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WORST SHAPES OF IMPERFECTIONS FOR SPACE TRUSSES WITH MANY SIMULTANEOUSLY BUCKLING MEMBERS

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Abstract—Optimally designed truss type structures whose joints do not transmit moments often have a large number of coincident buckling modes. Each mode corresponds to the buckling of an individual member. Due to the interaction between various simultaneous modes, such trusses can be sensitive to imperfections. This problem is analysed using the Lyapunov–Schmidt–Koiter decomposition and asymptotic expansion technique. The shape of the imperfection that maximizes the load drop is determined from the postbifurcated equilibrium branch of the perfect structure on which the load drops most rapidly. It is shown that this branch is obtained by minimizing a homogeneous quadratic form subject to linear inequality constraints. The general theory is illustrated by several examples involving two- and three-dimensional trusses.

1. INTRODUCTION

Lattice type structures are not only common for applications on the ground such as lattice roofs, radio antennae, crane booms, dishes of radio telescopes and lattice domes; but are also being envisioned for applications in space, including lattice columns, communications platforms, radio-astronomy dishes, solar panels, reflectors and other structures. With the advent and increasing availability of higher strength materials, buckling often becomes a critical design consideration. This is especially true for applications in space, due to the combination of large structural dimensions, small loads and the importance of weight minimization (although the limitation of deflections is also an essential consideration in this case).

Often such structures are optimized so that, as the applied load is increased, a number of members reach their buckling load at the same time. Buckling of any member corresponds to an eigenmode for the system as a whole. Thus, an optimized structure can have a large number of coincident eigenmodes.

It has long been established (Koiter, 1945) that multiple coincident eigenmodes can lead to high imperfection sensitivity of the structure's load carrying capacity. For example, the load carrying capacity of a cylinder in axial compression can drop by a factor of five or more due to imperfections. Lattice structures can also be sensitive to imperfections (Wright, 1965, 1966a; Castaño, 1989; Britvec, 1973; Britvec and Davister, 1985): a lattice dome in Bucharest failed at an estimated load of about 40% of the theoretical buckling load for a perfect dome (Wright, 1965, 1966a). The high imperfection sensitivity of lattice domes was further confirmed by experimental results reported in Wright (1966b). Lattice columns are also very imperfection sensitive when the overall column buckling and member buckling occur at approximately the same level of applied load (Thompson and Hunt, 1973; Crawford and Benton, 1980; Elyada, 1985). In this paper a general methodology for determining the imperfection sensitivity for space trusses with multiple coincident modes involving buckling of individual members in the elastic range is presented. The advantages of the approach presented here over previous studies on this subject are: (1) it provides the worst shape of imperfection, as well as a simple relationship between the magnitude of the imperfection (of the worst shape) and the corresponding drop in load carrying capacity of the structure; (2) it is applicable for any truss type structure; and (3) it is computationally efficient and avoids severely ill-conditioned calculations.

The approach is based on the decomposition and asymptotic expansion technique that was pioneered by the mathematicians Lyapunov and Schmidt around the turn of the century, and also later (apparently independently) by Koiter (1945) who applied the methodology to structural problems. For the reader's convenience, the main general results from this Lyapunov–Schmidt–Koiter (LSK) decomposition and asymptotic expansion are reviewed in Appendix A. Results for the worst shape of imperfection (Koiter, 1976; Triantafyllidis and Peek, 1992) are also summarized in this appendix.

2. RESULTS FOR A SINGLE MEMBER

In order to determine the postbuckling behavior of the structure, it is first necessary to establish a single valued and smooth expression for the strain energy of an individual member as a function of member deformation parameters. One possibility would be to use the member elongation, $e_{(m)}$ for member m, as the deformation parameter. However, a plot of axial force versus elongation exhibits a sudden change in slope when the member buckles. This means that second and higher derivatives of this strain energy function for the member do not exist at this point, and renders the asymptotic expansion of the LSK approach inapplicable. To overcome this problem, an additional out-of-straight member deformation parameter $w_{(m)}$ is introduced, which arises naturally in the solution for the postbuckling behavior of a single member by the LSK approach.

More specifically, for the case of a member with constant cross-sectional properties, the transverse displacements (in a coordinate frame that remains aligned with the member endpoints) can be expressed as

$$W(X) = w_{(m)} \sin\left(\frac{\pi X}{L_{(m)}}\right) + \hat{W}(X), \tag{1}$$

where $L_{(m)}$ denotes the length of the member before deformation, and $\hat{W}(X)$ must satisfy the orthogonality condition,

$$\int_0^{L_{(m)}} \sin\left(\frac{\pi X}{L_{(m)}}\right) \hat{W}(X)\, dX = 0. \tag{2}$$

Note that upper case symbols U or W denote displacements that vary along the length of the member as a function of X, whereas lower case symbols $w_{(m)}$ or $e_{(m)}$ denote scalar deformation parameters. Following the LSK approach, the member strain energy for specified member elongation $e_{(m)}$ and out-of-straight deformation $w_{(m)}$ is minimized by an appropriate choice of $U(X)$ and $\hat{W}(X)$, where $U(X)$ denotes the axial displacements within the member and $\hat{W}(X)$ must satisfy the orthogonality condition (eqn (2)). This leads to a unique and adequately smooth strain energy function, $\phi_{(m)}(e_m, w_{(m)}, \bar{W}_{(m)})$, where $\bar{W}_{(m)}$ represents the imperfection of the member. At this point, the perfect structure is considered so that $\bar{W}_{(m)} = 0$. Using a formulation for slender columns (in the sense that shear deformations are negligible) but arbitrarily large displacements (Appendix B), yields the following results for this member strain energy $\phi_{(m)}$ and its derivatives evaluated at $w_{(m)} = \bar{W}_{(m)} = 0$:

$$\phi_{(m)}(e,0,0) = \frac{1}{2}\left(\frac{EA}{L}\right)_{(m)} e^2, \quad \phi_{(m),e}(e,0,0) = \left(\frac{EA}{L}\right)_{(m)} e, \quad \phi_{(m),ee}(e,0,0) = \left(\frac{EA}{L}\right)_{(m)}, \tag{3}$$

$$\phi_{(m),w}(e,0,0) = 0, \quad \phi_{(m),ew}(e,0,0) = 0. \tag{4}$$

At the bifurcation point for the member under axial compression, $e_{(m)} = -e_{c(m)}$, where $e_{c(m)} \equiv \pi^2 (s_{(m)})^2 L_{(m)}$ is the member shortening at buckling, and $s_{(m)} \equiv (I/A)^{1/2}_{(m)}/L_{(m)}$ is the slenderness ratio, the following expressions are obtained:

$$\phi^c_{(m),ww} = 0, \quad \phi^c_{(m),eww} = \frac{\pi^2}{2}\left(\frac{EA}{L^2}\right)_{(m)} (1 + O(\varepsilon_c^2)), \quad \phi^c_{(m),www} = 0, \tag{5}$$

$$\phi^{c}_{(m),wwww} = \frac{3\pi^4}{4}\left(\frac{EA}{L^3}\right)_{(m)}(1+\tfrac{1}{2}\varepsilon_{c(m)})(1+O(\varepsilon_c^2)), \tag{6}$$

where $\varepsilon_{c(m)} = \pi^2(s_{(m)})^2$ is the magnitude of the axial strain at criticality.

Equations (3)–(6) contain all the information that is needed in the subsequent analysis of the overall system. They are based on an exact formulation for a compressible column (Appendix B). However, the factor $1+O(\varepsilon_c^2)$ will be dropped in the subsequent analysis since ε_c^2 is very small compared to unity in most applications. Using the corresponding exact expressions given in Appendix B would introduce no additional difficulty into the analysis. However, approximations introduced by neglecting shear deformations of the buckled members would remain, and are probably of no less importance than the $O(\varepsilon_c^2)$ terms in eqns (5b) and (6).

3. ANALYSIS OF OVERALLL SYSTEM (PERFECT CASE)

Let v denote the collection of all joint displacement vectors for the structure, and $w = (\ldots, w_{(m)}, \ldots)$ be the collection of all member out-of-straight deformations. All these quantities can be collected into a vector $u \equiv (v, w)$, which fully defines the configuration of the structure. If the loads are applied at the joints only, and their magnitude and direction is a function of λ only, the total potential energy can be written as

$$\phi(u, \lambda, \bar{u}) = \phi(v, w, \lambda, \bar{u}) = \sum_m \phi_{(m)}(e_{(m)}, w_{(m)}, \bar{W}_{(m)}) - P(\lambda)v, \tag{7}$$

where $P(\lambda)$ is a linear operator such that $P(\lambda)v$ is the work done by the applied loads through joint displacements v, $\bar{u}$ describes the imperfections for the entire structure, and the summation is carried out over all members m in the structure. In evaluating the derivatives (or variations) of this potential energy, the following functional dependencies must be considered:

$$e_{(m)} = e_{(m)}(v_{(m)}), \quad v_{(m)} = v_{(m)}(v), \quad w_{(m)} = w_{(m)}(w), \quad \bar{W}_{(m)} = \bar{W}_{(m)}(\bar{u}), \tag{8}$$

where $v_{(m)}$ is the difference† in the joint displacement vectors at each end of member m. The Gateaux derivatives (see Appendix C for definition) of these quantities are

$$v_{(m),v}\delta v = \delta v_{(m)}, \quad w_{(m),w}\delta w = \delta w_{(m)}, \qquad e_{(m),v}\delta v = \frac{1}{l}x_{(m)}\cdot\delta v_{(m)}, \tag{9a,b, 10}$$

$$e_{(m),vv}\delta_2 v\delta_1 v = \frac{1}{l}\delta_2 v_{(m)}\cdot\delta_1 v_{(m)} - \frac{1}{l^3}(x_{(m)}\cdot\delta_2 v_{(m)})(x_{(m)}\cdot\delta_1 v_{(m)}), \tag{11}$$

where δv, $\delta_1 v$, etc. and δw, $\delta_1 w$, etc. denote variations in the displacements v and w respectively; $\delta v_{(m)}$ is the difference of the variations in joint displacement at each end of member m; $\delta w_{(m)}$ is the variation in the out-of-straight deformation of member m; $x_{(m)}$ is the difference in joint coordinates at each end of member m in the deformed configuration; and $l = (x_{(m)}\cdot x_{(m)})^{1/2}$ is the deformed length of member m.

With the aid of eqns (7)–(9), the Gateaux derivatives of the potential energy can be evaluated:

$$\phi_{,v}\delta v = \sum_m N_{(m)}e_{(m),v}\delta v - P(\lambda)\delta v, \qquad \phi_{,w}\delta w = \sum_m \phi_{(m),w}\delta w_{(m)}, \tag{12, 13}$$

$$\phi_{,vv}\delta_2 v\delta_1 v = \sum_m \phi_{(m),ee}(e_{(m),v}\delta_2 v)(e_{(m),v}\delta_1 v) + \sum_m N_{(m)}e_{(m),vv}\delta_2 v\delta_1 v, \tag{14}$$

†Such differences in displacements at each end of the member should be taken by defining a beginning and end for each member, and always subtracting the displacement at the beginning from that at the end.

$$\phi_{,vw}\delta w\delta v = \sum_m \phi_{(m),ew}\delta w_{(m)}(e_{(m),v}\delta v), \qquad \phi_{,ww}\delta_2 w\delta_1 w = \sum_m \phi_{(m),ww}\delta_2 w_{(m)}\delta_1 w_{(m)}, \tag{15, 16}$$

$$\phi_{vww}\delta_2 w\delta_1 w\delta v = \sum_m \phi_{(m),eww}\delta_2 w_{(m)}\delta_1 w_{(m)}(e_{(m),v}\delta v), \tag{17}$$

$$\phi_{,www}\delta_3 w\delta_2 w\delta_1 w = \sum_m \phi_{(m),www}\delta_3 w_{(m)}\delta_2 w_{(m)}\delta_1 w_{(m)}, \tag{18}$$

$$\phi_{,wwww}\delta_4 w\delta_3 w\delta_2 w\delta_1 w = \sum_m \phi_{(m),wwww}\delta_4 w_{(m)}\delta_3 w_{(m)}\delta_2 w_{(m)}\delta_1 w_{(m)}, \tag{19}$$

where $N_{(m)} = \phi_{(m),e}$ is the axial force in member m.

For the perfect structure with no out-of-straight deformations of the members, substituting eqns (3) and (4) into eqns (13) and (15) yields

$$\phi_{,w}(v, 0, \lambda, 0)\delta w = 0, \quad \phi_{,vw}(v, 0, \lambda, 0)\delta w\delta v = 0. \tag{20a,b}$$

In view of eqn (20a) the equilibrium conditions, $\phi_{,u}\delta u = 0$ ($\phi_{,v}\delta v + \phi_{,w}\delta w = 0$), are satisfied if $w = 0$ and $\phi_{,v}\delta v = 0$. Therefore the principal solution must be of the form:

$$u = \overset{0}{u}(\lambda) = (\overset{0}{v}(\lambda), 0), \quad e_{(m)} = \overset{0}{e}_{(m)}(\lambda), \quad N_{(m)} = \overset{0}{N}_{(m)}(\lambda) \tag{21}$$

and the equations for the buckling modes $\overset{(i)}{u} \equiv (\overset{(i)}{v}, \overset{(i)}{w})$ reduce to

$$\phi_{,vv}\overset{(i)}{v}\,\delta v = 0\,\forall\,\delta v, \quad \phi_{,ww}\overset{(i)}{w}\,\delta w = 0\,\forall\,\delta w. \tag{22}$$

If the operator $\phi_{,vv}$ becomes singular, global buckling modes would develop. This case is not considered here. Instead it is assumed that $\phi_{,vv}$ remains positive definite, while a set of members $m \in M$ reach their buckling load at $\lambda = \lambda_c$. In this case the following relations apply for all members $m \in M$:

$$\overset{0}{e}_{(m)}(\lambda_c) = -e_{c(m)}, \quad \overset{0}{N}_{(m)}(\lambda_c) = -N_{c(m)}, \quad \phi_{(m),ww} = 0, \tag{23}$$

where $N_{c(m)} = \pi^2(s^2EA)_{(m)}$ is the buckling load for member m. From the last condition, and eqn (16) it is seen that a typical buckling mode involves out-of-straight deformations of some member $i \in M$ only, and can be written as

$$\overset{(i)}{u} = (0, \overset{(i)}{w}), \quad \overset{(i)}{w} = (\ldots, \overset{(i)}{w}_{(m)}, \ldots), \quad \overset{(i)}{w}_{(m)} = 0\,\forall\,m \neq i. \tag{24}$$

For convenience, the mode identifier i is taken to be the number of the member that buckles for mode i. The mode normalization condition, eqn (A2), reduces to

$$\left[\frac{d}{d\lambda}\phi_{,ww}(\overset{0}{v}(\lambda), 0, \lambda, 0)\right]_{\lambda=\lambda_c}\overset{(i)}{w}\,\overset{(j)}{w} = \frac{\pi^2\dot{N}_{(i)}}{2L_{(i)}}(\overset{(i)}{w}_{(i)})^2\delta_{ij} = -\phi_0\delta_{ij}, \tag{25}$$

where a dot placed above any quantity indicates that this quantity should be evaluated on the principal branch (as a function of λ only), then differentiated with respect to λ and evaluated at $\lambda = \lambda_c$; δ_{ij} is the Kronecker delta†; and ϕ_0 is an arbitrary positive constant. Choosing ϕ_0 determines the values of $\overset{(i)}{w}_{(i)}$. In order to facilitate the interpretation of the algebraic results, this normalization is defined via the properties of a reference member, which may, but need not, correspond to any of the members in the structure. Thus

† There are no implied summations for repeated indices throughout this paper.

$$\phi_0 = \frac{\pi^2 N_{c(0)} L_{(0)}}{2\lambda_c}, \quad \overset{(i)}{w}_{(i)} = \sqrt{L_{(i)} L_{(0)} N_{c(0)} / (-\lambda_c \dot{N}_{(i)})}, \tag{26}$$

where any member properties identified by a subscript zero in parentheses correspond to properties of the reference member. For linear prebuckling behavior, $-\lambda_c \dot{N}_{(i)} = N_{c(i)}$, and therefore $\overset{(i)}{w}_{(i)} = L_{(i)}$, for any member i whose properties coincide with those of the reference member.

It follows from eqns (24), (18) and (5a) that $\phi^c_{,uuu}\overset{(i)}{u}\,\overset{(j)}{u}\,\overset{(k)}{u}$ vanishes for all i, j, k. This indicates a symmetric bifurcation. Higher order postbuckling coefficients ϕ_{ijkl} defined in eqn (A6) (Appendix A) are therefore required to determine the directions and curvatures of the bifurcated equilibrium branches. The postbuckling coefficients in turn depend on second order displacements, $\overset{(ij)}{u} = (\overset{(ij)}{v}, \overset{(ij)}{w})$. To determine these quantities, the procedure outlined in Appendix A is followed: first the space of admissible displacements, A, is decomposed as the linear span of spaces A_0 and $\hat{A}$, where A_0 is the space spanned by the eigenvectors and $\hat{A}$ is a complementary space, such that the zero vector is the only element that is common to both A_0 and $\hat{A}$. The obvious choice for this complementary space is

$$\hat{A} = \{u = (v, w) : w_{(m)} = 0 \,\forall\, m \in M\}. \tag{27}$$

With the aid of eqns (20) and (24), the general equation for the second order displacements, eqn (A5), reduces to

$$(\phi^c_{,vww}\overset{(i)}{w}\,\overset{(j)}{w} + \phi^c_{,vv}\overset{(ij)}{v})\delta v + (\phi^c_{,www}\overset{(i)}{w}\,\overset{(j)}{w} + \phi^c_{,ww}\overset{(ij)}{w})\delta w = 0 \,\forall\, (\delta v, \delta w) \in \hat{A}, \tag{28}$$

where $\overset{(ij)}{u} = (\overset{(ij)}{v}, \overset{(ij)}{w}) \in \hat{A}$ are the desired second order displacements. The variation with respect to w together with eqns (16) and (18), and the observation that $\phi_{(m),ww} > 0$ for $m \notin M$ leads to $\overset{(ij)}{w} = 0$. The variation with respect to v, and application of eqns (5), (17) and (26) leads to the conclusion that $\overset{(ij)}{v} = 0$ for $i \neq j$, and $\overset{(ii)}{v}$ are determined from

$$(N_{0(i)} e_{(i),v} + \phi^c_{,vv}\overset{(ii)}{v})\delta v = 0, \tag{29}$$

where

$$N_{0(i)} \equiv \phi^c_{(i),eww}(\overset{(i)}{w}_{(i)})^2 = \frac{\phi_0}{(-\dot{e}_{(i)})}. \tag{30}$$

Thus the second order displacements $\overset{(ii)}{v}$ represent the joint displacements due to an initial tension $N_{0(i)}$ in the member i, as calculated by linearization about the critical point. (The initial tension is treated as if it were caused by an infinitesimal thermal contraction.) By the assumption that local buckling modes develop before the global modes, positive definiteness of $\phi^c_{,vv}$ is assured. Indeed, this operator represents the tangent stiffness matrix of the structure with out-of-straight member deformations restrained. All second order displacements can be obtained with only one factorization of this tangent stiffness matrix.

For the calculation of the fourth order postbuckling coefficients, ϕ_{ijkl}, eqn (A6) reduces to

$$\phi_{ijkl} = \{\phi^c_{,wwww}\overset{(i)}{w}\,\overset{(j)}{w}\,\overset{(k)}{w}\,\overset{(l)}{w} + \phi^c_{,vww}\overset{(i)}{w}(\overset{(j)}{w}\,\overset{(kl)}{v} + \overset{(k)}{w}\,\overset{(lj)}{v} + \overset{(l)}{w}\,\overset{(jk)}{v})\}/\phi_0. \tag{31}$$

Evaluating eqn (31) with the aid of eqns (5), (6), (17) and (19) yields

$$\phi_{ijkl} = a_{ik}\delta_{ij}\delta_{kl} + a_{il}\delta_{ik}\delta_{lj} + a_{ij}\delta_{il}\delta_{jk}, \tag{32}$$

where

$$a_{ij} \equiv \{ \tfrac{1}{3}\phi_{(i),wwww}(\overset{(i)}{w}_{(i)})^4\delta_{ij} + \phi_{(i),eww}(\overset{(i)}{w}_{(i)})^2 \overset{(jj)}{e}_{(i)} \} \quad \phi_0 = \frac{1}{(-\dot{N}_{(i)})} \{ \overset{(jj)}{N}_{(i)} + \tfrac{1}{2}N_{0(i)}\varepsilon_{c(i)}\delta_{ij} \} \tag{33a,b}$$

and

$$\overset{(jj)}{N}_{(i)} = N_{0(i)}\delta_{ij} + \left(\frac{EA}{L}\right)_{(i)} e_{(i),v} \overset{(jj)}{v}$$

represents the change in axial force in member i associated with the second order displacements for the buckling of member j. Symmetry of the coefficients a_{ij} can be verified by substituting $\delta v = \overset{(ii)}{v}$ into eqn (29). However, in general neither the matrix $[a_{ij}]$ nor the matrix $[-a_{ij}]$ is positive definite.

Having calculated the required postbuckling coefficients, attention is now focused on the bifurcated equilibrium branches for the perfect structure. By application of eqns (A3) and (A4), these can be written as:

$$\lambda(\xi) = \lambda_c + \tfrac{1}{2}\lambda_2\xi^2 + O(\xi^3),$$
$$v(\xi) = \overset{(0)}{v}(\lambda(\xi)) + \tfrac{1}{2}\xi^2 \sum_{i \in M} \overset{(ii)}{v}(\alpha_i)^2 + O(\xi^3), \quad w_{(i)}(\xi) = \xi\alpha_i + O(\xi^3), \tag{34a–c}$$

where ξ is the path parameter, and λ_2 and α_i must be solutions to

$$\alpha_i\left(-\lambda_2 + \sum_{j \in M} a_{ij}(\alpha_j)^2\right) = 0 \,\forall\, i \in M, \quad \sum_{i \in M} (\alpha_i)^2 = 1. \tag{35a,b}$$

Solutions to these equations can readily be obtained as follows: (1) Partition the set of modes M into two mutually exclusive and collectively exhaustive sets M_0 and M_1, and take $\alpha_i = 0 \,\forall\, i \in M_0$; (2) solve the system of linear equations

$$\sum_{j \in M_1} a_{ij} y_j = 1 \,\forall\, i \in M_1, \tag{36}$$

for y_j, $j \in M_1$; and (3) obtain the solutions for this partition as

$$\alpha_i = \pm\sqrt{y_i \Big/ \left(\sum_{j \in M_1} y_j\right)}, \quad \lambda_2 = 1 \Big/ \left(\sum_{j \in M_1} y_j\right). \tag{37}$$

The solution is real only when all y_i are of the same sign.

Regarding the stability of the bifurcated equilibrium branches, applying eqn (A7) leads to the incremental stability matrix

$$B_{ij} = (-\lambda_2 + \sum_{k \in M_1} a_{ik}(\alpha_k)^2)\delta_{ij} \,\forall\, (i, j) \in (M_0 \times M_0) \cup (M_0 \times M_1) \cup (M_1 \times M_0)$$
$$= 2a_{ij}\alpha_i\alpha_j \,\forall\, (i, j) \in M_1 \times M_1. \tag{38a,b}$$

This matrix must be positive definite to ensure stability of the bifurcated branches. Alternatively positive semidefiniteness is a necessary condition for stability. This in turn requires

$$-\lambda_2 + \sum_{j \in M_1} a_{ij}(\alpha_j)^2 \geq 0 \,\forall\, i \in M_0. \tag{39}$$

Violation of this last condition for some $i \in M_0$ involves loss of stability for the member i while it remains straight. This means that the member must be carrying an axial load in excess of its buckling load. Therefore eqn (39) will be referred to as the local, or member stability condition. Solutions which violate this condition will be said to be locally unstable. Locally stable solutions are not necessarily stable (since B_{ij} still can have negative eigenvalues corresponding to eigenvectors χ_i with $\chi_i = 0\,\forall\, i \in M_0$). Indeed all bifurcated branches on which the load drops away from the bifurcation point ($\lambda_2 < 0$) are unstable.

Of particular interest is the postbuckling branch for which λ_2 is a minimum, since this also provides the worst imperfection shape (see Appendix B). One way of determining this branch is to find all postbuckling branches with the aid of eqns (36) and (37), and pick the real solution for which λ_2 is a minimum. This method quickly becomes impractical, since the number of such solutions increases exponentially with the number of potentially interacting modes. The preferable alternative is to solve the minimization problem given in eqn (A10). After substituting for the coefficients ϕ_{ijkl} from eqn (32), this reduces to:

minimize

$$\lambda_2 = \sum_{i \in M} \sum_{j \in M} a_{ij} x_i x_j, \tag{40a}$$

subject to

$$\sum_{i \in M} x_i = 1, \quad x_i \geqslant 0\,\forall\, i \in M \tag{40b,c}$$

where $x_i = (\alpha_i)^2$. Despite the simple appearance of the minimization problem, it may have many local minima, since the matrix of coefficients a_{ij} is not positive (or negative) definite. Indeed any solution of eqns (36) and (37) is a potential local minimum. Fortunately, however, the numerical examples of Section 5 suggest that the actual number of local minima is much smaller.

It can be shown, using the Kuhn–Tucker conditions for optimality of the solution to eqns (40) (see Appendix D), that the postbuckling branch on which the load drops most rapidly is locally stable in the sense of eqn (39). More generally, local minima for eqns (40) correspond to locally stable bifurcated equilibrium branches, and *vice versa*.

4. IMPERFECT STRUCTURE AND WORST IMPERFECTION SHAPE

The imperfections considered can be written as $\bar{u} = \varepsilon\bar{\bar{u}}$, where ε is the scalar magnitude of the imperfection and $\bar{\bar{u}}$ is the shape of the imperfection, which will be normalized in some fashion. The following types of imperfections are considered: (1) errors in the joint coordinates in the reference configuration; (2) member misfit (a member is too long or too short); (3) curved initial geometry of the members; and (4) eccentricities at the joints. For the present case, when local buckling modes develop before the global modes, it is found that the leading order effects are due to the imperfections (3) and (4) only. These effects lead to a drop in load carrying capacity of the structure which is of the order $\varepsilon^{2/3}$. The other imperfections [(1) and (2)], on the other hand, have effects which are of the order ε or higher, and will therefore not be included. The analysis provides results which are asymptotically exact for small imperfections.

In view of these considerations, the imperfection $\bar{u}$ can be characterized as

$$\bar{u} = \bar{W} = (\ldots, \bar{W}_{(m)}, \ldots) = \varepsilon\bar{\bar{W}} = \varepsilon(\ldots, \bar{\bar{W}}_{(m)}, \ldots), \tag{41}$$

where $\bar{W}_{(m)} = \bar{W}_{(m)}(X)$ define a transverse geometric imperfection for member m, with $\bar{W}_{(m)} = 0\,\forall\, X$ representing a straight member with zero eccentricities at the ends (see Appendix B for details). The projections of the imperfection shape [eqn (A9b)] which determine the postbuckling behavior of the structure, are given by

$$\bar{\bar{\xi}}_i = -\frac{1}{\lambda_c \phi_0} \phi^c_{,w\bar{W}} \bar{\bar{W}} \overset{(i)}{w} = -\frac{1}{\lambda_c \phi_0} \phi^c_{(i),w\bar{W}} \bar{\bar{W}}_{(i)} \overset{(i)}{w}_{(i)} = \frac{N_{c(i)}}{\sqrt{-\lambda_c \dot{N}_{(i)} N_{c(0)} L_{(i)} L_{(0)}}} \bar{\bar{w}}_{(i)}, \tag{42a–c}$$

where

$$\bar{\bar{w}}_{(i)} = \frac{2}{L_{(i)}} \int_0^{L_{(i)}} \sin\left(\frac{\pi X}{L_{(i)}}\right) \bar{\bar{W}}_{(i)}(X)\,\mathrm{d}X \tag{43}$$

is the amplitude of an equivalent sinusoidal imperfection.

The worst shape of imperfection corresponds to $\bar{\bar{\xi}}_i = \alpha_i$, and hence to

$$\bar{\bar{w}}_{(i)} = \frac{\sqrt{(-\lambda_c \dot{N}_{(i)}) N_{c(0)} L_{(i)} L_{(0)}}}{N_{c(i)}} \alpha_i . \tag{44}$$

It can readily be verified that the imperfection norm

$$\|\bar{u}\| = \|\bar{W}\| = \left\{ \sum_m \frac{(N_{c(m)})^2}{(-\lambda_c \dot{N}_{(m)}) N_{c(0)} L_{(m)} L_{(0)}} \frac{2}{L_{(m)}} \int_0^{L_{(m)}} (\bar{W}_{(m)}(X))^2\,\mathrm{d}X \right\}^{1/2} \tag{45}$$

satisfies the conditions outlined in the last paragraph of Appendix A, and leads to a worst imperfection shape that involves sinusoidal imperfections of amplitude $\bar{\bar{w}}_{(i)}$ given by eqn (44) for the members $i \in M_1$, and no imperfection for the other members.

In summary, the procedure for finding the worst imperfection shape, and the corresponding load drop is as follows: (1) obtain the solution for the principal branch using eqn (12) and locate the bifurcation point λ_c at which members $m \in M$ reach their buckling load. (2) Calculate the second order joint displacements from eqns (29) and (30). (3) Obtain the coefficients a_{ij} from eqn (33). (4) Solve the quadratic programming problem, eqn (40). (5) Calculate $\alpha_i = \pm\sqrt{x_i}$. Finally, (6) the worst imperfection shape is given by eqn (44), and the corresponding load drop for any given amplitude of imperfection by eqn (A10).

5. EXAMPLES

5.1. *Description of structures considered*

Typical examples considered are shown in Fig. 1 for planar trusses and Fig. 2 for space trusses. Therein members in compression are shown as continuous lines, whereas those in tension are dashed. The member flexural rigidities, EI, are chosen such that all members in compression reach their buckling load simultaneously. Consequently the number of potentially interacting modes in every truss equals the number of its members under compression in the principal solution.

The dimensions of the structures analysed are as follows: All members for the two-dimensional trusses of Fig. 1 have length L, except the diagonal members for the rectangular truss in Fig. 1b, which are of length $\sqrt{2}L$. For the three-dimensional trusses of Fig. 2, the chord members (i.e. those parallel to the z-axis) are of length L, and the cross-section is such that the chords fall on a circle of radius L. Thus the length of the ties (i.e. the members parallel to the x–y plane) are $\sqrt{3}L$ for the triangular cross-section of Fig. 2a, and $\sqrt{2}L$ for the ties parallel to the x or y axes in Fig. 2b.

The trusses are constructed from identical unit cells. However, the number of unit cells and their arrangement varies: for example, the hexagonal truss of Fig. 1a has a radius of $2L$. Hexagonal trusses of radii L and $3L$ (as shown in Table 1) are also analysed. The joint numbering system for these follows the same pattern established in Fig. 1a. Tables 1 and 2 also illustrate the various arrangements of unit cells considered for the rectangular truss of Fig. 1b, as well as the "antennae" of Fig. 2.

The displacement boundary conditions for the two-dimensional structures are illustrated in Fig. 1. For the three-dimensional structures with n bays, all joints in the plane through points A_n, B_n and C_n are restrained in the z-direction. Sufficient restraint in the x- and y-directions is also provided to prevent rigid body motions. Since these x and y restraints do not produce any reactions, their exact configuration does not influence the results reported here. The applied loads and reactions are shown in Figs 1 and 2. They do not

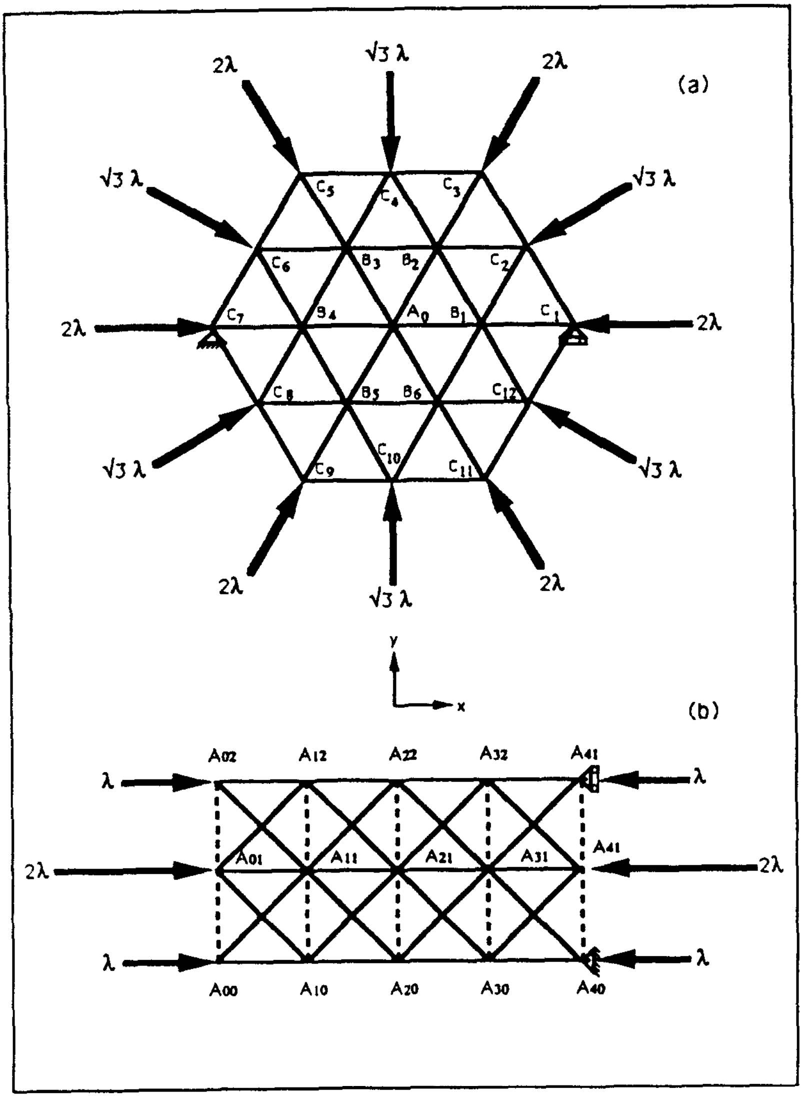


Fig. 1. Two-dimensional example trusses.

change direction. However the magnitude of the reactions shown in Figs 1 and 2 applies only for the principal solution.

All members have the same axial rigidity, EA. However, to simplify the prebuckling solutions, member sites that are shared by two unit cells (e.g. member sites on lines A_{01}–A_{11}, A_{10}–A_{12}, A_{20}–A_{22} and A_{30}–A_{32} in Fig. 1b, and member sites on planes through points A_1, B_1 and C_1 in Fig. 2) are occupied by two identical members. This results in identical stress states in each unit cell for the principal solution, regardless of the overall size of the structure. Although the hexagonal trusses can also be constructed from unit cells, here only one member occupies each member site; identical prebuckling forces, $N_{(m)} = -\lambda$, in all members are achieved by appropriate choice of the external loads.

For the purpose of mode normalization, the length of the reference member is taken to be $L_{(0)} = L, \sqrt{2}L, L, \sqrt{2}L$ for the structures of Figs 1a,b and 2a,b respectively, and the reference member buckling load is taken as $N_{c(0)} = \lambda_c = EA\varepsilon_c$, where ε_c, the buckling strain of the reference member, controls the slenderness of the members. For the hexagonal truss, ε_c also represents the buckling strain in all the members. For other structures, the member buckling strains differ, but are still approximately proportional to ε_c.

Following the discussion in Section 4 (see also Appendix A), the worst imperfection shape for a given truss is determined by the bifurcated equilibrium branch of the perfect

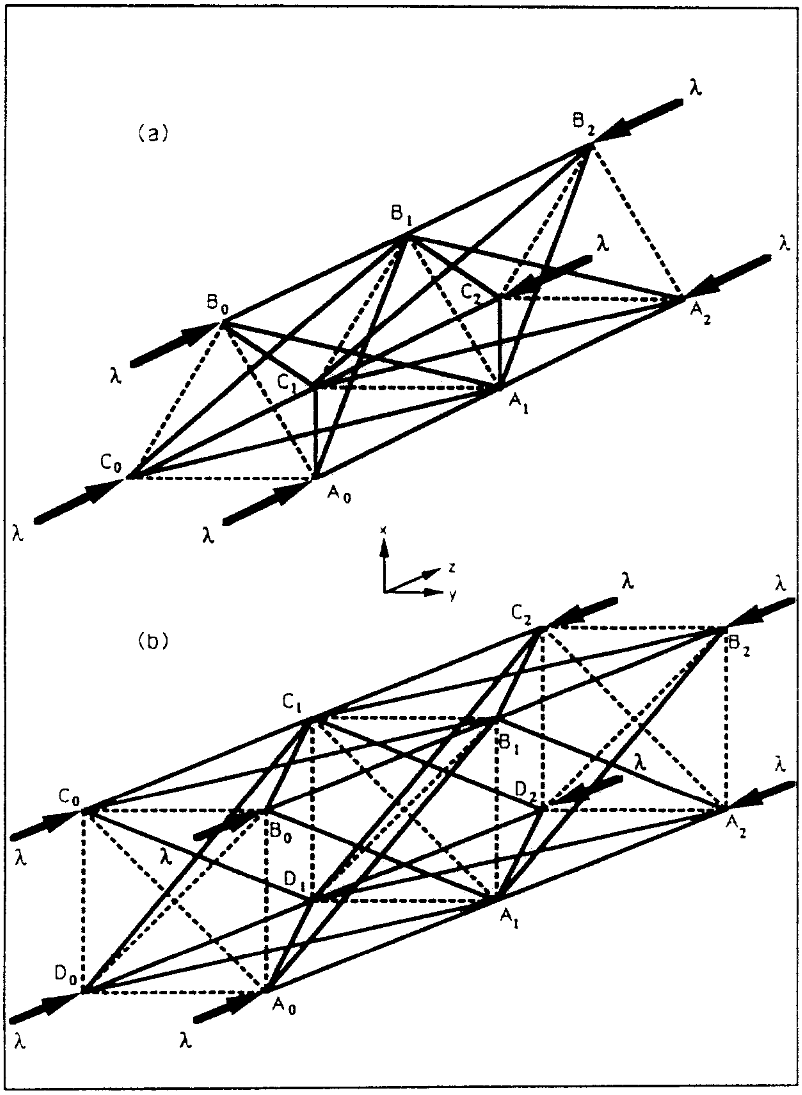


Fig. 2. Three-dimensional example trusses.

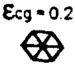
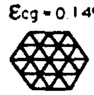
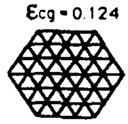
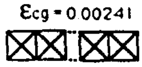
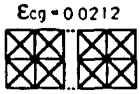
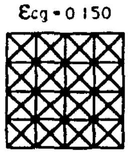
structure for which λ_2 is a minimum. This will be referred to as the critical bifurcated branch. It follows from dimensional analysis considerations alone, that the dimensionless curvature of the critical bifurcated branch, λ_2/λ_c, depends only on ε_c (and not on EA or L).

5.2. *Computational strategies used*

Since the structures considered exhibit essentially linear prebuckling behavior (as long as $\varepsilon_c < 0.1$), a standard Newton Iteration procedure provides rapidly converging solutions on the principal branch of the perfect structure. Such solutions can also readily be extended to the point where global buckling modes develop (i.e. $\phi_{,vv}$ becomes singular). The value of the reference strain λ/EA at the point where the first global buckling mode develops is shown in Tables 1 and 2 as ε_{cg}. As expected, ε_{cg} is seen to depend on the slenderness of the structure as a whole. (Recall that, in contrast, ε_c determines the slenderness of individual members.) Only trusses for which $\varepsilon_c < \varepsilon_{cg}$ are considered, so that local buckling occurs before the global bifurcation load is reached, and $\phi_{,vv}$ remains positive definite.

To find the bifurcated equilibrium branch of the perfect structure with the minimum curvature $\lambda_2(\lambda_2 < 0)$, two different strategies are used: a direct approach and an optimization approach. Although it cannot be asserted with 100% certainty that these methods provide the global minimum, there is every indication that for the examples considered, the solutions obtained are indeed globally minimal.

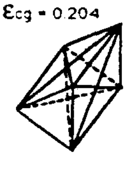
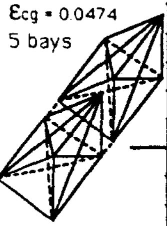
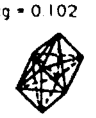
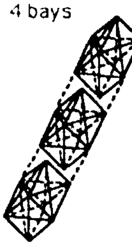
Table 1. Results for two-dimensional examples

$\varepsilon_{cg} = 0.249$ 	$\varepsilon_c = 0001$	$\lambda_2/\lambda_c = -3.29$	mode A_0B_1, B_3B_4 (+ sym. modes in x, y)
	$\varepsilon_c = 001$	$\lambda_2/\lambda_c = -3.31$	mode A_0B_1, B_3B_4 (+ sym. modes in x, y)
	$\varepsilon_c = 01$	$\lambda_2/\lambda_c = -3.46$	mode A_0B_1, B_3B_4 (+ sym. modes in x, y)
	$\varepsilon_c = 05$	$\lambda_2/\lambda_c = -4.30$	mode A_0B_1, B_3B_4 (+ sym. modes in x, y)
	$\varepsilon_c = 1$	$\lambda_2/\lambda_c = -5.97$	mode A_0B_5, B_2B_3 (+ sym. modes in x, y)
	$\varepsilon_c = 2$	$\lambda_2/\lambda_c = -27.86$	mode A_0B_2 (+ sym. modes in x, y)
$\varepsilon_{cg} = 0.149$ 	$\varepsilon_c = 0001$	$\lambda_2/\lambda_c = -2.88$	mode B_1C_2, B_2C_2/B_2C_4, B_3C_4 (+ sym.)
	$\varepsilon_c = 001$	$\lambda_2/\lambda_c = -2.89$	mode B_1C_2, B_2C_2/B_2C_4, B_3C_4 (+ sym.)
	$\varepsilon_c = 01$	$\lambda_2/\lambda_c = -3.05$	mode B_1C_2, B_2C_2/B_2C_4, B_3C_4 (+ sym.)
	$\varepsilon_c = 05$	$\lambda_2/\lambda_c = -3.96$	mode B_1C_2, B_2C_2/B_2C_4, B_3C_4 (+ sym.)
	$\varepsilon_c = .1$	$\lambda_2/\lambda_c = -8.98$	mode B_1B_6, C_1C_2 (+ sym. modes in x, y)
	$\varepsilon_c = .2$	$\lambda_2/\lambda_c =$	mode global
$\varepsilon_{cg} = 0.124$ 	$\varepsilon_c = 0001$	$\lambda_2/\lambda_c = -2.88$	mode C_1D_2, C_2D_2 (+ sym. modes in x, y)
	$\varepsilon_c = 001$	$\lambda_2/\lambda_c = -2.89$	mode C_1D_2, C_2D_2 (+ sym. modes in x, y)
	$\varepsilon_c = 01$	$\lambda_2/\lambda_c = -3.05$	mode C_1D_2, C_2D_2 (+ sym. modes in x, y)
	$\varepsilon_c = 05$	$\lambda_2/\lambda_c = -3.95$	mode C_1D_2, C_2D_2 (+ sym. modes in x, y)
	$\varepsilon_c = 1$	$\lambda_2/\lambda_c = -16.26$	mode D_1D_2 (+ sym. modes in x, y)
	$\varepsilon_c = 2$	$\lambda_2/\lambda_c =$	mode global
$\varepsilon_{cg} = 0.00241$  1 × 16 bays	$\varepsilon_c = 0001$	$\lambda_2/\lambda_c = -227.82$	mode 5 members (optimization)
	$\varepsilon_c = 001$	$\lambda_2/\lambda_c = -362.79$	mode 6 members (optimization)
	$\varepsilon_c = 01$	$\lambda_2/\lambda_c =$	mode global
	$\varepsilon_c = 05$	$\lambda_2/\lambda_c =$	mode global
	$\varepsilon_c = 1$	$\lambda_2/\lambda_c =$	mode global
	$\varepsilon_c = 2$	$\lambda_2/\lambda_c =$	mode global
$\varepsilon_{cg} = 0.0212$  2 × 8 bays	$\varepsilon_c = 0001$	$\lambda_2/\lambda_c = -52.20$	mode $A_{70}A_{80}$, $A_{71}A_{80}$ (+ sym. modes in x
	$\varepsilon_c = 001$	$\lambda_2/\lambda_c = -54.26$	mode $A_{70}A_{80}$, $A_{71}A_{80}$ (+ sym. modes in x
	$\varepsilon_c = 01$	$\lambda_2/\lambda_c = -93.15$	mode $A_{70}A_{80}$, $A_{71}A_{80}$ (+ sym. modes in x
	$\varepsilon_c = 05$	$\lambda_2/\lambda_c =$	mode global
	$\varepsilon_c = 1$	$\lambda_2/\lambda_c =$	mode global
	$\varepsilon_c = 2$	$\lambda_2/\lambda_c =$	mode global
$\varepsilon_{cg} = 0.150$ 	$\varepsilon_c = 0001$	$\lambda_2/\lambda_c = -10.11$	mode 7 members (optimization)
	$\varepsilon_c = 001$	$\lambda_2/\lambda_c = -10.18$	mode 7 members (optimization)
	$\varepsilon_c = 01$	$\lambda_2/\lambda_c = -10.92$	mode 7 members (optimization)
	$\varepsilon_c = 05$	$\lambda_2/\lambda_c = -15.71$	mode 6 members (optimization)
	$\varepsilon_c = 1$	$\lambda_2/\lambda_c = -32.18$	mode 6 members (optimization)
	$\varepsilon_c = 2$	$\lambda_2/\lambda_c =$	mode global

The direct approach is based on finding solutions to eqn (36) for various choices of the set M_1. Since the number of possible sets M_1 grows exponentially with the number of potentially interacting modes, it quickly becomes impractical to consider all possible sets M_1. Furthermore, it was observed that the critical postbuckling branch typically involves buckling of only a few members. Therefore, only sets M_1 that involve buckling of up to five members are considered. Even then, the number of possible sets M_1 (a sum of binomial coefficients) can become staggeringly high. For expediency, the number of possible sets M_1 is therefore further limited to 500,000 starting with those that involve buckling of the fewest members.

As an alternative to the above direct approach, the quadratic programming algorithm by Schittkowski (1986) is used to solve the problem defined by eqns (40). This proved very efficient even for rather large numbers of interacting modes (up to 100). For all the examples considered, several initial guesses lead to the same final optimal solution. Moreover, for problems with a relatively low number of interacting modes (up to about 40) where the direct approach is feasible, the results from the optimization method are always in agreement with those from the direct approach. In other cases (identified in Tables 1 and 2 by the word optimization in parentheses), none of the 500,000 solutions calculated by the direct procedure were real and locally stable, and it was necessary to rely on the results from the optimization approach alone.

Table 2. Results for three-dimensional examples

ε_{cg} = 0.204 	ε_c = .000001	λ_2/λ_c = -4.19	mode A_0A_1,B_0A_1,C_0A_1 (+ sym modes)
	ε_c = .00001	λ_2/λ_c = -4.19	mode A_0A_1,B_0A_1,C_0A_1 (+ sym modes)
	ε_c = .0001	λ_2/λ_c = -4.20	mode A_0A_1,B_0A_1,C_0A_1 (+ sym. modes)
	ε_c = .001	λ_2/λ_c = -4.23	mode A_0A_1,B_0A_1,C_0A_1 (+ sym modes)
	ε_c = .01	λ_2/λ_c = -4.59	mode A_0A_1,B_0A_1,C_0A_1 (+ sym modes)
	ε_c = .1	λ_2/λ_c = -73.75	mode A_0C_1,B_0A_1,C_0B_1 / A_0B_1,B_0C_1,C_0A
ε_{cg} = 0.0474 5 bays 	ε_c = .000001	λ_2/λ_c = -30.51	mode 3 members (optimization)
	ε_c = .00001	λ_2/λ_c = -30.52	mode 5 members (optimization)
	ε_c = .0001	λ_2/λ_c = -30.57	mode 5 members (optimization)
	ε_c = .001	λ_2/λ_c = -31.10	mode 5 members (optimization)
	ε_c = .01	λ_2/λ_c = -37.71	mode A_4A_5,B_4A_5,C_4A_5 (+ sym modes)
	ε_c = .1	λ_2/λ_c = -58.69	mode global
	ε_c = .000001	λ_2/λ_c = -63.41	mode $A_9A_{10},B_9A_{10},C_9A_{10}$ (+sym modes)
	ε_c = .00001	λ_2/λ_c = -63.45	mode $A_9A_{10},B_9A_{10},C_9A_{10}$ (+sym modes)
	ε_c = .0001	λ_2/λ_c = -63.85	mode $A_9A_{10},B_9A_{10},C_9A_{10}$ (+sym modes)
ε_{cg} = 0.0126 10 bays	ε_c = .001	λ_2/λ_c = -68.19	mode $A_9A_{10},B_9A_{10},C_9A_{10}$ (+sym modes)
	ε_c = .01	λ_2/λ_c = -292.2	mode A_7A_8, A_8A_9, A_9A_{10} (+sym modes)
	ε_c = .1	λ_2/λ_c =	mode global
ε_{cg} = 0.102 	ε_c = .000001	λ_2/λ_c = -52.23	mode A_0D_1,C_0D_1,D_0D_1
	ε_c = .00001	λ_2/λ_c = -52.24	mode A_0D_1,C_0D_1,D_0D_1
	ε_c = .0001	λ_2/λ_c = -52.29	mode A_0D_1,C_0D_1,D_0D_1
	ε_c = .001	λ_2/λ_c = -52.79	mode A_0D_1,C_0D_1,D_0D_1
	ε_c = .01	λ_2/λ_c = -58.51	mode A_0D_1,C_0D_1,D_0D_1
	ε_c = .1	λ_2/λ_c = -8,914	mode A_0D_1,C_0D_1
ε_{cg} = 0.0237 4 bays 	ε_c = .000001	λ_2/λ_c = -219.73	mode A_3D_4,C_3D_4,D_3D_4
	ε_c = .00001	λ_2/λ_c = -219.81	mode A_3D_4,C_3D_4,D_3D_4
	ε_c = .0001	λ_2/λ_c = -220.62	mode A_3D_4,C_3D_4,D_3D_4
	ε_c = .001	λ_2/λ_c = -229.06	mode A_3D_4,C_3D_4,D_3D_4
	ε_c = .01	λ_2/λ_c = -374.63	mode A_3D_4,C_3D_4,D_3D_4
	ε_c = .1	λ_2/λ_c =	mode global
	ε_c = .000001	λ_2/λ_c = -443.10	mode A_7D_8,C_7D_8,D_7D_8
	ε_c = .00001	λ_2/λ_c = -443.55	mode A_7D_8,C_7D_8,D_7D_8
	ε_c = .0001	λ_2/λ_c = -448.16	mode A_7D_8,C_7D_8,D_7D_8
	ε_c = .001	λ_2/λ_c = -500.54	mode A_7D_8,C_7D_8,D_7D_8
ε_{cg} = 0.00818	ε_c = .01	λ_2/λ_c =	mode global
8 bays	ε_c = .1	λ_2/λ_c =	mode global

5.3. *Discussion of results*

The results for all 12 trusses considered are shown in Tables 1 and 2 for various values of the parameter ε_c. For small values of ε_c (slender members), the curvature of the critical bifurcated branch, $-\lambda_2/\lambda_c$, is seen to approach an asymptotic value (independent of ε_c). This corresponds to the case investigated by Britvec (1973) and Britvec and Davister (1985) in which compressibility of the members is neglected. Indeed, for a selection of the examples considered by Britvec (1973), the current analysis gives results that are in agreement with his. Even for essentially incompressible members, the slenderness of the structure as a whole has a strong effect on the postbuckling behavior and imperfection sensitivity, with λ_2/λ_c being larger when the structure as a whole is more slender.

For larger ε_c (stubbier members), λ_2/λ_c increases in magnitude, and it becomes very large as the local buckling load approaches the global (i.e. as $\varepsilon_c \to \varepsilon_{cg}$). This is not surprising, since $\phi_{,cc}$ becomes singular in this case, resulting [via eqn (29)] in large second order displacements, which in turn [via eqn (33)] produce postbuckling coefficients a_{ii} of large magnitude.

The dependence of λ_2/λ_c on ε_c is further illustrated in Fig. 3 for a 1×4 bay rectangular truss made of four identical square unit cells each of side $L_{(0)}$ and a 1×1 bar truss of dimensions $L_{(0)} \times 4L_{(0)}$. Global bifurcation for these structures occurs at $\varepsilon_{cg} = 0.0375$ for the 1×4 bay truss, and at $\varepsilon_{cg} = 0.0309$ for the somewhat more flexible 1×1 bay truss of

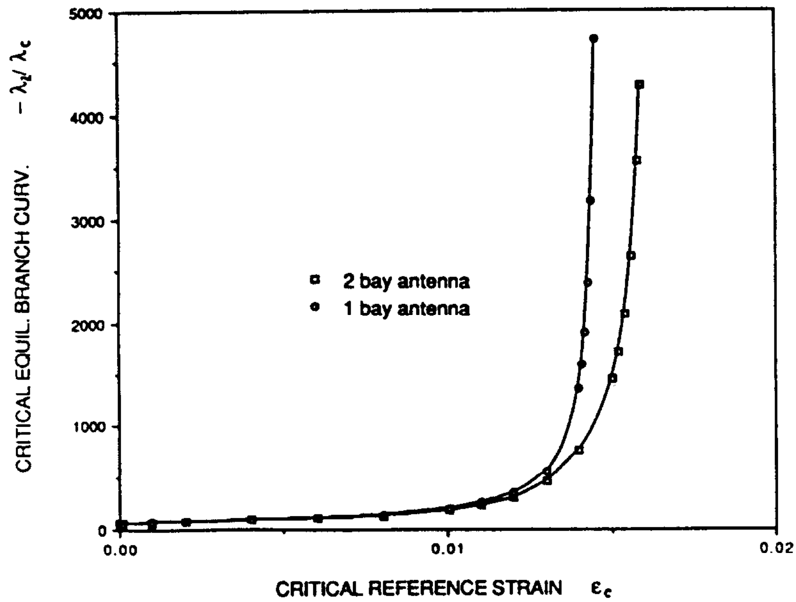


Fig. 3. Results for rectangular trusses of overall dimensions $L_{(0)} \times 4L_{(0)}$.

the same overall dimensions. Clearly λ_2/λ_c is seen to approach infinity as the local buckling load approaches the global one. This implies that the imperfection sensitivity becomes infinite under such conditions. It must be borne in mind, however, that this result applies only for infinitesimally small imperfections. Actual imperfections are finite, and so is the drop in load carrying capacity they produce. Hence the limits of applicability of the asymptotic results must become vanishingly small as the local modes approach the global. Under such circumstances, the interaction between local and global modes should be explicitly included in the analysis, before any meaningful conclusions in regard to imperfection sensitivity can be drawn.

Similar results are observed in Fig. 4 for a 2 bay space truss of the type shown in Fig. 2a with an equilateral triangular section of sidelength $L_{(0)}$ and an overall length $10L_{(0)}$, and

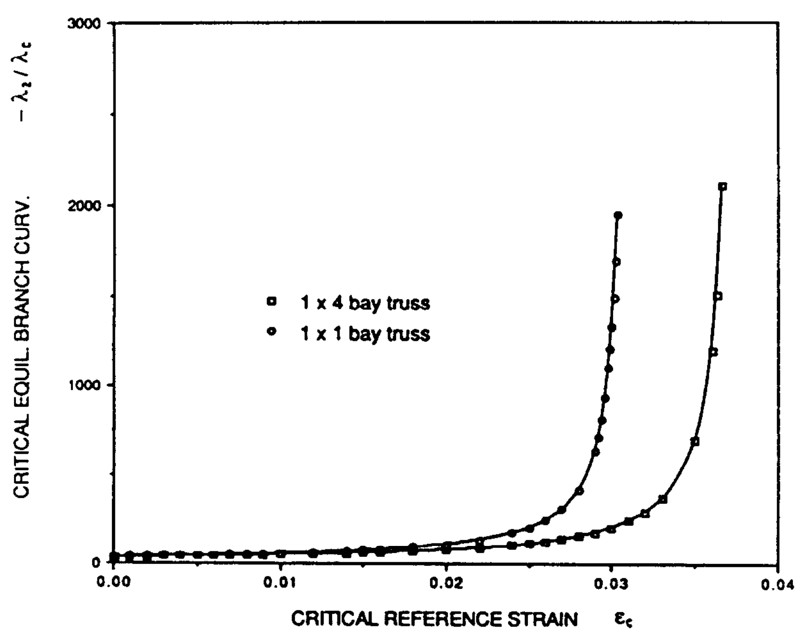


Fig. 4. Results for triangular section antennae of overall dimensions $L_{(0)} \times 10L_{(0)}$.

a 1 bay space truss of the same overall dimensions. Global buckling for these structures occurs at $\varepsilon_{cg} = 0.0165$ and $\varepsilon_{cg} = 0.0148$. Again, for the same overall dimensions, the structure with a larger number of bays has the higher global buckling load. For essentially incompressible member behavior on the other hand λ_2/λ_c seems not to depend on the number of bays that a given structure of fixed overall dimensions is divided into.

The members that buckle on the critical bifurcated branch of the structure are listed in the last column of Tables 1 and 2 as the "mode". In most cases a given mode is seen to persist for a range of values of ε_c. However, changes in the mode do occur: there seems to be a tendency towards buckling of a smaller number of members as ε_c increases towards ε_{cg}, but changes in mode that are not accompanied by changes in the number of members that buckle also occur, for example for the hexagonal truss of radius L.

6. CONCLUSIONS

A methodology has been established based on the Lyapunov–Schmidt–Koiter decomposition and asymptotic expansion, by which the worst imperfection shape and the corresponding load drop for any truss-type structure with multiple eigenmodes that involve buckling of individual members can be determined. This problem reduces to a quadratic programming problem, in which the number of unknowns is equal to the number of interacting modes. Although the computational effort for finding the global minimum with 100% certainty grows exponentially with the number of modes, it appears from the examples considered that in most cases the global minimum can be found much more efficiently. The results are asymptotically exact for small magnitudes of the imperfection.

The method is effective for essentially incompressible member behavior [the case considered by Britvec (1973) and Britvec and Davister (1985)], as well as for compressible members. The imperfection sensitivity increases with increasing member compressibility (stubbier members), and becomes infinite as the local member buckling load approaches the bifurcation point for global buckling. Under such conditions, the range of validity of the current analysis is expected to become vanishingly small; an analysis in which interaction of local and global modes is explicitly considered should be used.

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APPENDIX A: GENERAL RESULTS FOR A MULTIPLE SYMMETRIC BIFURCATION

This appendix summarizes general results from the Lyapunov–Schmidt–Koiter asymptotic postbuckling theory that are used in this paper. These results apply when: (1) The structure is elastic with a sufficiently smooth potential energy function $\phi = \phi(u, \lambda, \bar{u})$, where $u \in A$ is the displacement field, λ is the load parameter, and $\bar{u} \in \bar{A}$ represents an imperfection. (2) The sets of admissible displacement and imperfection fields, A and $\bar{A}$ respectively, are vector spaces. (3) A smooth principal solution $\overset{0}{u}(\lambda)$ exists which vanishes at $\lambda = 0$ and satisfies the equilibrium condition, $\phi_{,u}(\overset{0}{u}(\lambda), \lambda, 0)\delta u = 0$, for all $\delta u \in A$, and $\lambda \in [0, \lambda_c^+)$, where $\lambda_c^+ > \lambda_c$. (4) Equilibrium states on the principal branch must be stable for $\lambda \in [0, \lambda_c)$, in the sense that the bilinear operator, $\phi_{,uu}(\overset{0}{u}(\lambda), \lambda, 0)$, is positive definite. (5) At criticality ($\lambda = \lambda_c$) this stability operator has a finite dimensional nullspace, A_0. (6) The basis vectors for this nullspace are buckling modes denoted by $\overset{(i)}{u}$, and satisfy

$$\phi^c_{,uu}\overset{(i)}{u}\delta u = 0 \;\forall\, \delta u \in A, \quad \phi^c_{,u\lambda}\overset{(i)}{u} = 0, \tag{A1}$$

where the superscript c on the potential energy ϕ and its Gateaux derivatives denotes evaluation at criticality. (7) The buckling modes are orthonormalized so that

$$\left[\frac{\mathrm{d}}{\mathrm{d}\lambda}\,\phi_{,uu}(\overset{0}{u}(\lambda), \lambda, 0)\overset{(i)}{u}\overset{(j)}{u}\right]_{\lambda=\lambda_c} = -\phi_0\delta_{ij}, \tag{A2}$$

where ϕ_0 is any positive constant, and can be chosen to achieve a convenient normalization.

The bifurcation is said to be symmetric if

$$\phi^c_{,uuu}\overset{(i)}{u}\overset{(j)}{u}\overset{(k)}{u} = 0$$

for all combinations of modes, i, j, k. In this case the solution for the bifurcated equilibrium branches of the perfect structure is of the form

$$\lambda(\xi) = \lambda_c + \tfrac{1}{2}\lambda_2\xi^2 + O(\xi^3), \quad u(\xi) = \overset{0}{u}(\lambda(\xi)) + \xi\sum_i \alpha_i\overset{(i)}{u} + \tfrac{1}{2}\xi^2\sum_{i,j}\alpha_i\alpha_j\overset{(ij)}{u} + O(\xi^3), \tag{A3}$$

where ξ is a path parameter which is approximately equal to the distance of the solution to the principal branch, $\overset{(ij)}{u}$ are second order displacements to be defined in eqn (A5), and the parameters α_i and λ_2 are solutions to

$$-\frac{1}{2}\lambda_2\alpha_i + \frac{1}{6}\sum_{j,k,l}\phi_{ijkl}\alpha_j\alpha_k\alpha_l = 0, \quad \sum_i(\alpha_i)^2 = 1, \tag{A4}$$

where the postbuckling coefficients ϕ_{ijkl} are determined as follows: (1) Decompose the space of admissible displacements A into the space A_0 which is spanned by the buckling modes $\overset{(i)}{u}$, and a complementary space $\hat{A}$, defined so that A_0 and $\hat{A}$ span the space A, but the zero vector is the only element that is common to both A_0 and $\hat{A}$. (2) Find second order displacements, $\overset{(ij)}{u} \in \hat{A}$ such that

$$(\phi^c_{,uuu}\overset{(i)}{u}\overset{(j)}{u} + \phi^c_{,uu}\overset{(ij)}{u})\delta\hat{u} = 0, \quad \forall\, \delta\hat{u} \in \hat{A}. \tag{A5}$$

Finally, (3) the postbuckling coefficients can be calculated from

$$\phi_{ijkl} = \{\phi^c_{,uuuu}\overset{(i)}{u}\overset{(j)}{u}\overset{(k)}{u}\overset{(l)}{u} + \phi^c_{,uuu}\overset{(i)}{u}(\overset{(j)}{u}\overset{(kl)}{u} + \overset{(k)}{u}\overset{(jl)}{u} + \overset{(l)}{u}\overset{(jk)}{u})\}/\phi_0 = \{\phi^c_{,uuuu}\overset{(i)}{u}\overset{(j)}{u}\overset{(k)}{u}\overset{(l)}{u} - \phi^c_{,uu}(\overset{(ij)}{u}\overset{(kl)}{u} + \overset{(ik)}{u}\overset{(jl)}{u} + \overset{(il)}{u}\overset{(jk)}{u})\}/\phi_0. \tag{A6a,b}$$

The bifurcated equilibrium branch defined by eqn (A3) is stable in the vicinity of the bifurcation point, if the incremental stability matrix

$$B_{ij} = -\lambda_2\delta_{ij} + \sum_{k,l}\phi_{ijkl}\alpha_k\alpha_l \tag{A7}$$

is positive definite.

For most problems the potential energy ϕ can be written as a sum of two terms: the first depends on the displacements u only, and represents the strain energy of the structure; the other term represents the potential energy of the loads, and can be written as $-\lambda\Delta(u)$, where $\Delta(.)$ is a linear function. If the structure is loaded by a single point load of magnitude λ and constant direction, then $\Delta = \Delta(u)$ represents the deflection under the load. Budiansky (1974) refers to Δ as the generalized load shortening. Using his approach to evaluate this quantity on any of the bifurcated branches for the perfect structure yields:

$$\Delta = \Delta(u(\xi)) = \Delta_0 + \tfrac{1}{2}\phi_0\xi^2 + O(\xi^3), \tag{A8}$$

where Δ_0 is the generalized load shortening evaluated on the principal equilibrium branch at $\lambda = \lambda(\xi)$.

Next consider an imperfection $\bar{u} = \varepsilon\bar{\bar{u}}$, where ε is the scalar imperfection amplitude, and $\bar{\bar{u}}$ is the shape of imperfection, normalized so that

$$\sum_i (\bar{\bar{\xi}}_i) = 1, \quad \text{where} \quad \bar{\bar{\xi}}_i \equiv -\frac{1}{\lambda_c \phi_0} \phi_{,u\bar{u}} \bar{\bar{u}} \overset{(i)}{u}. \tag{A9a,b}$$

For small magnitudes of the imperfection ε, the behavior in the vicinity of the bifurcation point depends only on the projections $\bar{\bar{\xi}}_i$. The largest load drop (defined as $\lambda_c - \lambda_L$, where λ_L is the value of the load parameter at the first limit point for the imperfect structure) for a given (small) amplitude of imperfection occurs for $\bar{\bar{\xi}}_i = \alpha_i$, where α_i is the solution to eqns (A4) for which λ_2 is smallest. Assuming that this λ_2 is negative, the corresponding normalized load drop is

$$\left(\frac{\lambda_c - \lambda_L}{\lambda_c}\right) = \frac{3}{2}\left(\frac{-\lambda_2 \varepsilon^2}{\lambda_c}\right)^{1/3} + O(\varepsilon). \tag{A10}$$

So far only the worst shape of the projections $\bar{\bar{\xi}}_i$ have been determined. In general there is an infinite set of imperfection shapes $\bar{\bar{u}}$, all of which have the same projections $\bar{\bar{\xi}}_i$. In order to obtain a unique solution for the imperfection shape, it is necessary to define an imperfection norm, such that for any $\bar{\bar{\xi}}_i$ satisfying eqn (A9a): (1) all $\bar{\bar{u}}$ satisfying eqns (A9b) also satisfy the condition $\|\bar{\bar{u}}\| \geq 1$, and (2) there exists a unique $\bar{\bar{u}}$ which satisfies both eqns (A9b) and the condition $\|\bar{\bar{u}}\| = 1$. This last $\bar{\bar{u}}$ is the worst imperfection shape if $\bar{\bar{\xi}}_i$ are worst values for the projections of the imperfection shape.

APPENDIX B: POSTBUCKLING ANALYSIS OF A COMPRESSIBLE COLUMN

Surprisingly, an exact initial postbuckling analysis for an elastic compressible column appears not to be available. The closest that could be found is the analysis by Britvec (1973), in which an unnecessary assumption regarding the distribution of slope along the column is made. The reason for this may be that a substantial amount of algebra is involved. Furthermore, the results merely confirm that for slender columns compressibility effects are not important in the postbuckling range. The assumption that the column is compressible in the prebuckling range, but freezes axially once the critical load is reached (Kondoh and Atluri, 1985) is well justified. Indeed, for smaller values of the slenderness ratio, when compressibility effects might play some role, shear deformation will also become important, and perhaps more so. Nevertheless, for the sake of completeness, a brief summary of the formulation and final results from the postbuckling analysis of a compressible column with specified end displacements (but no restraint against end rotations) by the LSK technique is given here.

Planar deformation in the X–Z plane is considered, where the X-axis always passes through the endpoints of the member where the loads are applied. The initial and deformed geometry of the centroidal axis of the column are written as

$$\mathbf{R}(X) = X\mathbf{e}_X + \bar{W}(X)\mathbf{e}_Z, \quad \mathbf{r}(X) = \{X(1+e/L) + U(X)\}\mathbf{e}_X + \{\bar{W}(X) + W(X)\}\mathbf{e}_Z \tag{B1}$$

respectively, where $\mathbf{e}_X$, $\mathbf{e}_Y$, and $\mathbf{e}_Z$ are the unit vectors in the X, Y and Z directions. Member quantities defined in the main body of the paper are not redefined here, and the subscripts in parentheses identifying the member to which they pertain are omitted. Thus, for example, e denotes the specified member elongation. The set of admissible displacements A includes all displacement functions (U, W) that vanish at the endpoints, $X = O, L$. For the moment, the imperfection $\bar{W}$ is also assumed to vanish at the endpoints. The measures of axial and bending deformation, ε and κ, are taken to be

$$\varepsilon = \frac{\|\mathbf{r}'\| - \|\mathbf{R}'\|}{\|\mathbf{R}'\|}, \quad \kappa = \frac{\theta' - \bar{\theta}'}{\|\mathbf{R}'\|}, \quad \bar{\theta}' = \frac{(\mathbf{R}' \times \mathbf{e}_Y)\cdot\mathbf{R}''}{\|\mathbf{R}'\|^2}, \quad \theta' = \frac{(\mathbf{r}' \times \mathbf{e}_Y)\cdot\mathbf{r}''}{\|\mathbf{r}'\|^2}, \tag{B2}$$

where a prime denotes differentiation with respect to X; and $\bar{\theta} = \bar{\theta}(X)$ and $\theta = \theta(X)$ denote the rotation of the longitudinal axis of the column before and after deformation respectively. (The rotation produced by the loads is $\theta - \bar{\theta}$.)

The potential energy of the column is given by

$$\phi[U, W, e, \bar{W}] = \int_0^L \left(\frac{EA}{2}\varepsilon^2 + \frac{EI}{2}\kappa^2\right)\|\mathbf{R}'\|\,dX, \tag{B3}$$

where $EI = s^2L^2EA$ is the flexural rigidity of the column. The arguments of this potential energy functional are enclosed in square brackets, to distinguish the functional $\phi[\ldots]$ from the reduced potential energy function $\phi(\ldots)$ defined later, for which the arguments are enclosed in parentheses instead of brackets. Stationarity of the potential energy with respect to U and W leads to

$$\frac{1}{\|\mathbf{r}'\|}(EI\kappa)'\mathbf{n} - EA\varepsilon\mathbf{t} = \text{constant}, \quad \mathbf{t} = \frac{1}{\|\mathbf{r}'\|}\mathbf{r}', \quad \mathbf{n} = \mathbf{t} \times \mathbf{e}_Y, \tag{B4}$$

which coincides with the equation that would be obtained from equilibrium considerations, if the axial force and bending moment at any point in the column are taken to be $EA\varepsilon$ and $EI\kappa$, respectively. This illustrates the advantage of using the deformation measures, ε and κ, defined in eqn (B2). Other deformation measures would produce much the same final results as long as strains in the column are small, but they would also lead to different (and potentially more complicated) equilibrium and/or constitutive equations.

The principal solution is $U = W = 0$.

The first bifurcation occurs at $e = -e_c = -\varepsilon_c L$, where the buckling strain, ε_c, is given by

$$\varepsilon_c = \tfrac{1}{2}(1-\sqrt{1-4\pi^2 s^2}). \tag{B5}$$

This exact expression is within a factor of $(1+O(\varepsilon_c))$ of the approximation $\varepsilon_c = \pi^2 s^2$ used in Sections 2 and 3. The buckling mode is

$$\overset{(1)}{U}(X) = 0, \quad \overset{(1)}{W}(X) = \sin(\pi X/L). \tag{B6}$$

In following the LSK technique the set of admissible displacements A is decomposed as the linear span of the eigenspace A_0 and a complementary space $\hat{A}$ as follows:

$$A_0 = \{(U, W): U = 0, W(X) = w\sin(\pi X/L)\}, \quad \hat{A} = \left\{(U, W): \int_0^L W(X)\sin(\pi X/L)\,dX = 0\right\}, \tag{B7a,b}$$

where w is the scalar out-of-straight deformation of the member. Using this decomposition, the reduced potential energy can be defined as

$$\phi(e, w, \bar{W}) = \min_{(\hat{U},\hat{W})\in\hat{A}} \phi[\hat{U}, w\overset{(1)}{W} + \hat{W}, e, \bar{W}]. \tag{B8}$$

Although the authors are not aware of any closed form solution to this minimization problem, it is possible to resort again to an asymptotic expansion to obtain the first few terms in a series representation of the reduced potential energy, or equivalently, to obtain the derivatives of the reduced potential energy evaluated at the bifurcation point. After a substantial amount of algebra (some of which was done with the aid of a computer and the MACSYMA program) the results obtained for the derivatives of the reduced potential energy evaluated at $w = \bar{W} = 0$ are those given in eqns (3) and (4). These results are exact. Additional derivatives of the reduced potential energy are given in eqns (5) and (6). Where these results contain a factor $1+O(\varepsilon_c^2)$, the corresponding exact results are:

$$\phi^c_{,eww} = \frac{\pi^2 EA}{2L^2}\frac{1-2\varepsilon_c}{(1-\varepsilon_c)^2}, \quad \phi^c_{,wwww} = \frac{\pi^4 EA}{8L^3}\frac{6-21\varepsilon_c+12\varepsilon_c^2}{(1-\varepsilon_c)^4}, \quad \phi^c_{,w\bar{W}}\bar{W} = -\frac{\pi^2 EA\varepsilon_c}{L^2}\int_0^L \bar{W}(X)\sin(\pi X/L). \tag{B9a–c}$$

The approximations in eqns (5) and (6) are within a factor of $(1+O(\varepsilon_c^2))$ of these exact values. Further approximation in eqn (6) would be possible while maintaining accuracy to within a factor of $(1+O(\varepsilon_c))$. However, in this case, the results for the postbuckling behavior of the column would reduce to those for the moderate deformation theory, which predicts that axial load remains constant with increasing displacements after buckling occurs.

Finally, some discussion with regard to the assumption that the imperfection $\bar{W}$ vanishes at the endpoints is in order: since the formulation is based on arbitrarily large imperfections, it remains valid for an imperfection that contains rapid changes in $\bar{W}$ in the vicinity of the endpoints. Thus end eccentricities can also be represented, and eqn (43) is still valid for calculating the equivalent sinusoidal imperfection. Note however that this analysis applies for link elements (i.e. the elements with length equal to the eccentricities that provide the connection between the column end points and the point of load application) consisting of a short segment of the column. If these links are to be replaced by rigid links, an alternative formulation is required, since this introduces a nonlinear essential boundary condition, as a result of which the set of admissible displacements of the centroid of the cross-section no longer form a vector field. Such an alternative formulation leads to the same result as eqns (B9) and (43) to within a factor of $(1+O(\varepsilon_c))$.

APPENDIX C: DEFINITION AND NOTATION FOR GATEAUX DERIVATIVES

Let $F: (U\times V) \to W$ be a mapping, where U, V and W are vector spaces. Assuming suitable smoothness,† the limit

$$F_{,u}\delta u \equiv \lim_{\varepsilon\to 0}\frac{F(u+\varepsilon\,\delta u, v)-F(u,v)}{\varepsilon} = \left[\frac{d}{d\varepsilon}F(u+\varepsilon\,\delta u, v)\right]_{\varepsilon=0} \tag{C1}$$

exists, and is linear in δu. Thus the Gateaux derivative $F_{,u}$ is a linear operator. For fixed δu, another mapping $(F_{,u}\delta u): (U\times V)\to W$ can then be considered and the above definition can be reapplied to define higher Gateaux derivatives. For example,

$$F_{,uv}\delta v\,\delta u \equiv (F_{,u}\delta u)_{,v}\delta v = \left[\frac{\partial^2}{\partial\eta\,\partial\varepsilon}F(u+\varepsilon\,\delta u, v+\eta\,\delta v)\right]_{\varepsilon=\eta=0},$$

$$F_{,uu}\delta_2 u\,\delta_1 u \equiv (F_{,u}\delta_1 u)_{,u}\delta_2 u = \left[\frac{\partial^2}{\partial\eta\,\partial\varepsilon}F(u+\varepsilon\,\delta_1 u+\eta\,\delta_2 u, v)\right]_{\varepsilon=\eta=0}. \tag{C2}$$

Note that the nth Gateaux derivative is an n-linear operator. All n arguments always follow the Gateaux derivative, except that any scalar arguments (corresponding to the case when U and/or V is the set of real numbers) are

† Norms are required on the spaces U, V, W to define the notions of smoothness and existence of the limit.

omitted from the list. These may be included as scalar multipliers at any other location, since in this case the Gateaux derivative reduces to the usual partial derivative. The order of the subscripts is always opposite to that of the corresponding arguments of the n-linear operator.

Although no norms appear explicitly in this definition, addressing existence of the Gateaux derivative does require norms. Frechet derivatives could also have been used throughout this paper. However their definition does require the explicit use of norms, and is less suggestive of their meaning or method of evaluation. That is the only reason why the Gateaux derivative is preferred here.

APPENDIX D: PROOF OF LOCAL STABILITY

Suppose the inequality constraint in eqn (40c) is replaced by $x_i \geq b_i$, and that these inequality constraints are active for all $i \in M_0$. If b_i is increased for some $i \in M_0$ then the inequality constraint becomes more restrictive and the minimizing λ_2 should increase. Thus a necessary condition for optimality is $\partial\lambda_2/\partial b_i \geq 0$. This derivative is given by $\partial\lambda_2/\partial b_i = -\mu_i$, where μ_i $(i \in M_0)$ is a Lagrange multiplier, which is determined by requiring that the Lagrangian,

$$L = \sum_{i \in M} \sum_{j \in M} a_{ij} x_i x_j + \mu\left(\sum_{i \in M} x_i - 1\right) + \sum_{i \in M_0} \mu_i (x_i - b_i), \tag{D1}$$

be stationary with respect to x_i $(i \in M)$, and the Lagrange multipliers μ_i $(i \in M_0)$, and μ. Stationarity with respect to x_i $(i \in M_1)$ recovers eqn (35a), and leads to $\mu = -2\lambda_2$. With this result, stationarity of the Lagrangian with respect to x_i $(i \in M_0)$ produces,

$$2\left(\sum_{j \in M_1} a_{ij} x_j - \lambda_2\right) = -\mu_i = \frac{\partial \lambda_2}{\partial b_i} \geq 0 \quad \forall\, i \in M_0. \tag{D2}$$

Thus the local stability condition, eqn (39), holds for the postbuckling branch which minimizes λ_2, as required.