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Omissions in a Closed Survey. By DE VOLSON WOOD, Prof. of C. E. University of Michigan.

It is well known that, if certain omissions are made in a *Closed Survey*, they may be supplied from the remaining notes. I propose in this article to investigate the subject analytically.

I wish at the outset to express my acknowledgments to Mr. Cleveland Abbe, formerly a tutor in the University, and now connected with the United States Coast Survey, for the free use which he has permitted me to make of his notes upon this subject. His analysis covered nearly the whole ground, so that I have only to make slight additions and changes, and add geometrical illustrations, to put it in the form in which it is now presented.

In order to convert the bearings taken in the field into angles, which will be more convenient for our computations, we have

The bearing	N β°E	is equivalent to the angle		β°
" "	S β°E	"	" "	$180^\circ-\beta^\circ$
" "	S β°W	"	" "	$180^\circ+\beta^\circ$
" "	N β°W	"	" "	$360^\circ-\beta^\circ$

and conversely an angle β being given we have when

$\beta^\circ < 90^\circ$ and $> 0^\circ$	the bearing is	N β°E.
$\beta^\circ > 90^\circ$ and $< 180^\circ$	" "	S $180^\circ-\beta^\circ$E.
$\beta^\circ < 270^\circ$ and $> 180^\circ$	" "	S $\beta^\circ-180^\circ$W.
$\beta^\circ > 270^\circ$ and $< 360^\circ$	" "	N $360^\circ-\beta^\circ$W.

Let l_1; l_2; l_3; &c., be the lengths of the known sides.
x_1; x_2 " " unknown sides.
b_1; b_2; b_3; &c., the known angles as found from the known bearings as given above.
β_1; β_2; . . the unknown angles.
D = the algebraic sum of all the known departures.
L = the algebraic sum of all the known latitudes.

Then for a closed field we evidently have

$$\left.\begin{array}{l}\Sigma \text{ Dep.} = l_1 \sin b_1 + l_2 \sin b_2 + \ . \ . \ l_n \sin b_n = 0 \\ \Sigma \text{ Lat.} = l_1 \cos b_1 + l_2 \cos b_2 + \ . \ . \ l_n \cos b_n = 0\end{array}\right\} \quad . \quad (1)$$

From which any two quantities may generally be found when all the others are known. We have several cases.

CASE I.

Let One Bearing be Omitted.

Letting l=the length of any side, and we have from (1)

$$\left.\begin{array}{l}l \sin. \beta_1 = \text{D} \\ l \cos. \beta = \text{L}_1\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$\therefore \sin. \beta_1 = \frac{D}{l} \cos. \beta_1 = \frac{L}{l}$. . . . . (3)

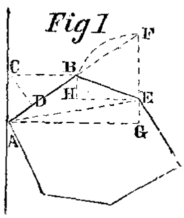
and β_1 obtained from each of those must be the same. We also have tang. $\beta_1 = \frac{D}{L}$, which should give the same value as each of the others.

Eliminating l from (2) gives

$D \cos. \beta - L \sin. \beta = 0$ . . . . . (4)

These may be deduced geometrically; thus, let the bearing of AB, Fig. 1, be unknown.

Fig 1



Then CB=D. AC=L, and,

$\sin. CAB = \sin. \beta_1 = \frac{CB}{AB} = \frac{D}{l}$

$\cos. CAB = \cos. \beta_1 = \frac{AC}{AB} = \frac{L}{l}$

$\tan. CAB = \tan. \beta_1 = \frac{CB}{AC} = \frac{D}{L}$

From C let fall CD, perpendicular to AB. Then

$CD = CB \sin CBD = CB \cos. CAD = D \cos. \beta_1$

Also $CD = CA \sin CAD = L \sin. \beta_1$

By subtracting these we have $D \cos. \beta_1 - L \sin. \beta_1 = 0$

All these are the same as the analytical results.

Case II.

Let One Distance be Omitted.

Then Eq. (1) gives, if b be any angle,

$x_1 \sin. b = D$ . . . . (5)

$x_1 \cos. b = L$ . . . . (6)

$\therefore x_1 = \frac{D}{\sin. b} = \frac{L}{\cos. b}$

Squaring (5) and (6) and adding, gives

$$x^2 = D^2 + L^2$$

In Fig. 1, $AB = CA \div \cos CAB$ or $x_1 = \frac{L}{\cos. b}$

also $AB = CB \div \sin CAB$ or $x_1 = \frac{D}{\sin. b}$

also $AB^2 = AC^2 + BC^2$,

or $x_1^2 = L^2 + D^2$.

Case III.

The Length and Bearing of One Side Omitted.

From (1) we have

$$\left.\begin{array}{l} x_1 \sin \beta_1 = D \\ x_1 \cos \beta_1 = L \end{array}\right\} \quad . \quad . \quad . \quad . \quad (7)$$

From these we readily find

$$\tan \beta_1 = \frac{D}{L}$$

$$x_1 = \sqrt{D^2 + L^2}$$

These results have already been illustrated geometrically.

CASE IV.

The Length of One Side and Bearing of Another Wanting.

For this Eq. (1) gives

$$x_1 \sin b_1 + l_2 \sin \beta_2 = D \quad . \quad . \quad (8)$$

$$x_1 \cos b_1 + l_2 \cos \beta_2 = L \quad . \quad . \quad (9)$$

Multiply Eq. (8) by $\cos b_1$ and (9) by $\sin b_1$ and subtract the results, and we find

$$l_2 \sin (\beta_2 - b_1) = D \cos b_1 - L \sin b_1 \quad . \quad . \quad (10)$$

from which we may find two values for $\beta_2 - b_1$, viz: $\theta°$ and $180° - \theta°$.

$$\therefore \quad \beta_2 = \theta + b_1 \text{ and } 180° - \theta° + b_1$$

From (8) and (9) we find

$$x_1 = \frac{D}{\sin b_1} - \frac{l_2}{\sin b_1} \sin \beta_2 \quad . \quad . \quad . \quad (11)$$

$$x_1 = \frac{L}{\cos. b_1} - \frac{l_2}{\cos. b_1} \cos. \beta_2$$

in which if both values of β_2 be substituted, we will obtain two values of x_1.

From (10) we find that β_2 (and therefore x_1) is imaginary when

$$\frac{D \cos. b_1 - L \sin. b_1}{l_2} > 1.$$

If $\beta_2 - b_1 = 90°$ there is only one solution.

If the second member of (10) is zero, we have $\beta_2 = b_1$; or the two courses coincide in direction, and we have from (8) or (9)

$$x_1 + l_2 = \frac{D}{\sin. b_1}$$

and the case is essentially the same as II.

To show this geometrically, we first observe that the two courses may be adjacent or separated. If separated, we may transfer the sides so as to bring the desired ones adjacent.

In fig. 1, let the length of AB and bearing of BE be wanting. Then by Case III compute the bearing and length of a line which joins A and E. Lay off the line AB according to the given bearing. Then with E as a centre and radius equal BE, describe an arc which will intersect AB in two points, B and F; hence there will be two lengths, AB and AF; and two bearings, one for BE, the other for FE. If the arc BF is tangent to AB, there will be but one solution; and if the arc does not touch AB, the solution is impossible, and the survey is not closed.

In the figure draw HE and AG perpendicular, and BH and EG parallel to AC. Then

$$\text{AG} = \text{CB} + \text{HE},$$

or

$$\text{D} = x_1 \sin.\, b - l_2 \sin.\, \beta_2.$$

The sign of sin. β_2 is minus because the angle is obtuse. From this equation we find

$$x_1 = \frac{\text{D}}{\sin.\, b_1} - \frac{l_2}{\sin.\, b_1} \sin.\, \beta_2 \text{ as before.}$$

CASE V.

Let the Lengths of Two Sides be Wanting.

Equation (1) gives

$$x_1 \sin.\, b_1 + x_2 \sin.\, b_2 = \text{D} \quad . \quad . \quad . \quad . \quad (12)$$

$$x_1 \cos.\, b_1 + x_2 \cos.\, b_2 = \text{L} \quad . \quad . \quad . \quad . \quad (13)$$

Multiply (12) by cos. b_2 and (13) by sin. b_2 and take the difference, and we obtain

$$x_1 = \frac{\text{D} \cos.\, b_2 - \text{L} \sin.\, b_2}{\sin.\, (b_2 - b_1)} \quad . \quad . \quad (14)$$

Similarly,

$$x_2 = \frac{\text{D} \cos.\, b_1 - \text{L} \sin.\, b_1}{\sin.\, (b_2 - b_1)} \quad . \quad . \quad (15)$$

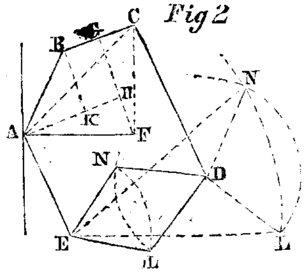
If these results are negative, the lines are imaginary and the solution is impossible—in other words, it is not a closed survey.

If $b_2 = 180 + b_1$, the unknown lines are parallel, and the denominator becomes zero; but the numerator also becomes zero at the same time—see (Eq. 4)—hence $x_2 = \frac{0}{0}$, which is undeterminate.

When a solution is possible there is no ambiguity.

In fig. 2, let the lengths of AB and BC be unknown. By Case III compute AC. Then draw AB and BC with the known bearings, and their point of intersection will give the point B, and AB will be the required length of one line, and BC of the other. If the bearing of BC should cause it to run in the opposite direction, it would not meet AB, and the solution would be impossible.

Fig 2



Draw AF perpendicular, and CF parallel to AM; AH parallel to BC, and FB perpendicular to BC. Then

$$\text{BAH} = b_2 - b_1.$$

$$\text{BK} = \text{GH} = x_1 \sin.\, (b_2 - b_1).$$

$$\text{FG} = \text{CF} \sin.\, \text{BCF} = \text{L} \sin.\, b_2.$$

$$\text{FH} = \text{AF} \cos.\, \text{AFH} = \text{D} \cos.\, b_2.$$

$$\therefore \text{GH} = \text{FG} - \text{FH} =$$

$$\text{L} \sin.\, b_2 - \text{D} \cos.\, b_2 = x_1 \sin.\, (b_2 - b_1).$$

$$\therefore x_1 = \frac{\text{D} \cos.\, b_2 - \text{L} \sin.\, b_2}{-\sin.\, (b_2 - b_1)} \text{ as before.}$$

If AE and CD are the unknown lengths, and are parallel, it is evident that the bearings and lengths of all the other lines may be the same for all lengths of these lines; provided only that they have a constant difference, hence they are indeterminate.

Case VI.

Let the Bearings of Two Sides be Wanting.

Then $$l_1 \sin\beta_1 + l_2 \sin\beta_2 = D \qquad (16)$$
$$l_1 \cos\beta_1 + l_2 \cos\beta_2 = L \qquad (17)$$

These give $$l_1 \sin\beta_1 = D - l_2 \sin\beta_2.$$
$$l_1 \cos\beta_1 = L - l_2 \cos\beta_2.$$

Squaring these and adding gives
$$l_1^2 = D^2 + L^2 + l_2^2 - 2\, l_2 (D \sin\beta_2 + L \cos\beta_2).$$

Similarly $$l_2^2 = D^2 + L^2 + l_1^2 - 2\, l_1 (D \sin\beta_1 + L \cos\beta_1).$$

In the first of these we might substitute $\cos\beta_2 = \sqrt{1 - \sin^2\beta_2}$ and reduce, but the expression would be very lengthy. Instead of that we will make

$$D = m \cos\theta \qquad (18)$$
$$L = m \sin\theta \qquad (19)$$

and we shall find

$$\sin(\theta + \beta_2) = \frac{m^2 + l_2^2 - l_1^2}{2\, m\, l_1} \qquad (20)$$

$$\sin(\theta + \beta_1) = \frac{m^2 - (l_2^2 - l_1^2)}{2\, m\, l_1} \qquad (21)$$

From which we find two values for $(\theta + \beta_2)$ and two for $(\theta + \beta_1)$; hence there are two solutions. If $\theta + \beta_2 = 90°$ or $180°$, there is but one solution, and the two bearings will equal each other.

In fig. 2, let EL and DL be the sides whose bearings are unknown. With E and D as centres and radii equal the lengths of the sides describe arcs, and the points where they intersect will be the corners of the field. As they will generally intersect in two points, there will generally be two solutions. EN or EL will be one side; and DN or DL the other. If the arcs are tangent to each other there will be but one solution, and the two sides will coincide in direction.

As an illustration of the manner of using these equations, take the following examples, found in Gillespie's Surveying. In Article 441 of his Surveying we have given

$$D = -5314{\cdot}34587 \qquad l_2 = +4621{\cdot}5$$
$$L = -1405{\cdot}32477 \qquad b_1 = 198°$$

This problem belongs to Case IV.

Now apply Eq. (10).

	$D \cos b_1$	$= +5054{\cdot}244$
	$L \sin b_1$	$= +\ \ 434{\cdot}269$
	log of dif.	$= 3{\cdot}664641$
	$\log b_2$	$= 3{\cdot}664783$
$\therefore$	$\log \sin(\beta_2 - b_1)$	$= 9{\cdot}999858$
$\therefore$	$\beta_2 - b_1$	$= 88° 32'$ or $91° 28'$
$\therefore$	β_2	$= 286° 32'$ or $289° 28'$

Corresponding bearing is N. 73° 28′ W., or N. 70° 32′ W.

Now use Eq. (11).

Log sin β_2 =	n 9·981662	or n 9·974436
Log l_2 =	3·664783	3·664783
Sum of log =	n 3·646445	n 3·639219
Log sin b_1 =	n 9·489982	n 9·489982
Log D =	n 3·725450	n 3·725450
D cosec b_1 =	+ 17197·6	+ 17197·6
l_2 sin β_2 cosec b_1 =	14337·16	14100·58
∴ x_1 =	+ 2860·44	or + 3097·02

For an example under Case VI, take the one in Article 448 of Gillespie. We have

D = + 1479·75010 $\qquad l_1 = + 2400$

L = — 2303·26591 $\qquad l_2 = + 2860$

Apply Eqs. (18), (19), (20), and (21).

Log. m sin. θ	n 3·362344			l_2^2	+ 8179600
log. sin. θ	n 9·924967	log. m	3·437377	l_1^2	+ 5760000
log. m cos. θ	3·170189	log. m^2	6·874754		
					+ 2419600
θ =	302° 43′ 8·7″			m^2	+ 7494700

log. $(m_1^2 + l_2^2 - l_1^2)$	6·996262	log. $(m^2 - l_2^2 + l_1^2)$	6·705445
av. comp. 2 m_2	6·261613		6·261613
av. comp. l_2	6 543634	av. comp. l_1	6·619789
sin. $(\theta + \beta_2)$	9·801509	sin. $(\theta + \beta_1)$	9·586847
∴ $\theta + \beta_2$ =	39° 16′ 59·2″ or 140° 43′ 0·8″	∴ $\theta + \beta$ =	22° 43′ 22·6″ or 157° 16′ 37·4″
θ =	302 43 8·7 or 302 43 8·7		302 43 8·7 or 302 43 8·7
β_2 =	96 33 50·5 or 197 59 52·1	β_1 =	80 0 13·9 or 214 33 28·7

Bear'g = S 83° 26′ 9·5″ E or S 17° 59′ 52·1″ W. Br'g. = N 80° 0′ 13·9″ E or S 34° 33′ 28·7″ W

MECHANICS, PHYSICS, AND CHEMISTRY.

Steam Engine Economy.

From the Lond. Mechanics' Magazine, Jan., 1864.

A patent case has recently been tried at Washington, U. S., which has elicited certain statements so remarkable in their bearing on the system of steam engine construction adopted of late in the American Navy, that we cannot pass it without notice. The case in question is simple enough in itself. A Mr. Mattingly, of Washington, sued a steamboat company for a share of the savings effected by a peculiar form of cut-off valve, the patent right in which he held, and had sold to the company, taking a share of the savings effected as the pecuniary consideration. The defence set up was singular enough. The complainant asserted, and as he believed proved, that the saving in fuel amounted to over 30 per cent., and he demanded a verdict in accordance with his statement. The defendants, however, called Mr. Benjamin Isher-