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On the Limit Cycles of $y'' + \mu F(y') + y = 0$

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The classical investigations of the Liénard-type equation,

$$x'' + \mu F(x') + x = 0, \tag{1}$$

and its derivative,

$$x'' + \mu f(x)x' + x = 0, \tag{1a}$$

by Levinson and Smith, and others, (see Cesari [1]) have proven the existence of at least one limit cycle of (1) or (1a) under various hypotheses. These hypotheses have invariably included the requirement that F be, in some sense, monotone for x' sufficiently large. Recently Hockstadt and Stephan [2], Comstock [3], and D'heedene [4] have studied the limit cycles of

$$x'' + \mu \sin x' + x = 0 \tag{2}$$

and have, by different devices, shown that (2) has an infinite number of limit cycles, although $F(x') = \sin x'$ is not monotonic. By modifying D'heedene's proof we show that (1) has an infinite number of limit cycles for a wide class of oscillatory functions.

The fact that (1) may have a number of oscillations if $F(x')$ has a number of oscillations has been noted by several others. Stoker [5] had conjectured that (2) has an infinite number of limit cycles. Ponzo and Wax [6] have a proof of the existence of limit cycles which could be used to prove the existence of a nested set of limit cycles for $F(x')$ oscillatory, provided F also grows as $|x'| \to \infty$. Thus their work would not apply to (2).

We consider the phase plane portrait of (1), with an $F(x')$ that is odd and oscillatory and has an infinite number of zeros at x_i', $i = 0, 1, 2,\ldots$. Our procedure is to observe that certain curves in the phase plane must cross the $x' = 0$ axis for some negative value, a, of x and continue around to cross the same axis for some positive value, b, of x. If $b^2 - a^2 = 0$ the corresponding curve, by the oddness of $F(x')$ constitutes a closed curve. Looking at the integral of F, under suitable hypotheses on F, we show that $b^2 - a^2$ changes sign an infinite number of times and thus must be zero an infinite number of times. This is essentially the procedure that D'heedene used for the specific case $F(x') = \sin x'$ [4].

We consider the phase plane,

$$x' = y \qquad \text{and} \qquad y' = -\mu F(y) - x, \tag{3}$$

and letting,

$$u(x, y) = \tfrac{1}{2}(x^2 + y^2), \tag{4}$$

we have

$$\frac{du}{dx} = x + y\frac{dy}{dx} = -\mu F(y). \tag{5}$$

We now consider the class of functions $F(y)$ which satisfy the hypothesis,

H_1 : $F(y)$ is a continuous, differentiable, odd, bounded and oscillatory function of y defined on $-\infty < y < \infty$, with $yF(y) > 0$ for y sufficiently small and >0, with $|F| \leqslant A < \infty$ and $|F'| \leqslant B < \infty$.

Since (3) is invariant under $x \to -x$, $y \to -y$ we need only consider $y \geqslant 0$. Following D'heedene we call the curve

$$\mu F(y) = -x \tag{6}$$

the characteristic curve, and we assume that (6) defines x as a single valued function of y. From (3) any solution crosses the characteristic (6) with zero slope and has positive slope to the left and negative slope to the right. Thus, any solution to (3) is a single valued function $x_-(y)$ to the left of (6) and a single valued function $x_+(y)$ to the right. We define

$$g(y) = \frac{dx^-}{dy} - \frac{dx^+}{dy}, \tag{7}$$

and observe that $g(y) > 0$ for $y > 0$. We now proceed with our results.

LEMMA 1. *For each point $(-\mu F(y^*), y^*)$ on the characteristic curve* (6) *there exists points $(a, 0)$ and $(b, 0)$, $a < b$ such that the orbit through $(-\mu F(y^*), y^*)$ goes through these points.*

Proof. The proof follows exactly from the corresponding result in D'heedene [4] and will not be repeated in detail. We sketch merely an outline. Construct two semicircles which form inner and outer bounds for the orbit in the following manner. Centered on the x axis with radius y^*, so that the semicircles have their horizontal peak at the same height y^* as the orbit peak, let one semicircle be centered at $(|\mu A|, 0)$ and the other at $(-|\mu A|, 0)$, where $A = \max F(y)$. Thus the equation of these semicircles is

$$y^2 + (x \pm |\mu A|)^2 = (y^*)^2.$$

Consider the slope of these curves:

$$\frac{dy}{dx} = \frac{-x \mp |\mu A|}{y}.$$

Noting that by construction the semicircles are centered to the right and the left respectively of the orbit peak and comparing the slopes with (3), we see that these semicircles do form inner and outer bounds as shown in Fig. 1.

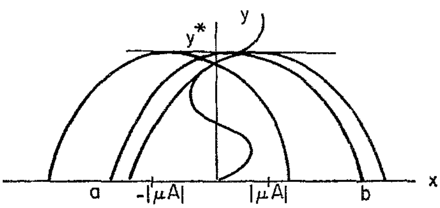

FIGURE 1

We let

$$\Delta(y^*) = \tfrac{1}{2}(b^2 - a^2), \tag{8}$$

and note from (4) that

$$\Delta(y^*) = u(b, 0) - u(a, 0). \tag{9}$$

We also note that if for some y_0, $\Delta(y_0) = 0$ then by the oddness of the phase plane orbit, for that y_0 this orbit is closed and thus a periodic solution. With this in mind we show

THEOREM 1. *Let the positive zeros of $F(y)$ be $y_1, y_2, \ldots, y_n$. Then if $g'(y) > 0$, F satisfies H_1, and the magnitude of the integral $\int_{y_{n-1}}^{y_n} F(y)\, dy$ is a nondecreasing function of n then*

$$\operatorname{sgn} \Delta(y_n) = (-1)^n \operatorname{sgn} \mu. \tag{10}$$

Proof. In (9) let $y^* = y_n$. Then from (5) we see

$$\begin{aligned}\Delta(y_n) &= u(0, b) - u(0, a) \\ &= -\mu \int_0^b F(y)\, dx - \mu \int_a^0 F(y)\, dx \\ &= -\mu \int_{y_n}^0 F(y) \frac{dx_+}{dy}\, dy - \mu \int_0^{y_n} F(y) \frac{dx_-}{dy}\, dy \\ &= -\mu \int_0^{y_n} F(y)\, g(y)\, dy, \end{aligned} \tag{11}$$

using (7). Thus,

$$\Delta(y_n) = -\mu \sum_{k=1}^{n} \int_{y_{k-1}}^{y_k} F(y)\, g(y)\, dy.$$

By construction $g(y) > 0$ and by hypothesis $g'(y) > 0$, so that $g(y)$ is continuous and monotone increasing. Also by hypothesis

$$\int_{y_{l-1}}^{y_l} F(y)\, dy = -(1 + \epsilon_l) \int_{y_{l-2}}^{y_{l-1}} F(y)\, dy, \tag{12}$$

where $\epsilon_l \geqslant 0$ and

$$\int_0^{y_1} F(y)\, dy > 0. \tag{13}$$

Now consider the case where n is even.

$$\begin{aligned}\Delta(y_n) = &-\mu \left\{ \int_0^{y_1} F(y)\, g(y)\, dy + \int_{y_1}^{y_2} F(y)\, g(y)\, dy \right\} \\ &-\mu \left\{ \int_{y_2}^{y_3} F(y)\, g(y)\, dy + \int_{y_3}^{y_4} F(y)\, g(y)\, dy \right\} + \cdots \\ &-\mu \left\{ \int_{y_{n-2}}^{y_{n-1}} F(y)\, g(y)\, dy + \int_{y_{n-1}}^{y_n} F(y)\, g(y)\, dy \right\} \\ = &+\mu\{(1 + \epsilon_2)\, g(\bar{y}_2) - g(\bar{y}_1)\} \int_0^{y_1} F(y)\, dy \\ &+\mu\{(1 + \epsilon_4)\, g(\bar{y}_4) - g(\bar{y}_3)\} \int_{y_2}^{y_3} F(y)\, dy + \cdots \\ &+\mu\{(1 + \epsilon_n)\, g(\bar{y}_n) - g(\bar{y}_{n-1})\} \int_{y_{n-2}}^{y_{n-1}} F(y)\, dy, \end{aligned}$$

where we have used a mean value theorem to remove the $g(y)$ from the integral. From (12), (13) and the hypothesis on g, $\operatorname{sgn} \Delta(y_n) = \operatorname{sgn} \mu$. For n odd

$$\begin{aligned}\Delta(y_n) = &-\mu g(\bar{y}_1) \int_0^{y_1} F(y)\, dy \\ &-\mu \left\{ g(\bar{y}_2) \int_{y_1}^{y_2} F(y)\, dy + g(\bar{y}_3) \int_{y_2}^{y_3} F(y)\, dy \right\} + \cdots \\ &-\mu \left\{ g(\bar{y}_{n-1}) \int_{y_{n-2}}^{y_{n-1}} F(y)\, dy + g(\bar{y}_n) \int_{y_{n-1}}^{y_n} F(y)\, dy \right\} \\ &-\mu g(\bar{y}_1) \int_0^{y_1} F(y)\, dy \\ &-\mu \{(1 + \epsilon_3)\, g(\bar{y}_3) - g(\bar{y}_2)\}(1 + \epsilon_3)^{-1} \int_{y_2}^{y_3} F(y)\, dy + \cdots \\ &-\mu \{(1 + \epsilon_n)\, g(\bar{y}_n) - g(\bar{y}_{n-1})\}(1 + \epsilon_n)^{-1} \int_{y_{n-1}}^{y_n} F(y)\, dy.\end{aligned}$$

Again each term is greater than zero so, for n odd

$$\operatorname{sgn} \Delta(y_n) = -\operatorname{sgn} \mu. \qquad \text{Q.E.D.}$$

The crucial element to the above theorem is the hypothesis that $g'(y) > 0$. If we can show that $g'(y) > 0$ for all μ then we will be able to prove the result that we want, namely that (1) has an infinite number of limit cycles. To this end we state two lemmas

LEMMA 2. *If $F'(y)$ is bounded for all y*

$$|F'(y)| \leqslant B < \infty, \tag{14}$$

then for $|\mu| < 2/B$ we have $g'(y) > 0$, for any y.

Proof. This result is an obvious minor modification of the first half of D'heedene's Theorem 1 [4], with μ in his theorem replaced by μB. We will not repeat that algebra in detail. Very briefly we consider the proof. Let

$$\xi(y) = -x_-(y) - \mu F(y) \qquad \text{and} \qquad \eta(y) = x_+(y) + \mu F(y), \tag{15}$$

and substituting in (7)

$$g(y) = y/\xi + y/\eta.$$

Differentiating and rearranging we have

$$g'(y) = \frac{\xi + \eta}{\xi^3 \eta^3} \left[y^2(\xi^2 - \xi\eta + \eta^2) + \xi^2\eta^2 + \mu y \xi\eta(\eta - \xi) F'(y) \right]. \tag{16}$$

Observing that ξ and η are >0, only the sign of the bracket is significant. Using (14) and completing the square on the bracket in (16) we obtain

$$\frac{\xi^3\eta^3}{\xi+\eta} g'(y) \geqslant \left[y \mid \xi - \eta \mid - \frac{\mid \mu B \mid}{2} \xi\eta\right]^2 + [y^2 + (1 - \mu^2 B^2/4)\, \xi\eta]\, \xi\eta,$$

which is positive under our hypotheses.

LEMMA 3. *For orbits with y_n sufficiently large that*

$$y_n > 2 \mid \mu A \mid \{\mid \mu B \mid + 1 + (\mu^2 B^2 + 2 \mid \mu B \mid)^{1/2}\}, \tag{17}$$

where

$$\mid F(y) \mid \leqslant A < \infty, \tag{18}$$

we have $g'(y) > 0$ for $0 \leqslant y \leqslant y_n$.

Proof. The proof of this lemma is given as the proof of Theorem 2 in D'heedene [4], with obvious modifications. We will not repeat the algebra. The essence of the proof is as follows.

One again needs to determine the sign of (16). Completing the square in a different way we get

$$g'(y) = \frac{\xi+\eta}{\xi^3\eta^3} [y^2(\xi-\eta)^2 + y^2\xi\eta + \xi^2\eta^2 - \xi\eta\mu y(\xi - \eta) F'(y)].$$

Ignoring the first term in brackets we get

$$\frac{\xi^2\eta^2}{\xi+\eta} g'(y) \geqslant [y^2 + \xi\eta - \mid \mu B \mid y \mid \xi - \eta \mid]. \tag{19}$$

Using the semicircles from the proof of Lemma 1 we can bound $x_+(y)$ and $x_-(y)$, and thus, through (15) and (18) we can bound the difference $\mid \xi - \eta \mid$ to be

$$\mid \xi - \eta \mid \leqslant 4 \mid \mu A \mid. \tag{20}$$

Bounding $x_+(y)$ and $x_-(y)$ in terms of these semicircles, whose peaks, remember, are at $(\pm \mid \mu A \mid, y_n)$ we have

$$\xi, \eta \geqslant (y_n^2 - y^2)^{1/2} - 2 \mid \mu A \mid. \tag{21}$$

Solving (21) for y and using (20) we treat (19) as a quadratic in y, looking at the discriminant. The result is (17).

This enables us to state

THEOREM 2. *For all finite μ, if $F(y)$ satisfies the hypothesis of Theorem* 1, (1) *has an infinite number of limit cycles.*

Proof. By hypothesis there are an infinite number of y_n which satisfy (17) so that there are an infinite number of y_n which satisfy Theorem 1. Then by continuity of the function (8) there are an infinite number of zeros of $\Delta(y^*)$ and the theorem is proved.

For large μ Theorem 2 states only that for large enough values of y_n there is a limit cycle corresponding to every other y_n. For small μ, the use of Lemma 2 insures that there is a limit cycle for each pair of y_n.

This result goes beyond a conjecture in [3], wherein it was conjectured that if the peak amplitudes of $F(y)$ decreased for large y there would not be an infinite number of limit cycles. Lemma 3 shows that, for *finite* μ, this need not be true, that the amplitude of the peaks may decrease if the spacing of the zeros increase rapidly enough. However, since Lemma 3 requires μ to be bounded, the conjecture of [3] appears to be true in order to treat (1) as a singular perturbation problem.

We note that the hypotheses on $F(y)$ could be weakened if there were some way to determine the growth rate of $g(y)$.

References

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