Maximum-entropy-type Lyapunov functions for robust stability and performance analysis*

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Abstract: We present two Lyapunov functions that ensure the unconditional stability and robust performance of a modal system with uncertain damped natural frequency. Each Lyapunov function involves the sum of two matrices, the first being the solution to the so-called maximum-entropy equation and the second being a constant auxiliary portion. The significant feature of these Lyapunov functions is that the guaranteed robust stability region is independent of the weighting matrix, while the performance bounds are relatively tight compared to alternative approaches. Thus, these Lyapunov functions are less conservative than standard bounds that tend to be highly sensitive to the choice of state space basis.

Keywords: Maximum-entropy function; robust stability; robust performance

1. Introduction

The maximum-entropy approach to robust control was specifically developed to address the problem of modal uncertainty in flexible structures [2, 5, 6, 18, 19]. The rationale for this approach was based upon insights from the statistical analysis of lightly damped structures [20]. Despite favorable comparisons to other approaches [9, 10, 12, 13] and experimental application [11], the basis and meaning of the approach remain mostly empirical and largely obscure. The purpose of this paper is to make significant progress in developing a rigorous foundation for this approach.

Besides the statistical modal analysis techniques of [20], a variety of formulations have been put forth for justifying the maximum-entropy approach. To reproduce certain covariance phenomena of uncertain

multimodal systems (decorrelation, incoherence, and equipartition; see [20]), a multiplicative white-noise model was invoked [18, 19]. The specific model chosen was interpreted in the sense of Stratonovich, thus entailing a critical correction term in the covariance equation due to the conversion from Stratonovich to Ito calculus. The Stratonovich model was itself based upon a limiting process in which the parameter entropy increased, thus suggesting the name "maximum-entropy" control. White-noise models as a basis for robust control are discussed in [1].

An alternative justification for the maximum-entropy model was given in [14] in terms of positive real transfer functions. This attempt was motivated by the observation that in the limit of high modal frequency uncertainty the maximum-entropy controller assumed a rate dissipative structure [18, 19]. An alternative attempt to justify the maximum-entropy model was given in [17], where a covariance averaging approach [16] was used to show that if the state covariance is averaged over uncertain modal frequencies possessing a Cauchy distribution, then the resulting averaged covariance satisfies the maximum-entropy covariance model.

Although the various formulations of maximum-entropy theory lend considerable insight into the nature of the approach, there remains a significant gap between this approach and more conventional techniques, such as H_∞ theory. The missing link, in our opinion, is the lack of a Lyapunov function that guarantees the robust stability of the closed-loop control system. In this regard it was long suspected that such a Lyapunov function would be unconventional, that is, unlike those arising in H_∞ theory. This view arose from the fact that the maximum-entropy controllers were often robust to large perturbations in the damped natural frequencies, that is, the imaginary part of the eigenvalues. Such perturbations are highly structured, and thus are often treated conservatively by conventional small-gain-type bounds.

The goal of the present paper is to provide a Lyapunov function basis for the maximum-entropy covariance model for the case of modal frequency uncertainty. In fact, in this special case, we provide two alternative Lyapunov functions along with the corresponding performance bounds. Each Lyapunov function involves the sum of two matrices, the first being the solution to the maximum-entropy equation (see equation (22)) and the second being a constant auxiliary portion. This construction is similar to the parameter-dependent Lyapunov function technique developed in [15] except that in the present paper the auxiliary portion is constant, that is, independent of the uncertainty.

The maximum-entropy equation (22) differs fundamentally from alternative robustness tests such as those given in [3, 4]. Specifically, whereas the modified Lyapunov functions in [3] involve additional nonnegative-definite terms in the Lyapunov equation, the maximum-entropy equation entails an *indefinite* modification. This distinction appears to play a critical role with respect to the way in which the maximum-entropy equation deals with the change in basis induced by the input and weighting matrices.

While this paper potentially provides a Lyapunov function foundation for the maximum-entropy control approach, our results are limited to open-loop analysis. Future research will focus on robust stability of the closed-loop system for the controllers given in [2, 5, 6, 9–13, 18–20]. Furthermore, although the techniques used to construct the Lyapunov functions for the maximum-entropy equation are limited to modal frequency uncertainty, they appear to be generalizable to larger classes of uncertainty. Nevertheless, for structures with modal frequency uncertainty [2, 5, 6, 9–13, 18, 19], these results have practical ramifications.

2. Robust stability and performance problems

Let $\mathscr{U} \subset \mathbb{R}^{n \times n}$ denote a set of perturbations ΔA of a given nominal dynamics matrix $A \in \mathbb{R}^{n \times n}$. It is assumed that A is asymptotically stable and that $0 \in \mathscr{U}$.

Robust stability problem. Determine whether the linear system

$$\dot{x}(t) = (A + \Delta A) x(t), \quad t \in [0, \infty),$$ (1)

is asymptotically stable for all $\Delta A \in \mathscr{U}$.

Robust performance problem. For the disturbed linear system

$$\dot{x}(t) = (A + \Delta A)x(t) + Dw(t), \quad t \in [0, \infty), \tag{2}$$

$$z(t) = Ex(t), \tag{3}$$

where $w(\cdot)$ is a zero-mean d-dimensional white-noise signal with intensity I_d, determine a performance bound β satisfying

$$\mathcal{T}(\mathcal{U}) \triangleq \sup_{\Delta A \in \mathcal{U}} \limsup_{t \to \infty} E\{\|z(t)\|_2^2\} \leq \beta. \tag{4}$$

For convenience, define the $n \times n$ nonnegative-definite matrices $R \triangleq E^\mathrm{T}E$ and $V \triangleq DD^\mathrm{T}$. The following result is immediate. For a proof, see [3].

Lemma 2.1. *Suppose $A + \Delta A$ is asymptotically stable for all $\Delta A \in \mathcal{U}$. Then*

$$\mathcal{T}(\mathcal{U}) = \sup_{\Delta A \in \mathcal{U}} \mathrm{tr}(Q_{\Delta A}R) = \sup_{\Delta A \in \mathcal{U}} \mathrm{tr}(P_{\Delta A}V), \tag{5}$$

where $Q_{\Delta A} \in \mathbb{R}^{n \times n}$ and $P_{\Delta A} \in \mathbb{R}^{n \times n}$ are the unique, nonnegative-definite solutions to

$$0 = (A + \Delta A)Q_{\Delta A} + Q_{\Delta A}(A + \Delta A)^\mathrm{T} + V \tag{6}$$

and

$$0 = (A + \Delta A)^\mathrm{T}P_{\Delta A} + P_{\Delta A}(A + \Delta A) + R. \tag{7}$$

Conditions for robust stability and robust performance are developed in the following theorem. Let $\mathcal{N}^n$ and $\mathcal{S}^n$ denote the sets of $n \times n$ nonnegative-definite and symmetric matrices, respectively.

Theorem 2.2. *Let $\Omega_0 : \mathcal{N}^n \to \mathcal{S}^n$, and suppose there exists $P \in \mathcal{N}^n$ satisfying*

$$0 = A^\mathrm{T}P + PA + \Omega_0(P) + R. \tag{8}$$

Furthermore, let $P_0 : \mathcal{U} \to \mathcal{S}^n$ and $R_0 \in \mathcal{S}^n$ be such that $R_0 \leq R$,

$$\Delta A^\mathrm{T}P + P\Delta A \leq \Omega(P, \Delta A) + R_0, \quad \Delta A \in \mathcal{U}, \tag{9}$$

and

$$P + P_0(\Delta A) \geq 0, \quad \Delta A \in \mathcal{U}, \tag{10}$$

where

$$\Omega(P, \Delta A) \triangleq \Omega_0(P) - [(A + \Delta A)^\mathrm{T}P_0(\Delta A) + P_0(\Delta A)(A + \Delta A)]. \tag{11}$$

Then

$$(R - R_0, A + \Delta A), \quad \Delta A \in \mathcal{U}, \tag{12}$$

is detectable if and only if

$$A + \Delta A, \quad \Delta A \in \mathcal{U}, \tag{13}$$

is asymptotically stable. In this case, the following statements are true. If $\gamma < 1$ is such that $R_0 \leq \gamma R$, then

$$P_{\Delta A} \leq \frac{1}{1 - \gamma}(P + P_0(\Delta A)), \quad \Delta A \in \mathcal{U}, \tag{14}$$

where $P_{\Delta A}$ satisfies (7), and

$$\mathcal{T}(\mathcal{U}) \leq \frac{1}{1 - \gamma}[\mathrm{tr}(PV) + \sup_{\Delta A \in \mathcal{U}} \mathrm{tr}(P_0(\Delta A)V)]. \tag{15}$$

In addition, if there exists $\bar{P}_0 \in \mathscr{S}^n$ such that

$$P_0(\varDelta A) \leq \bar{P}_0, \tag{16}$$

then

$$\mathscr{T}(\mathscr{U}) \leq \frac{1}{1-\gamma} \operatorname{tr}[(P + \bar{P}_0)V]. \tag{17}$$

Proof. Note that, for all $\varDelta A \in \mathscr{U}$, (8) is equivalent to

$$
\begin{aligned}
0 &= (A + \varDelta A)^{\mathsf{T}}(P + P_0(\varDelta A)) + (P + P_0(\varDelta A))(A + \varDelta A) + \Omega_0(P) + R \\
&\quad - [(A + \varDelta A)^{\mathsf{T}} P_0(\varDelta A) + P_0(\varDelta A)(A + \varDelta A)] - (\varDelta A^{\mathsf{T}} P + P \varDelta A) \\
&= (A + \varDelta A)^{\mathsf{T}}(P + P_0(\varDelta A)) + (P + P_0(\varDelta A))(A + \varDelta A) + R - R_0 + R'_0,
\end{aligned} \tag{18}
$$

where

$$
\begin{aligned}
R'_0 &\triangleq \Omega_0(P) + R_0 - [(A + \varDelta A)^{\mathsf{T}} P_0(\varDelta A) + P_0(\varDelta A)(A + \varDelta A)] - (\varDelta A^{\mathsf{T}} P + P \varDelta A) \\
&= \Omega(P, \varDelta A) + R_0 - (\varDelta A^{\mathsf{T}} P + P \varDelta A).
\end{aligned}
$$

Hence, (18) has a solution $P \in \mathscr{N}^n$ for all $\varDelta A \in \mathscr{U}$. Thus, if the detectability condition (12) holds for all $\varDelta A \in \mathscr{U}$, then it follows from [21, Theorem 3.6] that $(R - R_0 + R'_0, A + \varDelta A)$ is detectable, $\varDelta A \in \mathscr{U}$. It now follows from (18) and [21, Lemma 12.2] that $A + \varDelta A$ is asymptotically stable, $\varDelta A \in \mathscr{U}$. Conversely, if $A + \varDelta A$ is asymptotically stable for all $\varDelta A \in \mathscr{U}$, then (12) is immediate.

Now, subtracting $(1 - \gamma) \cdot (7)$ from (18) yields

$$
\begin{aligned}
0 &= (A + \varDelta A)^{\mathsf{T}}(P + P_0(\varDelta A) - (1 - \gamma)P_{\varDelta A}) + (P + P_0(\varDelta A) - (1 - \gamma)P_{\varDelta A})(A + \varDelta A) \\
&\quad + R'_0 - R_0 + \gamma R, \quad \varDelta A \in \mathscr{U},
\end{aligned} \tag{19}
$$

or, since $A + \varDelta A$ is asymptotically stable for all $\varDelta A \in \mathscr{U}$ and $R_0 \leq \gamma R$, (19) implies that, for all $\varDelta A \in \mathscr{U}$,

$$
\begin{aligned}
P + P_0(\varDelta A) - (1 - \gamma)P_{\varDelta A} &= \int_0^\infty \mathrm{e}^{(A + \varDelta A)^{\mathsf{T}} t}[R'_0 + \gamma R - R_0]\mathrm{e}^{(A + \varDelta A)t}\,\mathrm{d}t \\
&\geq \int_0^\infty \mathrm{e}^{(A + \varDelta A)^{\mathsf{T}} t} R'_0\,\mathrm{e}^{(A + \varDelta A)t}\,\mathrm{d}t \\
&\geq 0,
\end{aligned}
$$

which implies (14).

Next, using (14), it follows from (5) that

$$
\begin{aligned}
\mathscr{T}(\mathscr{U}) &= \sup_{\varDelta A \in \mathscr{U}} \operatorname{tr}(D^{\mathsf{T}} P_{\varDelta A} D) \leq \frac{1}{1 - \gamma} \sup_{\varDelta A \in \mathscr{U}} \operatorname{tr}[D^{\mathsf{T}}(P + P_0(\varDelta A))D] \\
&= \frac{1}{1 - \gamma}\left[\operatorname{tr}(PV) + \sup_{\varDelta A \in \mathscr{U}} \operatorname{tr}(P_0(\varDelta A)V) \right],
\end{aligned}
$$

which yields (15). Furthermore, using (16) it follows that

$$
\begin{aligned}
\mathscr{T}(\mathscr{U}) &\leq \frac{1}{1 - \gamma}\left[\operatorname{tr}(PV) + \sup_{\varDelta A \in \mathscr{U}} \operatorname{tr}(P_0(\varDelta A)V) \right] \leq \frac{1}{1 - \gamma}[\operatorname{tr}(PV) + \operatorname{tr}(\bar{P}_0 V)] \\
&= \frac{1}{1 - \gamma}\operatorname{tr}[(P + \bar{P}_0)V]. \qquad \square
\end{aligned}
$$

Remark 2.3. Theorem 2.2 is a generalization of Theorem 3.1 of [15]. Specifically, the bound in [15] is required to hold for all nonnegative-definite matrices, whereas in Theorem 2.2 equation (9) need only hold for the solution P of (8). Furthermore, in [15], $R_0 = 0$.

Remark 2.4. Inequality (9) is equivalent to

$$(A + \Delta A)^\mathrm{T}(P + P_0(\Delta A)) + (P + P_0(\Delta A))(A + \Delta A) + R - R_0 \le 0,$$

which shows that $V(x) = x^\mathrm{T}(P + P_0(\Delta A))x$ is a Lyapunov function corresponding to $A + \Delta A$. In constructing this Lyapunov function, the matrix P can be viewed as a *predictor* term, $P_0(\Delta A)$ provides a *corrector* term, and $P_\mathrm{T} \triangleq P + P_0(\Delta A)$ is the *total* Lyapunov matrix.

Remark 2.5. If $P_0(\Delta A)$ is independent of ΔA, then by choosing $\bar{P}_0 = P_0(\Delta A)$ it follows that (15) is identical to (17).

3. Application to the maximum-entropy covariance model

Now we specialize to the case in which $\mathscr{U}$ is given by

$$\mathscr{U} \triangleq \left\{ \Delta A \in \mathbb{R}^{n \times n}: \Delta A = \sum_{i=1}^r \sigma_i A_i, \ |\sigma_i| \le \delta_i, \ i = 1, \ldots, r \right\}, \tag{20}$$

where $\delta_i > 0$ and the matrices $A_i \in \mathbb{R}^{n \times n}$, which represent the uncertainty structure, are the given skew-symmetric matrices, that is, $A_i + A_i^\mathrm{T} = 0$, $i = 1, \ldots, r$. In addition, we assume that $A + A^\mathrm{T} < 0$. This formulation can be viewed as the representation of a dissipative system (such as a flexible structure) with energy-conserving perturbations. This property can be seen by means of the Lyapunov function $V(x) = x^\mathrm{T}x$ whose decay rate is independent of σ_i. Thus, $A + \Delta A$ is uniformly asymptotically stable even for arbitrarily time-varying $\sigma_i(t)$. For simplicity, however, we confine our analysis to constant parameter uncertainty. In addition, although the system is robustly stable for time-varying parameter uncertainties, the performance bounds we obtain via Theorem 2.2 are valid only for the case of constant parameter uncertainty.

We now introduce a specific choice of $\Omega_0(P)$ that is motivated by the maximum-entropy covariance model. Specifically, as in [18] we choose

$$\Omega_0(P) = \sum_{i=1}^r \delta_i^2(\tfrac{1}{2}A_i^{2\mathrm{T}}P + A_i^\mathrm{T}PA_i + \tfrac{1}{2}PA_i^2). \tag{21}$$

First we prove that with this choice of $\Omega_0(P)$ equation (8) has a unique solution. Then we show that, when $r = 1$, equation (8) has an asymptotic solution for $\delta_1 \to \infty$.

Proposition 3.1. *Assume that $A + A^\mathrm{T} < 0$, $A_i + A_i^\mathrm{T} = 0$, and $\delta_i \ge 0$, $i = 1, \ldots, r$. Then there exists a unique matrix $P \in \mathbb{R}^{n \times n}$ satisfying*

$$0 = A^\mathrm{T}P + PA + \sum_{i=1}^r \delta_i^2(\tfrac{1}{2}A_i^{2\mathrm{T}}P + A_i^\mathrm{T}PA_i + \tfrac{1}{2}PA_i^2) + R. \tag{22}$$

Furthermore, P is nonnegative-definite.

Proof. Applying the "vec" operator [7] to (22) yields

$$0 = \mathscr{A}^\mathrm{T} \operatorname{vec} P + \operatorname{vec} R, \tag{23}$$

where

$$\mathscr{A} \triangleq (A \oplus A) + \sum_{i=1}^r \tfrac{1}{2} \delta_i^2 (A_i \oplus A_i)^2$$

and $\oplus$ and later $\otimes$ denote Kronecker sum and product, respectively. Since $A + A^T < 0$, it follows that $(A \oplus A) + (A \oplus A)^T = (A + A^T) \oplus (A + A^T) < 0$. In addition, the assumption that A_i is skew-symmetric implies that $A_i \oplus A_i$ is also skew-symmetric and thus $(A_i \oplus A_i)^2 \le 0$, $i = 1, \ldots, r$. Thus, $\mathscr{A} + \mathscr{A}^T < 0$, which implies that $\mathscr{A}$ is asymptotically stable. Thus, (23) yields $P = \mathrm{vec}^{-1}(-\mathscr{A}^{-T} \mathrm{vec}\, R)$. This proves existence and uniqueness.

Next, we show that P is nonnegative-definite. Note that since $-\mathscr{A}^{-T} = \int_0^\infty e^{\mathscr{A}^T t}\, dt$, we can write

$$P = \mathrm{vec}^{-1}\left(\int_0^\infty e^{\mathscr{A}^T t} \mathrm{vec}\, R\, dt\right). \tag{24}$$

After some manipulation (24) can be written as

$$P = \mathrm{vec}^{-1}\left(\int_0^\infty \exp\left(t\left[\sum_{i=1}^r\left(\frac{A}{r} + \tfrac{1}{2}\delta_i^2 A_i^2\right)^T \oplus \sum_{i=1}^r\left(\frac{A}{r} + \tfrac{1}{2}\delta_i^2 A_i^2\right)^T + \sum_{i=1}^r \tfrac{1}{2}\delta_i^2 (A_i \otimes A_i)^T\right]\right)\mathrm{vec}\, R\, dt\right). \tag{25}$$

Now, using the exponential product formula it follows that

$$P = \mathrm{vec}^{-1}\left(\int_0^\infty \lim_{m\to\infty}\left[\exp\left(\frac{t}{n}\left[\sum_{i=1}^r\left(\frac{A^T}{r} + \tfrac{1}{2}\delta_i^2 A_i^{2T}\right) \oplus \sum_{i=1}^r\left(\frac{A^T}{r} + \tfrac{1}{2}\delta_i^2 A_i^{2T}\right)\right)\right]\right)\right.$$
$$\left. \times \prod_{i=1}^r \exp\left(\frac{\delta_i^2 t}{2m}(A_i^T \otimes A_i^T)\right)\right]^m \mathrm{vec}\, R\, dt\right). \tag{26}$$

For simplicity, we assume $r = 1$. If $r > 1$ only minor modifications are needed. First fix m and let $R_{(0)} \triangleq R$; define the series $Z_{(j)}, R_{(j)}, j = 0, 1, \ldots, m - 1$, by

$$\mathrm{vec}\, Z_{(j+1)}(t) \triangleq e^{(\delta_1^2 t/2m)(A_1 \otimes A_1)^T} \mathrm{vec}\, R_{(j)}(t) = \mathrm{vec}\sum_{k=0}^\infty \frac{1}{k!}\left(\frac{\delta_1^2 t}{2m}\right)^k A_1^{kT} R_{(j)}(t) A_1^k,$$

$$\mathrm{vec}\, R_{(j+1)}(t) \triangleq \exp\left(\frac{t}{m}\left(A + \frac{\delta_1^2}{2}A_1^2\right) \oplus \left(A + \frac{\delta_1^2}{2}A_1^2\right)^T\right)\mathrm{vec}\, Z_{(j+1)}(t)$$

$$= \mathrm{vec}\exp\left(\frac{t}{m}\left(A + \frac{\delta_1^2}{2}A_1^2\right)^T\right)Z_{(j+1)}(t)\exp\left(\frac{t}{m}\left(A + \frac{\delta_1^2}{2}A_1^2\right)\right).$$

It is obvious that both $Z_{(j)}(t)$ and $R_{(j)}(t)$ are nonnegative-definite matrices for all $j = 0, 1, \ldots, m - 1$ and $t \ge 0$. Finally, since m is arbitrary, it can be shown that

$$P = \mathrm{vec}^{-1}\left(\int_0^\infty \lim_{m\to\infty}\mathrm{vec}\, R_{(m)}\, dt\right) = \int_0^\infty \lim_{m\to\infty} R_{(m)}\, dt \ge 0. \qquad \square$$

Next we show that (22) with $r = 1$ has an asymptotic solution for $\delta_1 \to \infty$. First, we need the following definition and lemma.

Definition 3.2. For $F \in \mathbb{R}^{n \times n}$, the smallest nonnegative integer k such that rank $(F^k) = $ rank (F^{k+1}) is called the *index* of F and is denoted by $\mathrm{Ind}\,(F)$ [8].

Remark 3.3. If F is invertible, $\mathrm{Ind}\,(F) = 0$. Also $\mathrm{Ind}\,(0) = 1$. We adopt the convention that $0^0 = 1$ [8].

Definition 3.4. A matrix $F \in \mathbb{R}^{n \times n}$ is called EP [8] if either F is invertible or there exists an orthogonal matrix $U \in \mathbb{R}^{n \times n}$ and an invertible matrix $F_1 \in \mathbb{R}^{m \times m}$, where $m \le n$, such that

$$F = U\begin{bmatrix} F_1 & 0 \\ 0 & 0 \end{bmatrix}U^T.$$

Remark 3.5. If F is EP, then $\text{Ind}(F) \le 1$, and the group inverse $F^\#$ of F is given by [8]

$$F^\# = U \begin{bmatrix} F_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^{\mathrm{T}}.$$

Lemma 3.6. Let $A, B \in \mathbb{R}^{n \times n}$, where $A + A^{\mathrm{T}} < 0$ and B is an EP matrix. Then

$$\text{Ind}(AB) = \text{Ind}(B). \tag{27}$$

Proof. Since B is an EP matrix, Remark 3.5 implies that $\text{Ind}(B) \le 1$. Hence, we consider two cases.

(1) Suppose $\text{Ind}(B) = 0$, so that B is invertible. Since $A + A^{\mathrm{T}} < 0$, it follows that A is asymptotically stable and hence invertible. Therefore, AB is invertible and thus $\text{Ind}(AB) = 0$.

(2) Suppose $\text{Ind}(B) = 1$, and let $\text{rank}(B) = n - r$, where $r \ge 1$. Since B is an EP matrix, there exists an orthogonal matrix U and a matrix D_B such that $B = UD_BU^{\mathrm{T}}$, where

$$D_B = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_1 \in \mathbb{R}^{(n-r) \times (n-r)}, \ \det(B_1) \ne 0.$$

Since $\text{rank}(AB) = n - r$, it suffices to show that the zero eigenvalue of AB has multiplicity r.

By writing $U^{\mathrm{T}}AU$ in the form

$$A' \triangleq U^{\mathrm{T}}AU = \begin{bmatrix} A'_{11} & A'_{12} \\ A'_{21} & A'_{22} \end{bmatrix},$$

where $A'_{11} \in \mathbb{R}^{(n-r) \times (n-r)}$, $A'_{22} \in \mathbb{R}^{r \times r}$, $A'_{12} \in \mathbb{R}^{(n-r) \times r}$, $A'_{21} \in \mathbb{R}^{r \times (n-r)}$, we have

$$U^{\mathrm{T}}AUD_B = \begin{bmatrix} A'_{11}B_1 & 0 \\ A'_{21}B_1 & 0 \end{bmatrix}.$$

Consequently, the characteristic polynomial of AB is

$$\det(\lambda I - AB) = \det(\lambda I - U(U^{\mathrm{T}}AU D_B)U^{\mathrm{T}}) = \det(\lambda I - U^{\mathrm{T}}AU D_B)$$

$$= \det \begin{bmatrix} \lambda I_{n-r} - A'_{11}B_1 & 0 \\ -A'_{21}B_1 & \lambda I_r \end{bmatrix} = \lambda^r \det(\lambda I_{n-r} - A'_{11}B_1). \tag{28}$$

Equation (28) implies that the zero eigenvalue of AB has at least multiplicity r.

The final step is to show that $A'_{11}B_1$ has no zero eigenvalue or, equivalently, $\det(A'_{11}B_1) \ne 0$. Since $A + A^{\mathrm{T}} < 0$, it follows that $U^{\mathrm{T}}(A + A^{\mathrm{T}})U < 0$, that is, $A' + A'^{\mathrm{T}} < 0$. Thus, $A'_{11} + (A'_{11})^{\mathrm{T}} < 0$, which implies that A'_{11} is asymptotically stable. Therefore, we have $\det(A'_{11}) \ne 0$. Noting

$$\det(A'_{11}B_1) = \det(A'_{11})\det(B_1) \ne 0$$

completes the proof. $\square$

For convenience, we define

$$\Lambda \triangleq (A^{\mathrm{T}} \oplus A^{\mathrm{T}})^{-1}(A_1^{\mathrm{T}} \oplus A_1^{\mathrm{T}})^2. \tag{29}$$

Lemma 3.7. Let $A, A_1 \in \mathbb{R}^{n \times n}$, where $A + A^{\mathrm{T}} < 0$ and $A_1 + A_1^{\mathrm{T}} = 0$. Then $\text{Ind}(\Lambda) = 1$.

Proof. Since A_1 is skew-symmetric, it follows that $A_1 \oplus A_1$ is also skew-symmetric. Thus, $(A_1 \oplus A_1)^2$ is symmetric (actually, it is negative-semidefinite) and hence is EP. In addition, it is obvious that $A_1 \oplus A_1$ is singular. Thus, $\text{Ind}(A_1^{\mathrm{T}} \oplus A_1^{\mathrm{T}})^2 = 1$. Furthermore, since $A + A^{\mathrm{T}} < 0$ implies $(A \oplus A) + (A^{\mathrm{T}} \oplus A^{\mathrm{T}}) < 0$ and equivalently implies $(A \oplus A)^{-1} + (A^{\mathrm{T}} \oplus A^{\mathrm{T}})^{-1} < 0$, it follows from Lemma 3.6 that $\text{Ind}(\Lambda) = 1$. $\square$

We are now ready to prove the existence of an asymptotic solution of equation (8) when $r = 1$. For notational convenience, we replace $\delta_1^2/2$ by α.

Proposition 3.8. *Let* $A, A_1 \in \mathbb{R}^{n \times n}$, $R \in \mathcal{N}^n$ *and* $\alpha \geq 0$. *Furthermore, assume that* $A + A^T < 0$, $A_1 + A_1^T = 0$, *and let* $P_\alpha \in \mathcal{N}^n$ *be the unique, nonnegative-definite solution to*

$$0 = A^1 P + PA + \alpha(A_1^{2T} P + 2A_1^T PA_1 + PA_1^2) + R. \tag{30}$$

Then $P_\infty \triangleq \lim_{\alpha \to \infty} P_\alpha$ *exists and is given by*

$$P_\infty = \mathrm{vec}^{-1}[(I - AA^\#)(A^T \oplus A^T)^{-1}(-\mathrm{vec}\, R)]. \tag{31}$$

Proof. Applying the vec operator to equation (30) yields

$$0 = [(A^T \oplus A^T) + \alpha(A_1^T \oplus A_1^T)^2]\mathrm{vec}\, P + \mathrm{vec}\, R,$$

so that

$$\mathrm{vec}\, P = [I + \alpha\Lambda]^{-1}(A^T \oplus A^T)^{-1}(-\mathrm{vec}\, R),$$

and we can write P_∞ as

$$\mathrm{vec}\, P_\infty = \lim_{\alpha \to \infty} (I + \alpha\Lambda)^{-1}(A^T \oplus A^T)^{-1}(-\mathrm{vec}\, R)$$

$$= \lim_{\alpha \to \infty} \left[\alpha\left(\frac{1}{\alpha}I + \Lambda\right)\right]^{-1}(A^T \oplus A^T)^{-1}(-\mathrm{vec}\, R)$$

$$= \lim_{z \to \infty} z(zI + \Lambda)^{-1}(A^T \oplus A^T)^{-1}(-\mathrm{vec}\, R).$$

Now since $\mathrm{Ind}(\Lambda) = 1$, it follows from [8, Theorem 7.6.2] that the above limit exists and is given by $\mathrm{vec}\, P_\infty = (I - \Lambda\Lambda^\#)(A^T \oplus A^T)^{-1}(-\mathrm{vec}\, R)$, which yields (31). $\square$

For the following result, define the commutator $[F, G] \triangleq FG - GF$.

Lemma 3.9. *Let* $A, A_1 \in \mathbb{R}^{n \times n}$, $R \in \mathcal{N}^n$. *Furthermore, suppose that* $A + A^T < 0$, $A_1 + A_1^T = 0$, *and let* $P_\infty \in \mathcal{N}^n$ *be given by* (31). *Then* P_∞ *satisfies*

$$[A_1^T, P_\infty] = 0. \tag{32}$$

Proof. Since A_1 is skew-symmetric, we have

$$\mathrm{vec}\,[A_1^T, P_\infty] = \mathrm{vec}\,(A_1^T P_\infty + P_\infty A_1) = (A_1^T \oplus A_1^T)\mathrm{vec}\, P_\infty$$

$$= (A_1^T \oplus A_1^T)(I - \Lambda\Lambda^\#)(A^T \oplus A^T)^{-1}(-\mathrm{vec}\, R), \tag{33}$$

where Λ is defined by (29). Since, by Lemma 3.7, $\mathrm{Ind}(\Lambda) = 1$, it follows from Remark 3.5 that Λ and $\Lambda^\#$ can be expressed in the form

$$\Lambda = V\begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix}V^{-1}, \qquad \Lambda^\# = V\begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix}V^{-1},$$

where $\det(C) \neq 0$. Writing $V = [V_1 \; V_2]$, the identity

$$\Lambda V = V\begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix}$$

implies that $\Delta V_2 = 0$. Consequently, $(A_1^{\mathsf{T}} \oplus A_1^{\mathsf{T}})^2 V_2 = 0$, and, since $\mathrm{Ind}(A_1^{\mathsf{T}} \oplus A_1^{\mathsf{T}}) = 1$, it follows that $(A_1^{\mathsf{T}} \oplus A_1^{\mathsf{T}}) V_2 = 0$. Therefore, equation (33) can be written as

$$\mathrm{vec}[A_1^{\mathsf{T}}, P_\infty] = (A_1^{\mathsf{T}} \oplus A_1^{\mathsf{T}})\left(I - V\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} V^{-1}\right)(A^{\mathsf{T}} \oplus A^{\mathsf{T}})^{-1}(- \mathrm{vec}\, R)$$

$$= (A_1^{\mathsf{T}} \oplus A_1^{\mathsf{T}})\left(V\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} V^{-1}\right)(A^{\mathsf{T}} \oplus A^{\mathsf{T}})^{-1}(- \mathrm{vec}\, R)$$

$$= (A_1^{\mathsf{T}} \oplus A_1^{\mathsf{T}})[0 \; V_2] V^{-1}(A^{\mathsf{T}} \oplus A^{\mathsf{T}})^{-1}(- \mathrm{vec}\, R)$$

$$= [0 \, (A_1^{\mathsf{T}} \oplus A_1^{\mathsf{T}}) \, V_2] V^{-1}(A^{\mathsf{T}} \oplus A^{\mathsf{T}})^{-1}(- \mathrm{vec}\, R) = 0.$$

As a result, $[A_1^{\mathsf{T}}, P_\infty] = 0$. $\square$

Remark 3.10. If P is symmetric, A_1 is skew-symmetric, then it can be shown that $[A_1^{\mathsf{T}}, [A_1^{\mathsf{T}}, P_\infty]] = 0$ if and only if $[A_1^{\mathsf{T}}, P_\infty] = 0$. This fact is of interest since (21) can be written as

$$\Omega_0(P) = \sum_{i=1}^{r} \tfrac{1}{2}\delta_i^2 [A_i^{\mathsf{T}}, [A_i^{\mathsf{T}}, P]].$$

Thus, if $r = 1$ and $\delta_1 \to \infty$, then $[A_1^{\mathsf{T}}, [A_1^{\mathsf{T}}, P_\infty]] \to 0$. Note $(\delta_1^2/2)\,[A_1^{\mathsf{T}}, [A_1^{\mathsf{T}}, P_\infty]] = -(A^{\mathsf{T}} P_\infty + P_\infty A + R) = -\mathrm{vec}^{-1}[(A_1^{\mathsf{T}} \oplus A_1^{\mathsf{T}})^2 [(A^{\mathsf{T}} \oplus A^{\mathsf{T}})^{-1}(A_1^{\mathsf{T}} \oplus A_1^{\mathsf{T}})^2]^* (A^{\mathsf{T}} \oplus A^{\mathsf{T}})^{-1} \mathrm{vec}\, R]$.

4. The choice of corrector term P_0

Now we propose a corrector term P_0 for the case of general skew-symmetric matrices $A_i \in \mathbb{R}^{n \times n}$, $i = 1, \ldots, r$, where $r \geq 1$. For a symmetric matrix B, define $|B| \triangleq \sqrt{B^2}$.

Proposition 4.1. *Assume $A + A^{\mathsf{T}} < 0$, $A_i + A_i^{\mathsf{T}} = 0$, and $\delta_i \geq 0$, $i = 1, \ldots, r$. Let $P \in \mathcal{N}^n$ satisfy (22) and let*

$$\beta \geq \max\left\{\sum_{i=1}^{r} \mu_i, \; -\lambda_{\min}(P)\right\},\tag{34}$$

where, for $i = 1, \ldots, r$,

$$\mu_i \triangleq \lambda_{\max}((\delta_i |[A_i^{\mathsf{T}}, P]| - \tfrac{1}{2}\delta_i^2 [A_i^{\mathsf{T}}, [A_i^{\mathsf{T}}, P]])(- A - A^{\mathsf{T}})^{-1}).$$

If $P_0(\Delta A) \triangleq \beta I_n$, then (9) and (10) are satisfied with $R_0 = 0$ and $\mathcal{U}$ given by (20).

Proof. By substituting $P_0(\Delta A) = \beta I_n$ into (9) with $R_0 = 0$ and letting $G = \sqrt{- A^{\mathsf{T}} - A}$, we have

$$\Omega(P, \Delta A) + R_0 - (\Delta A^{\mathsf{T}} P + P \Delta A)$$

$$= \beta(- A^{\mathsf{T}} - A) - \sum_{i=1}^{r} \sigma_i [A_i^{\mathsf{T}}, P] + \sum_{i=1}^{r} \tfrac{1}{2}\delta_i^2 [A_i^{\mathsf{T}}, [A_i^{\mathsf{T}}, P]]$$

$$\geq \beta(- A^{\mathsf{T}} - A) - \sum_{i=1}^{r} \delta_i |[A_i^{\mathsf{T}}, P]| + \sum_{i=1}^{r} \tfrac{1}{2}\delta_i^2 [A_i^{\mathsf{T}}, [A_i^{\mathsf{T}}, P]]$$

$$= G\{\beta I_n - \sum_{i=1}^{r} G^{-1}(\delta_i |[A_i^{\mathsf{T}}, P]| - \tfrac{1}{2}\delta_i^2 [A_i^{\mathsf{T}}, [A_i^{\mathsf{T}}, P]]) G^{-1}\} G$$

$$\geq G\{\beta I_n - \sum_{i=1}^{r} \lambda_{\max}(G^{-1}(\delta_i |[A_i^{\mathsf{T}}, P]| - \tfrac{1}{2}\delta_i^2 [A_i^{\mathsf{T}}, [A_i^{\mathsf{T}}, P]]) G^{-1}) I_n\} G$$

$$= G\{\beta I_n - \sum_{i=1}^{r} \lambda_{\max}((\delta_i|[A_i^T, P]| - \tfrac{1}{2}\delta_i^2 [A_i^T, [A_i^T, P]])(-A - A^T)^{-1})I_n\}G$$

$$= G\{\beta I_n - \sum_{i=1}^{r} \mu_i I_n\}G$$

$$\geq 0,$$

which proves (9). Finally, it is obvious that $P + P_0(\Delta A) = P + \beta I_n \geq \lambda_{\min}(P)I_n + \beta I_n \geq 0$, so that (10) is satisfied. $\square$

Henceforth, we confine our attention to the special case $r = 1$ and

$$A = \begin{bmatrix} -\eta & \omega \\ -\omega & -\eta \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{35}$$

where $\eta > 0$ and $\omega \in \mathbb{R}$. For notational convenience, we adopt the traditional symbol J for A_1. In this case $\Omega_0(P)$ given by (21) has the form

$$\Omega_0(P) = \delta_1^2 (\tfrac{1}{2} J^{2T} P + J^T PJ + \tfrac{1}{2}PJ^2). \tag{36}$$

Note that $J^T = -J$ and $J^2 = -I_2$, where I_2 denotes the 2×2 identity matrix.

Proposition 4.2. *Assume that R is positive-definite and let P satisfy*

$$0 = A^T P + PA + \delta_1^2 (\tfrac{1}{2} J^{2T} P + J^T PJ + \tfrac{1}{2}PJ^2) + R, \tag{37}$$

let $\gamma < 1$, and define

$$P_0(\Delta A) \triangleq (1 - \gamma) J^T PJ - \gamma P, \quad \Delta A \in \mathcal{U}. \tag{38}$$

Then (9) and (10) are satisfied with $R_0 = \gamma R$. Furthermore, the performance bound (15) is given by

$$\mathcal{T}(\mathcal{U}) \leq \operatorname{tr}(V)\operatorname{tr}(P). \tag{39}$$

Proof. Clearly, (10) is satisfied. Secondly, since

$$AJ = JA, \quad JJ^T = J^T J = I_2, \quad J^T \Omega_0(P) J = -\Omega_0(P),$$

and P satisfies (37), it follows that

$$\Omega_0(P) + R_0 - [(A + \sigma_1 J)^T P_0 + P_0(A + \sigma_1 J)] - \sigma_1(J^T P + PJ)$$

$$= \Omega_0(P) + R_0 - [(1 - \gamma)(A^T J^T PJ + J^T PJA) + \sigma_1(1 - \gamma)(J^T J^T PJ + J^T PJJ)$$

$$\qquad - \gamma(A^T P + PA) - \sigma_1 \gamma(J^T P + PJ)] - \sigma_1(J^T P + PJ)$$

$$= \Omega_0(P) + R_0 - (1 - \gamma) J^T(A^T P + PA)J + \gamma(A^T P + PA)$$

$$= \Omega_0(P) + R_0 - (1 - \gamma) J^T(-\Omega_0(P) - R)J + \gamma(-\Omega_0(P) - R)$$

$$= R_0 - \gamma R + (1 - \gamma) J^T RJ$$

$$\geq 0.$$

Finally, we have

$$\mathcal{T}(\mathcal{U}) \leq \frac{1}{1 - \gamma}[\operatorname{tr}(PV) + \operatorname{tr}(P_0 V)] = \operatorname{tr}(PV) + \operatorname{tr}(J^T PJV)$$

$$= \operatorname{tr}[P(V + JVJ^T)] = \operatorname{tr}(V)\operatorname{tr}(P). \quad \square$$

Remark 4.3. Note that unlike the parameter-dependent Lyapunov function used in [15] for the Popov criterion, the auxiliary portion $P_0(\Delta A)$ given by (38) is independent of σ_1. Therefore, this auxiliary portion $P_0(\Delta A)$ guarantees robust stability with respect to time-varying $\sigma_1(t)$. This robust stability property was already shown at the beginning of this section by means of the Lyapunov function $V(x) = x^\mathsf{T} x$.

Remark 4.4. Since by Proposition 3.1, equation (37) has a solution for all $\delta_1 > 0$, it follows that robust stability is guaranteed for arbitrary σ_1, that is, not necessarily bounded by δ_1.

Remark 4.5. It is easy to show that $\operatorname{tr}(P) = (1/2\eta) \operatorname{tr}(R)$ and $P_T = P + P_0 = (1 - \gamma)(J^\mathsf{T} P J + P)$ $= (1 - \gamma) \operatorname{tr}(P) I_2$. Thus, (39) becomes

$$\mathscr{T}(\mathscr{U}) \le \frac{1}{2\eta} \operatorname{tr}(V) \operatorname{tr}(R). \tag{40}$$

Thus, the performance bound (39) is independent of δ_1. Furthermore, it is easy to check that P_T satisfies the equation

$$0 = A^\mathsf{T} P_T + P_T A + J^\mathsf{T} R J + R. \tag{41}$$

We now present an alternative choice of $P_0(\Delta A)$.

Proposition 4.6. *Let*

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}, \qquad R = \begin{bmatrix} R_{11} & R_{12} \\ R_{12} & R_{22} \end{bmatrix} > 0$$

satisfy (37) *and let* $P_0(\Delta A) \triangleq \mu I_2$, *where*

$$\mu \triangleq \frac{\sqrt{\delta_1^2 + \delta_1^4}}{2\eta} \sqrt{(P_{22} - P_{11})^2 + (2P_{12})^2}. \tag{42}$$

Then (9) *and* (10) *are satisfied with* $R_0 = 0$. *Furthermore, the performance bound* (15) *is given by*

$$\mathscr{T}(\mathscr{U}) \le \operatorname{tr}(PV) + \mu \operatorname{tr}(V). \tag{43}$$

Proof. Since $P \ge 0$ and $P_0(\Delta A) \ge 0$, $\Delta A \in \mathscr{U}$, it follows that (10) is satisfied. Next, to show that (9) is true, recall that $\Omega_0(P)$ is given by equation (36). Therefore,

$$\Omega_0(P) - [(A + \sigma_1 J)^\mathsf{T} P_0 + P_0(A + \sigma_1 J)] - \sigma_1(J^\mathsf{T} P + PJ)$$

$$= \delta_1^2(-P + J^\mathsf{T} P J) - \mu(A^\mathsf{T} + A) - \sigma_1(J^\mathsf{T} P + PJ)$$

$$= 2\mu\eta I_2 + \delta_1^2(-P + J^\mathsf{T} P J) - \sigma_1(J^\mathsf{T} P + PJ)$$

$$= 2\mu\eta I_2 + S \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} S^\mathsf{T}$$

$$= S \begin{bmatrix} 2\mu\eta + \lambda_1 & 0 \\ 0 & 2\mu\eta + \lambda_2 \end{bmatrix} S^\mathsf{T},$$

where $\lambda_1 = -\lambda_2 = \sqrt{\sigma_1^2 + \delta_1^4} \sqrt{(P_{22} - P_{11})^2 + (2P_{12})^2}$ are the eigenvalues of $\delta_1^2(-P + J^\mathsf{T} P J) - \sigma_1(J^\mathsf{T} P + PJ)$ and S is a 2×2 orthogonal matrix. Choosing μ according to (42) implies that $2\mu\eta + \lambda_1 \ge 0$ and $2\mu\eta + \lambda_2 \ge 0$. Thus, (9) is satisfied. Finally, the performance bound (15) has the form

$$\mathscr{T}(\mathscr{U}) \le \operatorname{tr}[(P + P_0(\Delta A))V] = \operatorname{tr}(PV) + \mu \operatorname{tr}(V). \qquad \square$$

Remark 4.7. As in [3, 4] the robust performance bounds (40) and (43) are only valid for constant uncertainty σ_1.

Before we present a numerical example, we shall illustrate some important aspects of P given by equation (37). The analytical solution for (37) yields

$$P_{11} + P_{22} = \frac{1}{2\eta}(R_{11} + R_{22}), \qquad P_{11} - P_{22} = \frac{1}{\alpha}\left[\frac{\eta + \delta_1^2}{2}(R_{11} - R_{22}) - \omega R_{12}\right],$$

$$2P_{12} = \frac{1}{\alpha}\left[\frac{\omega}{2}(R_{11} - R_{22}) + (\eta + \delta_1^2)R_{12}\right],$$

where $\alpha \triangleq (\eta + \delta_1^2)^2 + \omega^2$. For large δ_1, it is easy to see that

$$P_{11} - P_{22} \sim \frac{1}{2\delta_1^2}(R_{11} - R_{22}), \qquad 2P_{12} \sim \frac{1}{\delta_1^2}R_{12}$$

and

$$\lim_{\delta_1 \to \infty} [A_1^T, P] = \lim_{\delta_1 \to \infty}\begin{bmatrix} -2P_{12} & P_{11} - P_{22} \\ P_{11} - P_{22} & 2P_{12} \end{bmatrix} = 0,$$

which agrees with Lemma 3.9. Hence, $P_{11} - P_{22}$ and P_{12} both approach zero as $\delta_1 \to \infty$. These properties are the so-called equipartition (modal energy equilibration) and incoherence (modal decorrelation) phenomena [17, 20]. Since

$$\bar{\mu} \triangleq \lim_{\delta_1 \to \infty} \mu = \frac{1}{2\eta}\sqrt{\left(\frac{R_{11} - R_{22}}{2}\right)^2 + R_{12}^2},$$

the performance bound given by (43) approaches a *(finite)* constant as $\delta_1 \to \infty$. Furthermore, since $\lim_{\delta_1 \to \infty} P_{11} = \lim_{\delta_1 \to \infty} P_{22} = (1/4\eta)\,\mathrm{tr}(R)$, it follows that

$$\lim_{\delta_1 \to \infty} \mathrm{tr}(PV) + \mu\,\mathrm{tr}(V) = \left(\frac{1}{4\eta}\,\mathrm{tr}(R) + \bar{\mu}\right)\mathrm{tr}(V).$$

We now compare the performance bounds given by (39) and (43) for large values of δ_1. Denoting $\mathscr{T}_1 = \mathrm{tr}(V)\,\mathrm{tr}(P)$ and $\mathscr{T}_2 = \mathrm{tr}(PV) + \mu\,\mathrm{tr}(V)$, it can be shown using $R_{12}^2 < R_{11}R_{22}$ that

$$\lim_{\delta_1 \to \infty} \mathscr{T}_1 - \mathscr{T}_2 = \frac{\mathrm{tr}(V)}{2\eta}\left[\frac{R_{11} + R_{22}}{2} - \sqrt{\left(\frac{R_{11} - R_{22}}{2}\right)^2 + R_{12}^2}\right] = \frac{\mathrm{tr}(V)}{2\eta}\lambda_{\min}(R) > 0. \tag{44}$$

Finally, if $\det R = 0$, then $\lim_{\delta_1 \to \infty} \mathscr{T}_1 = \lim_{\delta_1 \to \infty} \mathscr{T}_2 = (1/2\eta)\,\mathrm{tr}(V)\,\mathrm{tr}(R)$.

5. Numerical examples

Example 5.1. Let us consider a lightly damped system with $\zeta = 0.02$, $\omega_n = 2$, $\eta = \zeta\omega_n$, $\omega = \sqrt{1 - \zeta^2}\omega_n$,

$$A = \begin{bmatrix} -\eta & \omega \\ -\omega & -\eta \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

and let

$$R = \begin{bmatrix} 2\beta & 0 \\ 0 & 2 \end{bmatrix},$$

where $\beta > 0$. For robust stability, we compare our result to the approach of [22]. For $R \neq 2I_2$ we must use a congruence transformation in order to apply the theorem in [22]. Hence, we transform

$$A^{\mathsf{T}}P + PA + R = 0 \tag{45}$$

to obtain

$$\hat{A}^{\mathsf{T}}\hat{P} + \hat{P}\hat{A} + 2I_2 = 0,$$

where $\hat{A} \triangleq S^{-1}AS$, and S is the congruence transformation matrix such that $S^{\mathsf{T}}RS = 2I_2$. As was mentioned in Remark 4.3, this system is robustly stable for all $\sigma_1 \in \mathbb{R}$. This follows from [22] by taking $\beta = 1$, that is, $R = 2I_2$, so that equation (45) has the solution $P = (1/\eta)I_2$. Therefore, in the notation of [22], $P_1 \triangleq \frac{1}{2}(J^{\mathsf{T}}P + PJ) = 0$, and thus the singular values of P_1 are all zero. As a result, the robust stability region is $|\sigma_1| < \infty$.

Now consider the case $\beta \gg 0$. Following the same procedure mentioned above, we have $|\sigma_1| \leq \delta_1 \sim (2/\omega\beta)(\eta^2 + \omega^2)$ as $\beta \to \infty$. Thus, for large β the approach of [22] becomes highly conservative. The reason for this conservatism is the similarity transformation of the skew-symmetric matrix J which was effectively imposed by the choice $R \neq 2I_2$. In the new basis, the matrix J is transformed to $S^{-1}JS$, which is no longer skew-symmetric.

Example 5.2. Consider the same system in Example 5.1 except with

$$R = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix},$$

and for robust performance, let

$$V = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

First, the robust stability region found by using the same technique as in the previous example is $|\sigma_1| < 1.37$, an extremely conservative result. As in the previous example, the reason for this conservatism is due to the similarity transformation of the skew-symmetric matrix J. In the new basis, the matrix J is transformed to $S^{-1}JS$, which is no longer skew-symmetric.

Next, let us compare the robust performance bound given by equation (39) in Proposition 4.2 with the bound suggested by Bernstein and Haddad [3]. According to (39) the performance bound is $\mathcal{J}(\mathcal{U}) \leq (1/2\eta)\,\mathrm{tr}\,(R) = 98.50$, which is valid for all $\sigma_1 \in \mathbb{R}$. In [3] the stability region and performance bound can be found by solving

$$A^{\mathsf{T}}P_A + P_A A + \Lambda + R = 0 \tag{46}$$

and by determining the values of σ_1 such that

$$\sigma_1(A_1^{\mathsf{T}}P_A + P_A A_1) \leq \Lambda, \tag{47}$$

where Λ is a nonnegative-definite matrix. First, letting $\Lambda = kI_2$, where $k \geq 0$, it can be shown that the solution to equation (46) is $P_A = P + (k/2\eta)I_2$, where P is the solution to (45) with

$$R = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Therefore, we have the performance bound $\mathcal{J}(\mathcal{U}) \leq \mathrm{tr}\,(PV) + (k/2\eta)\,\mathrm{tr}\,(V)$ with robust stability region $|\sigma_1| \leq k/\lambda_{\max}(J^{\mathsf{T}}P + PJ)$ (see Fig. 1). Alternatively, choosing $\Lambda = 0.53R$ yields the robust stability region $-2.57 \leq \sigma_1 \leq 0.37$ which yields the symmetric stability region $|\sigma_1| \leq 0.37$. For this robust stability region the performance bound $\mathcal{J}(U) \leq 118.20$ (see Fig. 2).
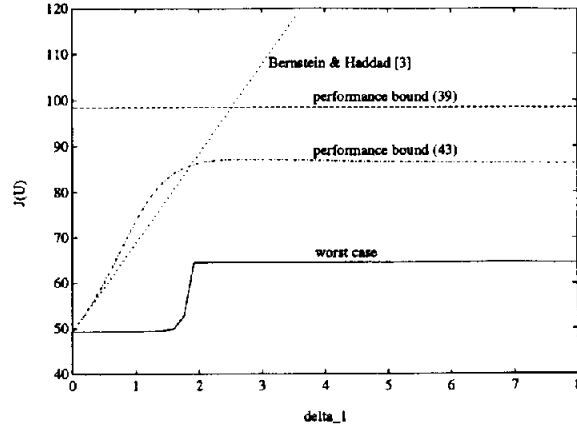
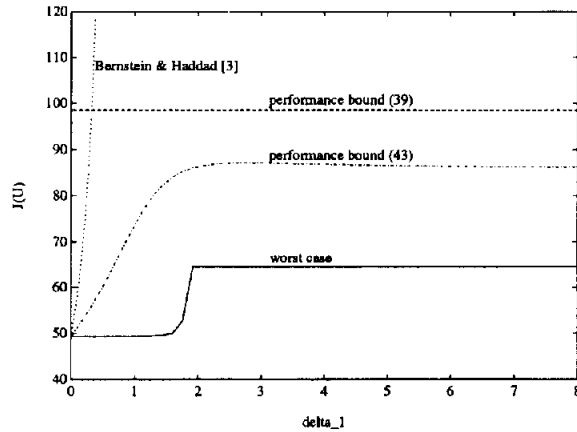Fig. 1. Comparison of different robust performance bounds

Fig. 2. Comparison of different robust performance bounds

6. Discussion and conclusions

As was shown in Propositions 4.2 and 4.6, the maximum-entropy-type Lyapunov functions correctly predict unconditional robust stability for arbitrary coordinates and thus, effectively, for an arbitrary state space basis. In addition, the performance bounds predicted by the maximum-entropy Lyapunov function are comparatively tight, even for large δ_1, whereas the bound of [3] is extremely conservative and highly coordinate-dependent. The problem of choosing an appropriate basis may be relatively benign if robust stability analysis is performed independently of robust performance analysis. That is, for robust stability analysis one can arbitrarily choose the state space basis to produce the best estimate of the robust stability region without regard to robust performance. However, in the problem of robust controller *synthesis* the basis is not arbitrary but rather is dictated by the weighting matrices V and R. Thus, the fact that the maximum-entropy-type Lyapunov functions provide robust stability and performance bounds that are only slightly affected by the choice of V and R appears to be a desirable feature for robust controller synthesis. This may explain the favorable results obtained in [2, 5, 6, 18, 19].

Acknowledgment

We wish to thank Jonathan How for noting Remark 4.5.

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