JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 8, 423-444 (1964)

A Study of Second Order Nonlinear Systems

Y. S. LIM

Bell Telephone Laboratories, Whippany, New Jersey

AND

L. F. KAZDA

University of Michigan, Ann Arbor, Michigan

Submitted by C. L. Dolph

This paper studies a class of second order forced oscillation systems with nonlinear damping and nonlinear restoring forces. The boundedness of the solutions is obtained by recognizing the behavior of the solution path in the phase plane. All solutions are shown to converge to a unique solution provided the characteristics of the nonlinear terms are sufficiently smooth. The convergence of solutions is investigated by studying the asymptotic stability of a second order linear equation with time-varying parameters formed from the nonlinear system. Based on the Second Mcthod of Lyapunov, two different V functions are used to obtain different sufficient conditions on the asymptotic stability of the linear system.

I. INTRODUCTION

In the last two decades, a considerable amount of work has been given to the study of second-order nonlinear differential systems [1-4]. This paper is to study a class of second-order differential systems of the form

$$\ddot{x} + f(\dot{x}) + g(x) = e(t) . \tag{1}$$

Figure 1 shows a control system having a particular form of nonlinear damping and nonlinear feedback of the type described by Eq. (1), namely, with $g(x) = kg_1(x)$ and $e(t) = ke_1(t)$.

It is the purpose of this paper to investigate (a) the boundedness of $x(t)$ and $\dot{x}(t)$ when $e(t)$ is bounded, and (b) the conditions for asymptotic stability. The technique by Loud [4], which was to construct a bounded region, is improved and extended to (1) in Section II. Section III studies the stability problem of a second-order linear differential equation by means of Lyapunov functions. Sufficient conditions on the time-varying parameters are obtained for asymptotic stability. These results are used in Section IV to establish the convergence theorems which show that, under certain conditions on $f(\dot{x})$, $g(x)$, and $e(t)$, the steady state solution will be independent of the initial conditions.

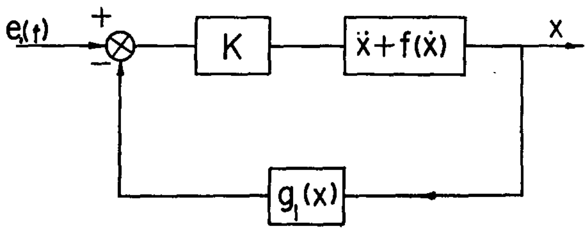

FIG. 1. Nonlinear control system

II. The Boundedness of Solutions

The scheme described below is to construct a closed curve in the phase plane such that every solution of (1) will ultimately move to the inside of it. This requires $f(\dot{x})$, $g(x)$, and $e(t)$ to have certain characteristics. Furthermore, if the forcing function is periodic, the existence of periodic solutions will immediately follow with the aid of Brower's fixed point theorem.

THEOREM I. *Let*

(1) *$f(y)$, $g(x)$ and $e(t)$ be continuous,*

(2) *$g(x)$ be a monotonically increasing function that satisfies the Lipschitz condition,*

(3) $g(0) = 0$, $f(0) = 0$

and if there exist positive constants b, c, and E such that

(4) $f(y)/y \geqslant b, \quad g(x)/x \geqslant c, \quad | e(t) | \leqslant E$

then for any solution of (1) when $t > t_0$ (t_0 depending on the particular solution),

$| x(t) | \leqslant I$

$$I = \frac{E}{c} \qquad \text{for} \qquad \frac{c}{b^2} < \tfrac{1}{4}$$

$$= \frac{E}{c}\left[1 + \sqrt{\frac{4c}{b^2} - 1} \cdot \exp\left(-\frac{\pi}{2\sqrt{(4c/b^2) - 1}}\right)\right] \qquad \text{for} \qquad \frac{c}{b^2} \geqslant \tfrac{1}{4}$$

$| \dot{x}(t) | \leqslant J = 2E/b.$

REMARKS. For certain special cases, namely, when $g(x) = x$, $f(y)$ and $e(t)$ possess first derivatives, and $z = \dot{x}$, Eq. (1) reduces to the form

$$\ddot{z} + f(z)\,\dot{z} + g(z) = e(t)\,,$$

which can be found discussed frequently in the literature. In general, however, such a reduction is impossible. The general form as given by (1) and which is discussed in this paper requires only the continuity of $f(y)$, $g(x)$, and $e(t)$ in order for the existence of solutions (not necessarily unique). The functions $f(y)$ and $g(x)$ may be represented as shown respectively in Fig. 2 and Fig. 3. If $g_1(0) \neq 0$ and $f_1(0) \neq 0$, these difficulties can be removed by letting

$$g(x) = g_1(x) - g_1(0) \qquad \text{and} \qquad f(y) = f_1(y) - f_1(0);$$

then let

$$e(t) = e_1(t) - g_1(0) - f_1(0)\,.$$

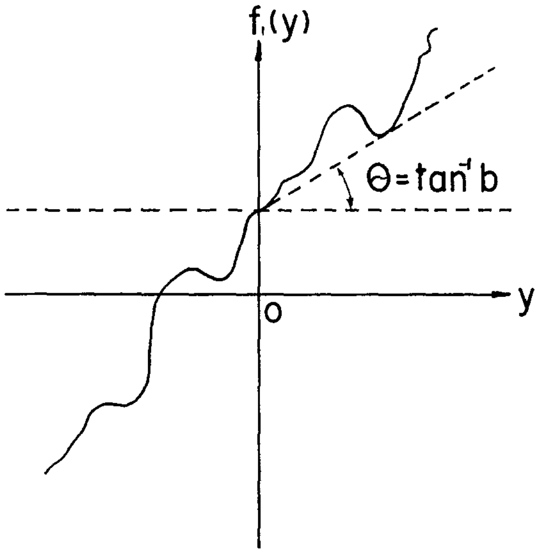

FIG. 2. Characteristics of $f_1(y)$

PROOF. Letting $y = \dot{x}$ Eq. (1) can be written in the following form:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -g(x) - f(y) + e(t) \end{aligned} \tag{2}$$

and

$$\frac{dy}{dx} = -\frac{f(y)}{y} - \frac{g(x)}{y} + \frac{e(t)}{y}\,. \tag{3}$$

Now consider the following equation

$$\dot{x} = y$$
$$\dot{y} = -by + (E/|y|)y - g(x) \tag{4}$$

and

$$\frac{dy}{dx} = -b + \frac{E}{|y|} - \frac{g(x)}{y} \tag{5}$$

Since $f(y)/y \geqslant b$ and $|e(t)| \leqslant E$, dy/dx in (3) does not exceed dy/dx in (5). Then, if a closed region can be found in which the solutions of (4) will be ultimately bounded, the boundedness of the solutions of (2) will be readily established.

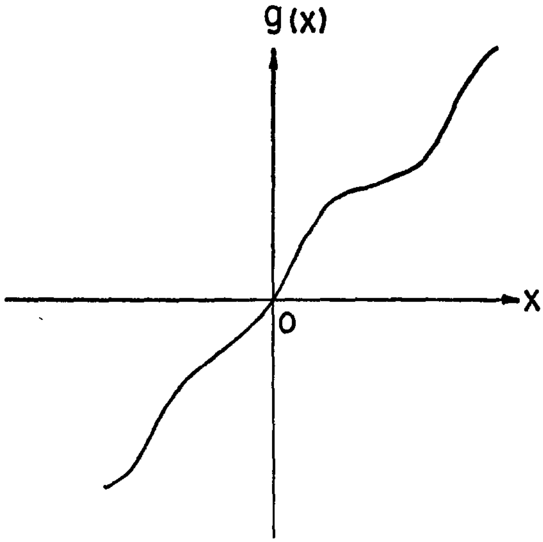

FIG. 3. Characteristics of $g(x)$

Let $\alpha(\alpha < 0)$ and $\beta(\beta > 0)$ be such that $g(\alpha) = -E$ and $g(\beta) = E$. α and β are uniquely determined because of the monotonic property of $g(x)$. Then $-E/c \leqslant \alpha < 0$, and $0 < \beta < E/c$.

Now in Eq. (5) let $dy/dx = 0$

$$-by = g(x) - E \qquad \text{for} \qquad y > 0 \tag{6}$$
$$-by = g(x) + E \qquad \text{for} \qquad y < 0\,. \tag{7}$$

These two curves are shown in Fig. 4. To the left of the curve (6), $dy/dx > 0$, and therefore the solution curve must move upward and to the right until it reaches a maximum of y when crossing. After crossing, it must move downward and cross the line $x = \beta$ before it reaches the x-axis at some $x > \beta$. Furthermore, for $x > \beta$, as y decreases and x increases, dy/dx becomes increasingly negative for $y > 0$. The solution path must move concave downward toward the x-axis. After crossing the x-axis, dy/dx becomes decreasingly positive for $x > \beta$ and $y < 0$. The solution path now moves concave toward the curve (7).

A similar set of statements can be made regarding the solutions of (4) in the half plane $y < 0$.

Let C_1 be the solution path which starts at (β, y_1) with $y_1 > 2E/b$, passes through $(x_2, 0)$ and then intersects with the line $x = \alpha$ at (α, y_3).

Let

$$G(x) = \int_0^x g(x)\, dx.$$

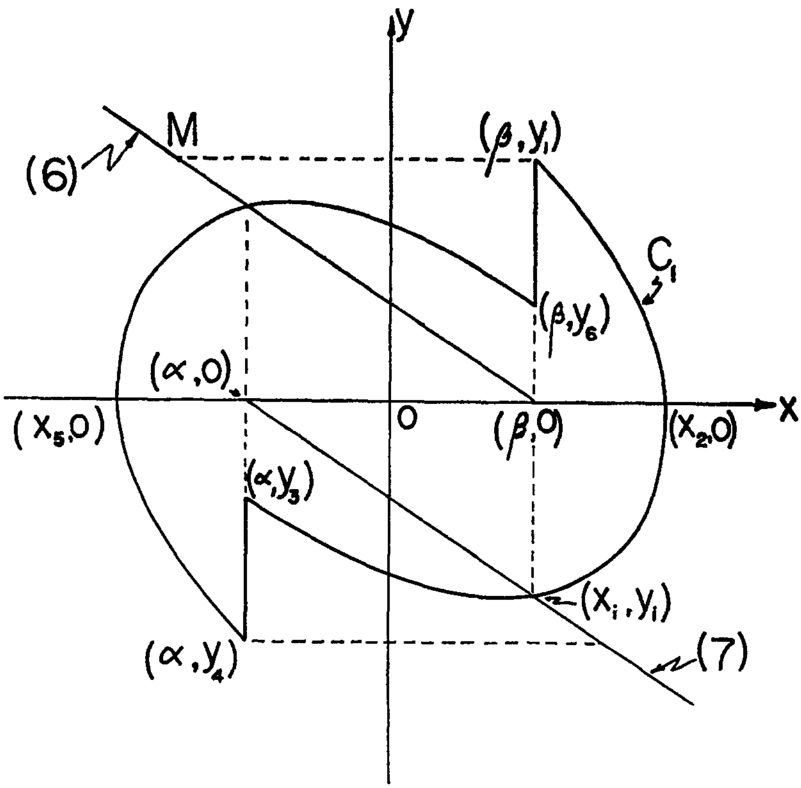

FIG. 4. Phase-plane curve

Integrating (5) from (β, y_1) to $(x_2, 0)$, and from $(x_2, 0)$ to (x_i, y_i), which is the minimum point of C_1, on the curve (7), one has

$$\tfrac{1}{2} y_1^2 = b \int_\beta^{x_2} y\,dx + G(x_2) - G(\beta) - E(x_2 - \beta) \tag{8}$$

$$\frac{-1}{2} y_i^2 = b \int_{x_2}^{x_i} y\,dx + G(x_i) - G(x_2) + E(x_i - x_2). \tag{9}$$

If $x_i < \beta$, then $g(x_i) < g(\beta) = E$, so that

$$y_i = -\frac{g(x_i) + E}{b} > -\frac{2E}{b}, \qquad |y_i| < \frac{2E}{b} \leqslant y_1 .$$

If $x_i > \beta$, one observes that the integrals in (8) and (9) represent the areas between the solution curves and the x-axis, and because the solution curve is concave, they are greater than the areas of the triangles formed by replacing the solution curves by straight lines. That is

$$b \int_\beta^{x_2} y\,dx > \frac{b}{2} y_1(x_2 - \beta),$$

and

$$b \int_{x_2}^{x_i} y\,dx > \frac{b}{2} |y_i| (x_2 - x_i) .$$

Since $x_i > \beta$, from (7),

$$|y_i| > \frac{g(\beta) + E}{b} = \frac{2E}{b}$$

so

$$b \int_{x_2}^{x_i} y\,dx > E(x_2 - x_i).$$

In addition, utilizing the relations

$$y_1 > \frac{2E}{b}, \qquad G(x_i) - G(\beta) > 0,$$

gives

$$\tfrac{1}{2}(y_1^2 - y_i^2) \geqslant \frac{b}{2} y_1(x_2 - \beta) + G(x_i) - G(\beta) - E(x_2 - \beta)$$

$$> \frac{b}{2}\left(y_1 - \frac{2E}{b}\right)(x_2 - \beta) > 0. \tag{10}$$

Hence for $y_1 > 2E/b$, $y_1 > |y_i|$. Now consider the case when $y_1 = 2E/b$.

Suppose now that $x_i > \beta$, then it follows that $|y_i| > [g(\beta) + E]/b = 2E/b$ by (7). However, by (10), $y_1^2 - y_i^2 > 0$ or $|y_i| < y_i = 2E/b$ which is a contradiction. Hence $x_i < \beta$ and $|y_i| < 2E/b$. Thus, it has been shown that for all points in arc C_1, the inequality $|y| < y_1$ with $y_1 \geqslant 2E/b$ is true. A similar statement can be made for the corresponding arc C_2 starting at (α, y_4) with $y_4 \leqslant -2E/b$ to (β, y_6) as shown in Fig. 4.

Now let R be the region bounded by a closed curve Γ. Γ consists of C_1 and C_2 with $y_1 = |y_4| = 2E/b$, and the straight line portions from (β, y_6) to (β, y_1) and from (α, y_3) to (α, y_4).

Consider outside Γ any solution of (5). Let this solution first cross $x = \beta$ at $y = y_{10}$. The uniqueness fo this solution is guaranteed by $g(x)$ satisfying the Lipschitz condition. Then this solution will cross the line $x = \beta$ at $y = y_{11}, y_{12}, y_{13} \cdots$ after consecutive turns through the lower half plane and back to the upper half plane. By (10), $y_{10}, y_{11}, y_{12} \cdots$ form a monotonically decreasing sequence as long as they are not less than $2E/b$. If, as $t \to \infty$, the solution still could not enter R, then y would have a limit and the solution would have a limit cycle outside of Γ. But this contradicts the statement $|y| < y_1$ for $y_1 \geqslant 2E/b$. Therefore the solution will spiral into R in finite time.

Since dy/dx in (3) does not exceed dy/dx in (5), the solution curve of (2) can cross that of (4) only from left to right. Hence all solutions of (2) are carried into R. Once the solution curve enters R or either it starts in R, it will remain permanently in R.

The boundary of R in y is clearly $|y| \leqslant J = 2E/b$. The maximum of $|x|$ is the larger of x_2 and $|x_5|$. However, the bound for x can be improved by choosing a starting point of the solution curve as close to the point M on the curve (6) as possible. Let a solution curve of (4) start at $(0, 2E/b)$ instead at $(\beta, 2E/b)$ and intersect the x-axis at x_2', and let a solution curve of the following system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -by - cx + E \end{aligned} \tag{11}$$

$$\frac{dy}{dx} = -b - \frac{cx}{y} + \frac{E}{y} \tag{12}$$

start also at $(0, 2E/b)$ and intersect the x-axis at x_7. For $y > 0$ and $x > 0$, dy/dx in (5) never exceeds dy/dx in (12). Hence $x_2' \leqslant x_7$. The same reasoning can be applied to the corresponding curve in the lower half plane, so that both x_2' and $|x_5'|$ are not greater than x_7 which is considered to be the bound of x.

If $4c/b^2 < 1$, the solution of (11) with initial condition $(0, 2E/b)$ is

$$x = (E/c) + D_1 e^{v_1 t} + D_2 e^{v_2 t}$$

where

$$p_1 = -\frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 - c}, \qquad p_2 = -\frac{b}{2} - \sqrt{\left(\frac{b}{2}\right)^2 - c},$$

$$D_1 = -\frac{E}{2c}\left[1 - \sqrt{1 - \frac{4c}{b^2}}\right], \qquad D_2 = -\frac{E}{2c}\left[1 + \sqrt{1 - \frac{4c}{b^2}}\right].$$

Since $p_1 < 0$, $p_2 < 0$, $D_1 < 0$, and $D_2 < 0$ then $x_7 < E/c$. Hence

$$x_2' \leqslant x_7 < E/c \qquad \text{for} \qquad 4c/b^2 < 1 \tag{13}$$

If $4c/b^2 \geqslant 1$, the solution curve of (11) starting at $(0, 2E/b)$ intersects the x-axis at x_7 which is obtained as

$$x_7 = \frac{E}{c}\left[1 + \sqrt{\frac{4c}{b^2} - 1} \cdot \exp\left(-\frac{\pi}{2\sqrt{(4c/b^2) - 1}}\right)\right]$$

Hence

$$x_2' \leqslant x_7 = \frac{E}{c}\left[1 + \sqrt{\frac{4c}{b^2} - 1} \cdot \exp\left(-\frac{\pi}{2\sqrt{(4c/b^2) - 1}}\right)\right]. \tag{14}$$

Figure 5 shows the bound for x vs. c with $E = b = 1$. This completes the proof of this theorem.

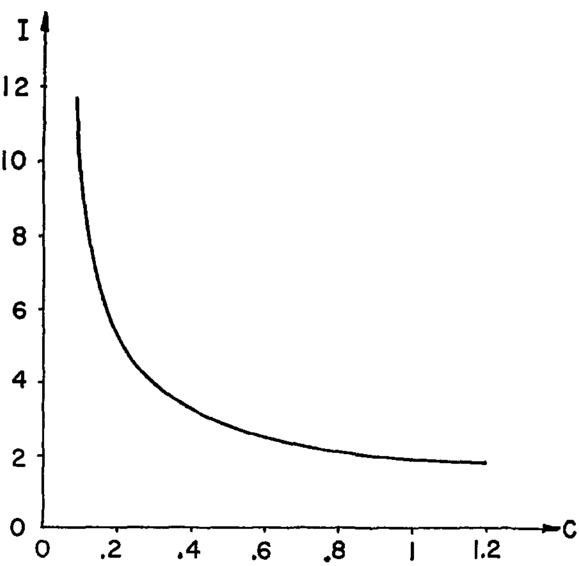

FIG. 5. Bound of x when $E = 1$, $b = 1$

Having constructed a closed region from which no solution path can leave, the existence of periodic solutions will follow immediately if the forcing function is periodic. Let (x_0, y_0) be any initial point in R of the $t = 0$ plane. Then $x = x(t, x_0, y_0)$ and $y = y(t, x_0, y_0)$ continue to exist as functions of t, x_0, and y_0. In particular, in the plane $t = T$, the point

$$x = x(T, x_0, y_0) \qquad y = y(T, x_0, y_0)$$

defines a transformation of a closed region into itself. This transformation is continuous since the right members of (2) are continuous and the solution paths are all continuous. By the Brouwer's fixed point theorem, there exists at least one fixed point (x^*, y^*). If a solution starts at (x^*, y^*), the transformation will carry it back to (x^*, y^*) at T. Since $e(t + T) = e(t)$, a second transformation from T to $2T$ is identical to the first one. Hence a solution $x(t + T)$, $y(t + T)$ with initial point at (x^*, y^*) will be the same as the solution $x(t)$, $y(t)$ with the same initial point, i.e.,

$$x(t) = x(t + T) = x(t + nT),$$

$n =$ an integer. This shows the system as defined by Eq. (1) to have at least one periodic solution with period T.

For autonomous systems, i.e., $e(t) = 0$, by Theorem 1 R vanishes. Under these conditions the system is asymptotically stable in the sense that any disturbances produced within the system will tend to zero. However, this is not true in nonautonomous systems. For example, consider the system shown in Fig. 1 but which contains a sinusoidal input, a linear plant and a nonlinear feedback such that it is described by a Duffing equation of the form

$$\ddot{x} + b\dot{x} + x + ax^3 = E \sin \omega t.$$

This equation may have three periodic solutions which corresponds to two stable fixed points, and one unstable. In Section III, the existence of only one stable fixed point is to be shown subject to further conditions imposed on $f(y)$ and $g(x)$.

III. Stability of Second Order Linear System

The differential system considered is of the form

$$\ddot{u} + p(t)\,\dot{u} + q(t)\,u = 0 \tag{15}$$

which is equivalent to

$$\begin{aligned} \dot{u} &= v \\ \dot{v} &= -\,p(t)\,v - q(t)\,u \end{aligned} \tag{16}$$

where $p(t)$ and $q(t)$ are assumed to be continuous and $q(t) \geqslant q_1 > 0$. The Second Method of Lyapunov is used to study the stability problem. Two V functions are considered.

(A) Consider the function[1]

$$V = \tfrac{1}{2} Au^2 + uv + \tfrac{1}{2} Bv^2 \tag{17}$$

where A and B are constants, and $A > 0, AB > 1$. Then V is definite positive.

$$\dot{V} = -\{q(t)\, u^2 + [Bq(t) + p(t) - A]\, uv + [Bp(t) - 1]\, v^2\}$$

or

$$\dot{V} = -\left[q\left(u + \frac{Bq + p - A}{2q} v\right)^2 + \left(Bp - 1 - \frac{(p + Bq - A)^2}{4q}\right) v^2\right]. \tag{18}$$

Since $q(t) \geqslant q_1 > 0$, and if the coefficient of v^2 is

$$\left[Bp - 1 - \frac{(p + Bq - A)^2}{4q}\right] \geqslant \epsilon, \tag{19}$$

where ϵ is a small positive fixed quantity, then $\dot{V}$ is definite negative, V is a Lyapunov function and the solution of (15) is asymptotically stable. Equation (19) when rearranged gives

$$p^2 - 2Bpq + B^2q^2 - 2Ap - 2(AB - 2)\, q + A^2 + 4q\epsilon \leqslant 0,$$

or

$$(p - Bq - A)^2 - 4(AB - 1 - \epsilon)\, q \leqslant 0, \tag{20}$$

which represents in the $p - q$ plane the region bounded by a parabola. From (19),

$$p \geqslant \frac{1}{B}\left(1 + \epsilon + \frac{(p + Bq - A)^2}{4q}\right) > 0,$$

so the parabola must lie only in the first quadrant of the $p - q$ plane. The equation of the parabola has two terms on the right side, namely,

$$p = p_l + p_c, \qquad p_l = Bq + A \quad \text{(straight line)},$$

$$p_c = \pm 2 \sqrt{AB - 1 - \epsilon}\, \sqrt{q} \text{ (parabola)}.$$

[1] Starzinsky [7] used the same function but a different approach. His result is incomplete and too restrictive.

The stability region is also bounded by the line $q = q_1$ as shown in Fig. 6.

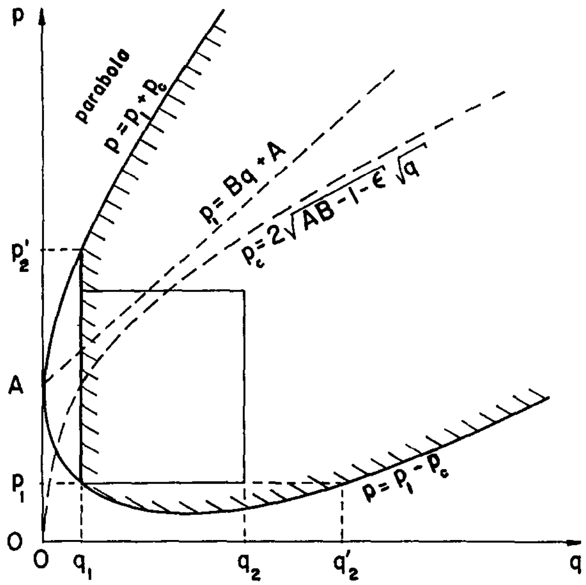

FIG. 6. Stability region

Consider the case

$$0 < p_1 \leqslant p(t) \leqslant p_2 , \qquad 0 < q_1 \leqslant q(t) \leqslant q_2$$

which represents the region bounded by a rectangle. In order to see the relationship between these parameters for which the rectangle will be located in the stability region, let a parabola pass through (q_1 , p_1) and some point (q_2' , p_1), $q_2' \geqslant q_2$, as shown in Fig. 7. These two points will determine the constants A and B. Then

$$p_1 = Bq_1 + A - 2\sqrt{AB - 1 - \epsilon}\,\sqrt{q_1} \tag{21}$$

$$p_1 = Bq_2' + A - 2\sqrt{AB - 1 - \epsilon}\,\sqrt{q_2'} \tag{22}$$

$$p_2' = Bq_1 + A + 2\sqrt{AB - 1 - \epsilon}\,\sqrt{q_1}\,. \tag{23}$$

From (21) and (22),

$$B_{\pm} = \frac{2}{(\sqrt{q_2'} - \sqrt{q_1})^2}\left[p_1 \pm \sqrt{p_1^2 - (\sqrt{q_2'} - \sqrt{q_1})^2\,(1 + \epsilon)}\right] \tag{24}$$

$$A_{\pm} = \frac{(\sqrt{q_2'} + \sqrt{q_1})^2}{4} B_{\pm} + \frac{1+\epsilon}{B_{\pm}} \tag{25}$$

$$(\sqrt{BA - 1 - \epsilon})_{\pm} = \frac{\sqrt{q_2'} + \sqrt{q_1}}{2} B_{\pm} \tag{26}$$

q_2' must be such that (for $A_{\pm}$ and $B_{\pm}$ are real values)

$$p_1^2 \geqslant (\sqrt{q_2'} - \sqrt{q_1})^2 (1 + \epsilon). \tag{27}$$

The $\pm$ sign indicates that there are two sets of B and A, i.e., two parabolas passing through the same points (q_1, p_1) and (q_2', p_1). Subtracting (21) from (23) and by (26),

$$\begin{aligned} p_2'(q_1, p_1, q_2', \epsilon)_{\pm} &= p_1 + 4\sqrt{AB - 1 - \epsilon}\,\sqrt{q_1} \\ &= p_1 + 4\frac{(\sqrt{q_2'} + \sqrt{q_1})\sqrt{q_1}}{(\sqrt{q_2'} - \sqrt{q_1})^2} \\ &\quad \times \left[p_1 \pm \sqrt{p_1^2 - (\sqrt{q_2'} - \sqrt{q_1})^2 (1+\epsilon)}\right]. \end{aligned} \tag{28}$$

Evidently, p_2' on the parabola of B_+ and A_+ is greater than p_2' on the parabola of B_- and A_-, i.e.,

$$p_2'(q_1, p_1, q_2', \epsilon)_+ > p_2'(q_1, p_1, q_2', \epsilon)_- .$$

It is apparent that as q_2' increases in (28), $p_2'(q_1, p_1, q_2', \epsilon)_+$ decreases. When q_2' increases to such a value that the equal sign in (27) holds, the two parabolas converge to a single one. On the other hand, as q_2' decreases, $p_2'(q_1, p_1, q_2', \epsilon)_+$ increases (see Fig. 7). Since q_2' is not allowed to be smaller than q_2, then a maximum p_2' can be obtained by taking the parabola passing through (q_1, p_1) and (q_2, p_1) with $B = B_+$ and $A = A_+$. For stability consideration, the conditions

$$p_2 \leqslant p_2'(p_1, q_1, q_2, \epsilon)_+ \tag{29}$$

$$p_1 \geqslant (\sqrt{q_2} - \sqrt{q_1})\sqrt{(1+\epsilon)} \tag{30}$$

must be satisfied.

Now consider an $\epsilon = 0$ parabola, and let this parabola pass through the points (q_1, p_1) and (q_2, p_1) with

$$B = B(p_1, q_1, q_2, 0)_+ \qquad \text{and} \qquad A = A(p_1, q_1, q_2, 0)_+ .$$

It is evident from (28) that

$$p_2'(p_1, q_1, q_2, 0)_+ > p_2'(p_1, q_1, q_2, \epsilon)_+ > p_2 .$$

Since

$$\lim_{\epsilon \to 0} B(q_1, p_1, q_2, \epsilon)_+ = B(q_1, p_1, q_2, 0)_+$$

$$= \frac{2}{(\sqrt{q_2} - \sqrt{q_1})^2} \left[[p_1 + \sqrt{p_1^2 - (\sqrt{q_2} - \sqrt{q_1})^2} \right]$$

$$\lim_{\epsilon \to 0} A(q_1, p_1, q_2, \epsilon)_+ = A(q_1, p_1, q_2, 0)_+$$

$$= \frac{(\sqrt{q_2} + \sqrt{q_1})^2}{4} B_+ + \frac{1}{B_+(q_1, p_1, q_2, 0)}$$

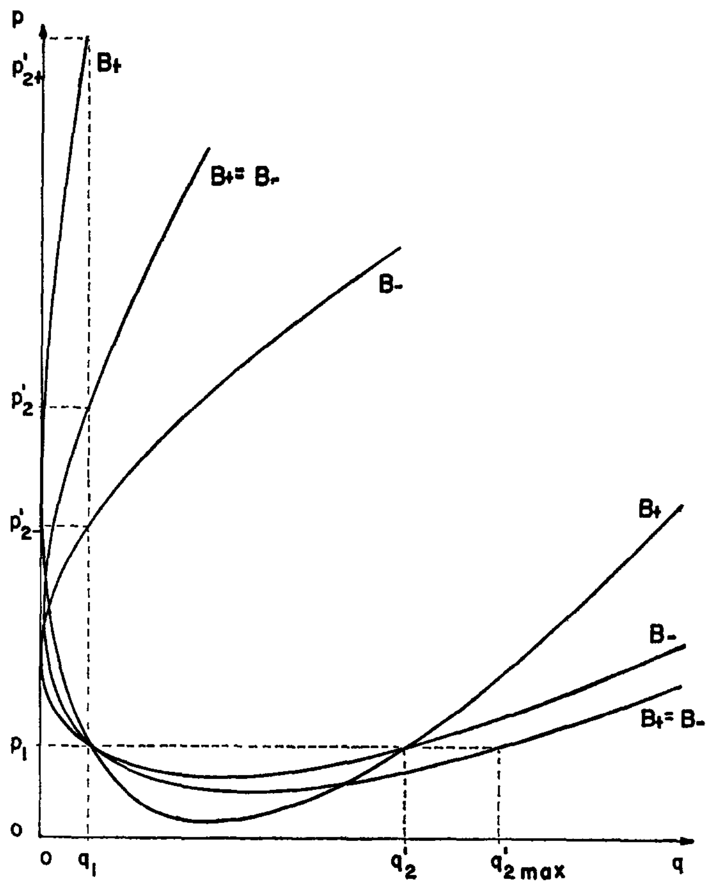

FIG. 7. $p - q$ plane representation

and

$$\lim_{\epsilon \to 0} p_2'(q_1, p_1, q_2, \epsilon)_+ = p_2'(q_1, p_1, q_2, 0)_+$$

$$= \frac{p_1(5q_1 + q_2 + 2\sqrt{q_1 q_2}) + 4(q_1 + \sqrt{q_1 q_2})\sqrt{p_1^2 - (\sqrt{q_2} - \sqrt{q_1})^2}}{(\sqrt{q_2} - \sqrt{q_1})^2}$$

then, for a given $\delta = p_2'(q_1, p_1, q_2, 0)_+ - p_2$, an ϵ can always be chosen so small such that

$$p_2'(q_1, p_1, q_2, 0)_+ - p_2'(q_1, p_1, q_2, \epsilon)_+ < \delta.$$

Hence one concludes that the conditions for stability are

$$p_2 < \frac{p_1(5q_1 + q_2 + 2\sqrt{q_1 q_2}) + 4(q_1 + \sqrt{q_1 q_2})\sqrt{p_1^2 - (\sqrt{q_2} - \sqrt{q_1})^2}}{(\sqrt{q_2} - \sqrt{q_1})^2} \tag{31}$$

and

$$p_1 > \sqrt{q_2} - \sqrt{q_1}. \tag{32}$$

The latter follows immediately from (30):

$$p_1 > (\sqrt{q_2} - \sqrt{q_1})\sqrt{1 + \epsilon} > (\sqrt{q_2} - \sqrt{q_1}).$$

Relation (31) gives the restriction on p_2 explicitly in terms of p_1, q_1, and q_2. Conditions will now be found which gives the restriction on q_2 explicitly in terms of q_1, p_1, and p_2. Only the $\epsilon = 0$ parabola is considered here since it is the limiting case.

$$p_1 = Bq_1 + A - 2\sqrt{AB - 1}\sqrt{q_1} \tag{33}$$

$$p_1 = Bq_2' + A - 2\sqrt{AB - 1}\sqrt{q_2'} \tag{34}$$

$$p_2 = Bq_1 + A + 2\sqrt{AB - 1}\sqrt{q_1}. \tag{35}$$

From (33) and (35)

$$B_\pm = \frac{p_1 + p_2}{4q_1} \pm \frac{1}{2q_1}\sqrt{p_1 p_2 - 4q_1}$$

$$\sqrt{AB - 1} = \frac{p_2 - p_1}{4\sqrt{q_1}}.$$

From (33) and (34)

$$\sqrt{q_2'} = \frac{2\sqrt{AB-1}}{B} - \sqrt{q_1} = \sqrt{q_1}\left[\frac{2(p_2-p_1)}{(p_2+p_1) \pm 2\sqrt{p_1p_2-4q_1}} - 1\right]. \tag{36}$$

Since $q_2' > q_1$, the term inside the bracket must be greater than 1, i.e.,

$$\frac{2(p_2-p_1)}{p_2+p_1 \pm 2\sqrt{p_1p_2-4q_1}} - 1 > 1,$$

or

$$-2p_1 > \pm 2\sqrt{p_1p_2-4q_1}.$$

This inequality is true only when the $-$ sign on the right side is utilized. Then there is only one parabola, i.e., only one q_2' for each fixed p_2. It can be shown that when

$$p_2 = p_1 + \frac{8q_1}{p_1} + 4\sqrt{q_1}$$

in (36), q_2' is maximum, and

$$q'_{2\max} = (\sqrt{q_1} + p_1)^2.$$

Therefore the condition for stability is

if

$$p_2 > p_1 + \frac{8q_1}{p_1} + 4\sqrt{q_1},$$

then

$$q_2 < \left[\frac{2(p_2-p_1)}{p_2+p_1-2\sqrt{p_1p_2-4q_1}} - 1\right]^2 q_1 \tag{37}$$

if

$$p_2 < p_1 + \frac{8q_1}{p_1} + 4\sqrt{q_1},$$

then

$$q_2 < (p_1 + \sqrt{q_1})^2. \tag{38}$$

Relation (38) follows from the fact that one can take the parabola passing through $(q_1, p_1 + (8q_1/p_1) + 4\sqrt{q_1})$ instead of the one passing through (q_1, p_2).

(B) Consider the function

$$V = u^2 + \frac{1}{q_{(t)}} v^2, \qquad 0 < a_1 \leqslant q_{(t)} \leqslant a_2 . \tag{39}$$

Then

$$\dot{V} = -\frac{\dot{q}_{(t)} + 2p(t)\, q_{(t)}}{q_{(t)}^2} v^2 . \tag{40}$$

Now if

$$\frac{\dot{q} + 2pq}{q^2} \geqslant \epsilon \qquad \text{or} \qquad p \geqslant \frac{\epsilon q}{2} - \frac{\dot{q}}{2q}, \tag{41}$$

then $\dot{V}$ is semidefinite negative. This implies that the solution is stable only. In addition, if p is also bounded from above, i.e., $p \leqslant a_3 < \infty$, then the solution is asymptotically stable. In this case it requires the existence of $\dot{q}_{(t)}$, whereas for the V function in (A) this is not necessary. The equal sign in (41) defines a surface in the three dimension space, in which p is allowed only in one side of the surface.

To show the asymptotic stability under these conditions, it is sufficient to show that as $t \to \infty$, $V \to 0$. Suppose that instead V approaches a position constant V_0 as $t \to \infty$. Let $V_0 < V < V_0 + \epsilon_1$ for $t > t_1$, $\epsilon_1 > 0$. Since $0 < 1/a_2 \leqslant 1/q \leqslant 1/a_1$, then

$$u^2 + \frac{1}{a_2} v^2 \leqslant u^2 + \frac{1}{q} v^2 \leqslant u^2 + \frac{1}{a_1} v^2 .$$

V is a nonincreasing function so that

$$V_0 < V \leqslant u^2 + \frac{1}{a_1} v^2 \qquad \text{and} \qquad u^2 + \frac{1}{a_2} v^2 \leqslant V < V_0 + \epsilon_1 .$$

Hence, the solution path for all $t > t_1$ must be circulating in the region bounded by the two ellipses:

$$L_1 : u^2 + \frac{1}{a_2} v^2 = V_0 + \epsilon_1$$

$$L_2 : u^2 + \frac{1}{a_1} v^2 = v_0$$

as shown in Fig. 8.

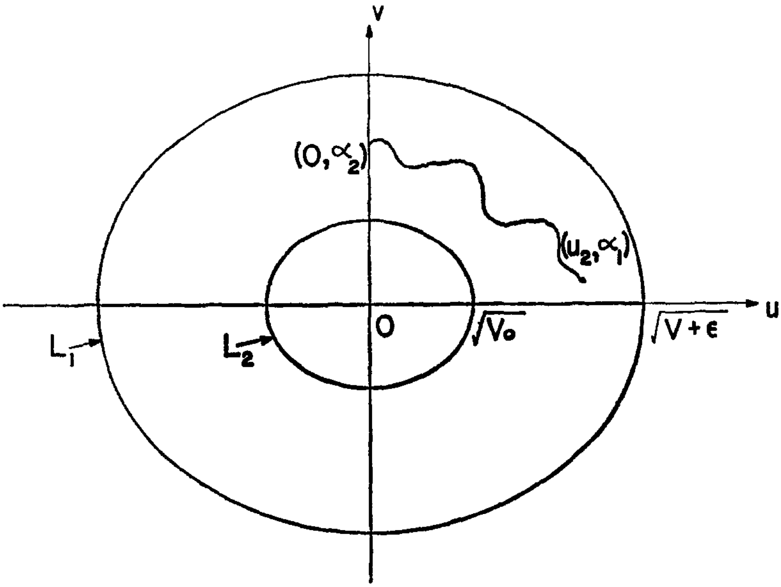

FIG. 8. $u - v$ plane representation

Assume a solution path pass $(0, \alpha_2)$ at t_1 and (u_2, α_1) at t_2, with $\alpha_2 > \alpha_1 > 0$. Integrating (15) from t_1 to t_2, one has

$$\int_{t_1}^{t_2} \dot{v} dt = - \int_{t_1}^{t_2} [p(t) v + q(t) u] dt.$$

Since

$$p(t) \leqslant a_3, \qquad v \leqslant \sqrt{(V_0 + \epsilon_1) a_2} = \alpha_3, \qquad q \leqslant a_2$$

and

$$u \leqslant \sqrt{V_0 + \epsilon_1} = \beta_1,$$

so that

$$\alpha_2 - \alpha_1 = \int_{t_1}^{t_2} (pv + qu) dt \leqslant (a_3\alpha_3 + a_2\beta_1)(t_2 - t_1)$$

then

$$t_2 - t_1 \geqslant \frac{\alpha_2 - \alpha_1}{a_3\alpha_3 + a_2\beta_1} > 0$$

and

$$\dot{V} \leqslant - \epsilon\alpha_1^2 < 0 \qquad \text{for} \qquad t_1 < t \leqslant t_2 .$$

Hence

$$\Delta V = V_1 - V_2 \geqslant \epsilon\alpha_1^2 \frac{\alpha_2 - \alpha_1}{a_3\alpha_3 + a_2\beta_1} > 0.$$

This shows that as long as the solution path lies outside L_2, V decreases at least by $\epsilon\alpha_1^2(\alpha_2 - \alpha_1)/(a_3\alpha_3 + a_2\beta_1)$ in each arc such as shown in Fig. 8. Now since the solution considered has an infinite number of such arcs, V would have to eventually become negative, which is a contradiction. Hence $V_0 = 0$, or $V \to 0$ as $t \to \infty$, and the solution is asymptotically stable.

IV. Convergence of Solutions of the Nonlinear System

The results in Section III will be used now to establish two theorems.

Theorem II. *Let*

(1) $f(y)$, $g(x)$ and $e(t)$ *be continuous,*

(2) $f'(y)$ exists and $0 < p_1 \leqslant f'(y) \leqslant p_2$,

(3) $g'(x)$ *exists and* $0 < q_1 \leqslant g'(x) \leqslant q_2$,

then all solutions of (1) converge to a unique (steady state) solution provided relations (31) and (32) or (37) and (38) hold.

Proof. Let $x_1(t)$ and $x_2(t)$ be any two distinct solutions,

$$\ddot{x}_1 + f(\dot{x}_1) + g(x_1) = e(t) \tag{42}$$

$$\ddot{x}_2 + f(\dot{x}_2) + g(x_2) = e(t) . \tag{43}$$

Subtracting (42) from (43),

$$\ddot{x}_2 - \ddot{x}_1 + \frac{f(\dot{x}_2) - f(\dot{x}_1)}{\dot{x}_2 - \dot{x}_1}(\dot{x}_2 - \dot{x}_1) + \frac{g(x_2) - g(x_1)}{x_2 - x_1}(x_2 - x_1) = 0. \tag{44}$$

Letting

$$u = x_2 - x_1 , \qquad v = \dot{u} = \dot{x}_2 - \dot{x}_1 = y_2 - y_1$$

in the above equation, the following equation is obtained:

$$\ddot{u} + p(t)\,\dot{u} + q(t)\,u = 0 \tag{15}$$

where

$$p(t) = \frac{f(\dot{x}_2) - f(\dot{x}_1)}{\dot{x}_2 - \dot{x}_1} ,$$

and

$$q(t) = \frac{g(x_2) - g(x_1)}{x_2 - x_1}.$$

Now to prove the convergence of x is equivalent to showing that $u \to 0$ as $t \to \infty$. By assumptions (2) and (3), p and q have the following properties:

(a) p and q are continuous functions of t,

(b) $0 < p_1 \leqslant p \leqslant p_2$, $0 < q_1 \leqslant q \leqslant q_2$.

Then it follows immediately that

$$u(t) = x_2(t) - x_1(t) \to 0.$$

$$v(t) = \dot{x}_2(t) - \dot{x}_1(t) \to 0$$

as $t \to \infty$ provided relations (31) and (32) or (37) and (38) hold.

This completes the proof of Theorem II.

Assuming that the conditions in Theorem II are satisfied, one can also conclude:

(i) If $e(t) = 0$, and $g(0) = 0$, and $f(0) = 0$, $x = 0$ is a solution; then all other solutions converge to $x = 0$. Hence, it is asymptotically stable in the large.

(ii) If $e(t)$ is periodic, all solutions will approach a limit cycle (only one fixed point as defined in Section II).

(iii) If $e(t)$ varies in random fashion, then the steady state solution is independent of the initial conditions.

THEOREM III. *In addition to the assumptions in Theorem I, if*

(1) $g'(x)$ *is continuous and* $g'(x) \geqslant q_1 > 0$ *in* $|x| \leqslant I$,

(2) $g''(x)$ *exists and* $|g''(x)| \leqslant C_2$ *in* $|x| \leqslant I$,

(3) $f'(y)$ *exists and* $f'(y) \geqslant p_1 > 0$ *in* $|y| \leqslant J$,

then all solutions of (1) converge to a unique (steady state) solution provided $p_1^2 > 2EC_2/q_1$.

PROOF. Consider any two distinct solutions. Equations (44) and (15) are used again. By Theorem I, the solutions of (1) will be ultimately bounded in R. Since $f'(y)$ and $g'(x)$ exist, p and q each have upper bounds in R. Then according to (B) in Section III, if the condition $p + (\dot{q}/2q) > \epsilon q/2$ is satisfied $u \to 0$ as $t \to \infty$.

Now

$$\frac{\dot{q}}{2q} = \tfrac{1}{2}\frac{d}{dt}\log q = \tfrac{1}{2}\frac{d}{dt}\{\log[g(x_2) - g(x_1)] - \log(x_2 - x_1)\}$$

$$= \tfrac{1}{2}\left[\frac{g'(x_2)\,y_2 - g'(x_1)\,y_1}{g(x_2) - g(x_1)} - \frac{y_2 - y_1}{x_2 - x_1}\right].$$

Let x_3 be some intermediate values between x_2 and x_1, then

$$\frac{\dot{q}}{2q} = \frac{1}{2g'(x_3)}\left[y_2\frac{g'(x_2) - g'(x_3)}{x_2 - x_1} + y_1\frac{g'(x_3) - g'(x_1)}{x_2 - x_1}\right].$$

Since x_3 is in between x_2 and x_1, so that

$$\left|\frac{g'(x_2) - g'(x_3)}{x_2 - x_1}\right| \leqslant \left|\frac{g'(x_2) - g'(x_3)}{x_2 - x_3}\right| = |g''(x_4)| \leqslant C_2,$$

$$\left|\frac{g'(x_3) - g'(x_1)}{x_2 - x_1}\right| \leqslant \left|\frac{g'(x_3) - g'(x_1)}{x_3 - x_1}\right| = |g''(x_5)| \leqslant C_2.$$

By Theorem I,

$$|y_2| \leqslant \frac{2E}{p_1}, \qquad \text{and} \qquad |y_1| \leqslant \frac{2E}{p_1},$$

then

$$\left|\frac{\dot{q}}{2q}\right| \leqslant \frac{1}{2q_1}\left[|y_2|\cdot\left|\frac{g'(x_2) - g'(x_3)}{x_2 - x_1}\right| + |y_1|\cdot\left|\frac{g'(x_3) - g'(x_1)}{x_2 - x_1}\right|\right]$$

$$\leqslant \frac{1}{2q_1}\cdot\frac{2E}{p_1}\cdot[|g''(x_4)| + |g''(x_5)|] \leqslant \frac{2EC_2}{q_1 p_1}.$$

Now if $p_1^2 > 2EC_2/q_1$ or $p_1 > 2EC_2/q_1p_1$, let $k = p_1 - (2EC_2/q_1p_1)$. Since $q \leqslant \max g'(x)$ in R, and $p \geqslant p_1$, then an ϵ can be chosen so small such that

$$p - \frac{2EC_2}{q_1p_1} \geqslant p_1 - \frac{2EC_2}{q_1p_1} = k > \frac{\epsilon q}{2}.$$

Hence

$$p + \frac{\dot{q}}{2q} \geqslant p - \left|\frac{\dot{q}}{2q}\right| \geqslant p - \frac{2EC_2}{q_1p_1} > \frac{\epsilon q}{2}.$$

This completes the proof of this theorem.

If the conditions in this theorem are satisfied, the same conclusions as that following Theorem II can be made.

Note that the conditions in Theorem III do not depend on the upper bounds of $f'(y)$ and $g'(x)$ which are required in Theorem II. However, the bounds of $|e(t)|$ and $|g''(x)|$ are required in Theorem III but not in Theorem II.

Consider the Duffing equation

$$\ddot{x} + b\dot{x} + x + ax^3 = E \sin \omega t$$

with $0 < b < 2$, $a > 0$. The solutions are bounded as

$$|x(t)| \leqslant I = E\left[1 + \sqrt{\frac{4}{b^2} - 1}\, \exp\left(-\frac{\pi}{2\sqrt{(4/b^2) - 1}}\right)\right] \tag{45}$$

$$|\dot{x}(t)| \leqslant J = \frac{2E}{b}. \tag{46}$$

Since

$$g'(x) = 1 + 3ax^2 > 1\,, \qquad g''(x) = 6ax\,,$$

then $C_2 = 6aI$, and

$$\frac{2EC_2}{q_1} = 12aEI.$$

If a or E is sufficiently small such that the inequality

$$b^2 > \frac{2EC_2}{q_1} = 12aE^2\left[1 + \sqrt{\frac{4}{b^2} - 1}\, \exp\left(-\frac{\pi}{2\sqrt{(4/b^2) - 1}}\right)\right] \tag{47}$$

holds, then all solutions will converge to unique periodic solution having the same period as that of the forcing function. Otherwise it may have three periodic solutions or subharmonics.

CONCLUSIONS

In the last two decades numerous articles have been written on the behavior of nonlinear second-order differential equations. For the most part these articles treated very specific equations. This paper investigates the boundedness of the output variable and the time derivative of the output variable of a nonlinear second-order system containing a fairly general form of nonlinear damping and nonlinear restoring force under nonautonomous operation, the only restriction on the forcing function, $e(t)$, being that it is bounded. This paper may be therefore considered as an extension and generalization of some of the previous works.

The stability of solutions is investigated by studying the asymptotic stability of a second-order linear time-varying equation which is formed from the original nonlinear system equation. Utilizing Lyapunov's Second Method two different functions are used to obtain different sufficient conditions on the asymptotic stability of the linear system.

Acknowledgment

The material in this paper is based on the first part of the thesis submitted by Y. S. Lim in partial fulfillment for the Ph.D. degree in Electrical Engineering at the University of Michigan. The authors wish to thank Prof. W. Kaplan and Prof. L. Cesari for their valuable suggestions and criticism.

References

1. Levinson, N. On the existence of periodic solutions for second order differential equation with a forcing term. *J. Math. Phys.* **22**, 41-48 (1943).
2. Cartwright, M. L. and Littlewood, J. E. On the nonlinear differential equation of the second order. *Ann. Math.* (2) **45**, 723-737 (1944).
3. Reuther, G. E. H. On certain nonlinear differential equations with almost periodic solutions. *J Lond. Math. Soc.* **26**, 215 (1951).
4. Loud, W. S. Boundedness and convergence of solutions of
$$\ddot{x} + c\dot{x} + g(x) = e(t).$$
Duke Math. J. **24**, 63-72 (1957).
5. Kaplan, W. Stability theory. *Proc. Symp. Non-linear Circuit Analysis, Polytech. Inst. Brooklyn, New York*, pp. 3-21 (1956).
6. Cesari, L. "Asymptotic Behavior and Stability Problems in Ordinary Differential Equations." Springer, Berlin, 1959.
7. Starzhinskii, V. M. Sufficient conditions of stability of Mechanical systems with one degree of freedom. *Priklad. Mat. i Mekh.* **16**, 369-374 (1952).
8. Coddington, E. A. and Levinson, N. "Theory of Ordinary Differential Equations." McGraw-Hill, New York, 1955.