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The Hydrostatic Trough.

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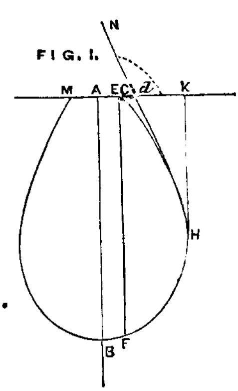
PROBLEM.—If a perfectly flexible and inextensible trough have its edges firmly fixed in a horizontal plane, and filled with a heavy fluid, it is required to find the equation of a transverse section.

Or, the problem may be stated in the following way:

FIG. 1.



If a perfectly flexible and inextensible arc, MBC, Fig. 1, be fixed at its ends, M and C, and filled with a heavy fluid; it is required to find its form under the pressure of the fluid. This I call "A Hydrostatic Trough," in distinction from the "Hydrostatic Arch" of Von Villarceaux.

The problem which I have proposed is that of a normally pressed arc. But few problems of such curves have, to my knowledge, ever been solved. The first one is very simple, being that of a ring placed horizontally under a fluid, in which case the curve is a circle; the second is the "Hydrostatic Arch" of Von Villarceaux, in which the arch is placed under a fluid so that the crown will be at a finite distance from the surface of the fluid. The third was investigated for a suspension bridge, by Mr. Robinson, and is published in the September Number of this Journal for 1863. In the following article I shall investigate the fourth one.

I will first investigate the general equations applicable to normally pressed arcs.

Let $f(x,y,z,)$ be the equation of the curve referred to rectangular co-ordinates,

T, the tension at any point,

X,Y,Z, the components of the impressed forces on a unit of length of the curve,

ds an element of the arc.

Then, Xds, Yds, Zds, are the components on an element of length.

$T\frac{dx}{ds}$ is the component of T on the axis of x, and

$d\,(T\frac{dx}{ds})$ is the component of the tension on ds due to the impressed forces on the element, consequently equal Xds; similarly for the others.

$$\left.\begin{aligned} \therefore\quad d\left(T\frac{dx}{ds}\right) &= Xds \\ d\left(T\frac{dy}{ds}\right) &= Yds \\ d\left(T\frac{dz}{ds}\right) &= Zds \end{aligned}\right\} \qquad (1)$$

These equations are applicable to any curve in which the forces are continuous functions of the co-ordinates. But in a normally pressed arc T *is constant;* for it cannot vary unless there be a tangential component of the applied forces; but there can be no tangential component when they are all normal.

Observing this, squaring and adding, and we have

$$T^2\left[\left(d\frac{dx}{ds}\right)^2+\left(d\frac{dy}{ds}\right)^2+\left(d\frac{dz}{ds}\right)^2\right]=(X^2+Y^2+Z^2)\,ds^2 \qquad (2)$$

Let

ρ = radius of curvature at point, $(x,y,z,)$

P = applied force on a unit of length, and equation (2) becomes

$$T^2\rho^2\,ds^2 = P^2\,ds^2;$$

$$\therefore P=\frac{T}{\rho}. \qquad (3)$$

Hence, the applied force on each unit of length will be inversely as the radius of curvature.

In Fig. 1, let

$D = AB =$ total depth of the trough,
$\rho_0 =$ the radius of curvature at the lowest point,
$i =$ angle Ndk which the tangent at any point makes with the axis x,
$i_0 =$ the value of i at the superior element,
$w =$ weight of unit of volume of the fluid.

Take the origin at A, the middle point of the span; x horizontal, and y vertical.

Since the pressure of a fluid is proportional to its depth, we have

$P = wy$; nence from (3) we have
$T = wy\rho$ for any point, and
$T = wD\rho_0$ for the lowest point.

$$\therefore \quad wy\rho = wD\rho_0. \qquad (4)$$

$$\therefore \quad \text{or } \rho = \frac{D\rho_0}{y}, \qquad (5)$$

Hence the radius of curvature at any point is inversely as the distance of the point below the surface of the fluid; hence at the upper end it is infinite.

The second member of (4) is constant; hence it is the equation of an hyperbola referred to its asymptotes, in which ρ is the abscissa and y the corresponding ordinate.

Substituting the value of the radius of curvature in (5), and we have

$$\pm\left(1+\frac{dy^2}{dx^2}\right)^{-\frac{3}{2}}\frac{d^2y}{dx^2} = \frac{y}{D\rho_0}.$$

The first integral is

$$\mp\left(1+\frac{dy^2}{dx^2}\right)^{-\frac{1}{2}} = \frac{y^2}{2D\rho_0} + C \qquad (6)$$

In the second integral I find that the *minus* sign before the parenthesis gives an imaginary result; hence we use the plus sign only.

Observing that $\frac{dy}{dx} = 0$ for $y = D$, and (6) gives

$$C = 1 - \frac{D}{2\rho_0} = \frac{2\rho_0 - D}{2\rho_0}.$$

$$\therefore \left(1+\frac{dy^2}{dx^2}\right)^{-\frac{1}{2}} = \frac{y^2}{2D\rho_0} + \frac{2\rho_0 - D}{2\rho_0}. \qquad (7)$$

For $y = 0$, $i = i_0$ and $\left(1+\frac{dy^2}{dx^2}\right)^{-\frac{1}{2}} = \cos. i_0$.

$$\therefore \cos. i_0 = \frac{2\rho_0 - D}{2\rho_0} = 1 - \frac{D}{2\rho_0}. \qquad (8)$$

$$\therefore \rho_0 = \frac{\frac{1}{2}D}{1-\cos. i_0} = \frac{D}{4\sin^2\frac{1}{2}i_0}. \qquad (9)$$

$$\begin{aligned} \text{Hence if } i_o &= 0, \ \rho_o = \infty \\ i_o &= 90°, \ \rho_o = \tfrac{1}{2} \text{ D} \\ i_o &= 180°, \ \rho_o = \tfrac{1}{4} \text{ D} \\ i_o &> 90°, \ \rho_o < \tfrac{1}{2} \text{ D} \\ i_o &< 90°, \ \rho_o > \tfrac{1}{2} \text{ D} \end{aligned}$$

$\rho_o = \frac{1}{4}$ D is the limit of ρ_o in the direction of its smallest value, but a limit it can never reach unless T is infinite; for the vertical component of T at the upper end must sustain one-half the fluid; or

$$w \int_o x \ dy = \text{T sin. } i_o. \qquad . \qquad . \qquad . \qquad (10)$$

but for $i_o = 180°$, sin. $i_o = 0$ ∴ T must be infinite if the first number is finite.

Some of the preceding expressions may be deduced in a more elementary way: thus, the total horizontal pressure of the fluid must equal the tension at the lowest point, together with the horizontal component at the upper end; or more generally, the horizontal components in one direction must equal those in the opposite direction. The horizontal pressure of the fluid is $\frac{1}{2} w \text{ D}^2$: the horizontal component of the tension at the upper end is T cos. i_o, and is negative when the angle is obtuse. We have then

$$\tfrac{1}{2} w\text{D}^2 = \text{T} - \text{T cos. } i_o.$$

$$\therefore \text{T} = \frac{\frac{1}{2} w \text{ D}^2}{1 - \text{cos. } i_o}. \qquad . \qquad . \qquad . \qquad (11)$$

Let $y_1 = \text{KH} =$ the vertical tangent ordinate.

The horizontal pressure between y_1 and D equals the tension at the lowest point.

$$\therefore \tfrac{1}{2} w \ (\text{D}^2 - y_1^2) = \text{T} = w\text{D}\rho_o. \qquad . \qquad . \qquad (12)$$

$$\therefore y_1 = \sqrt{\text{D}^2 - 2 \text{ D} \rho_o}. \qquad . \qquad . \qquad . \qquad (13)$$

Eliminating between (11), (12), and (13), and we find,

$$\rho_o = \frac{\frac{1}{2}\text{D}}{1 - \text{cos. } i_o} \text{ which is the same as (9).}$$

If $i_o = 90°$ Equation (11) gives $\text{T} = \frac{1}{2} w\text{D}^2$
and " (9) " $\rho_o = \frac{1}{2}$ D, and this
in " (13) " $y_1 = 0$, and this
in " (12) " $\text{T} = \frac{1}{2} w\text{D}^2$ as before.

If i_o is less than 90°, we have

$$\tfrac{1}{2} w \text{ D}^2 + \text{T cos. } i_o = \text{T}.$$

$$\therefore \text{T} = \frac{\frac{1}{2} w \text{ D}^2}{1 - \text{cos. } i_o} \text{ which is the same as (11).}$$

Now returning the differential expression we observe that (6) gives

$$(dx^2 + dy^2)^{\frac{1}{2}} = ds = \frac{2 \rho_o \text{ D}dy}{\sqrt{4 \text{ D}^2\rho_o^2 - (2 \text{ D}\rho_o \text{ C} + y^2)^2}} \qquad (14)$$

$$\text{or} \quad dx = \frac{2 \text{ D } \rho_o \text{ C} + y^2}{\sqrt{4 \text{ D}^2 \rho_o^2 - (2 \text{ D } \rho_o \text{C} + y^2)^2}} dy. \qquad . \qquad (15)$$

To find where x is a maximum, make $\frac{ds}{dy}=1$ in (14) or $\frac{dx}{dy}=$ 0 in (12); in either case we find

$$y=y_1=\sqrt{D(-2\rho_o+D,)}$$ which is the same as (13).

Equations (14) and (15) may be integrated by means of elliptic functions. Substitute the value of C and make *

$$c^2=\frac{D}{4\rho_o}=\sin^2.\ \theta\ \therefore \sin.\ \theta=\tfrac{1}{2}\sqrt{\frac{D}{\rho_o}}. \qquad (16)$$

$$y=D\cos.\ \varphi\ \therefore\ dy=-D\sin.\ \varphi d\varphi. \qquad (17)$$

These in (14) give

$$s=\int\frac{-2\rho_o D^2\sin.\ \varphi d\varphi}{\sqrt{D^2\rho_o{}^2-(2D\rho_o-D^2+D^2\cos.^2\varphi)^2}}$$

Make $\cos.^2\varphi=1-\sin.^2\varphi$, develop and reduce, and we find

$$s=\int\frac{-2\rho_o D^2\sin.\ \varphi d\varphi}{\sqrt{4\rho_o D^3(1-\frac{D}{4\rho_o}\sin.^2\varphi)\sin.^2\varphi}}$$

$$=\sqrt{\rho_o D}\int_o^\varphi\frac{-d\varphi}{\sqrt{1-c^2\sin.^2\varphi}}. \qquad (18)$$

The substitutions in (15) will give

$$x=\int\frac{-[D(2\rho_o-D)+D\cos.^2\varphi]\,d\varphi}{2\sqrt{\rho_o D}\sqrt{1-\frac{D}{4\rho_o}\sin.^2\varphi}}$$

$$=-2\sqrt{\rho_o D}\left[\frac{\frac{1}{2}-\frac{D}{4\rho_o}\sin.^2\varphi}{\sqrt{1-\frac{D}{4\rho_o}\sin.^2\varphi}}\right]d\varphi$$

By adding and subtracting 1 to the numerator, it may be put under the following form, viz:

$$x=-2\sqrt{\rho_o D}\left[-\tfrac{1}{2}\int_o^\phi\frac{d\phi}{\sqrt{1-c^2\sin.^2\phi}}+\int_o^\phi(1-c^2\sin.^2\varphi)d\varphi\right] \qquad (19)$$

The complete integral of (18) is

$$s=\sqrt{\rho_o D}\ \mathrm{F}_{1(c)}. \qquad (20)$$

The general integral of (19) is

$$x=2\sqrt{\rho_o D}\left[-\tfrac{1}{2}\mathrm{F}_{(c,\phi)}+\mathrm{E}_{(c,\phi)}\right] \qquad (21)$$

in which we compute the arc from the lowest joint. I have changed the sign of x, for the curve is symmetrical with the axis y; and hence the positive value will equal the negative one.

* See Legendre's Fonctions Elliptique, Chap. III.

For the half span AC, we have from (17) and (21)

$$y=0, \therefore \varphi = 90^\circ$$

$$x=x_0=2\sqrt{\rho_0 \text{D}}\left[-\tfrac{1}{2}\text{F}_{1(c_1)}+\text{E}_{1(c_1)}\right]. \qquad (22)$$

Equation (21) is the equation of the curve, and by means of it and (17) it may be constructed. We see that the *form* is independent of the weight of the fluid.

In problems of this kind the data would naturally be the length of arc and span, but with such data the problem can only be solved by successive approximations; for c, which is a function of D in (21), cannot be found explicitly. But if D and ρ_0 are known we may obtain a direct solution.

For example, let $\rho_0 = \frac{16}{49}\text{D}$.

Then from (16) $c=\sin.\theta=\frac{7}{8} \therefore \theta=61^\circ\ 3'$.

In the second volume of Legendre's *Fonctions Elliptique*, Table VIII, we find for $\theta=61\frac{01}{20}$ that $\text{F}_{1(c_1)}=2{\cdot}18566$.

Hence the total length is

$$2\text{ s}=2\sqrt{\frac{16}{49}\text{D}^2}\times 2{\cdot}18566= 2{\cdot}52912\text{ D}.$$

From the same table we find

$$\text{E}_{\lambda_1}\left(61\tfrac{01}{20}\right) = 1{\cdot}20106.$$

Then (20) gives $2x=6{\cdot}2474$ D . . (23)

for the total span at the upper end.

To construct the curve, we will call $\theta=61^\circ$. From equations (21) and (17) and Table IX, we find

for	$\varphi = 0^\circ$	$y = 1{\cdot}000$ D	$x = 0{\cdot}000$
	$\varphi = 20^\circ$	$y = 0{\cdot}940$ D	$x = 0.190$ D
	$\varphi = 40^\circ$	$y = 0{\cdot}766$ D	$x = 0{\cdot}325$ D
	$\varphi = 50^\circ$	$y = 0{\cdot}643$ D	$x = 0{\cdot}354$ D
	$\varphi = 53^\circ$	$y = 0{\cdot}602$ D	$x = 0{\cdot}357$ D
	$\varphi = 54^\circ$	$y = 0{\cdot}587$ D	$x = 0{\cdot}361$ D
	$\varphi = 55^\circ$	$y = 0{\cdot}574$ D	$x = 0{\cdot}360$ D
	$\varphi = 70^\circ$	$y = 0{\cdot}342$ D	$x = 0{\cdot}308$ D
	$\varphi = 90^\circ$	$y = 0{\cdot}000$ D	$x = 0{\cdot}125$ D

By means of these values, I have constructed Fig. 1.

For the maximum abscissa we have from (13) $y_1=\frac{1}{7}\text{D}\sqrt{17}=0{\cdot}5890$ D, which in (17) gives $\varphi=53^\circ\ 54'$, which agrees well with the table above. This value of φ in (21) will give x.

For the inclination at the upper end we have from (8), $i_0=111^\circ\ 18'$.

For the tension, (4) gives $\text{T} = \frac{16}{49}w\text{ D}^2$.

For the total weight of the fluid, we have $2\,\text{T} \sin. i_o = \frac{32}{49} w\, \text{L}^2 \times 0{\cdot}93789$; hence the total area of the curve is $\frac{32}{49}\text{D}^2 \times 0{\cdot}93789$.

FIG. 2.

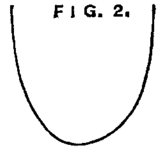
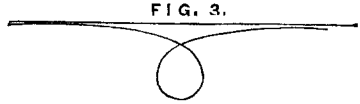


Fig. 2 is the curve when the upper ends are vertical.

Suppose $\rho^\circ = \frac{1}{4}\,\text{D}$; then we will have $\varphi = 90^\circ$, and this in (21) gives $x = -\infty$; hence the axis of x is an asymptote to the curve, as shown in Fig. 3.

FIG. 3.



To find ρ_o when the upper ends terminate in the origin. For this x and y are zero, and (21) gives

$\frac{1}{2}\,\text{F}_{1(c,)} = \text{E}_{1(c,)}$ from which c must be found. Table VIII, in Legendre, gives $c = \sin.\ 65^\circ\ 18'$ nearly, and observing that $c = \frac{1}{2}\sqrt{\frac{\text{D}}{\rho_o}}$ and we find

$$\rho_o = 0{\cdot}303\,\text{D nearly.}$$

To find the evolute of the curve.

Let x, and y, be the co-ordinates of the evolute. Then we readily find

$$-x_{\prime} = -x + \rho \sin.\ i.$$
$$y_{\prime} = y - \rho \cos.\ i.$$

which combined with the preceding equations so as to eliminate x, y, ρ, and i, will give the equation of the evolute.

The question naturally arises whether this principle is applicable to shells filled with a fluid, and if so it may give important information on the construction of jugs and bottles. Conceive a shell to be generated by the revolution of one of these curves about the axis of y: the shell generated will be of uniform thickness. Now, intersect this shell with a plane through the axis of y, and examine the pressure along this intersection. We at once see that the pressure is greatest at the lowest point, but the thickness is uniform and hence does not fulfill the conditions.

If we suppose that the total resisting surface in all horizontal intersecting planes in the same, we meet with a similar difficulty, for then the shell would be thinner (if in the form of a jug) below the middle of the depth than at the top, but the pressure would be greater at the latter than at the former point. In the case of bottles we must consider the strain in different planes; hence our investigation gives no light upon their proper construction.

In the early part of this article I intimated that but few curves of normally pressed arcs have been investigated; but it may be well to add that if a string be drawn upon (or around) a perfectly smooth curved surface the tension will be constant and the pressure normal;

but in this case the curve is known *à priori.* Such curves are called *Geodesic*, and their form depends upon that of the surface about which they are drawn.

The Actual State of the Works on the Mont Cenis Tunnel, and Description of the Machinery Employed. By THOMAS SOPWITH, JR.

From the Lond. Civ. Eng. and Arch. Jour., March, 1864.

This tunnel would form the completing link of the Victor Emmanuel Railway, and be the means of putting France and Italy in direct railway communication. The railway on the French side was already opened to St. Michael in Savoy, and on the Italian side to Susa in Piedmont. When the whole line was completed the mails and traffic with India might perhaps be advantageously transferred from Marseilles to some Italian port, as the Mediterranean Sea transit would thus be materially shortened.

During the last twenty years, many routes had been surveyed and recommended for crossing the great barrier of the Alps. Of these, that by the Mont Cenis was generally considered the most feasible; and that it was only a question, whether the mountain should be crossed by a series of inclines, or whether a tunnel should be made. In 1857, Messrs. Sommeiller, Grandis, and Grattoni, brought before public notice a new system of boring by machinery, instead of by hand labor. A government commission was appointed to report upon it, and to see if it could be applied to the boring of the tunnel under Mont Cenis. Their report was favorable, and M. Sommeiller and his partners were shortly afterwards charged with the execution of the work.

The ends only were available for attack, it being impossible, as was known from the first to sink shafts. It was feared that the ventilation would seriously retard, or altogether prevent, the completion of the tunnel: but this fear was uncalled for, as the artificial ventilation of collieries overcame greater natural difficulties, and the ventilating current passed through a longer distance, than could possibly be required in this tunnel. M. Sommeiller also proposed to use compressed air for driving the machinery, and calculated that on its escape, a volume of fresh air would be supplied adequate to the requirements of the workmen. The tunnel at the Modane, or French side, was of the following dimensions:—25 ft. 3½ ins. wide at the base, 26 ft. 2¾ ins. wide at the broadest part, and 24 ft. 7 ins. in height; the arch being a semicircle nearly. At Bardonnèche, the height was increased 11¾ inches. The exact length between the ends was 7·5932 miles. The present ends would not be the permanent entrances, as it was intended that a curved gallery should leave the tunnel at the north side, 415 yards from the end, and at the south side, 277 yards.

At Modane, the tunnel, was built entirely with stone; at Bardonnèche, for the greater part, the side walls only were of stone, and the remainder of brick. The Bardonnèche end was 434 feet higher than that at Modane. For one-half the length of the tunnel, therefore, from Modane to the middle, the gradient would be 1 in 45½; the other side