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Eulerian numbers, tableaux, and the Betti numbers of a toric variety

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Abstract

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Let Σ denote the Coxeter complex of S_n, and let $X(\Sigma)$ denote the associated toric variety. Since the Betti numbers of the cohomology of $X(\Sigma)$ are Eulerian numbers, the additional presence of an S_n-module structure permits the definition of an isotypic refinement of these numbers. In some unpublished work, DeConcini and Procesi derived a recurrence for the S_n-character of the cohomology of $X(\Sigma)$, and Stanley later used this to translate the problem of combinatorially describing the isotypic Eulerian numbers into the language of symmetric functions. In this paper, we explicitly solve this problem by developing a new way to use marked sequences to encode permutations. This encoding also provides a transparent explanation of the unimodality of Eulerian numbers and their isotypic refinements.

Introduction

The Coxeter complex Σ of the symmetric group S_n is the arrangement of convex cones defined by the intersections of the hyperplanes associated with the root system A_{n-1}. In some unpublished work, DeConcini and Procesi derived a recurrence for the S_n-character of the cohomology of the toric variety associated with Σ, presumably for the purpose of computing the Betti numbers of the S_n-isotypic components. Stanley gave a reformulation of this problem in the language of symmetric functions in [6], but still lacked a combinatorial description of these 'isotypic' Betti numbers.

An intriguing combinatorial aspect of this problem (mentioned by Stanley) is the fact that the kth Betti number of the overall cohomology is the $(k+1)$th Eulerian number, i.e., the number of $w \in S_n$ with k descents. Thus, the isotypic

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Betti numbers would represent some sort of refinement of Eulerian numbers. Furthermore, the machinery of algebraic geometry (namely, the hard Lefschetz Theorem) would imply that these numbers must be symmetric and unimodal. (Although Brenti has shown that the unimodality of these sequences can also be established by elementary methods [1].)

In this paper, we give an explicit description of these Betti numbers in terms of certain types of Young tableaux with marked entries. The fundamental idea which leads to this description is a new way to use marked sequences to encode permutations. This coding has several other interesting implications. For example, it yields a new interpretation of the Eulerian numbers which transparently explains their unimodality. Moreover, this transparency passes through to the isotypic Betti numbers and explains their unimodality as well. Also, we will see that S_n acts in a natural way on the set of marked sequences that encode permutations, and this action turns out to be isomorphic to the action of S_n on the cohomology of the toric variety.

The paper is organized as follows. In Section 1, we describe the encoding algorithm which converts a permutation into a marked sequence. In Section 2, we show (Lemma 2.1) how this encoding can be used to prove that sets of permutations that satisfy certain closure properties have unimodal generating functions. In Section 3 we analyse the action of S_n on the set of marked sequences so that we may apply Stanley's symmetric function characterization of the cohomology in Section 4. Finally, we describe the isotypic Betti numbers in Theorem 4.2, and we also describe (Theorem 4.3) a related family of parameters (first considered by Stanley) that refine a derangement analogue of the Eulerian numbers.

1. The code of a permutation

For any nonnegative integer sequence $\alpha \in \mathbb{N}^n$, let $S^+(\alpha)$ denote the positive content of α, i.e., the set of positive integers occurring in α. We will say that α is *k-admissible* if $S^+(\alpha) = \{1, 2, \ldots, k\}$. (In particular, a sequence of 0's is said to be 0-admissible.) If α is k-admissible for some $k \geqslant 0$, we will say that α is *admissible*. Thus, 210320 is a 3-admissible sequence, whereas 103130 is not admissible.

Let $m_j(\alpha)$ denote the number of occurrences of the integer j in α. We define a *marked sequence* to be a pair (α, f) consisting of a sequence $\alpha \in \mathbb{N}^n$ and a map $f: S^+(\alpha) \to \mathbb{N}$ such that $1 \leqslant f(j) < m_j(\alpha)$ for $j \in S^+(\alpha)$. The *index* of (α, f) is defined by setting

$$\operatorname{ind}(\alpha, f) = \sum_{j \in S^+(\alpha)} f(j).$$

Note that there are no marked sequences (α, f) in case $m_j(\alpha) = 1$ for any $j > 0$.

It is convenient to represent a marked sequence (α, f) by decorating α. More precisely, for each $j \in S^+(\alpha)$, let us replace one of the j's in α by the symbol $\hat{\jmath}$ so that there are $f(j)$ occurrences of j to the left of the $\hat{\jmath}$. For example, if $\alpha = 12012122$ and $f(1) = 2$, $f(2) = 1$, we may represent (α, f) by the sequence $1201\hat{2}\hat{1}22$.

An admissible marked sequence will be referred to as a *code*. For example, there are 6 codes of length 3:

$$000, \quad 01\hat{1}, \quad 10\hat{1}, \quad 1\hat{1}0, \quad 1\hat{1}1, \quad 11\hat{1};$$

their indices are 0, 1, 1, 1, 1, 2, respectively. Also, there are 9 positive codes of length 4:

$$1\hat{1}11, \quad 11\hat{1}1, \quad 111\hat{1}, \quad 1\hat{1}2\hat{2}, \quad 12\hat{1}\hat{2}, \quad 21\hat{1}\hat{2}, \quad 12\hat{2}\hat{1}, \quad 21\hat{2}\hat{1}, \quad 2\hat{2}1\hat{1};$$

the first and third have indices 1 and 3 and the remaining seven have index 2.

Regard S_n as the group of permutations of $1, 2, \ldots, n$. Recall that the *excedance* number of any $w \in S_n$ is the quantity $e(w) := |\{i: w(i) > i\}|$ [5]. We remark that the 6 permutations in S_3 have excedances 0, 1, 1, 1, 1, 2, and that among the 9 permutations in S_4 without fixed points (i.e., derangements), there are 7 with excedance 2, and one each with excedances 1 and 3.

We are now ready to state the main result of this paper.

Theorem 1.1. *There is a natural one-to-one correspondence* $w \overset{\phi}{\longmapsto} (\alpha, f)$ *between permutations* $w \in S_n$ *and codes* (α, f) *of length n such that*:

(a) *w is a derangement iff* $\phi(w)$ *is positive.*

(b) ϕ *maps the excedance to the index, i.e.,* $e(w) = \operatorname{ind} \phi(w)$.

We remark that it is possible to establish the purely enumerative content of this result by means of generating functions; it is our purpose here to give an explicit construction of a suitable map ϕ. Later we will use nonconstructive methods (i.e., generating functions) to prove a generalization in Section 3.

Proof. The correspondence ϕ will be obtained by iterating a *reduction algorithm* that takes as input a triple (w, α, f) consisting of a code (α, f) of length n and a nontrivial permutation w of the zero-set $\{i: \alpha_i = 0\}$ of α. The algorithm produces as output a new triple (w', α', f') of the same type, but with a smaller zero-set. Moreover, if α is k-admissible, then α' will be $(k+1)$-admissible, and f' will be obtained by extending f.

To define the reduction algorithm, let (w, α, f) be a suitable triple as described above, and assume that α is k-admissible. Define m to be the largest integer not fixed by w. (Such an integer exists since w is assumed to be nontrivial.) Let $C = (i_0, i_1, \ldots, i_{l-1})$ be the cycle of w containing m, labeled so that $m = i_0$, $w(i_0) = i_1$, $w(i_1) = i_2, \ldots, w(i_{l-1}) = i_0$. Determine the unique index r $(1 \leq r < l)$ for which $i_0 > i_1 > \cdots > i_r < i_{r+1}$ (subscripts taken mod l). Finally, let s be the least index ($r \leq s < l$) such that either $i_s > i_{s+1}$ or $i_{r-1} \leq i_{s+1}$. Thus $i_r, i_{r+1}, \ldots, i_s$ forms an increasing sequence of integers less than i_{r-1} and is maximal with respect to this property. For example, if $C = (97412586)$, then $r = 3$ and $s = 4$.

Case I: $s = l - 1$.

Define w' to be the permutation obtained by deleting C from w, and define α' to be the admissible sequence obtained by setting $\alpha'_i = k + 1$ for $i = i_0, \ldots, i_{l-1}$, and $\alpha'_i = \alpha_i$ otherwise. Extend the definition of the map f by setting $f(k+1) = e(C) = l - r$, and define $f' = f$. Note that $1 \leq l - r < l$ so this does define a legal mark.

Case II: $s < l - 1$.

Define w' to be the permutation obtained by deleting the subsequence $i_1, \ldots, i_s$ from C, i.e., replace the cycle with the cycle $C' = (i_0, i_{s+1}, \ldots, i_{l-1})$. Note that C' is of length ≥ 2, so this operation does not create new fixed points. Define α' to be the admissible sequence obtained by setting $\alpha'_i = k + 1$ for $i = i_1, \ldots, i_s$, and $\alpha'_i = \alpha_i$ otherwise. Finally, extend the map f via the rule

$$f(k+1) = e(C) - e(C') = \begin{cases} s - r & \text{if } i_s > i_{s+1}, \\ s - r + 1 & \text{if } i_{r-1} \leq i_{s+1}. \end{cases}$$

It is easy to check that $1 \leq f(k+1) < s$ so this does define a legal mark. For example, if $C = (97412586)$, then $C' = (9586)$, the positions of α' with $k + 1$'s would be 1, 2, 4, 7, and $f(k+1) = 2$.

Now, to define the correspondence ϕ, let $w \in S_n$ and form the triple $(w, \varepsilon, \emptyset)$, where $(\varepsilon, \emptyset)$ denotes the zero code. Iteratively apply the reduction algorithm to $(w, \varepsilon, \emptyset)$, thereby producing a sequence of triples $(w^{(i)}, \alpha^{(i)}, f^{(i)})$, $i = 0, 1, 2, \ldots$. The iterations will necessarily terminate as soon as we obtain a triple $(w^{(k)}, \alpha^{(k)}, f^{(k)})$ for which $w^{(k)}$ is the trivial permutation of the zero-set of $\alpha^{(k)}$. We then define $\phi(w) = (\alpha^{(k)}, f^{(k)})$. Since the algorithm does not create new fixed points, it follows that w is a derangement iff $\phi(w)$ is positive. Also note that the reduction algorithm is designed so that if (w, α, f) reduces to (w', α', f') then $e(w) + \operatorname{ind}(\alpha, f) = e(w') + \operatorname{ind}(\alpha', f')$. Thus, we have $e(w) = \operatorname{ind} \phi(w)$, as desired.

To complete the proof, we must verify that ϕ is a one-to-one correspondence. For this it suffices to show that the reduction algorithm is invertible. More precisely, let (w', α', f') be an arbitrary triple consisting of a $(k+1)$-admissible code (α', f') and a (possibly trivial) permutation w' of the zero set of α'. We will show that there is a unique triple (w, α, f) that reduces to (w', α', f').

To prove this, let $J = \{j_1 < \cdots < j_s\}$ be the set of positions occupied by $k + 1$'s in α'. Clearly (α, f) must be obtained from (α', f') by reassigning 0's to each position in J and restricting the domain of f' to $\{1, \ldots, k\}$. Thus, we need only to analyze the possibilities for w.

If j_s is larger than any of the indices not fixed by w', then (w, α, f) will reduce to (w', α', f') only if Case I of the algorithm applies. (In Case II, the largest nonfixed point $m = i_0$ of w is not reassigned in α'; it remains part of the zero-set.)

Therefore, the only possible choice for w is obtained by adding the cycle

$$C=(j_s, j_{s-1}, \ldots, j_{r+1}, j_1, j_2, \ldots, j_r) \quad (r=f'(k+1))$$

to w'. Moreover, it is easy to check that the triple (w, α, f) does serve as an inverse.

Otherwise, we may assume that the largest integer m not fixed by w' exceeds j_s. In that case, (w, α, f) will reduce to (w', α', f') only if Case II of the algorithm applies. Let $C'=(i_0, i_1, \ldots, i_{l-1})$ be the cycle of w' that includes $m=i_0$, and set $r=f'(k+1)$. Since m is not fixed by w', we have $l \geqslant 2$. If we define w to be the permutation obtained by replacing C' with the cycle C given by

$$C=\begin{cases}(i_0, j_s, \ldots, j_{r+2}, j_1, \ldots, j_{r+1}, i_1, \ldots, i_{l-1}) & \text{if } j_{r+1}>i_1,\\ (i_0, j_s, \ldots, j_{r+1}, j_1, \ldots, j_r, i_1, \ldots, i_{l-1}) & \text{if } j_{r+1}<i_1,\end{cases}$$

then we have $r=e(C)-e(C')$ and (w, α, f) reduces to (w', α', f'). To see that this is the only choice for w, note that it is necessary for the elements of J to be arranged as a subsequence of C in the order $j_s, \ldots, j_{t+1}, j_1, \ldots, j_t$, for some t. The relative sizes of j_{r+1} and i_1 thus determine a unique possibility for t. For example if $C'=(9586)$ and $J=\{1, 2, 4, 7\}$, then $C=(97421586)$, (97412586), or (91247586), according to the cases $r=1$, 2, or 3. □

As an illustration, consider the permutation $w=(296)(17358)(4)$. We obtain

$$(w, \varepsilon, \emptyset) \to ((17358)(4), 01000\hat{1}001) \to ((358)(4), 21000\hat{1}201)$$
$$\to ((4), 21303\hat{1}\hat{2}\hat{3}1),$$

and therefore, $\phi(w)=21303\hat{1}\hat{2}\hat{3}1$.

2. Unimodal applications

For any set of permutations $A \subseteq S_n$, let $F(A, q)$ denote the associated excedance generating function, i.e.,

$$F(A, q)=\sum_{w \in A} q^{e(w)}.$$

For example, $F(S_3, q)=1+4q+q^2$ and $F(D_4, q)=q+7q^2+q^3$, using D_n as an abbreviation for the set of derangements of $1, 2, \ldots, n$.

If $P(q)=\sum_k a_k q^k$ is any polynomial of degree r with lowest nonzero term of degree s, we will say that P is symmetric or unimodal when the coefficient sequence $(a_s, a_{s+1}, \ldots, a_r)$ satisfies these properties. We remark that it is well known and easy to prove that the set of symmetric unimodal polynomials in $\mathbb{N}[q]$ is multiplicatively closed.

In this section, we will show that the correspondence ϕ can be used to provide transparent proofs that for some permutation sets A, the polynomials $F(A, q)$ are symmetric and unimodal.

Let S be a set of codes of length n. We will say that S is *mark-invariant* if the membership of (α, f) in S depends only on α, i.e., $(\alpha, f) \in S$ if and only if $(\alpha, f') \in S$ for all α, f, f'. We will say that S is *shift-invariant* if for each admissible α with $m_0(\alpha) \geq 2$, the number of codes $(\alpha, f) \in S$ equals the number of codes $(\alpha^*, f^*) \in S$ with $\alpha_i^* = \alpha_i + 1$ $(1 \leq i \leq n)$. By abuse of notation, we will say that a set of permutations A is mark-invariant or shift-invariant whenever $\phi(A)$ satisfies these properties.

Lemma 2.1. *Let* $A \subseteq S_n$.

(a) *If* $\phi(A)$ *is a set of mark-invariant positive codes, then* $F(A, q)$ *is symmetric and unimodal with center of symmetry at* $n/2$.

(b) *If* $\phi(A)$ *is both mark-invariant and shift-invariant, then* $F(A, q)$ *is symmetric and unimodal with center of symmetry at* $(n-1)/2$.

Proof. If $\phi(A)$ is mark-invariant, Theorem 1.1 implies that

$$F(A, q) = \sum_{\alpha} F_\alpha(q), \qquad F_\alpha(q) := \sum_{f} q^{\mathrm{ind}(\alpha, f)},$$

where α ranges over the admissible sequences that occur in $\phi(A)$. Since the only constraints on f are the conditions $1 \leq f(j) < m_j(\alpha)$ for $j \in S^+(\alpha)$, we have

$$F_\alpha(q) = \prod_{j \in S^+(\alpha)} (q + q^2 + \cdots + q^{m_j(\alpha)-1}).$$

This shows that F_α is symmetric and unimodal and that it has a center of symmetry at

$$\sum_{j>0} m_j(\alpha)/2 = (n - m_0(\alpha))/2.$$

If the codes in $\phi(A)$ are all positive (part (a)), this shows that $F(A, q)$ is a sum of symmetric unimodal polynomials with a common center of symmetry at $n/2$.

Otherwise, if $\phi(A)$ is shift-invariant (part (b)) note that we have

$$F(A, q) = \sum_{\alpha} F_\alpha(q) + F_{\alpha^*}(q),$$

where α ranges over the admissible sequences in $\phi(A)$ with $m_0(\alpha) \geq 1$, and α^* denotes the admissible sequence obtained by setting $\alpha_i^* = \alpha_i + 1$. In particular, note that if $m_0(\alpha) = 1$, then $F_{\alpha^*}(q) = 0$. Since

$$F_\alpha(q) + F_{\alpha^*}(q) = (1 + q + \cdots + q^{m_0(\alpha)-1}) F_\alpha(q),$$

it follows that $F_\alpha + F_{\alpha^*}$ is symmetric and unimodal and has a center of symmetry at $(n - m_0)/2 + (m_0 - 1)/2 = (n-1)/2$. $\square$

It is an easy exercise to construct an injection that proves directly that any product of the form $\prod (q + q^2 + \cdots + q^{a_i})$ is unimodal. In conjunction with the correspondence ϕ, it would therefore be easy to give an explicit injective proof of the unimodality of $F(A, q)$ for any set A satisfying the above hypotheses.

Since the set D_n of derangements is obviously mark-invariant, we obtain the following.

Corollary 2.2. *$F(D_n, q)$ is symmetric and unimodal.*

An elementary (but indirect) proof of this result has also been given by Brenti [1].

For any $w \in S_n$, let $d(w) = |\{i: w(i) > w(i+1)\}|$ denote the number of descents in w. Recall that the Eulerian number $A(n, k+1)$ is the number of permutations $w \in S_n$ with $d(w) = k$. Less well known is the fact that $A(n, k+1)$ may also be interpreted as the number of $w \in S_n$ with $e(w) = k$. Moreover, the equivalence of these two interpretations can be established with a bijection [5, p. 23]. Thus, $qF(S_n, q)$ is the so-called Eulerian polynomial, i.e.,

$$qF(S_n, q) = \sum_k A(n, k)q^k.$$

Since S_n is obviously mark- and shift-invariant, Lemma 2.1 implies the next corollary.

Corollary 2.3. *$F(S_n, q)$ is symmetric and unimodal.*

Although the unimodality of Eulerian numbers is well known, this appears to be the first combinatorial proof. Another proof appears in [1]. The standard proofs in the literature (e.g., [2, p. 292]) rely on the fact that the roots of the Eulerian polynomial are real.

3. Permutations of codes

The symmetric group S_n acts naturally on the set of codes of length n as follows: $w \circ (\alpha, f) \mapsto (w\alpha, f)$, where $(w\alpha)_i = \alpha_{w^{-1}(i)}$ $(1 \leq i \leq n)$. Since the index is preserved by this action, we thereby obtain a (graded) permutation representation of S_n with graded components having Eulerian numbers as their degrees. Using the correspondence ϕ^{-1} as a cryptomorphism, one could represent this as an action of S_n on itself, but in this form it would appear less natural.

Let $\chi_{n,k}$ denote the character of the S_n-action on codes of index k. Thus $\chi_{n,k}(w)$ is the number of codes of index k fixed by w. To analyze the structure of these characters we will take advantage of the isomorphism between S_n-characters and symmetric functions afforded by the characteristic map [3].

Let $Z(S_n)$ denote the space of S_n-class functions, and let $\Lambda = \bigoplus_n \Lambda^n$ denote the $\mathbb{N}$-graded algebra of symmetric formal power series in the variables $x_1, x_2, \ldots$. The characteristic map is a linear isomorphism $\mathrm{ch}: Z(S_n) \to \Lambda^n$ defined as follows:

$$\mathrm{ch}\,\chi = \frac{1}{n!} \sum_{w \in S_n} \chi(w) p_{\nu(w)},$$

where $\nu(w) = (\nu_1, \nu_2, \ldots)$ denotes the cycle type of w, and $p_\nu = p_{\nu_1} p_{\nu_2} \cdots$ denotes a product of power-sum symmetric functions. Let h_n denote the characteristic of the trivial S_n-character. We remark that there is an explicit formula for the generating function $H(t) = \sum_{n \geq 0} h_n t^n$, namely,

$$H(t) = \prod_{i \geq 1} \frac{1}{1 - x_i t}.$$

See [3] for details.

Lemma 3.1. *We have*

$$\sum_{n,k \geq 0} \mathrm{ch}(\chi_{n,k}) q^k t^n = \frac{(1-q)H(t)}{H(qt) - qH(t)}.$$

Proof. The orbits of codes are indexed by pairs (μ, f) consisting of (1) a composition $\mu = (\mu_0, \mu_1, \ldots, \mu_l)$ of n into nonnegative integers such that $\mu_j \geq 2$ $(1 \leq j \leq l)$, and (2) a marking function f such that $1 \leq f(j) < \mu_j$ $(1 \leq j \leq l)$. The orbit indexed by (μ, f) consists of all codes (α, f) in which α is l-admissible and $m_j(\alpha) = \mu_j$. The stabilizer of any such code is a Young subgroup of S_n isomorphic to $S_\mu := S_{\mu_0} \times \cdots \times S_{\mu_l}$. Hence, the characters $\chi_{n,k}$ may be decomposed as follows:

$$\sum_{k \geq 0} q^k \chi_{n,k} = \sum_{(\mu, f)} q^{\mathrm{ind}(\mu, f)} \eta_\mu, \tag{1}$$

where η_μ denotes the character of the action of S_n on the left cosets of S_μ.

Since it is known that $\mathrm{ch}(\eta_\mu) = h_\mu = h_{\mu_0} \cdots h_{\mu_l}$ [3, I.7], it follows that for a fixed choice of μ, we have

$$\sum_f \mathrm{ch}(\eta_\mu) q^{\mathrm{ind}(\mu, f)} = h_{\mu_0} \prod_{i=1}^{l} (q + q^2 + \cdots + q^{\mu_i - 1}) h_{\mu_i}.$$

After collecting the contributions from each μ, we therefore obtain

$$\begin{aligned} \sum_{n,k \geq 0} \mathrm{ch}(\chi_{n,k}) q^k t^n &= \sum_{l \geq 0} \sum_{\mu} t^{\mu_0} h_{\mu_0} \prod_{i=1}^{l} (q + q^2 + \cdots + q^{\mu_i - 1}) h_{\mu_i} t^{\mu_i} \\ &= H(t) \sum_{l \geq 0} \Big(\sum_{m \geq 2} (q + q^2 + \cdots q^{m-1}) h_m t^m \Big)^l \\ &= H(t) \Big[1 - \sum_{m \geq 2} (q + q^2 + \cdots + q^{m-1}) h_m t^m \Big]^{-1}. \end{aligned}$$

After routine simplification, the claimed result follows. □

Note that if we restrict our attention to the positive codes, we obtain a permutation representation of total degree $|D_n|$ and of graded degree $F(D_n, q)$. Furthermore, essentially the same analysis as above applies, the only difference being that the relevant orbits (μ, f) have $\mu_0 = 0$. Therefore, if $\theta_{n,k}$ denotes the S_n-character associated with this action, we have

$$\sum_{n,k\geq 0} \mathrm{ch}(\theta_{n,k})q^k t^n = \left[1 - \sum_{m\geq 2} (q + q^2 + \cdots + q^{m-1})h_m t^m\right]^{-1},$$

which is equivalent to the following.

Lemma 3.2. *We have*

$$\sum_{n,k\geq 0} \mathrm{ch}(\theta_{n,k})q^k t^n = \frac{(1-q)}{H(qt) - qH(t)}.$$

Although the above results completely determine the characters $\chi_{n,k}$ and $\theta_{n,k}$, the following is a more explicit description of $\chi_{n,k}$. (We have used $[r] = 1 + q + \cdots + q^{r-1}$ as an abbreviation in what follows.)

Proposition 3.3. *If $\nu = (\nu_1, \ldots, \nu_l)$ is the cycle type of $w \in S_n$, then*

$$\sum_{k\geq 0} q^k \chi_{n,k}(w) = F(S_l, q)[\nu_1] \cdots [\nu_l].$$

Proof. Define a class-function $X_n \in Z(S_n)$ by setting $X_n(w) = F(S_l, q)[\nu_1] \cdots [\nu_l]$ for all w of cycle type $\nu = (\nu_1, \ldots, \nu_l)$. Since the characteristic map is injective it suffices to prove that $\mathrm{ch}(X_n) = \sum_k q^k \,\mathrm{ch}(\chi_{n,k})$.

From the definition of the characteristic map, we have

$$\sum_{n\geq 0} \mathrm{ch}(X_n)t^n = \sum_{\nu} \frac{1}{z_\nu} t^{|\nu|} F(S_{l(\nu)}, q) \prod_{i=1}^{l(\nu)} \frac{1 - q^{\nu_i}}{1 - q} p_{\nu_i},$$

summed over partitions ν, where z_ν denotes the size of the S_n-centralizer of any $w \in S_n$ of cycle type ν. Hence, the following property of Eulerian polynomials [2, p. 245],

$$\frac{F(S_l, q)}{(1-q)^{l+1}} = \sum_{k\geq 0} (k+1)^l q^k,$$

allows us to conclude that

$$\sum_{n\geq 0} \mathrm{ch}(X_n)t^n = (1-q) \sum_{k\geq 0} q^k \sum_{\nu} \frac{1}{z_\nu} t^{|\nu|} \prod_{i=1}^{l(\nu)} (k+1)(1 - q^{\nu_i})p_{\nu_i}. \tag{2}$$

However, as a function of power sums, $H(t)$ is the generating function of the cycles indices of the symmetric groups S_n, and so we have [3, I.2]

$$H(t) = \sum_{\nu} \frac{t^{|\nu|}}{z_\nu} p_\nu = \exp\left(\sum_{r\geq 1} t^r p_r / r\right).$$

Therefore, the inner sum in (2) can be identified as the image of $H(t)$ under the ring homomorphism of $\Lambda[[q, t]]$ determined by $p_r \mapsto (k+1)(1-q^r)p_r$ $(r \geq 1)$. Under this homomorphism, we have

$$H(t) \mapsto \exp\Big(\sum_{r\geq 1} (k+1)(1-q^r)p_r t^r/r\Big) = (H(t)/H(qt))^{k+1}.$$

Hence, (2) can be rewritten in the form

$$\sum_{n\geq 0} \mathrm{ch}(X_n)t^n = (1-q)\sum_{k\geq 0} q^k(H(t)/H(qt))^{k+1} = \frac{(1-q)H(t)}{H(qt)-qH(t)},$$

and so Lemma 3.1 implies $\mathrm{ch}(X_n) = \sum_k q^k\,\mathrm{ch}(\chi_{n,k})$. □

This result shows that the generating functions $\sum_k q^k \chi_{n,k}(w)$ are products of symmetric unimodal polynomials, and hence are themselves symmetric and unimodal. This observation could also be deduced directly from the interpretation of $\chi_{n,k}(w)$ as the number of w-invariant codes of index k; these codes are clearly mark- and shift-invariant, and so Lemma 2.1(b) applies. Another explanation of the unimodality that relies on representation theory will be given in the next section.

There does not seem to be an analogue of Proposition 3.3 for the characters $\theta_{n,k}$, i.e., the generating functions do not seem to factor in any significant way. In spite of this, it is interesting to note that since $\theta_{n,k}(w)$ is the number of w-invariant positive codes of index k, Lemma 2.1 implies that $\sum_k q^k\theta_{n,k}(w)$ is symmetric and unimodal.

Finally, we remark that the special case $w=1$ of Proposition 3.3 is equivalent to (the enumerative content of) Theorem 1.1(b). Considering the fact that the w-invariant codes are so easy to identify and that their index generating function has such a simple form, it would be interesting to find an extension of the correspondence ϕ that explains this proposition in general.

4. Marked tableaux and isotypic Eulerian numbers

Let X be the (complex) toric variety associated with the Coxeter complex Σ, as mentioned in the Introduction. The cohomology $H^*(X) = H^*(X, \mathbb{C})$ is nonzero only for even degrees less than $2n$, so the grading of $H^*(X)$ assumes the form

$$H^*(X) = H^0(X) \oplus H^2(X) \oplus \cdots \oplus H^{2n-2}(X). \tag{3}$$

We remark that the sequence of Betti numbers $\beta_k := \dim H^{2k}(X)$ $(0 \leq k < n)$ are known to be the Eulerian numbers (i.e., $\beta_k = A(n, k+1)$), and so the Poincaré polynomial $P(q) = \sum_k \beta_k q^k$ of $H^*(X)$ is in fact equal to $F(S_n, q)$ (cf. the remarks following Proposition 7.7. in [6]).

Since S_n acts as the Weyl group of A_{n-1}, it follows that there is a natural (graded) action of S_n on $H^*(X)$. It is therefore possible to refine the Poincaré polynomials according to the S_n-isotypic components of $H^*(X)$. More precisely, for each partiton λ of n let us define $\beta_k(\lambda)$ to be the multiplicity of the λth irreducible S_n-module in $H^{2k}(X)$, and let

$$P_\lambda(q) = \sum_k \beta_k(\lambda) q^k$$

denote the associated generating function. Note that

$$F(S_n, q) = \sum_{|\lambda|=n} f_\lambda P_\lambda(q), \tag{4}$$

where f_λ is the dimension of the λth S_n-module.

Let χ^λ denote the λth irreducible S_n-character, and let $s_\lambda = \text{ch}(\chi^\lambda)$ denote the associated Schur function. By adapting a recurrence derived by DeConcini and Procesi, Stanley [6, Proposition 7.7] has shown that the polynomials $P_\lambda(q)$ may be characterized via the expansion

$$\sum_\lambda t^{|\lambda|} P_\lambda(q) s_\lambda = \frac{(1-q)H(t)}{H(qt) - qH(t)}.$$

In view of Lemma 3.1, we may immediately deduce the following.

Proposition 4.1. *The (graded) S_n-module structure of $H^*(X)$ is isomorphic to the action of S_n on codes (graded by the index).*

Note that this implies that $H^*(X)$ carries a *permutation* representation of S_n; it would be interesting to explain this geometrically.

To describe the polynomials $P_\lambda(q)$ explicitly (a problem suggested by Stanley [6]), we will first define the tableau-analogue of a marked sequence.

Let λ be a partition of n into l parts, and let $D_\lambda = \{(i, j) \in \mathbb{Z}^2 : 1 \leq i \leq l, 1 \leq j \leq \lambda_i\}$ denote the diagram of λ. For our purposes, a *tableau* of shape λ will be a map $T : D_\lambda \to \mathbb{N}$ such that $T(i, j) \leq T(i, j+1)$ (nondecreasing rows) and $T(i, j) < T(i+1, j)$ (increasing columns). We will say that T is a *k-admissible* tableau if the positive range $S^+(T) := \{T(i, j) : T(i, j) > 0\}$ of T is the form $\{1, \ldots, k\}$ for some $k \geq 0$.

For example, using matrix-style coordinates, the following arrays both represent tableaux of shape $(5, 4, 2)$:

$$T_1 = \begin{matrix} 0 & 0 & 0 & 1 & 4 \\ 1 & 1 & 2 & 2 & \\ 2 & 4 & & & \end{matrix}, \qquad T_2 = \begin{matrix} 0 & 0 & 1 & 2 & 3 \\ 1 & 1 & 2 & 3 & \\ 2 & 2 & & & \end{matrix}.$$

However, note that T_1 is not admissible, whereas T_2 is 3-admissible.

Finally, we define a *marked tableau* to be a pair (T, f) consisting of an admissible tableau T and a map $f : S^+(T) \to \mathbb{N}$ such that $1 \leq f(j) < m_j(T)$ for

$j \in S^+(T)$, where $m_j(T)$ denotes the number of j's in T. The index of a marked tableau is defined in the same way as the index of a marked sequence. Note that there are no marked tableaux of the form (T, f) if $m_j(T) = 1$ for any $j > 0$.

Theorem 4.2. *For any partition λ, we have*

$$P_\lambda(q) = \sum_{(T,f)} q^{\mathrm{ind}(T,f)},$$

summed over all marked tableaux (T, f) of shape λ.

Following our conventions with codes, we may graphically represent a marked tableau (T, f) by replacing one occurrence of j in T with the symbol $\hat{j}$ so that there are $f(j)$ occurrences of j in columns to the left of $\hat{j}$. For example, the following is a list of the 6 marked tableaux of shape (4, 1, 1).

$$\begin{matrix} 0 & 0 & \hat{1} & \hat{2} \\ 1 \\ 2 \end{matrix} \qquad \begin{matrix} 0 & \hat{1} & 1 & \hat{2} \\ 1 \\ 2 \end{matrix} \qquad \begin{matrix} 0 & 1 & \hat{1} & \hat{2} \\ 1 \\ 2 \end{matrix}$$

$$\begin{matrix} 0 & \hat{1} & \hat{2} & 2 \\ 1 \\ 2 \end{matrix} \qquad \begin{matrix} 0 & \hat{1} & 2 & \hat{2} \\ 1 \\ 2 \end{matrix} \qquad \begin{matrix} 1 & \hat{1} & \hat{2} & \hat{3} \\ 2 \\ 3 \end{matrix}$$

In particular, Theorem 4.2 predicts that $P_{411}(q) = 3q^2 + 3q^3$. Note also that if λ has more than $(n+1)/2$ parts, then there are no marked tableaux of shape λ, and so $P_\lambda(q) = 0$.

Proof of Theorem 4.2. From (1), we have

$$\sum_{k \geq 0} q^k \chi_{n,k} = \sum_{(\mu,f)} q^{\mathrm{ind}(\mu,f)} \eta_\mu = \sum_{(\mu,f)} \sum_{\lambda} q^{\mathrm{ind}(\mu,f)} K_{\lambda\mu} \chi^\lambda,$$

where $K_{\lambda\mu}$ denotes the multiplicity of χ^λ in η_μ. Hence, Proposition 4.1 implies

$$P_\lambda(q) = \sum_{(\mu,f)} q^{\mathrm{ind}(\mu,f)} K_{\lambda\mu}.$$

Since $K_{\lambda\mu}$ is the number of tableaux T of shape λ and content μ (i.e., $m_i(T) = \mu_i$ for $i = 0, 1, \ldots$) [3, I.6], the result follows. □

There is a closely related family of polynomials $R_\lambda(q)$ that has been considered (but not explicitly described) by Stanley [6] and Brenti [1]. Stanley defined the polynomials R_λ by means of the expansion

$$\frac{1-q}{H(qt) - qH(t)} = \sum_\lambda t^{|\lambda|} R_\lambda(q) s_\lambda.$$

In view of Lemma 3.2, we may equivalently define R_λ via

$$\sum_{n,k\geq 0} q^k \theta_{n,k} = \sum_\lambda R_\lambda(q)\chi^\lambda.$$

By restricting the proof of Theorem 4.2 to orbits of positive codes, it is easy to deduce the following description of R_λ.

Theorem 4.3. *For any partition λ, we have*

$$R_\lambda(q) = \sum_{(T,f)} q^{\mathrm{ind}(T,f)},$$

summed over all positive marked tableaux (T,f) of shape λ.

Since the set of marked tableaux of shape λ is both mark- and shift-invariant, a direct application of Lemma 2.1 yields the following.

Corollary 4.4. *P_λ and R_λ are symmetric and unimodal with centers of symmetry at $(n-1)/2$ and $n/2$, respectively.*

The unimodality of R_λ was conjectured by Stanley and later proved by Brenti.

By the hard Lefschetz Theorem one knows that there is an action of $\mathrm{sl}_2(\mathbb{C})$ on $H^*(X)$ for which (3) is the weight space decomposition. This observation provides another explanation of the unimodality of Eulerian numbers. Moreover, as observed by Stanley [6], the fact that the actions of S_n and sl_2 on $H^*(X)$ commute also implies the unimodality of $P_\lambda(q)$. Another (more elementary) proof of this fact has been given by Brenti [1]. We note that the unimodality of the polynomials $\sum_k q^k\chi_{n,k}(w)$ from Section 3 can also be explained in these terms: The space of w-fixed points of $H^*(X)$ is sl_2-invariant (the actions commute), and thus must have a unimodal Poincaré polynomial.

As a final remark, we note that the correspondence ϕ, together with Schensted's correspondence [4], can be used to give a purely combinatorial explanation of identity (4). To see this, let $w \in S_n$ be a permutation, and let $(\alpha, f) = \phi(w)$ be the code of w. By applying the row-insertion algorithm of Schensted's correspondence to α, one may produce a pair of tableaux (T, Q) of the same shape in which the entries of T and α are the same and Q is standard. Thus,

$$w \xrightarrow{\phi} (\alpha, f) \to ((T,f), Q)$$

provides a bijection between S_n and pairs of consisting of a marked tableau (T,f) and a standard tableau Q of the same shape. This clearly explains (4). Also, by restricting this bijection to derangements, we obtain a combinatorial explanation of the identity [1, Proposition 6]

$$F(D_n, q) = \sum_\lambda f_\lambda R_\lambda(q).$$

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