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Brief Paper

Induced Norms for Sampled-data Systems*

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Key Words—Sampled-data systems; digital control; induced operator norms; performance analysis; discrete-time systems.

Abstract—In this paper, we consider a general linear interconnection of a continuous-time plant and a discrete-time controller via sample and hold devices. When the closed loop sampled-data feedback system is internally stable, bounded inputs produce bounded outputs. We present some explicit formulae for the induced norm of the closed loop system with $\mathscr{L}_\infty$ (i.e. peak value) and $\mathscr{L}_1$ (i.e. integral absolute) norms on the input and output signals.

1. *Introduction*

AN ANALYTICAL APPROACH for evaluating performance of linear feedback systems is through the use of system norms. Depending on the nature of the signals affecting the system and performance objectives, one can define a number of different system norms. Among the most commonly used norms for Finite Dimensional Linear Time-Invariant (FDLTI) systems are the following:

(1) the $\mathscr{H}_2$ norm—square root of the integral of the squared magnitude of the transfer function on the imaginary axis (or the unit circle in the discrete-time case). It measures the output power assuming that the input is a white Gaussian stochastic process of unit intensity;
(2) the $\mathscr{H}_\infty$ norm—the supremum of the magnitude of the transfer function evaluated on the imaginary axis (or the unit circle in discrete-time case). It measures the maximal energy gain;
(3) the $\mathscr{L}_1$ or $\mathscr{A}$ norm—integral of the absolute value of the impulse response. It measures the maximal peak gain.

All of these norms are quite useful in analyzing feedback systems in which the plant and the controller are linear time-invariant systems operating either both in the continuous-time or both in the discrete-time. This does not cover the class of *sampled-data systems* in which the plant operates in the continuous-time while the controller operates in the discrete-time and the plant and the controller are interfaced with each other using *analog-to-digital* (A/D) and *digital-to-analog* (D/A) converters. We will omit in this paper the fact that the A/D converter also involves *quantization in magnitude* of signals and are thus really nonlinear operators. We will treat them as ideal sampler and ideal hold devices. It is of interest to extend the notion of system norms for sampled-data systems.

Recently, many papers have appeared that deal with $\mathscr{H}_2$ and $\mathscr{H}_\infty$ norms for sampled-data systems. Bamieh and Pearson (1992), Başar (1991), Chen and Francis (1990), Kabamba and Hara (1990), Sun *et al.* (1991), Tadmor (1991) and Toivonen (1992) have considered $\mathscr{H}_\infty$ optimal control problems while Bamieh and Pearson (1991), Chen and Francis (1991b), Juan and Kabamba (1991) and Khargonekar and Sivashankar (1991) have considered the $\mathscr{H}_2$ optimal control problem for sampled-data systems. Keller and Anderson (1992) have worked on the related problem of discretization of continuous-time controllers. In this paper, we will present some formulae for the induced norms of sampled-data systems. A general interconnection of a continuous-time system (the plant) and a discrete-time system (the controller) with sample and hold operators will be considered. The key difference between analyzing a digital control system as a sampled-data system and as a discrete-time system is that the intersample behavior is taken into account directly in the former by treating the (exogenous) inputs and the (regulated) outputs as continuous-time signals. We will consider two different cases. In the first case, the input and output signal norm will be taken to be the $\mathscr{L}_\infty$ (peak value) norm and a formula for the induced norm of a sampled-data system will be given; in the second case we will give a similar result when the input and output signal norm is the $\mathscr{L}_1$ (integral absolute) norm. Using these two formulae, we can give an upper bound on the $\mathscr{L}_p$-induced norm of a stable sampled-data system for $1 < p < \infty$. This also shows that if a sampled-data feedback system is internally stable then it is input–output stable from the exogeneous inputs to the regulated variables.

As mentioned above, a major motivation for analyzing sampled-data systems stems from the need to deal with the intersample behavior of various signals. From this point of view, the $\mathscr{L}_\infty$-induced norm seems to be quite well suited. Consider for example the situation where the input is a disturbance signal and the output is tracking error. Then the $\mathscr{L}_\infty$-induced norm is exactly the maximal value of the amplitude of the output signal when the input is an arbitrary signal bounded in amplitude by one. Induced operator norms also play a major role in the robust stability and performance analysis and synthesis of sampled-data systems as shown in Sivashankar and Khargonekar (1991b).

In Section 2 we define our notation and set up the framework for sampled-data feedback system analysis. In Section 3 we derive a formula for the $\mathscr{L}_\infty$-induced norm of a sampled-data system. We also give an approximation to the $\mathscr{L}_\infty$-induced norm. It is shown that for a given sampled-data system we can obtain a Finite Dimensional Linear Shift-Invariant (FDLSI) discrete-time system whose input–output l_∞-induced norm approximates arbitrarily closely the $\mathscr{L}_\infty$-induced norm of the sampled-data system. A formula for the $\mathscr{L}_1$-induced norm of a sampled-data system is given in Section 4 and an upper bound for the $\mathscr{L}_p$-induced norm is derived in Section 5. This is followed by a simple numerical example to illustrate the formulae in Section 6. A preliminary version of this paper appeared in the *Proceedings of the American Control Conference* 1991 (see Sivashankar and Khargonekar (1991a)).

2. *Mathematical preliminaries*

2.1. *Signals, sequences and norms.* Let $\mathscr{C}^n$ denote the space of continuous functions from the time set $[0, \infty)$ to $\Re^n$, and let $\mathscr{PC}^n$ denote the space of piecewise-continuous functions from the time set $[0, \infty)$ to $\Re^n$ that are continuous from the left at every point except the origin. As usual,

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$\mathcal{L}_p^n[0, \infty)$ denotes the Lebesgue space of measurable functions f from $[0, \infty)$ to $\mathcal{R}^n$ which satisfy

$$\|f\|_{\mathcal{L}_p} := \left(\int_0^\infty \|f(t)\|^p \, dt\right)^{1/p} < \infty \quad \text{for} \quad 1 \le p < \infty,$$

and

$$\|f\|_{\mathcal{L}_\infty} := \operatorname{ess\,sup}_t \|f(t)\| < \infty \quad \text{for} \quad p = \infty,$$

where $\|\cdot\|$ is the vector norm on $\mathcal{R}^n$ defined as

$$\|f(t)\| := \begin{cases} \left(\sum_{i=0}^{n} |f_i(t)|^p\right)^{1/p} & \text{for} \quad 1 \le p < \infty, \\ \max_{i \in [1,n]} |f_i(t)| & \text{for} \quad p = \infty. \end{cases}$$

Similarly, in discrete-time $\mathcal{S}^n$ denotes the space of $\Re^n$-valued sequences defined on the time set $\{0, 1, 2, \ldots\}$, ℓ_p denotes the set of all sequences ξ in $\mathcal{S}^n$ which satisfy

$$\|\xi\|_{\ell_p} := \left(\sum_{k=0}^{\infty} \|\xi(k)\|^p\right)^{1/p} < \infty \quad \text{for} \quad 1 \le p < \infty,$$

and

$$\|\xi\|_{\ell_\infty} := \sup_k \|\xi(k)\| < \infty \quad \text{for} \quad p = \infty.$$

We will drop the superscript n in the subsequent sections when the dimension of the signal space is clear from the context.

Let $\mathcal{T}$ denote a bounded linear operator

$$\mathcal{T} : \mathcal{L}_p^{m_1} \to \mathcal{L}_p^{p_1} : w \mapsto z.$$

The $\mathcal{L}_p$-induced norm of $\mathcal{T}$ is defined as

$$\|\mathcal{T}\|_p := \sup\left\{\frac{\|z\|_{\mathcal{L}_p}}{\|w\|_{\mathcal{L}_p}} : w \in \mathcal{L}_p \text{ and } \|w\|_{\mathcal{L}_p} \neq 0\right\}. \quad (1)$$

2.2. *Sampled-data feedback systems.* Consider the sampled-data feedback system in Fig. 1. Here G is a FDLTI causal continuous-time plant, K is a FDLSI causal discrete-time controller, $w(t) \in \Re^{m_1}$ is exogenous input, $z(t) \in \Re^{p_1}$ is the regulated output, $u(t) \in \Re^{m_2}$ is the control input, and $y(t) \in \Re^{p_2}$ is the measurement output. The block labeled as S_T represents the sampling operator with time period T defined as follows

$$S_T : \mathcal{C}^{p_2} \to S^{p_2} : y \mapsto S_T y : (S_T y)(k) = y(kT).$$

The system block denoted by H_T represents the (zero-order) hold operator with time period T:

$$H_T : \mathcal{S}^{m_2} \to \mathcal{P}\mathcal{C}^{m_2} : \psi \mapsto H_T \psi$$
$$(H_T \psi)(t) = \psi(k), \quad kT < t \le (k+1)T.$$

Consider the following transfer function representation of G:

$$z = G_{11} w + G_{12} u,$$
$$y = G_{21} w + G_{22} u.$$

We will assume throughout this paper that G_{22} is strictly proper. This ensures well-posedness of the feedback system. In Fig. 1, notice that S_T acts on the measurement output y. So y must be (at least piecewise) continuous for this to make sense. To ensure this, it is sufficient to assume that G_{21} is strictly proper in which case y is continuous.

Let

$$G: \begin{cases} \dot{x} = Ax + B_1 w + B_2 u, \\ z = C_1 x + D_{11} w + D_{12} u, \\ y = C_2 x, \end{cases} \quad (2)$$

$$K: \begin{cases} \xi(k+1) = \Phi \xi(k) + \Gamma \eta(k), \\ \psi(k) = \Theta \xi(k) + \Upsilon \eta(k), \end{cases} \quad (3)$$

be the state space representations of the systems in Fig. 1. Let the state-dimension of G in (2) be n and that of K in (3) be $\hat{n}$. Notice that the direct feedthrough terms from w to y and from u to y are set to zero in the state-space representation of G to satisfy the conditions that G_{21} and G_{22} are strictly proper.

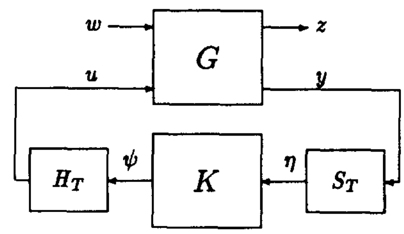

FIG. 1. Sampled-data system.

The feedback interconnection $(G, H_T K S_T)$ is called internally asymptotically stable if the associated unforced discrete-time system with the state

$$\begin{pmatrix} x(k) \\ \xi(k) \end{pmatrix} := \begin{pmatrix} x(kT) \\ \xi(k) \end{pmatrix},$$

is asymptotically stable.

In Fig. 1, since u is the output of a (zero-order) hold operator it follows that

$$x((k+1)T) = e^{AT} x(kT) + \int_0^T e^{A(T-s)} B_1 w(kT+s) \, ds + \phi(T) B_2 u(kT),$$

$$z(kT+t) = C_1 e^{At} x(kT) + \int_0^t [C_1 e^{A(t-s)} B_1 + D_{11} \delta(t-s)] \times w(kT+s) \, ds + [C_1 \phi(t) B_2 + D_{12}] u(kT),$$

where

$$\phi(t) = \int_0^t e^{A\tau} d\tau \quad \text{and} \quad t \in [0, T].$$

A compact way of writing the above system of equations for the input–output description of the plant G is as follows (Bamieh and Pearson (1992); Toivonen (1992))

$$x((k+1)T) = e^{AT} x(kT) + \bar{B}_1 w_k + \phi(T) B_2 u(kT),$$
$$z_k = \bar{C}_1 x(kT) + \bar{D}_{11} w_k + \bar{D}_{12} u(kT),$$

where $w_k(t) = w(kT+t)$, $z_k(t) = z(kT+t)$, $t \in [0, T]$, and $\bar{B}_1$, $\bar{C}_1$, $\bar{D}_{11}$ and $\bar{D}_{12}$ are linear operators defined as follows:

$$\bar{B}_1 : \mathcal{L}_p^{m_1}[0, T] \to \Re^n \quad \text{and} \quad \bar{B}_1 w = \int_0^T e^{A(T-s)} B_1 w(s) \, ds,$$

$$\bar{C}_1 : \Re^n \to \mathcal{L}_p^{p_1}[0, T] \quad \text{and} \quad (\bar{C}_1 x)(t) = C_1 e^{At} x,$$

$$\bar{D}_{11} : \mathcal{L}_p^{m_1}[0, T] \to \mathcal{L}_p^{p_1}[0, T] \quad \text{and}$$

$$(\bar{D}_{11} w)(t) = \int_0^t C_1 e^{A(t-s)} B_1 w(s) \, ds + D_{11} w(kT+t),$$

$$\bar{D}_{12} : \Re^{m_2} \to \mathcal{L}_p^{p_1}[0, T] \quad \text{and} \quad (\bar{D}_{12} u)(t) = [C_1 \phi(t) B_2 + D_{12}] u.$$

If the controller is given by (3) then it is easy to verify that the closed loop system with input w_k and output z_k and a combined state vector $(x'(kT) \ \xi'(k))'$ has the form (in packed matrix notation):

$$\mathcal{T} := \left[\begin{array}{c|c} F & E \\ \hline H & J \end{array}\right], \quad (4)$$

where

$$F := \begin{bmatrix} e^{AT} + \phi(T) B_2 \Upsilon C_2 & \phi(T) B_2 \Theta \\ \Gamma C_2 & \Phi \end{bmatrix},$$

$$E := \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix},$$

$$H := [\bar{C}_1 + \bar{D}_{12} \Upsilon C_2 \quad \bar{D}_{12} \Theta],$$

$$J := \bar{D}_{11}.$$

Thus the feedback system is internally asymptotically stable if and only if F has all its eigenvalues in the open unit disc.

We now set up some notations which will be used in Sections 3 and 4. Consider the closed loop system described in (4). Define

$$H(t) := (H^1(t) \quad H^2(t)) \quad \text{for} \quad t \in [0, T],$$

where

$$H^1(t) := C_1 e^{At} + (C_1 \phi(t) B_2 + D_{12}) Y C_2,$$

and

$$H^2(t) := (C_1 \phi(t) B_2 + D_{12}) \Theta.$$

The main results of this paper give explicit formulae for the $\mathcal{L}_\infty$- and the $\mathcal{L}_1$-induced norms of the closed loop sampled-data system in terms of G, K, and T.

3. *A formula for the $\mathcal{L}_\infty$-induced norm*

Consider the system given in Fig. 1 where G and K are as described by (2) and (3), respectively. Suppose the feedback system is internally asymptotically stable. We now state a result which shows that the $\mathcal{L}_\infty$-induced norm of the closed loop system is finite and gives a formula to evaluate it.

Theorem 3.1. Consider the system in Fig. 1, where G is a FDLTI causal continuous-time plant described by (2) and K is a FDLSI causal discrete-time controller described by (3). Suppose the closed loop system is internally asymptotically stable. Then the closed loop input-output operator

$$\mathcal{T} : \mathcal{L}_\infty \to \mathcal{L}_\infty : w \mapsto z,$$

is bounded and

$$\|\mathcal{T}\|_\infty = \max_{i \in \{1,\ldots,p_1\}} \max_{t \in [0,T]} \sum_{j=1}^{m_1} \left[\left\{ \sum_{k=0}^{\infty} \int_0^T \left| \left[H(t) F^k \begin{pmatrix} e^{As} B_1 \\ 0 \end{pmatrix} \right]_{ij} \right| \mathrm{d}s \right\} + \left\{ \int_0^t |[C_1 e^{As} B_1]_{ij}| \, \mathrm{d}s \right\} + |[D_{11}]_{ij}| \right], \quad (5)$$

where $[A]_{ij}$ represents the (i, j) entry of the matrix A.

Remarks. As one might observe, there are two distinct components in this formula. The first component involves closed loop system matrices as should be expected. The second component, however, depends only on the plant data-A, B_1, C_1, D_{11}. The reason for this somewhat unexpected term is as follows. In between the sampling instants there is no feedback and the closed loop system evolves according to the plant dynamics. Since we are dealing directly with inter-sample behavior in analyzing the $\mathcal{L}_\infty$-induced norm, presence of an "open loop" term should come as no surprise.

Proof of Theorem 3.1. Consider the ith output of the system in Fig. 1 at the time instant $kT + t$ where $t \in [0, T]$ and $k \in \{0, 1, 2, \ldots\}$:

$$z_i(kT + t) = \sum_{j=1}^{m_1} \left[\sum_{l=0}^{k-1} \int_0^T \left[H(t) F^{(k-1-l)} \begin{pmatrix} e^{A(T-s)} B_1 \\ 0 \end{pmatrix} \right]_{ij} w_j(lT + s) \, \mathrm{d}s + \int_0^t [C_1 e^{A(t-s)} B_1]_{ij} w_j(kT + s) \, \mathrm{d}s + [D_{11}]_{ij} w_j(kT + t) \right].$$

It then follows that

$$\|z(kT + t)\| \le \max_{i \in \{1,\ldots,p_1\}} \sum_{j=1}^{m_1} \left[\sum_{l=0}^{k-1} \int_0^T \left| \left[H(t) F^l \begin{pmatrix} e^{A(T-s)} B_1 \\ 0 \end{pmatrix} \right]_{ij} \right| \mathrm{d}s + \int_0^t |[C_1 e^{A(t-s)} B_1]_{ij}| \, \mathrm{d}s + |[D_{11}]_{ij}| \right] \|w\|_{\mathcal{L}_\infty},$$

where $\|\cdot\|$ (on the left hand side of the above inequality) is the "max" norm. Then,

$$\|z\|_{\mathcal{L}_\infty} \le \max_{t \in [0,T]} \max_{i \in \{1,\ldots,p_1\}} \sum_{j=1}^{m_1} \left[\sum_{l=0}^{\infty} \int_0^T \left| \left[H(t) F^l \begin{pmatrix} e^{As} B_1 \\ 0 \end{pmatrix} \right]_{ij} \right| \mathrm{d}s + \int_0^t |[C_1 e^{As} B_1]_{ij}| \, \mathrm{d}s + |[D_{11}]_{ij}| \right] \|w\|_{\mathcal{L}_\infty}. \quad (6)$$

This establishes the upper bound on the $\mathcal{L}_\infty$-induced norm:

$$\|\mathcal{T}\|_\infty \le \max_{t \in [0,T]} \max_{i \in \{1,\ldots,p_1\}} \sum_{j=1}^{m_1} \left[\sum_{l=0}^{\infty} \int_0^T \left| \left[H(t) F^l \begin{pmatrix} e^{As} B_1 \\ 0 \end{pmatrix} \right]_{ij} \right| \mathrm{d}s + \int_0^t |[C_1 e^{As} B_1]_{ij}| \, \mathrm{d}s + |[D_{11}]_{ij}| \right] =: \gamma_\infty,$$

which is finite because of internal asymptotic stability. This proves that the closed loop input–output operator $\mathcal{T}$ is bounded.

Now for a given $\epsilon > 0$, there exist $\hat{t} \in [0, T]$, $\hat{i} \in \{1, \ldots, p_1\}$ and $\hat{k} \in \{0, 1, 2, \ldots\}$ such that

$$\left| \gamma_\infty - \left(\sum_{j=1}^{m_1} \left[\sum_{l=0}^{\hat{k}-1} \int_0^T \left| \left[H(\hat{t}) F^l \begin{pmatrix} e^{A(T-s)} B_1 \\ 0 \end{pmatrix} \right]_{\hat{i}j} \right| \mathrm{d}s + \int_0^{\hat{t}} |[C_1 e^{A(\hat{t}-s)} B_1]_{\hat{i}j}| \, \mathrm{d}s + |[D_{11}]_{\hat{i}j}| \right] \right) \right| \le \epsilon. \quad (7)$$

Consider the $\hat{i}$th output of the feedback system, $z_{\hat{i}}$, in Fig. 1 at the time instant $\hat{k}T + \hat{t}$:

$$z_{\hat{i}}(\hat{k}T + \hat{t}) = \sum_{j=1}^{m_1} \left[\sum_{l=0}^{\hat{k}-1} \int_0^T \left[H(\hat{t}) F^{\hat{k}-1-l} \begin{pmatrix} e^{A(T-s)} B_1 \\ 0 \end{pmatrix} \right]_{\hat{i}j} w_j(lT + s) \, \mathrm{d}s + \int_0^{\hat{t}} [C_1 e^{A(\hat{t}-s)} B_1]_{\hat{i}j} w_j(\hat{k}T + s) \, \mathrm{d}s + [D_{11}]_{\hat{i}j} w(\hat{k}T + \hat{t}) \right].$$

Now choose

$$w_j(lT + s) = \begin{cases} \dfrac{\hat{w}}{|\hat{w}|} & \text{if } \hat{w} \text{ is nonzero} \\ 0 & \text{otherwise} \end{cases}$$

for $l = \{0, 1, \ldots, \hat{k} - 1\}$, $s \in [0, T]$ and $j \in \{1, 2, \ldots, m_1\}$,

where

$$\hat{w} := \left[H(\hat{t}) F^{\hat{k}-1-l} \begin{pmatrix} e^{A(T-s)} B_1 \\ 0 \end{pmatrix} \right]_{\hat{i}j},$$

and,

$$w_j(\hat{k}T + s) = \begin{cases} \dfrac{\bar{w}}{|\bar{w}|} & \text{if } \bar{w} \text{ is nonzero} \\ 0 & \text{otherwise} \end{cases}$$

for $s \in [0, \hat{t})$ and $j \in \{1, 2, \ldots, m_1\}$,

where $\bar{w} := [C_1 e^{A(\hat{t}-s)} B_1]_{\hat{i}j}$ and

$$w_j(\hat{k}T + \hat{t}) = \begin{cases} \dfrac{[D_{11}]_{\hat{i}j}}{|[D_{11}]_{\hat{i}j}|} & \text{if } [D_{11}]_{\hat{i}j} \text{ is nonzero} \\ 0 & \text{otherwise} \quad \text{for } j \in \{1, 2, \ldots, m_1\}. \end{cases}$$

Choosing the inputs w_j as described above we get,

$$z_{\hat{i}}(\hat{k}T + \hat{t}) = \sum_{j=1}^{m_1} \left[\sum_{l=0}^{\hat{k}-1} \int_0^T \left| \left[H(\hat{t}) F^l \begin{pmatrix} e^{A(T-s)} B_1 \\ 0 \end{pmatrix} \right]_{\hat{i}j} \right| \mathrm{d}s + \int_0^{\hat{t}} |[C_1 e^{A(\hat{t}-s)} B_1]_{\hat{i}j}| \, \mathrm{d}s + |[D_{11}]_{\hat{i}j}| \right]. \quad (8)$$

It now follows from (8), (7) and (6) that $\gamma_\infty \ge \|\mathcal{T}\|_\infty \ge (\gamma_\infty - \epsilon)$. This concludes the proof. ■

Consider the system in Fig. 2, where G and K are as described by (2) and (3), respectively. Let $T_N := T/N$ for some integer $N > 0$. Since the closed loop feedback system is internally stable, the closed loop operator

$$\mathcal{T}_N : l_\infty \to l_\infty : w_N \mapsto z_N,$$

is a bounded operator. Define the induced operator norm

$$\|\mathcal{T}_N\| := \sup\left\{\frac{\|z_N\|_{l_\infty}}{\|w_N\|_{l_\infty}} : \|w_N\|_{l_\infty} \neq 0\right\}.$$

We state a proposition next, which gives a way of approximating the $\mathcal{L}_\infty$-induced norm arbitrarily closely.

Proposition 3.2. Consider the feedback system in Fig. 2, where G and K are as defined in (2) and (3), respectively. Suppose the feedback system is internally asymptotically stable. Then

$$\lim_{N\to\infty} \|\mathcal{T}_N\| = \|\mathcal{T}\|_\infty. \tag{9}$$

The proof of the proposition is not given here and the interested reader may find it in Sivashankar and Khargonekar (1991a). A similar result is given in Dullerud and Francis (1992) for the case of stable G.

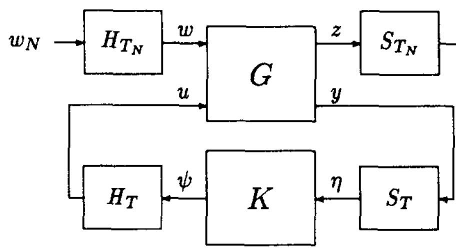

FIG. 2. Multi-rate approximation to a sampled-data system.

Notice that the operator $\mathcal{T}_N$ is a discrete-time system with two sampling periods T and T_N. So $\mathcal{T}_N$ can be represented as a *multi-rate linear discrete-time system*

$$\mathcal{T}_N = S_{T_N} G_{11} H_{T_N} + S_{T_N} G_{12} H_T Q S_T G_{21} H_{T_N},$$

where $Q := K(I - S_T G_{22} H_T K)^{-1}$ is a FDLSI (single rate with period T) discrete-time system. Now we can use the standard "lifting" techniques from literature (Jury and Mullin (1959); Khargonekar *et al.* (1985)) to reduce $\mathcal{T}_N$ to a FDLSI single-rate (sampling period T) system (see Dullerud and Francis (1992) for the explicit formulae). Since the "lifting" operation is system and signal norm preserving (Bamieh and Pearson (1992); Khargonekar *et al.* (1985); Toivonen (1992)), it follows that by calculating the l_∞-induced norm of this "lifted" discrete-time system we actually get $\|\mathcal{T}_N\|$.

It is clear that the approximation that we get using Proposition 3.2 is only a lower bound on the $\mathcal{L}_\infty$-induced norm. Using a finite term approximation to the infinite series and first order approximation to the integral in the formula for the $\mathcal{L}_\infty$-induced norm we can get other lower bounds. It is not clear as to which approximation is computationally more efficient. We can get some upper bounds on the $\mathcal{L}_\infty$-induced norm which may be quite conservative. With these upper and lower bounds one can derive an iterative algorithm to compute the $\mathcal{L}_\infty$-induced norm for a sampled-data systems. This is a subject for future research.

4. *A formula for the $\mathcal{L}_1$-induced norm*

Consider the system in Fig. 1, where G and K are as described by (2) and (3), respectively. Suppose the sampled-data feedback system is internally asymptotically stable. In this section we show that the $\mathcal{L}_1$-induced norm of the closed loop system is finite and derive a formula for it.

Theorem 4.1. Consider the system in Fig. 1, where G is a FDLTI causal continuous-time plant described by (2) and K is a FDLSI causal discrete-time controller described by (3). Suppose the closed loop system is internally asymptotically stable. Then the closed loop input–output operator

$$\mathcal{T} : \mathcal{L}_1 \to \mathcal{L}_1 : w \mapsto z,$$

is bounded and

$$\begin{aligned}\|\mathcal{T}\|_1 &= \max_{j\in\{1,\dots,m_1\}} \max_{s\in[0,T]} \sum_{i=1}^{p_1} \left[\left\{\sum_{m=1}^{\infty} \int_0^T \right.\right. \\ &\quad \times \left.\left|\left[H(t)F^{m-1}\begin{pmatrix} e^{A(T-s)}B_1 \\ 0\end{pmatrix}\right]_{ij}\right| \mathrm{d}t\right\} \\ &\quad + \left.\left\{\int_s^T |[C_1 e^{A(t-s)} B_1]_{ij}|\,\mathrm{d}t\right\} + |[D_{11}]_{ij}|\right],\end{aligned} \tag{10}$$

where $[A]_{ij}$ represents the (i, j) entry of the matrix A.

Remarks. Again, we observe that the formula has two components. The formula computes the "worst-case" $\mathcal{L}_1$ norm of the output z when the corresponding input $w(\|w\|_{\mathcal{L}_1} \le 1)$ is applied. As will be seen in the proof, the worst input is a Dirac delta function applied at some time $s \in [0, T]$ at some input channel. Thus, in the interval $[s, T]$, the sampled-data system evolves as an open loop system leading to the last two terms in (10).

Proof. Consider the ith output of the system, z_i at the time instant $kT + t$ in Fig. 1 where $k \in \{0, 1, 2, \dots\}$ and $t \in [0, T]$:

$$\begin{aligned}&z_i(kT+t) \\ &= \sum_{j=1}^{m_1}\left[\sum_{l=0}^{k-1}\left[H(t)F^{(k-1-l)}\int_0^T \begin{pmatrix} e^{A(T-s)}B_1 \\ 0\end{pmatrix}\right]_{ij} w_j(lT+s)\,\mathrm{d}s \right. \\ &\quad + \left.\int_0^t [C_1 e^{A(t-s)}B_1]_{ij} w_j(kT+s)\,\mathrm{d}s + [D_{11}]_{ij} w_j(kT+t)\right].\end{aligned}$$

It now follows that

$$\begin{aligned}&\int_0^T \sum_{i=1}^{p_1} |z_i(kT+t)|\,\mathrm{d}t \\ &\le \int_0^T \sum_{i=1}^{p_1}\sum_{j=1}^{m_1}\left[\sum_{l=0}^{k-1}\int_0^T \left|\left[H(t)F^{(k-1-l)}\begin{pmatrix} e^{A(T-s)}B_1 \\ 0\end{pmatrix}\right]_{ij}\right|\right. \\ &\quad \times |w_j(lT+s)|\,\mathrm{d}s + \int_0^t |[C_1 e^{A(t-s)}B_1]_{ij}| \\ &\quad \times \left. w_j(kT+s)|\,\mathrm{d}s + |[D_{11}]_{ij}|\,|w_j(kT+t)|\right]\mathrm{d}t.\end{aligned}$$

Then

$$\begin{aligned}\|z\|_{\mathcal{L}_1} &= \sum_{k=0}^{\infty}\int_0^T \sum_{i=1}^{p_1} |z_i(kT+t)|\,\mathrm{d}t \\ &\le \sum_{l=0}^{\infty}\int_0^T \sum_{j=1}^{m_1}\sum_{k=l+1}^{\infty}\sum_{i=1}^{p_1}\left[\int_0^T\right. \\ &\quad \times \left|\left[H(t)F^{(k-1-l)}\begin{pmatrix} e^{A(T-s)}B_1 \\ 0\end{pmatrix}\right]_{ij}\right|\mathrm{d}t \\ &\quad + \left.\int_s^T |[C_1 e^{A(t-s)}B_1]_{ij}|\,\mathrm{d}t + |[D_{11}]_{ij}|\right] |w_j(lT+s)|\,\mathrm{d}s \\ &\le \max_{j\in\{1,\dots,m_1\}}\max_{s\in[0,T]}\left\{\sum_{m=1}^{\infty}\sum_{i=1}^{p_1}\left[\int_0^T\right.\right. \\ &\quad \times \left|\left[H(t)F^{(m-1)}\begin{pmatrix} e^{A(T-s)}B_1 \\ 0\end{pmatrix}\right]_{ij}\right|\mathrm{d}t \\ &\quad + \left.\left.\int_s^T |[C_1 e^{A(t-s)}B_1]_{ij}|\,\mathrm{d}t + |[D_{11}]_{ij}|\right]\right\} \\ &\quad \times \left(\sum_{l=0}^{\infty}\int_0^T\sum_{j=1}^{m_1}|w_j(lT+s)|\,\mathrm{d}s\right).\end{aligned}$$

Let

$$\begin{aligned}\gamma_1 := \max_{j\in\{1,\dots,m_1\}}\max_{s\in[0,T]}\sum_{m=1}^{\infty}\sum_{i=1}^{p_1}&\left[\int_0^T\right. \\ &\times \left|\left[H(t)F^{(m-1)}\begin{pmatrix} e^{A(T-s)}B_1 \\ 0\end{pmatrix}\right]_{ij}\right|\mathrm{d}t \\ &+ \left.\int_s^T |[C_1 e^{A(t-s)}B_1]_{ij}\,|dt + |[D_{11}]_{ij}|\right].\end{aligned}$$

The constant γ_1 is finite because of internal asymptotic stability and we have shown that $\|\mathcal{T}\|_1 \leq \gamma_1$. This proves that the closed loop input–output operator $\mathcal{T}$ is bounded.

Now there exist $\hat{j} \in \{1, 2, \ldots, m_1\}$ and $\hat{s} \in [0, T]$ such that

$$\gamma_1 = \sum_{m=1}^{\infty} \sum_{i=1}^{p_1} \left[\int_0^T \left| \left[H(t)F^{(m-1)} \begin{pmatrix} e^{A(T-\hat{s})}B_1 \\ 0 \end{pmatrix} \right]_{i\hat{j}} \right| dt + \int_{\hat{s}}^T |[C_1 e^{A(t-\hat{s})}B_1]_{i\hat{j}}| \, dt + |[D_{11}]_{i\hat{j}}| \right]. \quad (11)$$

It is known that one can construct a sequence of functions $\{f_n\}$ such that $\|f_n\|_{\mathcal{L}_1} = 1$ and f_n converges to the Dirac delta function, $\delta(\hat{s})$ in the sense of distributions as $n \to \infty$. Now apply the approximate function f_n to the $\hat{j}$th input of the system in Fig. 1. Let z_{in} denote the corresponding ith output. Then it is not difficult to show that $z_{in} \to z_i$ where z_i is given by

$$z_i(\hat{s}) = [D_{11}]_{i\hat{j}},$$
$$z_i(t) = [C_1 e^{A(t-\hat{s})}B_1]_{i\hat{j}} \quad \text{for} \quad \hat{s} < t \leq T,$$
$$z_i(t) = \left[H(t-kT)F^k \begin{pmatrix} e^{A(T-\hat{s})}B_1 \\ 0 \end{pmatrix} \right]_{i\hat{j}}$$
$$\text{for} \quad kT < t \leq (k+1)T \quad \forall k \in \{1, 2, \ldots\}.$$

Clearly,

$$\|z\|_{\mathcal{L}_1} = \sum_{m=1}^{\infty} \sum_{i=1}^{p_1} \left[\int_0^T \left| \left[H(t)F^{(m-1)} \begin{pmatrix} e^{A(T-\hat{s})}B_1 \\ 0 \end{pmatrix} \right]_{i\hat{j}} \right| dt + \int_{\hat{s}}^T |[C_1 e^{A(t-\hat{s})}B_1]_{i\hat{j}}| \, dt + |[D_{11}]_{i\hat{j}}| \right].$$

For any $\epsilon > 0$, there exists n sufficiently large such that

$$\|\mathcal{T}\|_1 \geq \|z_{in}\| \geq \gamma_1 - \epsilon.$$

Thus, $\gamma_1 \geq \|\mathcal{T}\|_1 \geq \gamma_1 - \epsilon$ which completes the proof. ■

5. *An upper bound for the $\mathcal{L}_p$-induced norm*

Using the formulae developed for the $\mathcal{L}_\infty$- and the $\mathcal{L}_1$-induced norms, we give an upper bound for the $\mathcal{L}_p$-induced norm of a stable sampled-data system. The following theorem is a direct consequence of the Riesz convexity theorem (Stein and Weiss (1971); Chen and Francis (1991a)).

Theorem 5.1. Consider the sampled-data system given in Fig. 1 where the plant G and the controller K are as described in (2) and (3), respectively. Suppose the sampled-data feedback system is internally asymptotically stable. Then the closed loop input–output operator

$$\mathcal{T} : \mathcal{L}_p \to \mathcal{L}_p : w \mapsto z,$$

is bounded and

$$\|\mathcal{T}\|_p \leq \|\mathcal{T}\|_1^{1/p} \|\mathcal{T}\|_\infty^{1/q},$$

where $\|\mathcal{T}\|_\infty$ and $\|\mathcal{T}\|_1$ are given in (5) and (10), respectively and $(1/p) + (1/q) = 1$ (for $1 < p < \infty$).

6. *Example*

In this section, we will give a simple numerical example to illustrate the formulae developed in Sections 3 and 4.

Consider the plant G:

$$G: \begin{cases} \dot{x} = \begin{pmatrix} -a & -4 \\ 4 & -a \end{pmatrix} x + \begin{pmatrix} -1 \\ +1 \end{pmatrix} w + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u, \\ z = (1 \;\; 0)x + w, \\ y = (1 \;\; 1)x. \end{cases}$$

Here a is a real parameter and the sampling time period is $T = 2$. With a constant output feedback gain of $K = 0.5$, the eigenvalues $(\lambda_{1,2})$ of the system matrix F of the closed loop system are listed in Table 1. It is clear that the sampled-data system is internally asymptotically stable for all the values of the parameter a (listed in Table 1). We have computed the induced norm of the system for different values of the parameter a using standard numerical software and these are tabulated in Table 1. We used a finite term approximation for the infinite series in the formulae for numerical implementation.

TABLE 1. COMPARISON OF THE INDUCED NORMS

a	3	1.5	0.9	0.5	0.2
$\lambda_{1,2}$	0.117 0.003	0.04 ± 0.104j	0.023 ± 0.251j	0.007 ± 0.468j	−0.008 ± 0.78j
$\|\mathcal{T}\|_\infty$	1.398	1.784	2.29	3.458	8.717
$\|\mathcal{T}\|_1$	1.415	1.813	2.33	3.455	8.489
$\|\mathcal{T}_d\|$	1.287	1.352	1.421	1.599	2.462

Traditionally, sampled-data systems are analyzed by considering the feedback system only at the sampling instants. This is the same as using a sampler of period T at the output z and a hold operator of period T at the input w in Fig. 1. Using such a sample-hold equivalent of the sampled-data system, we get a FDLSI discrete-time system. We used standard numerical software to compute its ℓ_∞- and ℓ_1-induced norms. Note that for a FDLSI discrete-time system with scalar inputs and outputs the ℓ_∞-induced norm is equal to the ℓ_1-induced norm. The induced norm of this approximate system $\|\mathcal{T}_d\|$ for different values of the parameter a is also listed in Table 1. As expected, the numerical values for $\|\mathcal{T}\|_{\infty,1}$ are greater than those for $\|\mathcal{T}_d\|$ and hence our formula captures the inter-sample behavior in the system. We also notice that as the eigenvalues of the open-loop system matrix approach the imaginary axis, the induced norms $\|\mathcal{T}\|_\infty$ and $\|\mathcal{T}\|_1$ increase and the gap between these and the induced norm of the discrete-time approximation also widens significantly.

7. *Conclusion*

We have given explicit formulae for the $\mathcal{L}_\infty$- and $\mathcal{L}_1$-induced norms of a sampled-data system. We have also shown that the $\mathcal{L}_\infty$-induced norm of a sampled-data system can be approached as the limit of the norm of another multirate discrete-time system associated with the sampled-data system. One can now pose the problem of minimizing the $\mathcal{L}_\infty$- and $\mathcal{L}_1$-induced norm of the closed loop operator from w to z over all sampled-data controllers that provide internal stability. Some related works along these lines are reported in Dullerud and Francis (1992) and Bamieh *et al.* (1991).

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