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A Theory of Risk[1]

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A psychological theory of perceived risk is developed. The theory is formulated in terms of an ordering of options, conceived of as probability distributions with respect to risk. It is shown that, under the assumptions of the theory, the risk of an option is expressible as a linear combination of its mean and variance. The relationships to other theories of risk and preference are explored.

I. INTRODUCTION

The concept of risk has appeared in numerous investigations of decision making both as a descriptive and as an explanatory construct. No generally accepted definition of risk, however, has emerged from these investigations, nor have there been serious attempts to interrelate the various approaches to the study of risk. As a background for the present study, we sketch briefly the more salient directions of research.

One approach to the study of risk is exemplified in the work of Coombs and his associates, (Coombs, 1964; Coombs and Huang, 1969; Coombs and Meyer, 1968; Coombs and Pruitt, 1960, Pruitt, 1962). Coombs has explored the variables affecting the perception of the riskiness of gambles as well as the manner in which perceived risk affects preferences among them. Coombs' theory postulates that each individual has an ideal (or most preferred) risk level and that in choosing among lotteries with equal expectations the individual selects the lottery that is closest to his ideal risk level.

The subject of risk taking has been of great interest not only to students of decision making, but also to students of personality and social psychology. Indeed, the tendency to seek or avoid risk has been investigated in numerous studies in relation to other

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personality characteristics, situational variables, and group influences. For some reviews of this literature, see Cohen and Hansel (1956), Kogan and Wallach (1964, 1967), and Slovic (1964).

A very different approach to the study of risk can be found in the economic and the business literature that is concerned primarily with normative, rather than descriptive, issues. The problem of portfolio selection, for instance, is analyzed there in terms of the risks involved in each of the available courses of action; there risk is defined either in terms of the distribution of returns (e.g., Markowitz, 1959; Tobin, 1958) or in terms of properties of the utility function (e.g., Pratt, 1964).

The various approaches to the study of risk share three basic assumptions. 1. Risk is regarded as a property of options, (e.g., gambles, courses of action) that affects choices among them. 2. Options can be meaningfully ordered with respect to their riskiness. 3. The risk of an option is related in some way to the dispersion, or the variance, of its outcomes. (This last point was made as early as 1906 by the economist I. Fisher, and later restated by Allais (1953) in his critique of expected utility theory.) Beyond these basic assumptions, however, no general agreement concerning the nature of risk has been reached. Although various assumptions about the perception of risk have been introduced, they have not been derived from more basic principles, and they have typically been limited to restrictive contexts.

The present paper investigates the perception of risk from the standpoint of measurement theory. It provides a quantitative explication of the concept of risk in the form of a psychological theory. The theory is formulated in terms of the ordering of options, characterized as probability distributions, with respect to risk. In the next section we introduce several assumptions about the risk ordering and show that they yield a ratio scale measure of risk. The implications of the results and their relations to other theories of risk and preference are discussed in the last section.

II. Measurement of Risk

The theory is formulated in terms of a set $S = \{A, B, C, ...\}$ of probability distributions on the real line, interpreted as options or lotteries with monetary outcomes. Let $\circ$ denote convolution of probability distributions. Thus, if A and B are two discrete distributions with values $a_1, ..., a_m$ and $b_1, ..., b_n$ obtained with probabilities $p_1, ..., p_m$ and $q_1, ..., q_n$, respectively, then $A \circ B$ is the distribution with values $a_i + b_j$ obtained with probabilities $p_i q_j$ for $i = 1, ..., m, j = 1, ..., n$. Considerable care should be taken in the empirical interpretation of $\circ$. The distribution $A \circ B$, for example, corresponds to the lottery obtained by playing both A and B only if the two lotteries are independent. To illustrate, let A represent the lottery where one wins \$100 if heads comes up and loses \$100 if tails comes up, and let B represent the lottery where one wins \$50 if heads comes up and loses \$50 if tails comes up. Thus, $A = (100, \frac{1}{2}; -100, \frac{1}{2})$

and $B = (50, \frac{1}{2}; -50, \frac{1}{2})$, consequently, $A \circ B = (150, \frac{1}{4}; 50, \frac{1}{4}; -50, \frac{1}{4}; -150, \frac{1}{4})$. On the other hand, if the outcomes of the lotteries are both determined by a single toss of a coin, one must either win \$150 or lose \$150. Hence the distribution, $(150, \frac{1}{2}; -150, \frac{1}{2})$, that corresponds to this joint lottery does not equal $A \circ B$.

The key concept of the present theory is a binary relation of comparative risk denoted $\geqslant$. Thus, $A \geqslant B$ states that A is at least as risky as B, while $A \sim B$ states that A and B are equally risky. This relation can be obtained directly by asking an individual to judge which of any pair of options is riskier, or by asking him to assign a number to each option reflecting its perceived risk. Alternatively, the above relation can be inferred from preferences via an appropriate substantive theory. It should be emphasized, however, that the risk ordering, which is the subject matter of our theory, need not be related to the preference ordering in any simple way. The assumptions of the theory should, thus, be evaluated according to the selected operational definition of $\geqslant$ whatever its relations to preference may be.

The axioms of the theory are incorporated into the definition of a risk system. Let S be the set of all probability distributions on the real line, and let $\circ$ denote the binary operation of convolution. Clearly, $\circ$ is associative, commutative and closed. Let $\geqslant$ be a binary relation on S and define $A > B$ whenever $A \geqslant B$ but not $B \geqslant A$, and $A \sim B$ whenever $A \geqslant B$ and $B \geqslant A$. We use ϕ to denote the distribution where the value zero is obtained with probability one. Clearly, $A \circ \phi = A$ for any A.

The system $(S, \circ, \geqslant)$ is a *risk system* if it satisfies the following axioms for all A, B, C, in S,

A1 Weak Ordering. $\geqslant$ is connected and transitive.

A2 Cancellation. $A \geqslant B$ if and only if $A \circ C \geqslant B \circ C$.

A3 Solvability. (i) if $A \geqslant \phi$ for all A in S, then for any $A > B$ there exists some C in S such that $A \sim B \circ C$.

(ii) If $\phi > A$ for some A in S, then for any A in S there exists some B in S such that $A \circ B \sim \phi$.

A4 Archimedean. If $A > B > \phi$ then there exists some positive integer n such that $n * B \geqslant A$; where $n * B$ is defined inductively: $1 * B = B$, $n * B = [(n-1) * B] \circ B$.

The first axiom is the usual ordering assumption which states that S can be weakly-ordered with respect to risk. The second axiom asserts that the risk ordering is compatible with the operation, in the sense that order between any pair of distributions is preserved when a third distribution is convoluted with each of them. An equivalent version of the cancellation axiom is: $A \circ C \geqslant B \circ C$ if and only if $A \circ D \geqslant B \circ D$. The cancellation axiom asserts, therefore, that if A is judged to be riskier than B in one context (i.e., when combined with C), then it is judged to be riskier than B in any context. In particular, A2 implies that the risk ordering is independent of one's financial position. Although the assumption that the ordering of options is independent of financial position appears to be too restrictive with respect to the preference ordering, it appears to be much more satisfactory with respect to the risk ordering. Axioms 1 and 2 are the more interesting axioms from an empirical standpoint, since they capture basic ordinal properties that are readily testable.

The solvability axiom has two parts that apply to different types of risk ordering. In the case where all the elements of S are at least as risky as ϕ, solvability states that when A is riskier than B then the risk of A can always be matched by combining B with some appropriate C. In the case where S contains some elements that are less risky than ϕ, then solvability states that for any A one can always find some appropriate B such that $A \circ B$ and ϕ are equally risky. Finally, the Archimedean axiom is introduced to ensure that no option is infinitely riskier than any other one, provided both options are riskier than ϕ. Axioms 3 and 4, therefore, have a different status than Axioms 1 and 2 because of their existential nature and their more technical character.

The following representation theorem shows that if Axioms 1–4 hold, then one can construct an additive ratio scale that preserves the risk ordering.

THEOREM 1. *If* $(S, \circ, \succcurlyeq)$ *is a risk system then there exists a real-valued function,* R, *defined on* S, *such that for any* A, B *in* S

(i) $A \succcurlyeq B$ *if and only if* $R(A) \geqslant R(B)$.

(ii) $R(A \circ B) = R(A) + R(B)$.

(iii) *If* R' *is another function satisfying* (i) *and* (ii), *then* $R'(A) = \alpha R(A)$ *for some* $\alpha > 0$.

Proof. Two cases are considered: (a) $A \succcurlyeq \phi$ for all A in S, (b) $\phi \succ A$ for some A in S. In case (a) it is easy to verify that a risk system reduces to an extensive system of measurement (see Krantz, 1968; Suppes, 1951; Suppes and Zinnes, 1963) whence Theorem 1 follows from the standard representation theorem for extensive measurement. In fact, the only difference between the systems, in this case, lies in the existence of zero elements, but it follows from the present axioms, that if $A \succcurlyeq A \circ B$ then $B \sim \phi$.

In case (b) we show that the system $(S/\sim, \circ, \succcurlyeq)$ is a fully-ordered Archimedean group, where $S/\sim$ denotes the set of equivalence classes of S modulo $\sim$. Let $[A] = \{B \text{ in } S : A \sim B\}$ be the equivalence class containing A, and define $[A] \circ [B] = [A \circ B]$, and $[A] \succcurlyeq [B]$ if and only if $A \succcurlyeq B$. It is readily seen that both $\circ$ and $\succcurlyeq$ on $S/\sim$ are well-defined as they are independent of the representative elements that were chosen to define the equivalence classes. Clearly, $S/\sim$ is closed under $\circ$ which is, by definition, associative and commutative. Furthermore, $[\phi]$ is the identity element because $[A] \circ [\phi] = [A \circ \phi] = [A]$ for any A in S. Since for any A there exists a B such that $A \circ B \sim \phi$, by Axiom 3, we can define inverses by letting $[A]^{-1} = [B]$. To establish uniqueness, suppose $[A] \circ [B] = [A] \circ [C] = [\phi]$; hence, by Axiom 2, $[B] = [C]$ and the inverse is unique. Finally, $\geqslant$ is a full (or a total) order on $S/\sim$, which is Archimedean, by Axiom 4, and satisfies the cancellation axiom. Consequently, $(S/\sim, \circ, \geqslant)$ is a fully-ordered Archimedean group; hence, by Hölder's (1901) theorem (see Fuchs, 1963, p. 45) it is order-isomorphic to a subgroup of the additive group of real numbers with the natural ordering. Moreover, the isomorphism can be shown (see Fuchs, 1963, p. 46) to be unique up to multiplication by a positive real number, which completes the proof of Theorem 1.

Several comments about Theorem 1 are in order. First, it can be generalized by weakening some of the assumptions. In particular, one can use the results of Krantz (1968) and Luce and Marley (1968) to extend the theorem to the case where the operation is restricted to some subset of S. Furthermore, the present solvability and Archimedean axioms can be replaced by a weaker though more complicated axiom, formulated by Roberts and Luce (1968), who obtained necessary and sufficient conditions for extensive measurement. A necessary and sufficient condition for the existence of an additive risk scale can also be derived following an approach developed by Tversky (1967a, b). Second, the proof of Theorem 1 is not based on the fact that distributions are defined on the real line. In fact, the theorem applies to any set of probability distributions, defined on an arbitrary sample space, (e.g., the results of an election, the outcome of a game) provided it is closed under a well-defined operation of convolution.

The present system of risk is closely related to the classical extensive system that provides the axiomatic basis for the measurement of physical properties such as weight, length, and time. Such a system is based on an ordering of physical objects with respect to some property (e.g., weight) and on a physical operation of concatenation (e.g., placing two objects together on a pan balance). Under this interpretation of the ordering and the operation, the present axioms reduce to the well-known physical principles that govern the measurement of weight.

One essential difference, however, between a risk system and an extensive system is that the latter is typically nonnegative, in the sense that $A \circ B \geqslant A$ for all A and B, and the measurement scale is, consequently, nonnegative. The present system, on the other hand, admits negative values since the convolution of two distributions may be less risky than either one of them. The risk scale, therefore, may take both positive and negative values. In particular, since $R(A \circ \phi) = R(A) + R(\phi) = R(A)$ then $R(\phi) = 0$. Consequently, any option that is less risky than the status quo must have a negative risk value.

As a corollary to Theorem 1 we show how the risk of degenerate distributions (i.e., distributions where a single value is obtained with probability one) is determined. Let K_1 and K_2 denote degenerate distributions which yield the values k_1 and k_2, respectively, with probability one. Since $K_1 \circ K_2$ is a degenerate distribution that yields the value $k_1 + k_2$ with probability one, we can identify each degenerate distribution K with its value k. By the additivity of the risk scale $R(K_1 \circ K_2) = R(K_1) + R(K_2)$, hence $R(k_1 + k_2) = R(k_1) + R(k_2)$ for any real k_1, k_2. Consequently, it can be shown that $R(K) = \beta k$ for some real β. If $\beta = 0$, then $R(K) = 0$ for any degenerate distribution K, and hence the risk measure is translation-invariant in the sense that $R(A \circ K) = R(A) + R(K) = R(A)$ for all A. If $\beta \neq 0$, then degenerate distributions are bound to have negative or positive risk values.

These aspects of the theory are somewhat at variance with ordinary usage of risk according to which there is no negative risk, and sure-things (i.e., degenerate distributions) are regarded as having no risk. To illustrate the significance of negative risks in our theory, note that the gamble $A = (200, \frac{1}{2}; -200, \frac{1}{2})$ is likely to be judged (by most people) as riskier than the gamble $B = (300, \frac{1}{2}; -100, \frac{1}{2})$, which is constructed by adding a sure-thing to A. Consequently, the option of receiving 100 for sure must have negative risk.

The above example suggests that the addition of a positive sure-thing to an option cannot increase its risk. Stated as a formal assumption we obtain

A5 POSITIVITY. If K is a degenerate distribution with $k > 0$ then $A \succcurlyeq A \circ K$ for all A in S.

In the following discussion, we introduce additional assumptions about the risk ordering and study the constraints they impose on the risk scale. Let $E(A)$ denote the expectation of A, and let tA denote the distribution obtained by multiplying all values of A by some real number t. (For continuous distributions the density function of tA has to be normalized.) Using this construct, we introduce

A6 MONOTONICITY. For all A, B in S with $E(A) = E(B) = 0$ and for any real $t > 1$

(i) $tA \succ A$

(ii) $A \succcurlyeq B$ if and only if $tA \succcurlyeq tB$.

Axiom 6 asserts that, for distributions with zero expectation, risk increases with multiplication by any $t > 1$, and that the risk ordering is preserved upon multiplication by a positive real number. Thus, part (i) of the monotonicity axiom asserts that any fair bet whose outcomes are dollars, for example, is riskier than the bet obtained from it by changing the dollars to nickels. Similarly, part (ii) states that the risk ordering between gambles is independent of the denomination of the payoffs (e.g., dollars, nickels) provided their expectations are zero. We regard part (i) of A6, like A5, as a necessary assumption for any theory of risk. On the other hand, part (ii) of A6, like A2, is a more powerful assumption that is both appealing and testable.

Our last axiom is more technical in nature. A sequence $\{A_n\}$ $n = 1, 2, \ldots$, of distributions is said to approach a limiting distribution A, whenever $\Pr(x \leqslant A_n \leqslant y)$ approaches $\Pr(x \leqslant A \leqslant y)$ as $n \to \infty$ for any real x, y.

A7 CONTINUITY. If $\{A_n\}$ approaches A then $R(A_n)$ approaches $R(A)$, provided $E(A_n) = E(A)$ and $V(A_n) = V(A)$ for all n, where $V(A)$ denotes the variance of A. Axiom 7 establishes the continuity of the risk scale in the sense that if all the distributions in the sequence have the same mean and variance then $\lim R(A_n) = R(A)$ whenever $\lim\{A_n\} = A$. To summarize the added assumptions we introduce a new definition. A risk system $(S, \circ, \succcurlyeq)$ is called a *regular risk system* if it satisfies Axioms 5, 6, and 7.

To study the properties of a regular risk system, select some A, B in S such that $E(A) = E(B) = 0$, and $V(A) > V(B)$. Let $A_n = (n * A)/\sqrt{n}$ and $B_n = (n * B)/\sqrt{n}$, see A4 for the definition of $*$. As $n \to \infty$ the sequences $\{A_n\}$ and $\{B_n\}$ approach, respectively, limiting distributions A' and B' which, by the central limit theorem, are normal with $E(A') = E(B') = 0$ and $V(A') = V(A) > V(B) = V(B')$. Since $A' = kB'$ for some $k > 1$, then, by Axiom 6(i), $A' \succ B'$ and by Theorem 1, $R(A') > R(B')$. By Axiom 7, however, there exists an n such that $R(A_n) > R(B_n)$, hence $A_n \succ B_n$. Consequently, by Axiom 6(ii), $n * A \succ n * B$, and by Axiom 2, $A \succ B$. Recalling Theorem 1, we write $A \succcurlyeq B$ if and only if $R(A) \geqslant R(B)$ if and only if $V(A) \geqslant V(B)$ for all A, B with zero expectation.

Thus, there exists a strictly increasing function, f, such that, for any A in S with $E(A) = 0$, $R(A) = f[V(A)]$. Note that the set of distributions with zero expectation is closed under $\circ$. Furthermore, it follows from the definition of variance and Theorem 1, respectively, that both V and R are additive over $\circ$. Hence, if $E(A) = E(B) = 0$

$$\text{then} \qquad R(A \circ B) = f[V(A \circ B)] = f[V(A) + V(B)],$$

$$\text{and} \qquad R(A \circ B) = R(A) + R(B) = f[V(A)] + f[V(B)].$$

It follows from the above functional equation that f is linear; thus, there exists some $\alpha > 0$ such that $R(A) = f[V(A)] = \alpha V(A)$, whenever $E(A) = 0$.

Finally, for any A in S, define A_0 by the equation $A = A_0 \circ [E(A)]$, where $[E(A)]$ is the degenerate distribution where the value $E(A)$ is obtained with probability one. Consequently,

$$R(A) = R(A_0 \circ [E(A)]) = R(A_0) + R([E(A)]) = \alpha V(A) + \beta E(A), \text{ since } E(A_0) = 0$$

and $V(A_0) = V(A)$, where $\beta \leqslant 0$, by Axiom 5. Since the unit of the risk scale is arbitrary we can divide the above equation by $\alpha - \beta$ and let $\theta = \alpha/\alpha - \beta$. We have thus established

THEOREM 2. *If $(S, \circ, \succcurlyeq)$ is a regular risk system, then there exists a unique* $0 < \theta \leqslant 1$

such that for all A, B in S (with finite expectations and variances) $A \geqslant B$ if and only if $R(A) \geqslant R(B)$ where $R(A) = \theta V(A) - (1 - \theta) E(A)$.

Thus, in a regular risk system, the risk ordering is generated by a linear combination of expectation and variance. Put differently, the risk of any option can be readily computed, once a single parameter, θ, is determined. Furthermore, θ is attainable from a single judgment of risk-equality between two distinct distributions, and its value determines the relative contribution of the expectation and the variance to the riskiness of an option.

The result established in Theorem 2 is surprisingly strong. Although the validity of the critical assumptions, A2 and A6(ii), is not self-evident, they seem compatible with our intuitions of risk. Furthermore, the cancellation axiom (A2), for example, is an appealing consistency principle that one may wish to impose on his risk ordering. In this sense, the cancellation axiom may have normative application as a guideline in establishing a risk ordering, similar to the application of axioms of expected utility theory in establishing a preference ordering.

The evaluation of the adequacy of the present theory is a difficult task, since our intuitions concerning risk are not very clear and a satisfactory operational definition of the risk ordering is not easily obtainable. Indeed, the existence of a meaningful risk ordering (that has been presupposed by all workers in the field) should not be taken for granted.

III. Risk and Preference

The present results provide a quantitative explication of risk by demonstrating that, under the assumptions of the theory, risk is expressible as a linear combination of expectation and variance. That expectation and variance, however, are two essential components of risk has long been realized by many investigators.

In his monograph entitled "Portfolio Selection," Markowitz (1959, p. 129) defines a portfolio (i.e., a probability distribution over monetary values) as *inefficient* if "it is possible to obtain higher expected (or average) return with no greater variability of return." Markowitz does not offer any formal rationale for his definition of efficiency; instead he describes various methods for finding efficient portfolios. To cast the above definition in the present framework, let T be a subset of S. A distribution A in T is inefficient (with respect to T) if there exists a B in T such that $R(A) > R(B)$ for any $0 < \theta \leqslant 1$. Thus, a portfolio is inefficient (in T) if it is riskier than some other portfolio for all admissible values of θ. Additional applications and further discussions of criteria for preference among options that are based only on their expectations and variances (or standard deviations) have been presented by Baumol (1963), Lintner (1965), Sharpe (1963), and Tobin (1958); see also Hanoch and Levy (1969).

We are led now to the more general and basic problem of the relationships between risk and preference. In the economic and business literature, it has been typically assumed that people wish to minimize risk, which can, therefore, be inferred from preference. Some psychologists, on the other hand, have argued that the relationships between risk and preference are considerably more complicated. Indeed, one advantage of an explicit theory of risk is that it enables one to investigate the relationships between preference and risk in detail.

A *preference function*, f, is any real-valued function on S such that for all A and B, $f(A) > f(B)$ if and only if A is preferred to B. A preference function is, therefore, an ordinal utility scale, i.e., a numerical scale that reflects a given preference order. A preference order is said to depend on the risk measure, or to be *R-dependent* if there exists a function g satisfying $f(A) = g[R(A)] = g[\theta V(A) - (1 - \theta) E(A)]$ for all A in S. Thus, preferences are R-dependent whenever any two equally-risky options are indifferent. A preference order is *VE-dependent* if it depends only on the variance and the expectation, i.e., if there exists a real-valued function, h, in two arguments such that $f(A) = h[V(A), E(A)]$ for all A in S. Thus, preferences are VE-dependent whenever any two options with equal variances and expectations are indifferent. Clearly, any preference order that is R-dependent is also VE-dependent, but the converse is not true. Markowitz's (1959) model for the selection of efficient portfolios provides one example of VE-dependent preferences.

The major theory of decision making under risk is the expected utility theory. According to this theory, which was formulated by Bernoulli (1738) and first axiomatized by Von Neumann and Morgenstern (1944), there exists a real-valued function, u, that assigns a utility to each outcome. An option A is preferred to option B, according to the theory, if and only if $E(u_A) > E(u_B)$, where $E(u_A)$ is the expectation of the distribution u_A whose values are the utilities of the corresponding values of A. What are the relationships between R-dependency, VE-dependency and the theory of expected utility?

We assume that utility theory is satisfied for any set of distributions and that the utility function is sufficiently regular so that it can be expanded in a power series. Using these assumptions, it is possible to show, following Borch (1963), Markowitz (1959), and Tobin (1958), that the quadratic utility function generates the only preference order that is VE-dependent, and that no utility function generates a preference order that is R-dependent.

THEOREM 3. *If expected utility theory holds, and* $u(a) = \sum_{n=0}^{\infty} t_n a^n$ *then*

(i) *a preference order is VE-dependent if and only if*

$$u(a) = t_0 + t_1 a + t_2 a^2,$$

(ii) *there exists no preference order that is R-dependent.*

Proof. (i) First, suppose $u(a) = t_0 + t_1a + t_2a^2$, hence

$$\begin{aligned} E(u_A) &= E(t_0 + t_1A + t_2A^2) \\ &= t_0 + t_1E(A) + t_2E(A^2) \\ &= t_0 + t_1E(A) + t_2[V(A) + E(A)^2] \\ &= h[V(A), E(A)] \end{aligned}$$

and hence the order is VE-dependent. Second, by assumption, $u(a) = \sum_{n=0}^{\infty} t_n a^n$, hence $E(u_A) = \sum_{n=0}^{\infty} t_n E(A^n)$. Assuming the preference order is VE-dependent, we can write $E(u_A) = h[V(A), E(A)] = h[E(A^2) - E(A)^2, E(A)]$. Hence $t_n = 0$ for any $n > 2$, in the power series expansion of u. Otherwise, we can always select some A, B with equal expectations and variances but unequal expected utilities, contrary to our assumption. Consequently, $E(u_A) = t_0 + t_1E(A) + t_2E(A^2)$ and $u(a) = t_0 + t_1a + t_2a^2$ which completes the proof of part (i).

To prove (ii), suppose there exists a preference order compatible with expected utility theory that is R-dependent. Hence,

$$\begin{aligned} E(u_A) = g[R(A)] &= g[\theta V(A) - (1 - \theta)\, E(A)], \qquad 0 < \theta \leqslant 1. \\ &= g[\theta E(A^2) - \theta E(A)^2 - (1 - \theta)\, E(A)]. \end{aligned}$$

But $E(u_A) = t_0 + t_1E(A) + t_2E(A^2)$, by part (i) since R-dependency implies VE-dependency. There is no g, however, satisfying the above equation. To demonstrate, let $E(A) = 0$, hence $g[\theta E(A^2)] = t_0 + t_2E(A^2)$ for any value of $E(A^2)$ and g must be linear. Letting $V(A) = 0$, however, yields $g[(\theta - 1)\, E(A)] = t_0 + t_1E(A) + t_2E(A^2)$. Hence $t_2 = 0$ contrary to Axiom 6(i) which completes the proof of the theorem.

The latter part of Theorem 3 shows that there is no utility function compatible with a preference order that is R-dependent. This result may be taken, therefore, as evidence either against the present theory of risk or against expected utility theory. Alternatively, one may accept both theories and reject the notion that preferences are R-dependent. It seems reasonable to suppose that preferences do not depend solely on risk, and that an individual may not be indifferent between two options that appear equally risky to him. Similarly, it seems plausible that an individual may perceive one option as riskier than another, although he may be indifferent between the two. This does not mean, of course, that risk and preference are unrelated, only that one ordering cannot be inferred from the order.

The first part of Theorem 3 shows that the only utility function that is compatible with a preference order that is VE-dependent is of the general form $u(a) = t_0 + t_1a + t_2a^2$. The quadratic utility function, however, is not very satisfactory. First, its domain must be bounded, since for u to be an increasing function,

a must be bounded by $-t_1/2t_2$ from above or below depending on whether t_2 is negative or positive. Second, it is inevitable decreasing marginal utility must obtain for suitably large amounts of money that the utility function be concave (i.e., $u'' < 0$) somewhere, hence $t_2 < 0$. In this case, however, the degree of risk aversion (defined by $-u''/u'$) increases with an increase in assets as pointed out by Pratt (1964, p. 132). Hence, according to this function, the cash-equivalence (or the minimal selling price) of a gamble decreases as one gets wealthier. This conclusion seems unacceptable on both empirical and theoretical counts.

Returning to the relationship between the quadratic utility function and the function $h(V, E)$, we note that for any fixed value of E, h is either an increasing or a decreasing function of V depending on whether $t_2 > 0$ or $t_2 < 0$. Stated formally, we obtain

COROLLARY. *Suppose a preference order is VE-dependent and also satisfies expected utility theory. Then the preference order of any set of options with equal expectation must either minimize or maximize variance.*

The method employed in proving Theorem 3 can also be used to obtain a more general result. Suppose the risk of A is expressible as some function of the first n (raw) moments of A. Theorem 3 can, then, be easily extended to show that if there exists a utility function that is compatible with the proposed risk measure, then it must be a polynomial function of degree n.

One aspect of distributions that does not appear in the present theory is skewness. It may nevertheless be true that some gambles with negative skewness appear riskier than gambles with positive skewness even when expectation and variance are held constant. If skewness does, indeed, play an essential role in determining the perception of risk, then the range of applicability of the present theory should be restricted, e.g., to symmetric or equally-skewed options. In this case, one may attempt to generalize the present theory in order to account for the effects of skewness (or of other such factors) on the perceived riskiness of options.

There have been several experimental studies (e.g., Coombs and Pruitt, 1960; Edwards, 1954; Lichtenstein, 1965, and Pollatsek, 1966) that examined the variance preference hypothesis. This hypothesis (which can be regarded as a special case of Coombs' unfolding theory) states that, for any given expectation level, each individual has an ideal, or a most preferred, variance level. In selecting among gambles with equal expectation, therefore, individuals select the gamble whose variance is closest to their ideal variance level. The preceding discussion showed, however, that the minimization or the maximization of variance are the only forms of variance preferences admissible under expected utility theory. Unfortunately, the empirical evidence for systematic variance preferences is inconclusive. Yet, to the extent that the data provide evidence for consistent preferences for intermediate variance levels, they also provide evidence against the theory of expected utility, contrary to the views expressed by some writers.

In addition to research on the relationships between risk and preference, there has been some research on the perception of risk per se. Recently, Coombs and Huang (1969) have developed a "psychophysical" theory of risk that expresses the riskiness of a gamble as a composite function of its components. The authors have limited their discussion to two-outcome gambles of the form $G = (y, p; z, 1 - p)$, where one wins y with probability p and z with probability $1 - p$. Three classes of transformations of such gambles, denoted α, β, and γ, were defined as follows. $\alpha(G)$ is a transformation that increases $V(G)$ by increasing the range of the outcomes, while leaving p and $E(G)$ unchanged; $\beta(G)$ is $G \circ b$, where b is some constant that is added to both outcomes; and $\gamma(G)$ is $c * G$, that is, the gamble obtained by c independent plays of the gamble G. Starting from the zero gamble $(0, p; 0, 1 - p)$, one applies to it the transformations α, β, and γ in turn. The result is a new gamble, G, that can be characterized by a triple (a, b, c), where a is the increase in variance due to the transformation α, b is the increase in expectation due to the transformation β, and c is the number of independent multiple plays of the resulting gamble. Coombs and Huang have proposed a distributive polynomial model for the measurement of risk. According to this model, there exist real-valued functions f_1, f_2, and f_3 such that the risk of (a, b, c) equals $[f_1(a) + f_2(b)] f_3(c)$. Letting $f_i(x) = t_i x$ for $i = 1, 2, 3$ with $t_1, t_3 > 0 > t_2$ yields

$$\begin{aligned}
\text{Risk}\,(G) &= \text{Risk}\,(a, b, c) \\
&= (t_1 a + t_2 b)\, t_3 c, \qquad \text{by hypothesis} \\
&= t_1 t_3 V(G) + t_2 t_3 E(G), \qquad \text{since} \quad V(G) = ac \quad \text{and} \quad E(G) = bc. \\
&= t\theta V(G) - t(1 - \theta)\, E(G), \qquad \text{where} \quad \theta = \frac{t_1}{t_1 - t_2}, \qquad t = t_3(t_1 - t_2).
\end{aligned}$$

Hence, for the special class of gambles considered by Coombs and Huang (1969), their theory reduces to the present theory under the above linearity assumption. Furthermore, the supporting empirical evidence for the distributive model presented by Coombs and Huang (1969), based on rank ordering of gambles with respect to risk, also provides empirical support for the present theory.

The present theory of risk has two interrelated objectives. First, it provides a testable psychological model for the measurement of risk. It is not, however, a fundamental measurement model in the classical sense, since it presupposes the representation of options as distributions on the real line. On the other hand, it bypasses the measurement problem altogether since, under the assumptions of the model, the risk scale, R, is known in advance except for a single parameter, θ.[3] Second,

[3] The present theory has been formulated in terms of probability distributions over monetary values. It is conceivable, however, that the present axioms will fail to hold for this representation of the outcomes but will be satisfied when applied to probability distributions over some other representation of the outcomes such as a nonlinear utility or subjective value scale. In the latter case, the empirical interpretation of the convolution operation becomes more involved, since

it provides an explication of risk, that is, it offers a quantitative concept of risk as a substitute for the more familiar, but vaguer, notion of degree of risk. Furthermore, the proposed concept of risk is not only precise, but it is also sufficiently similar to the intuitive concept, so that it may capture most of its essential characteristics. The use of Shannon and Weaver's (1949) information measure as a quantitative explication for the concept of the amount of information (see Roberts and Luce, 1968) is a case in point. In both instances one starts with an ordering of probability distributions with respect to some property (risk or information) and shows that, under certain assumptions, an essentially unique index, of risk or information, can be derived.

The value of the present model depends, therefore, on the empirical validity of the axioms, as well as on the theoretical usefulness of the derived measure.

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the utility scale has to be known in order to identify the lottery corresponding to the convolution of two distributions. It can be shown, however, that when Axioms 1–7 hold, the numerical representation of the outcomes is determined up to multiplication by a positive constant. Hence, if the axioms are not satisfied relative to the monetary scale, one may wish to search for that (essentially unique) utility scale relative to which the axioms are satisfied.

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