that we can restrict our attention to that set.

Let  $T' = \max \{ \sqrt{S/\alpha}, \tau \}$  and  $R' = \sqrt{A/\beta}$ . Also let  $M' = T'/R'$ ,  $M_1 = \lfloor M' \rfloor$  and  $M_2 = \lceil M' \rceil$ , where  $\lfloor x \rfloor$  denotes the largest integer less than or equal to  $x$ . We have the following result:

**Theorem 2.** Let  $M^*$  and  $R^*$  be the optimum values of  $M$  and  $R$ , respectively. Then

$$\text{Max}\{M_1, 1\} \leq M^* \leq M_2,$$

$$R^* = \text{Max} \left\{ \sqrt{\frac{S + M^*A}{M^*(\alpha M^* + \beta)}}, \frac{\tau}{M^*} \right\}. \quad (1)$$

**Proof.** See Appendix B.

At most two values of  $M$  satisfy inequality (1). For a given value of  $M$ , the objective function is convex in  $R$ . Therefore, the objective function has a unique minimum and the optimal solution can be obtained accordingly.

### 6. Conditions for optimality of synchronized production and delivery

The results in the previous Section can be used to analyze the impact of various parameters on the solution and its cost. In this Section, we use the results to characterize conditions in which the optimal solution has  $M^* = 1$ , i.e., production and delivery are perfectly synchronized. The results indicate how much production setup costs and setup times must be reduced in order to have an optimal solution that has “just-in-time” characteristics. Derivations are straightforward and are, therefore, omitted.

#### 6.1. Case 1: $0 \leq M' \leq 1$

Using the definitions of  $T'$  and  $R'$  it is straightforward to show that  $M' \leq 1$  iff

$$(a) \quad S \leq A(1-pD)/2pD, \text{ and}$$

$$(b) \quad s \leq (1-pD)(A/ph)^{0.5}/D.$$

The first condition says that the setup cost must be sufficiently small in comparison to the transportation cost. Note that the condition becomes tighter as  $pD$  increases. Thus, if the goal is to achieve synchronization, it is more important to reduce setup costs for products that consume a large portion of the machine capacity. The sec-

ond condition says that the setup time must be sufficiently small in comparison to the square root of the transportation cost in order for  $M=1$  to be optimal. For a given  $D$ , the constraint becomes tighter as the production time,  $p$ , increases. Thus, it is more important to reduce the setup times of products with long processing times.

6.2. Case 2:  $1 \leq M^* \leq 2$  and  $TC(1, R^*(1)) \leq TC(2, R^*(2))$

6.2.1. Subcase (a):

$$R^*(1) = \sqrt{(S+A)/(\alpha+\beta)},$$

$$R^*(2) = \sqrt{(S+2A)/2(2\alpha+\beta)}.$$

In this case, the condition  $TC(1, R^*(1)) \leq TC(2, R^*(2))$  can be simplified to  $A/(S+A) \geq pD$ . Writing  $S = \gamma A$  (i.e.,  $\gamma$  is the ratio of the setup cost to the transportation cost), this can be expressed as

$$\gamma \leq (1/pD) - 1.$$

Thus, to ensure that  $M^* = 1$ , the ratio of setup cost to transportation cost cannot exceed a value which is decreasing with the capacity utilization of the product. Once again, capacity utilization is a key factor.

6.2.2. Subcase (b):

$$R^*(1) = \sqrt{(S+A)/(\alpha+\beta)}, \quad R^*(2) = \tau/2.$$

The condition  $TC(1, R^*(1)) \leq TC(2, R^*(2))$  reduces to

$$(S+2A)/\tau + Dh\tau/2 \geq \sqrt{2[(S+A)Dh(1+pD)]^{0.5}}.$$

Note that the left hand side is convex in  $\tau$ . This condition will be satisfied for all  $\tau$  if it is satisfied when the left hand side achieves its minimum. Thus, making appropriate substitutions, a more restrictive but simpler condition is

$$A/(S+A) \geq pD,$$

which is the same as the condition given in subcase (a). Consequently  $M^* = 1$  over a broader

range of parameters in subcase (b) than in subcase (a).

6.2.3. Subcase (c):

$$R^*(1) = \tau, \quad R^*(2) = \sqrt{(S+2A)/2(2\alpha+\beta)}.$$

For this case, the condition becomes

$$\sqrt{2[(S+2A)Dh]^{0.5}} \geq (S+A)/\tau + Dh(1+pD)\tau/2.$$

The right hand side of this inequality is convex in  $\tau$ . Thus, if the inequality is not satisfied when the right hand side is minimum, it will never be satisfied. Making appropriate substitutions, this implies that in this case,  $M^* = 1$  will never be optimal if

$$A/(S+A) > pD.$$

Note the similarity of this condition to those mentioned earlier and the divergent implications of the condition in subcases (b) and (c). It is apparent that characteristics of the optimal solution depend heavily upon how the setup time affects the value of  $R^*(1)$  and  $R^*(2)$ .

6.2.4. Subcase (d):

$$R^*(1) = \tau, \quad R^*(2) = \tau/2.$$

In this instance, the condition simplifies to

$$2A\beta \geq \tau^2 \quad \text{or equivalently} \\ s \leq D(2Aph)^{0.5}(1-pD).$$

This says, among other things, that the setup time must be less than a value that increases as the square root of the transportation cost.

These results can be used by managers to determine where setup-related improvements should be focused in the quest for "just-in-time".

## 7. Extension to a fixed-charge-per-truck transportation cost

Initially, we assumed that the delivery cost ( $A$ ) is a constant independent of the delivery quantity. We now relax this assumption and assume



instead that the delivery cost ( $A$ ) is a constant independent of the delivery quantity if it is less than or equal to the truck capacity. This is a realistic representation of transportation costs if goods are being shipped at full-truckload rates, or under a contract with point-to-point charges. The truck capacity is defined as the maximum quantity a truck can deliver at a time. Thus, since demand is constant, there is a delivery interval corresponding to the truck capacity and we define  $\rho$  to be this interval. Therefore, if we make a delivery every  $R$  time units, the delivery cost for each delivery is  $A\lceil R/\rho \rceil$  where  $\lceil x \rceil$  is the smallest integer greater than or equal to  $x$ . The formulation becomes:

$$\begin{aligned} \text{Minimize: } & \frac{S}{T} + \alpha T + \beta R + \frac{1}{R} \left( A \lceil \frac{R}{\rho} \rceil \right) \\ \text{Subject to: } & T \geq \tau, \quad T = M \cdot R, \\ & M \geq 1, \quad \text{integer.} \end{aligned} \tag{SE}$$

**Lemma 1.** Let  $R^*$  be an optimal value of  $R$  in problem (3). Then,  $R^* \leq \rho$ .

**Proof.** See Appendix C.

Note that the last term in the objective function properly accounts for the transportation cost by charging  $A$  for each truck shipped, irrespective of how full it is. With such a cost structure, it is never optimal to ship more than one truck at a time (under the assumption of deterministic demand) because inventory costs will increase without a corresponding offset in transportation costs. Therefore, we can restrict our attention to values of  $R$  less than or equal to  $\rho$  and the formulation of this problem can be simplified as follows:

$$\begin{aligned} \text{Minimize: } & \frac{S}{T} + \alpha T + \beta R + \frac{A}{R}, \\ \text{Subject to: } & T \geq \tau, \quad R \leq \rho, \quad T = M \cdot R, \\ & M \geq 1, \quad \text{integer.} \end{aligned} \tag{SE-1}$$

The solution procedure for problem (SE) is similar to that of (S1), and is stated in the theorem below.

**Theorem 3.** Let

$$T' = \text{Max} \left\{ \sqrt{\frac{S}{\alpha}}, \tau \right\},$$

$$R' = \text{Min} \left\{ \sqrt{\frac{A}{\beta}}, \rho \right\},$$

$$M' = \frac{T'}{R'}, \quad R(M) = \sqrt{\frac{S + MA}{M(\alpha M + \beta)}},$$

$$M_1 = \text{Max} \left\{ \lceil \frac{\tau}{\rho} \rceil, \lfloor M' \rfloor \right\}, \quad M_2 = \lceil M' \rceil.$$

The optimal value of  $M$  for problem (SE) is either  $M_1$  or  $M_2$ . Also, the optimal value of  $R$  is either  $R^*(M_1)$  or  $R^*(M_2)$ , where

$$R^*(M) = \begin{cases} \tau/M & \text{if } R(M) < \tau/M, \\ R(M) & \text{if } \tau/M \leq R(M) \leq \rho, \\ \rho & \text{if } R(M) \geq \rho. \end{cases}$$

**Proof.** See Appendix D.

The solution procedure for this extension follows directly from Theorem 3.

### 8. Conclusions

We have investigated a deterministic, continuous-time production and delivery scheduling problem in which one component is produced in batches at a supplier and then transported to the customer for use. The objective is to minimize the average cost per unit time of production setup, inventory, and transportation costs. For this simple model, we have developed a procedure which provides optimal solutions. Our analysis also provides a basis for determining when it is optimal to synchronize production and delivery, and how much setup costs and setup times must be reduced in order for synchronization to be optimal. We have also extended the model to allow for a fixed-charge-per-truck transportation cost structure.

Further research is needed to consider multiple components produced on different machines, more general transportation cost structures and transportation arrangements, including consolidation of freight from multiple suppliers. In se-

quels, we investigate extensions to multiple components [3–5]. Although these extensions treat more realistic problems, the combinatorial aspects of these multi-component problems obscure the fundamental role of production setup costs and setup times in determining delivery frequencies. We believe that the model presented here provides some important insights that can guide practitioners in prioritizing setup improvements in the quest for “just-in-time”.

**Appendix A: Optimality of a positive integer  $M$ :**

We show that a non-integer  $M$  cannot be optimal. If  $M$  is not integer, the earliness of the production runs may differ. Suppose the first production run starts exactly  $pDR$  before the first delivery point. Then, the earliness of the  $(i + 1)$ th production run is as follows (Fig. A1):

$$e_{i+1} = (iM - \lfloor iM \rfloor)R + pD(\lceil iM \rceil + 1 - iM)R,$$

where  $\lfloor x \rfloor$  is the largest integer less than or equal to  $x$ . The components produced in the  $i$ th production run are depleted at time  $iT = iMR$ . The delivery point just before this time point is  $\lfloor iM \rfloor R$  and the inventory level at this time would be  $(iM - \lfloor iM \rfloor)DR$ . However, the delivery amount ( $DR$ ) must be available prior to shipment, so the  $(i + 1)$ th production run must start  $p[DR - (iM - \lfloor iM \rfloor)DR]$  before this delivery point. As a result, the total earliness of the  $i$ th production run is  $(iM - \lfloor iM \rfloor)R + pD(\lceil iM \rceil + 1 - iM)R$ .

We can simplify  $e_i$  as follows:

$$e_{i+1} = pDR + (1 - pD)(iM - \lfloor iM \rfloor)R.$$

Therefore, the average earliness is

$$\begin{aligned} E &= \lim_{I \rightarrow \infty} \frac{1}{I} \sum_{i=1}^I e_i, \\ &= \lim_{I \rightarrow \infty} \frac{1}{I} \sum_{i=1}^I [pDR + (1 - pD) \\ &\quad \times (iM - \lfloor iM \rfloor)R], \\ &= pDR + (1 - pD)R \lim_{I \rightarrow \infty} \frac{1}{I} \\ &\quad \times \sum_{i=1}^I (iM - \lfloor iM \rfloor), \\ &= pDR + (1 - pD)RX(M), \end{aligned}$$

where

$$X(M) \equiv \lim_{I \rightarrow \infty} \frac{1}{I} \sum_{i=1}^I (iM - \lfloor iM \rfloor).$$

The average earliness cost can be expressed as

$$\begin{aligned} DhE &= pD^2hR + Dh(1 - pD)RX(M), \\ &= \beta R + 2\alpha RX(M), \end{aligned}$$

where  $\alpha = \frac{1}{2}(1 - pD)Dh$  and  $\beta = pD^2h$ . As a result, the total objective function value is

$$\begin{aligned} TC(M, R) &= \frac{S}{MR} + \alpha MR \\ &\quad + (\beta + 2\alpha X(M))R + \frac{A}{R}. \end{aligned}$$

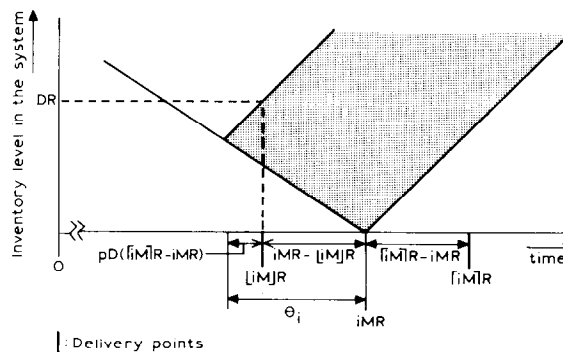


Fig. A1. Earliness of  $i$ th production run when  $M$  is non-integer.

Suppose  $M' = \lceil M \rceil$ .  
Then,

$$TC(M', R) = \frac{S}{M'R} + \alpha M'R + \beta R + \frac{A}{R}.$$

Thus,

$$TC(M, R) - TC(M', R) = \frac{S}{R} \left( \frac{1}{M} - \frac{1}{M'} \right) + \alpha R [M + 2X(M) - M'].$$

Since  $M' \geq M$ , if  $[M + 2X(M) - M'] \geq 0$ ,  $TC(M', R)$  is always less than  $TC(M, R)$ .

**Theorem 1.** Let  $M$  be non-integer but a positive rational number. Let

$$X(M) \equiv \lim_{I \rightarrow \infty} \frac{1}{I} \sum_{i=1}^I (iM - \lfloor iM \rfloor).$$

Let  $M' = \lceil M \rceil$ . Then,  $M + 2X(M) - M' \geq 0$ , and thus  $TC(M, R) - TC(M', R) > 0$  for any  $M$  and  $R$ .

**Proof.** Since  $M$  is rational,  $M = a/b$  for some positive integers  $a$  and  $b$ , where  $a$  and  $b$  are relatively prime. Also, it can be easily shown that  $X(M) = X(M - \lfloor M \rfloor)$ . Therefore, if  $M + 2X(M) - M' \geq 0$  for  $0 \leq M < 1$ ,  $M + 2X(M) - M' \geq 0$  for any positive  $M$ . Therefore, without loss of generality, we assume that  $M < 1$  and thus,  $a < b$  and  $M' = 1$ . Substituting  $a/b$  for  $M$ , we have

$$\begin{aligned} M + 2X(M) - M' &= \frac{a}{b} + 2 \lim_{I \rightarrow \infty} \frac{1}{I} \sum_{i=1}^I \left( \frac{ia}{b} - \lfloor \frac{ia}{b} \rfloor \right) - 1 \\ &= \frac{a}{b} + 2 \lim_{Y \rightarrow \infty} \frac{1}{Yb} \sum_{y=0}^{Y-1} \sum_{i=1}^b \left( \frac{(yb+i)a}{b} - \lfloor \frac{(yb+i)a}{b} \rfloor \right) - 1 \\ &= \frac{a}{b} + 2 \frac{1}{b} \sum_{i=1}^b \left( \frac{ia}{b} - \lfloor \frac{ia}{b} \rfloor \right) - 1 \\ &\quad \text{since } \frac{(yb+i)a}{b} - \lfloor \frac{(yb+i)a}{b} \rfloor = \frac{ia}{b} - \lfloor \frac{ia}{b} \rfloor, \\ &= \frac{a}{b} + 2 \frac{1}{b} \sum_{i=1}^{b-1} \left( \frac{ia}{b} - \lfloor \frac{ia}{b} \rfloor \right) - 1 \\ &\quad \text{since } a = \lfloor a \rfloor. \end{aligned}$$

Case 1.  $b$  is an odd integer. In this case:

$$\begin{aligned} &\sum_{i=1}^{b-1} \left( \frac{ia}{b} - \lfloor \frac{ia}{b} \rfloor \right) \\ &= \sum_{i=1}^{(b-1)/2} \left( \frac{ia}{b} - \lfloor \frac{ia}{b} \rfloor + \frac{(b-i)a}{b} - \lfloor \frac{(b-i)a}{b} \rfloor \right) \\ &= \sum_{i=1}^{(b-1)/2} \left[ a - \left( \lfloor \frac{ia}{b} \rfloor + \lfloor \frac{(b-i)a}{b} \rfloor \right) \right] \\ &= \sum_{i=1}^{(b-1)/2} \{ a - (a-1) \} \\ &\quad \text{since } \frac{ia}{b} - \frac{(b-i)a}{b} = a, \\ &= (b-1)/2. \end{aligned}$$

Case 2.  $b$  is an even integer. In this case:

$$\begin{aligned} &\sum_{i=1}^{b-1} \left( \frac{ia}{b} - \lfloor \frac{ia}{b} \rfloor \right) = \left[ 2 \sum_{i=1}^{(b-2)/2} \left( \frac{ia}{b} - \lfloor \frac{ia}{b} \rfloor \right) \right. \\ &\quad \left. + \frac{(b-i)a}{b} - \lfloor \frac{(b-i)a}{b} \rfloor \right] \\ &\quad + \frac{(b/2)a}{b} - \lfloor \frac{(b/2)a}{b} \rfloor, \\ &= (b-2)2 + \frac{1}{2} \text{ since } a \text{ is an odd number} \\ &\quad \text{and thus } (a/2) - \lfloor a/2 \rfloor = \frac{1}{2}, \\ &= (b-1)/2. \end{aligned}$$

As a result,

$$\begin{aligned} M + 2X(M) - M' &= \frac{a}{b} + 2 \frac{1}{b} \sum_{i=1}^{b-1} \left( \frac{ia}{b} - \lfloor \frac{ia}{b} \rfloor \right) - 1, \\ &= \frac{a}{b} + \frac{1}{b} (b-1) - 1, \\ &= (a-1)/b \geq 0 \quad \text{since } a \text{ is a} \\ &\quad \text{positive integer.} \end{aligned}$$

Consequently,

$$TC(M, R) - TC(M', R) > 0$$

for any  $M$  and  $R$ ,

and thus a non-integer  $M$  cannot be optimal.

**Appendix B: Proof of Theorem 2**

Let  $T' = \text{Max}\left\{\sqrt{\frac{S}{\alpha}}, \tau\right\}$  and  $R' = \sqrt{\frac{A}{\beta}}$ . Also let

$$TC(M,R) = \frac{S}{M \cdot R} + \alpha M \cdot R + \beta R + \frac{A}{R},$$

$$R(M) = \sqrt{\frac{S + M \cdot A}{M \cdot (\alpha M + \beta)}},$$

$$R^*(M) = \text{Max}\left\{R(M), \frac{\tau}{M}\right\},$$

where  $R(M)$  is the solution of  $\frac{\partial}{\partial R} TC(M,R) = 0$ . For a given  $M$ , the lower bound on the value of  $R$  is  $\frac{\tau}{M}$  because the capacity constraint requires that  $T = M \cdot R \geq \tau$ . Lemma 2 through 7 will be used in the proof of Theorem 2.

**Lemma 2.**  $R^*(M)$  minimizes  $\{TC(M,R) | MR \geq \tau\}$  for a given  $M \geq 1$ .

**Proof.** For a given  $M \geq 1$ ,

$$\frac{\partial^2}{\partial R^2} TC(M,R) = \frac{2(S + M \cdot A)}{MR^3} > 0 \quad \text{for } R > 0.$$

Therefore,  $TC(M,R)$  is convex in  $R$  for a given  $M \geq 1$ , and has a unique minimum at  $R = R(M)$ . Thus, if  $R < \tau/M$ , the constrained minimum is achieved at  $R = \tau/M$ . Therefore,  $R^*(M)$  minimizes  $TC(M,R)$ .

**Lemma 3.**  $TC(M,R(M))$  is increasing in  $M$  for

$$M \geq \sqrt{\frac{S\beta}{A\alpha}}, \text{ and decreasing in } M \text{ for } M \leq \sqrt{\frac{S\beta}{A\alpha}}$$

**Proof.**

$$TC(M,R(M)) = 2 \sqrt{\frac{(S + M \cdot A)(\alpha M + \beta)}{M}}$$

It is easily verified (from the first derivative) that

$(S + M \cdot A)(\alpha M + \beta)/M$  is increasing in  $M$  for  $M \geq \sqrt{\frac{S\beta}{A\alpha}}$ , and decreasing in  $M$  for  $M \leq \sqrt{\frac{S\beta}{A\alpha}}$ .

Therefore, Lemma 3 follows.

**Lemma 4.**  $TC\left(M, \frac{\tau}{M}\right)$  is a convex function of  $M$

and achieves its minimum at  $M = \frac{\tau}{R'}$ .

**Proof.**

$$TC\left(M, \frac{\tau}{M}\right) = \frac{S}{\tau} + \alpha\tau + \frac{\beta\tau}{M} + M \cdot \frac{A}{\tau}.$$

It is easily verified that  $\frac{\partial^2}{\partial M^2} TC\left(M, \frac{\tau}{M}\right) > 0$  and

$$\text{that } \frac{\partial}{\partial M} TC\left(M, \frac{\tau}{M}\right) = 0 \text{ at } M = \frac{\tau}{R'}.$$

**Lemma 5.**  $R(M)$  is decreasing in  $M$ .

**Proof.** It is easily verified that  $[R(M)]^2$  is decreasing in  $M$  (from the first derivative), and the result follows.

**Lemma 6.**  $M \cdot R(M)$  is increasing in  $M$ .

**Proof.** It is easily verified that  $M^2 \cdot [R(M)]^2$  is increasing in  $M$  (from the first derivative), and the result follows.

**Lemma 7.**  $R\left(\sqrt{\frac{S\beta}{A\alpha}}\right) = R'$ .

**Proof.**

$$\begin{aligned} [R(M)]^2 &= \frac{S + M \cdot A}{M \cdot (\alpha M + \beta)}, \\ &= \frac{A(S + M \cdot A)}{\beta\left(\frac{\alpha M^2}{\beta} \cdot A + M \cdot A\right)}. \end{aligned}$$

Therefore,

$$\left[ R \left( \sqrt{\frac{S\beta}{A\alpha}} \right) \right]^2 = \frac{A(S+M \cdot A)}{\beta(S+M \cdot A)} = \frac{A}{\beta} = [R']^2$$

and the result follows.

**Theorem 2.** Let  $T' = \text{Max} \left\{ \sqrt{\frac{S}{\alpha}}, \tau \right\}$  and

$R' = \sqrt{\frac{A}{\beta}}$ . Also let  $M' = \frac{T'}{R'}$ ,  $M_1 = \lfloor M' \rfloor$  and

$M_2 = \lceil M' \rceil$ . Let  $M^*$  and  $R^*$  be the optimum values of  $M$  and  $R$ , respectively. Then:

$$\text{Max}\{M_1, 1\} \leq M^* \leq M_2, \tag{A1}$$

$$R^* = \text{Max} \left\{ \sqrt{\frac{S+M^* \cdot A}{M^* \cdot (\alpha M^* + \beta)}}, \frac{\tau}{M^*} \right\}. \tag{A2}$$

**Proof.** For a given  $M$ , the objective function is convex in  $R$ . Therefore, the proof of (A2) is trivial and is omitted here. Let us now prove (A1).

Case 1:  $T' = \sqrt{\frac{S}{\alpha}}$ , which means  $\sqrt{\frac{S}{\alpha}} \geq \tau$  and

$$\text{thus } M' = \sqrt{\frac{S\beta}{A\alpha}}.$$

Subcase 1:  $M \geq M'$ . We will show that  $TC(M, R^*(M)) \geq TC(M_2, R^*(M_2))$ . Let us first show that  $R(M) \geq \frac{\tau}{M}$  for any  $M \geq M'$ .

$$[R(M)]^2 - \left[ \frac{\tau}{M} \right]^2 = \frac{1}{M} \left[ \frac{S+M \cdot A}{\alpha M + \beta} - \frac{\tau^2}{M} \right]$$

$$\text{since } R(M) = \sqrt{\frac{S+M \cdot A}{M \cdot (\alpha M + \beta)}}$$

$$\geq \frac{1}{M} \left[ \frac{S+M \cdot A}{\alpha M + \beta} - \frac{\left( \sqrt{\frac{S}{\alpha}} \right)^2}{M} \right]$$

$$\text{since } \sqrt{\frac{S}{\alpha}} \geq \tau$$

$$= \frac{1}{M} \left[ \frac{\alpha A \cdot M^2 - \beta S}{\alpha M (\alpha M + \beta)} \right] \geq 0$$

$$\text{since } M \geq \sqrt{\frac{S\beta}{A\alpha}}.$$

Therefore, for any  $M \geq M'$ ,  $R(M) \geq \frac{\tau}{M}$  and thus  $R^*(M) = R(M)$  by the definition of  $R^*(M)$ . As a result,

$$\begin{aligned} TC(M, R^*(M)) &= TC(M, R(M)), \\ &\geq TC(M_2, R(M_2)) \\ &\quad \text{by Lemma 3} \\ &= TC(M_2, R^*(M_2)). \end{aligned}$$

Subcase 2:  $M < M'$ . We will show that  $TC(M, R^*(M)) \geq TC(M_1, R^*(M_1))$ .

(i) If  $R^*(M_1) = R(M_1)$ , then for any  $M \leq M_1$ ,

$$\begin{aligned} TC(M, R^*(M)) &\geq TC(M, R(M)) \\ &\quad \text{by Lemma 2} \\ &\geq TC(M_1, R(M_1)) \\ &\quad \text{by Lemma 3} \\ &= TC(M_1, R^*(M_1)). \end{aligned}$$

(ii) If  $R^*(M_1) = \frac{\tau}{M_1}$ , the  $R(M) \leq \frac{\tau}{M}$  for  $M \leq M_1$  because  $M_1 \cdot R(M_1) \geq M \cdot R(M)$  by Lemma 6. Also,

$$\begin{aligned} M_1 &\leq \frac{\tau}{R(M_1)}, \\ &\leq \frac{\tau}{R(M')} \quad \text{by Lemma 5} \\ &= \frac{\tau}{R'} \quad \text{by Lemma 7.} \end{aligned}$$

Therefore, for any  $M \leq M_1$ ,

$$\begin{aligned}
 TC(M, R^*(M)) &= TC\left(M, \frac{\tau}{M}\right) \\
 &\geq TC\left(M_1, \frac{\tau}{M_1}\right) \\
 &\quad \text{by Lemma 4} \\
 &= TC(M_1, R^*(M_1)).
 \end{aligned}$$

As a result, from (i) and (ii),

$$\begin{aligned}
 TC(M, R^*(M)) \\
 \geq TC(M_1, R^*(M_1)) \quad \text{for } M \leq M_1.
 \end{aligned}$$

Consequently, when  $T' = \sqrt{\frac{S}{\alpha}}$ , we have  $M^* = M_1$  or  $M_2$ . However,  $M_1 = 0$  is infeasible and, hence,  $\text{Max}\{M_1, 1\} \leq M^* \leq M_2$ .

Case 2:  $T' = \tau$ , which means  $\sqrt{\frac{S}{\alpha}} \leq \tau$  and thus

$$M' = \tau \sqrt{\frac{\beta}{A}}.$$

Subcase 1:  $M \leq M'$ . We will show that  $TC(M, R^*(M)) \geq TC(M_1, R^*(M_1))$ . Let us first show that  $R(M) \leq \frac{\tau}{M}$

$$[R(M)]^2 - \left[\frac{\tau}{M}\right]^2 = \frac{1}{M} \left[ \frac{S + M \cdot A}{\alpha M + \beta} - \frac{\tau^2}{M} \right]$$

$$\text{since } R(M) = \sqrt{\frac{S + M \cdot A}{M \cdot (\alpha M + \beta)}},$$

$$\leq \frac{1}{M} \left[ \frac{M \cdot S + \tau^2 \cdot \beta}{M(\alpha M + \beta)} - \frac{\tau^2}{M} \right]$$

$$\text{since } M \leq \tau \sqrt{\frac{\beta}{A}}$$

$$= \frac{1}{M} \left[ \frac{S - \alpha \tau^2}{\alpha M + \beta} \right] \leq 0 \quad \text{since } \sqrt{\frac{S}{\alpha}} \leq \tau.$$

Therefore, for any  $M \leq M'$ ,  $R^*(M)$  is  $\frac{\tau}{M}$  and as a result,

$$\begin{aligned}
 TC(M, R^*(M)) &= TC\left(M, \frac{\tau}{M}\right), \\
 &\geq TC\left(M_1, \frac{\tau}{M_1}\right) \text{ by Lemma 4,} \\
 &= TC(M_1, R^*(M_1)).
 \end{aligned}$$

Subcase 2:  $M \geq M'$ . We will show that  $TC(M, R^*(M)) \geq TC(M_2, R^*(M_2))$ .

(i) If  $R^*(M_2) = R(M_2)$ , then for any  $M \geq M_2$ ,

$$\begin{aligned}
 TC(M, R^*(M)) &\geq TC(M, R(M)) \\
 &\geq TC(M_2, R(M_2))
 \end{aligned}$$

$$\text{since } M_2 \geq M' = \tau \sqrt{\frac{\beta}{A}} \geq \sqrt{\frac{S\beta}{A\alpha}},$$

and by Lemma 3,

$$= TC(M_2, R^*(M_2)).$$

(ii) If  $R^*(M_2) = \frac{\tau}{M_2}$ , then for any  $M \geq M_2$ ,

$$TC\left(M, \frac{\tau}{M}\right) \geq TC\left(M_2, \frac{\tau}{M_2}\right)$$

since  $M_2 \geq \frac{\tau}{R'}$  and by Lemma 4.

Therefore, for any  $M \geq M_2$  such that  $R^*(M) = \frac{\tau}{M}$

$$\begin{aligned}
 TC(M, R^*(M)) &= TC\left(M, \frac{\tau}{M}\right), \\
 &\geq TC\left(M_2, \frac{\tau}{M_2}\right) \text{ by Lemma 4,} \\
 &= TC(M_2, R^*(M_2)).
 \end{aligned}$$

For any  $M \geq M_2$  such that  $R^*(M) = R(M)$  (i.e.,  $R(M) \geq \frac{\tau}{M}$ ),

$$\begin{aligned}
 TC(M, R^*(M)) &= TC(M, R(M)) \\
 &= \frac{S}{M \cdot R(M)} + \alpha M \cdot R(M) + \beta R(M) + \frac{A}{R(M)}.
 \end{aligned}$$

Now,  $M \cdot R(M) \geq \tau \geq \sqrt{\frac{\beta}{A}}$ . Therefore,

$$\frac{S}{M \cdot R(M)} + \alpha M \cdot R(M) \geq \frac{S}{\tau} + \alpha \tau,$$

because  $\frac{S}{T} + \alpha T$  is a convex function of  $T$  and

achieves its minimum at  $T = \sqrt{\frac{\beta}{A}}$ . Moreover,

$R(M)$  is decreasing in  $M$  and  $R(M) \leq R(M_2) \leq \frac{\tau}{M_2} \leq \frac{\tau}{M'} = R'$ . Therefore,

$$\beta R(M) + \frac{A}{R(M)} \geq \beta R(M_2) + \frac{A}{R(M_2)},$$

because  $\beta R + \frac{A}{R}$  is a convex function of  $R$  and achieves its minimum at  $R = R'$ . Therefore, for any  $M \geq M_2$  such that  $R^*(M) = R(M)$ ,

$$\begin{aligned} TC(M, R^*(M)) &\geq TC\left(M_2, \frac{\tau}{M_2}\right) \\ &= TC(M_2, R^*(M_2)). \end{aligned}$$

As a result, from (i) and (ii),

$$\begin{aligned} TC(M, R^*(M)) &\geq TC(M_2, R^*(M_2)) \quad \text{for } M \geq M_2. \end{aligned}$$

Consequently, when  $T' = \tau$ ,  $M^* = M_1$  or  $M_2$ . However,  $M_1 = 0$  is infeasible, and hence,

$$\text{Max}\{M_1, 1\} \leq M^* \leq M_2.$$

### Appendix C: Proof of lemma 1

We prove that the optimal value of  $R$  is always less than  $\rho$  when there is a fixed-cost-per-truck-load transportation cost.

$$\text{Minimize: } \frac{S}{T} + \alpha T + \beta R + \frac{1}{R} \left( A \cdot \left\lceil \frac{R}{\rho} \right\rceil \right),$$

Subject to:  $T \geq \tau$ ,

$$T = M \cdot R,$$

$$M \geq 1, \quad \text{integer,}$$

**Lemma 1:** Let  $R^*$  be an optimal value of  $R$  in problem (SE). Then,

$$R^* \leq \rho.$$

**Proof.** Let

$$\begin{aligned} TC(M, R) &= \frac{S}{MR} + \alpha MR \\ &\quad + \beta R + \frac{1}{R} \left( A \cdot \left\lceil \frac{R}{\rho} \right\rceil \right). \end{aligned} \tag{A3}$$

Suppose  $R^* = a\rho + b$ , where  $a$  is an integer greater than or equal to 1 and  $b$  is a real value such that  $0 < b \leq \rho$ . Also, let  $M^*$  be the optimal value of  $M$  corresponding to  $R^*$ . Then,

$$\begin{aligned} TC(M^*, R^*) &= \frac{S}{M^* R^*} + \alpha M^* R^* + \beta R^* + \frac{(a+1)A}{a\rho + b}. \end{aligned} \tag{A4}$$

Let  $R' \equiv R^*/(a+1)$  and  $M' \equiv (a+1)M^*$ . Then  $(M', R')$  is also feasible to (SE) and  $M' R' = M^* R^*$ . As a result,

$$\begin{aligned} TC(M', R') &= \frac{S}{M' R'} + \alpha M' R' + \beta R' + \frac{A}{R'} \\ &= \frac{S}{M^* R^*} + \alpha M^* R^* + \beta R' + \frac{(a+1)A}{a\rho + b}, \\ &\leq \frac{S}{M^* R^*} + \alpha M^* R^* + \beta R^* + \frac{(a+1)A}{a\rho + b} \\ &\quad \text{since } R' < R^*, \end{aligned} \tag{A5}$$

which contradicts the fact that  $R^*$  is an optimal solution. Therefore,  $R^*$  must be less than  $\rho$ .

### Appendix D: Proof of Theorem 3

Theorem 3 states the form of the optimal solution for the extension to a fixed-charge-per-truck transportation cost. The problem is formulated as follows:

$$\text{Minimize: } \frac{S}{T} + \alpha T + \beta R + \frac{A}{R}, \tag{SE-1}$$

$$\text{Subject to: } T \geq \tau, \tag{A6}$$

$$R \leq \rho, \tag{A7}$$

$$T = M \cdot R, \tag{A8}$$

$$M \geq 1, \quad \text{integer.} \tag{A9}$$

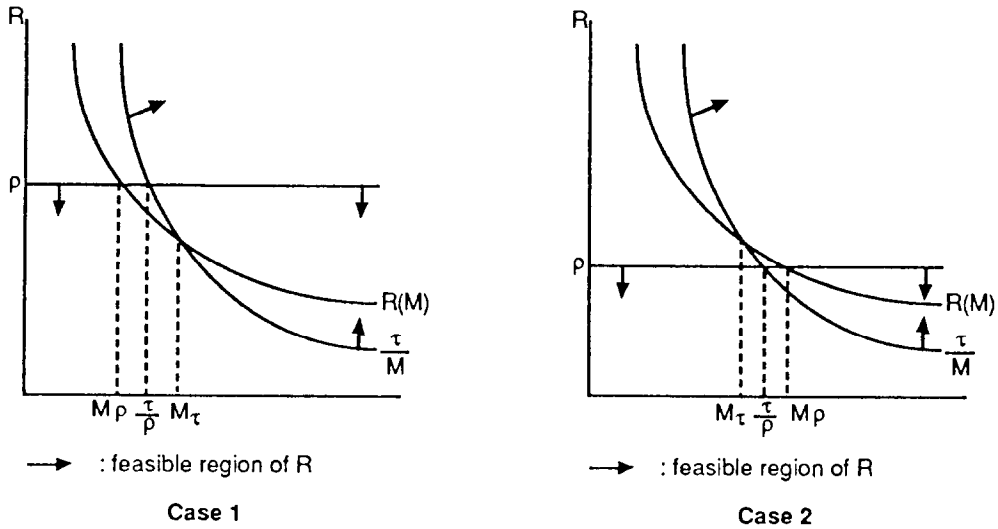


Fig. A2. Graphical illustration of cases 1 and 2.

**Theorem 3.** Let

$$T' = \text{Max} \left\{ \sqrt{\frac{S}{\alpha}}, \tau \right\},$$

$$R' = \text{Min} \left\{ \sqrt{\frac{A}{\beta}}, \rho \right\},$$

$$M' = \frac{T'}{R'}, \quad R(M) = \sqrt{\frac{S+MA}{M(\alpha M + \beta)}}$$

$$M_1 = \text{Max} \left\{ \lceil \frac{\tau}{\rho} \rceil, \lfloor M' \rfloor \right\}, \quad M_2 = \lceil M' \rceil.$$

The optimal value of  $M$  for problem (SE-1) is either  $M_1$  or  $M_2$ , and the respective optimal value of  $R$  is  $R^*(M_1)$  or  $R^*(M_2)$ , where

$$R^*(M) = \begin{cases} \frac{\tau}{M'} & \text{if } R(M) < \frac{\tau}{M'} \\ R(M), & \text{if } \frac{\tau}{M} \leq R(M) \leq \rho, \\ \rho & \text{if } R(M) \geq \rho. \end{cases}$$

**Proof.** First, we relax the integrality constraint on  $M$ . We refer to the resulting relaxed problem as RES. We will prove that

- (i) if the value of  $M$  is given, the objective function is convex in  $R$  and is minimized at  $R = R^*(M)$ ,
- (ii)  $TC(M, R^*(M))$  is monotonic in  $M$ ,
- (iii)  $M'$  is the optimal value of  $M$  for the relaxed problem (RES), and thus
- (iv)  $M_1$  or  $M_2$  is an optimal value of  $M$  for problem (SE-1).

The proof of (i) is trivial and omitted here.

*Proof of (ii).* By constraints (A6), (A7) and (A8),  $M$  must be greater than  $\frac{\tau}{\rho}$ . Let  $m_\tau$  be a value of  $M$

such that  $R(M) = \frac{\tau}{M}$  and  $m_\rho$  be a value of  $M$  such that  $R(M) = \rho$ . It can be easily verified that  $m_\tau$  is unique and exists. Moreover,

$$R(M) \leq \frac{\tau}{M'} \quad \text{if } M \leq m_\tau,$$

$$R(M) \geq \frac{\tau}{M'} \quad \text{if } M \geq m_\tau.$$

Analogously,  $m_\rho$  is unique and exists. Moreover,

$$R(M) \leq \rho, \quad \text{if } M \leq m_\rho,$$

$$R(M) \geq \rho, \quad \text{if } M \geq m_\rho.$$



Also, it can be easily proved that

$$\text{If } m_\rho \leq m_\tau, \quad \text{then } m_\rho \leq \frac{\tau}{\rho} \leq m_\tau,$$

$$\text{If } m_\tau \leq m_\rho, \quad \text{then } m_\tau \leq \frac{\tau}{\rho} \leq m_\rho.$$

As a result, we have the following cases (see Fig. A2).

Case 1:  $m_\rho \leq m_\tau$ :

$$\begin{aligned} R^*(M) &= \frac{\tau}{M} && \text{for } \frac{\tau}{\rho} \leq M \leq m_\tau, \\ &= R(M), && \text{for } M \geq m_\tau. \end{aligned}$$

Case 2:  $m_\rho \geq m_\tau$ :

$$\begin{aligned} R^*(M) &= \rho && \text{for } \frac{\tau}{\rho} \leq M \leq m_\rho, \\ &= R(M) && \text{for } M \geq m_\rho. \end{aligned}$$

If Case 1 holds,  $TC(M, R^*(M))$  is monotonic in  $M$  by Lemma 5. The monotonicity of  $TC(M, R^*(M))$  in Case 2 can be proved in a similar manner. As a result,  $TC(M, R^*(M))$  is monotonic in  $M$ .

*Proof of (iii).* It is obvious that  $T'$  and  $R'$  are optimal values of  $T$  and  $R$  for problem (RES).  $M'$  is the value of  $M$  which is determined correspondingly and it is therefore the optimal value of  $M$ .

By (ii) and (iii), (iv) is also true. ■

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