

FINITE ELEMENT METHOD
IN
MECHANICAL DESIGN

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INTRODUCTION. HISTORICAL REVIEW. NOTATION.
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W.J.A.

I. Introduction

The finite element method has greatly helped many engineers and scientists in their career fields. It is a tool that allows methodical solution for systems with complicated geometries and made of many varying materials. The user of finite element methods develops intuition about the fundamentals of mechanics, because he is forced to think in terms of matrix mappings for all phenomena. After becoming comfortable with these linear transformations, one gains a tremendous organizing effect in the thought process.

Many physical systems are discrete from the outset, e.g., truss systems, space frames, electrical and hydraulic circuits. These systems have been well handled by engineers in the past 50 years, are often called "networks" and can usually be studied by use of ordinary differential equations. Other physical systems are continuous in nature, e.g., plates, shells, flow problems and electrical fields. These systems have been handled with great difficulty in the past, are often called "field problems" and require the use of partial differential equations. The goal of most approximate engineering theories is to convert these field problems into something resembling a network problem, i.e. to discretize the field.

The process of discretization can be done by global or local interpolations of the field variables. Many methods such as Galerkin's method or the classical Rayleigh-Ritz method represent the field variables by a series of global functions, each of which must satisfy certain boundary conditions. On the other hand, the finite element and finite difference methods attempt to model the field variable only over small discrete regions. The local approach has many advantages in terms of reducing the complexity of the final set of matrix equations (the "bandwidth" is often smaller, and the boundary conditions can be more easily applied). The finite element method is currently the most successful method for discretization of a broad range of field problems. The finite difference method is still the favored method in many field flow problems, such as problems where an unknown fluid-solid interface occurs due to phase change. Global methods are still

preferred, however, by many people. There are those who feel that the "boundary solution" process has a bright future. This method can be used in conjunction with local methods to solve problems with infinite domains. ⁽¹⁾

The process of discretization can be done through the use of distorted or true modeling. There was a period during the 1950's and 1960's when civil engineers, in particular, attempted many distorted discrete models, e.g., a frame analogy for a slab. This was a dead end which probably delayed serious finite element modeling in civil engineering by several years. (Some reviewers believe distorted modeling to be a step on the way to finite element methods. This author, after having tried to do such modeling, feels it was a step backward.) Modern finite elements are usually based on true modeling, e.g., a plate element representing a plate. An exception is that some flat plate elements are yet being used to model curved shells. There is some question as to whether this is distorted modeling since, in the limit as elements become small, effects of curvature are included.

Occasionally, one encounters in nature a system which is partially discrete and partially continuous. The finite element method (unlike the finite difference method) has no problem with the assembly of a discrete model of such a system. Beam, truss and spring elements are often combined with plate and solid elements in finite element solutions.

II. Historical Review

The finite element method has grown in step with the development of the digital computer. Early theoretical work was done by Courant ⁽²⁾ on the torsion of noncircular shafts in 1943 and by Argyris ⁽³⁾ on aircraft structures in the late '40's and early '50's. The field was not really "ripe" at that time, however. The paper which had the largest practical impact on the field was "Stiffness and Deflection Analysis of Complex Structures," by Turner, Clough, Martin and Topp ⁽⁴⁾, appearing in the September, 1956 issue of Journal of Aeronautical Sciences. This paper rejected the idea of further use of distorted models and instead developed two-dimensional elements to model beams and sheet. A result was the constant strain triangle and a rectangular element. The constant strain (Turner) triangle survives today and is still an excellent tool for teaching the theory.

This early work on modeling structural systems was primarily supported by the aircraft industry, which had a very difficult geometry and insufficient classical methods available. Other technical fields, however, were busy working with systems which were naturally discrete and therefore concentrated on supporting technology such as mathematical assembly of stiffness matrices, equation solvers and eigenvalue solutions. One should therefore distinguish between the modeling aspects inherent in finite elements, which are unique, in contrast to all the supporting methodology developed in other fields, some of which dates back to the 1800's.

As civil structural geometries became more complex, civil engineers turned to finite element models in the 1960's, with such ferocity that at the present time, more of the public domain work, computer programs and text books are due to civil engineers than any other technical field.

Workers in field problems such as hydrodynamics, electricity and magnetism became serious about finite elements in the late 1960's and early 1970's. Each of these fields had already developed tremendous capability in numerical analysis and only needed to add the modeling ability.

At present, the new frontier is fluid mechanics and the interaction between fluids and solid components. The finite element theory may provide a new way to look at fluids problems since it is strongly topological in nature, and the layout of the grid on the fluid field has an impact on the convergence of results.

III. Examples of Finite Element Solutions

Typical finite elements are shown in Fig. 1. In each case, one needs to assume the internal displacement field (or temperature, flow velocity, etc.) and develop a model relating forces at the nodes to displacements at nodes. This is called the displacement method and is the most common approach in finite elements.

When the elements desired have been created, they are entered into an element "library" within a computer program. The user then assembles a model of a complicated structure by

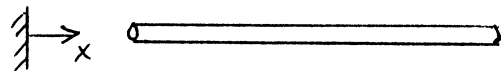


Fig. 1a. One-Dimensional Element.
(Truss, Beam ...)

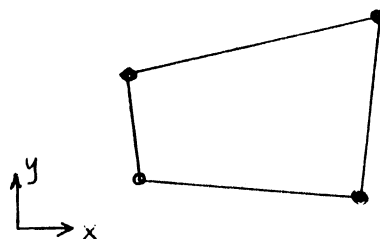


Fig. 1b. Two-Dimensional Element.
(Plane Stress, Plate ...)

using combinations of elements. The load ramp in Figure 2 is an all-welded aluminum structure, with steel grating on the bed. The picture shown is a 3/4 view of the assembled plate and beam model. The use of such computer graphics is extremely important to avoid errors in connecting the elements. The result of the particular study was to reduce the weight by 10% while maintaining the same load capacity.

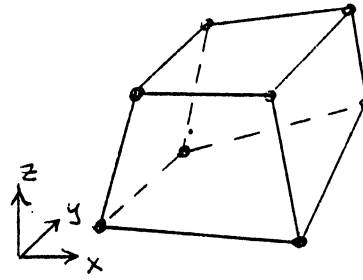


Fig. 1c. Three-Dimensional Element. (Solid).

A stress analysis was desired for a cylindrical tube with two slots cut in it as shown in Figure 3. The axial load caused high stresses at the ends of the slots. A contour plot of maximum shear strain on a "flat" between the slots is given in Figure 4. This showed an unacceptably high stress at the radius of the slots. After carrying out a parameter study on the width of the slot and the radius, the design was abandoned. The total study cost about \$4000, or approximately 1/2 as much as a photoelastic study would have cost. An interesting symmetry is present in the geometry, but the general purpose program SUPERB could not exploit it at the time, to allow solution of a smaller portion of the body. The program NASTRAN, MSC version, has "multiple point constraints," which can take advantage of this symmetry.

A problem for solidification of hot naphthalene as it is drawn down a cooled tube is shown in Figure 5. This work has been done by Antonio Valle for his doctoral dissertation in the Chemical Engineering Department at the University of Michigan. The finite elements used were very complicated and included temperature and convection efforts. The grid used in Figure 6 for the first quadrant resulted in the fluid convection shown in Figure 7. The solidification line involving a phase change is denoted.

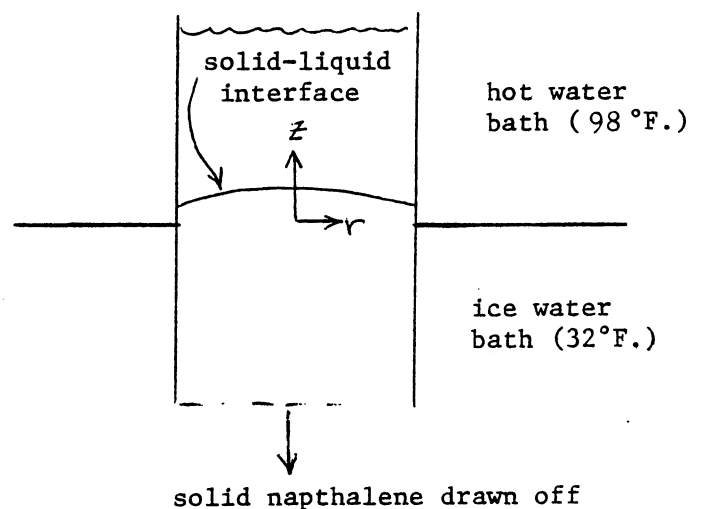
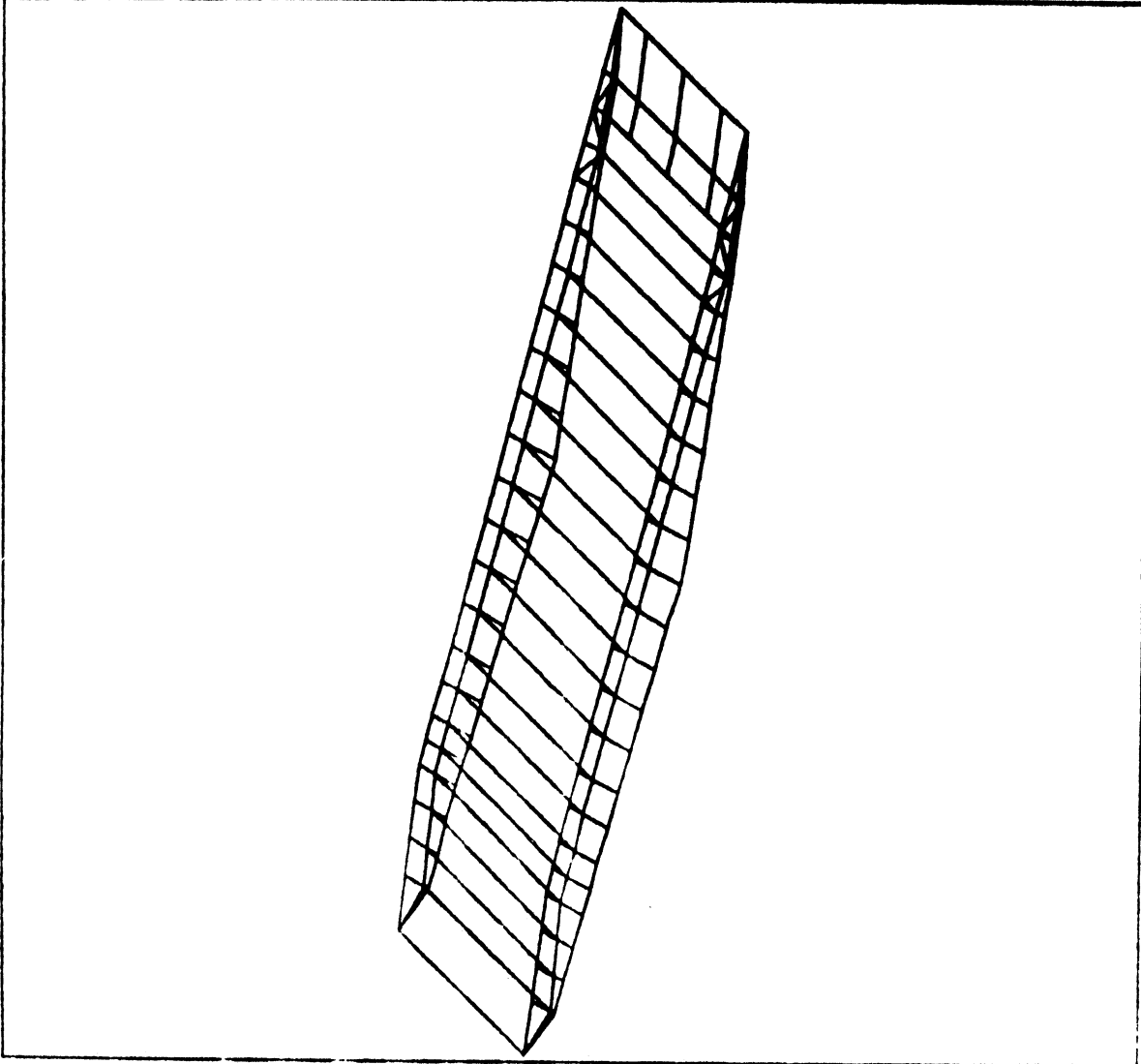


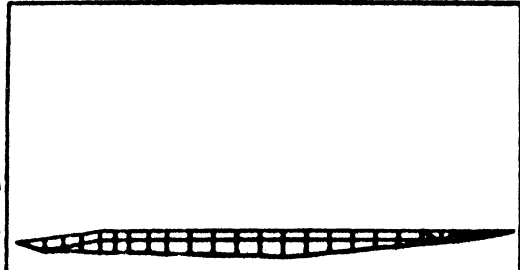
Fig. 5. Solidification of Napthalene.

BLOW-UP, REDRAW, OR NEXT PLOT?
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REG. 1



S-SECTION - 2

LOWER LEFT
 UPPER RIGHT
 Q-ROTATE 1
 R-ROTATE 2
 W-ALL AXIS
 M-CALL MTS
 N-NEW FILE
 O-CANDEL
 P-SAVE REG 1
 T-NEW MAGNIF
 U-NEW LOAD
 C-CALCOMP
 NODES= 153
 ELEM.= 329
 TYPES= 2
 LOAD = 0

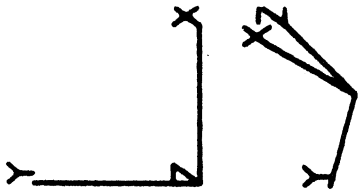


Fig. 2. Finite element mesh for beam-plate model of load ramp.

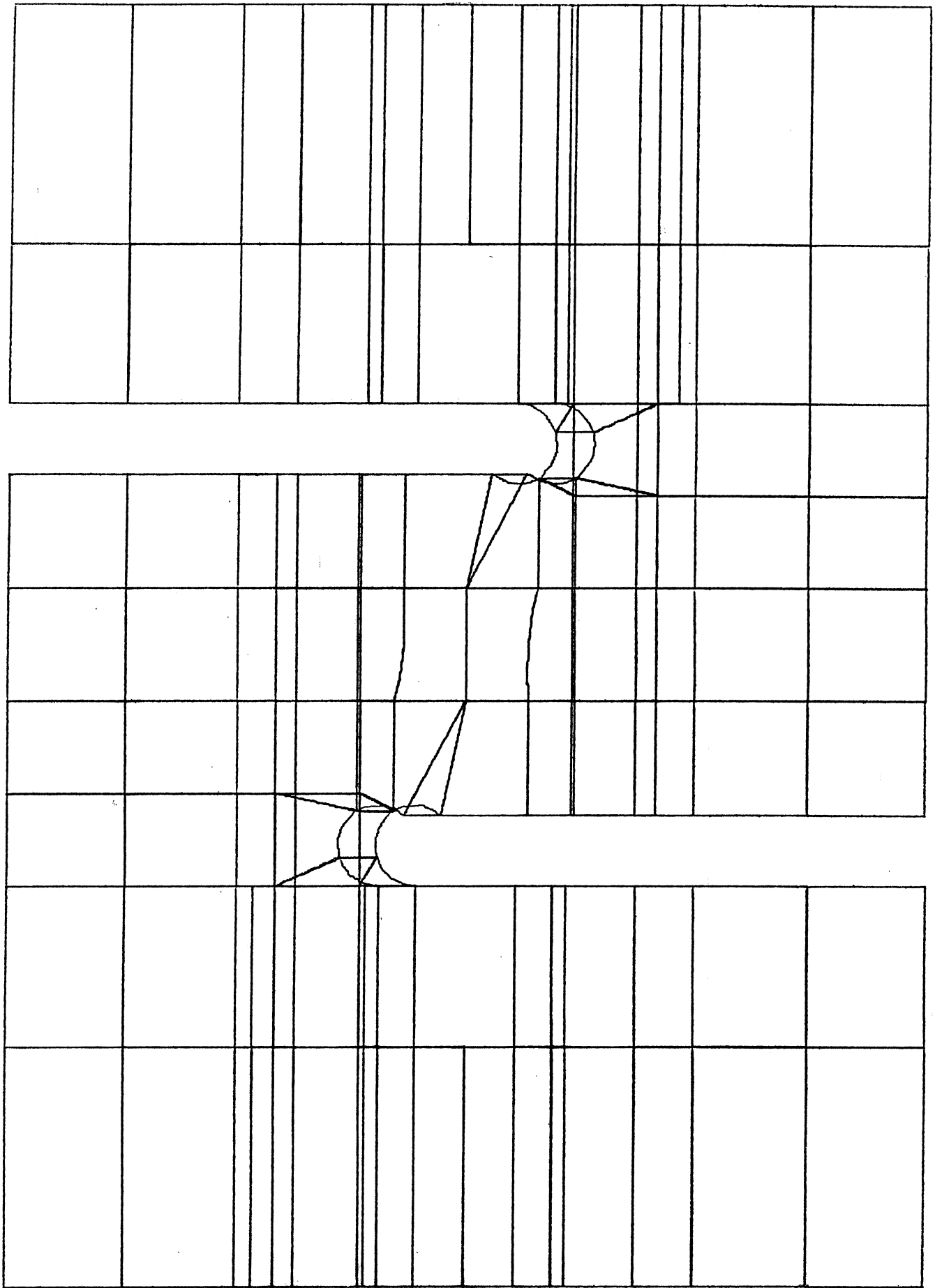
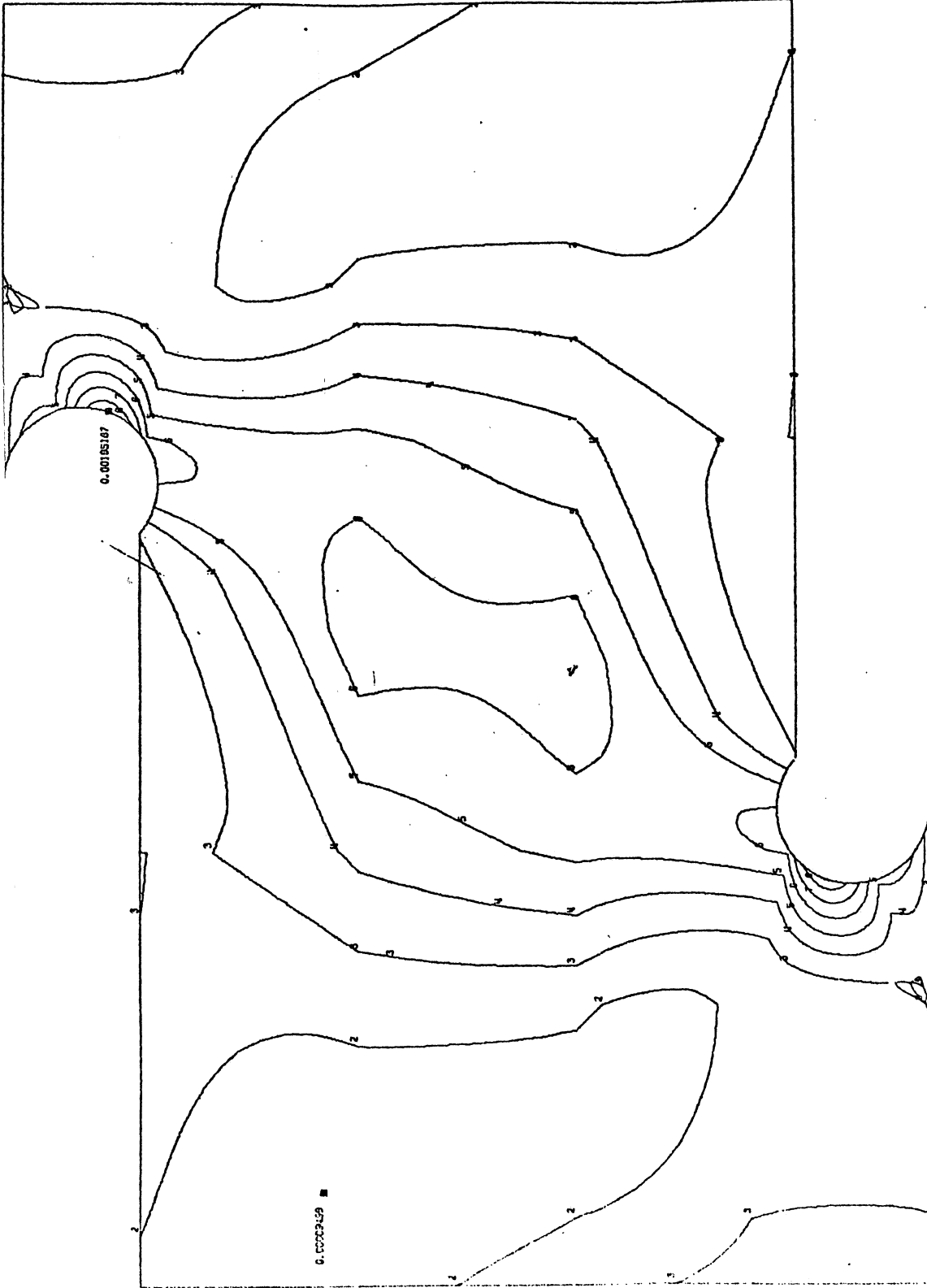


Fig. 3. Finite element mesh for a cylindrical tube with slots.

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CONTOUR NUMBER	CONTOUR LEVEL
1	0.00000000
2	0.00025000
3	0.00050000
4	0.00075000
5	0.00100000
6	0.00125000
7	0.00150000
8	0.00175000



ANALYSIS BY RA LETCHER AND HJ ANDERSON

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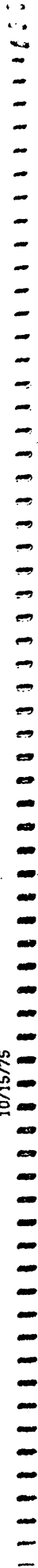
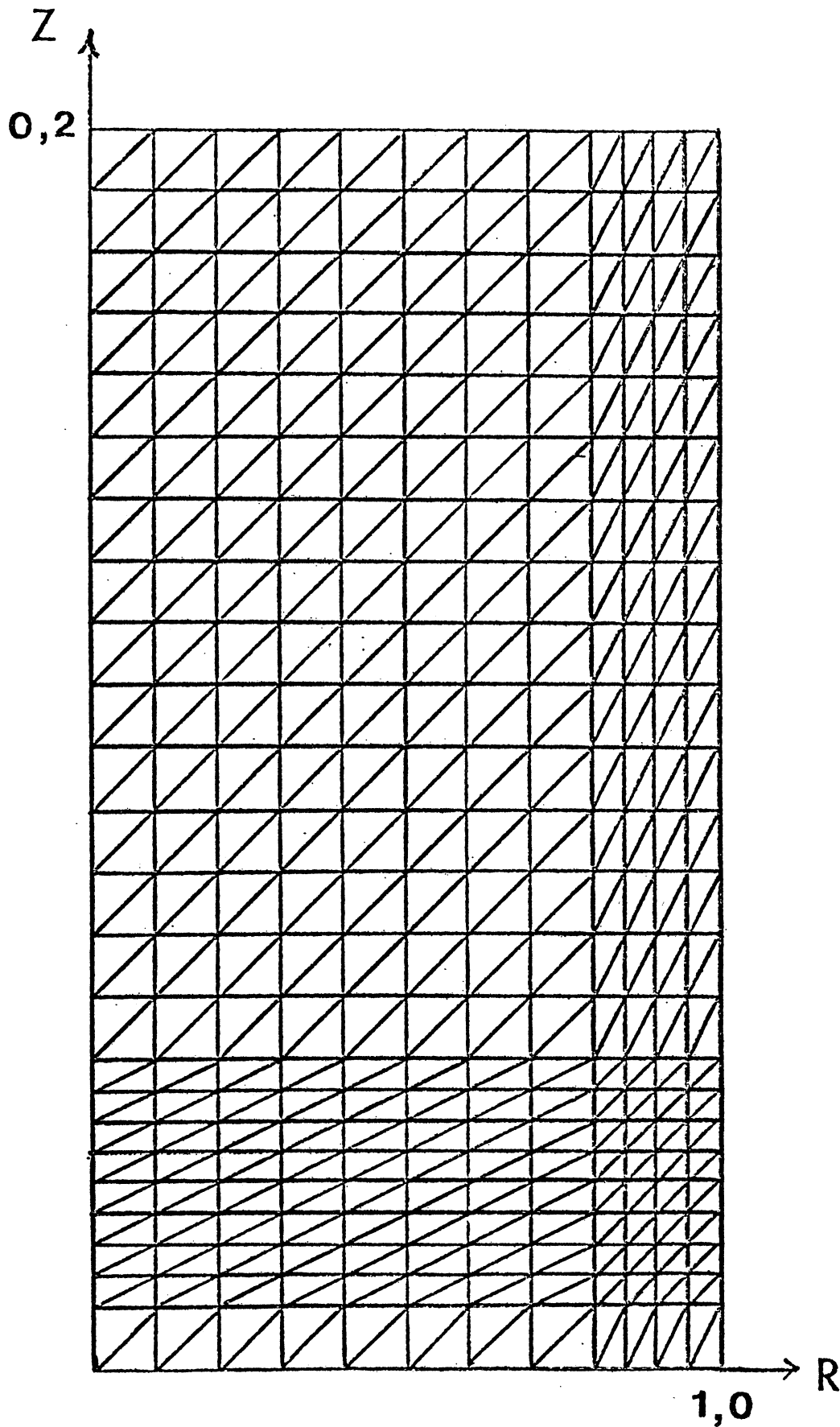
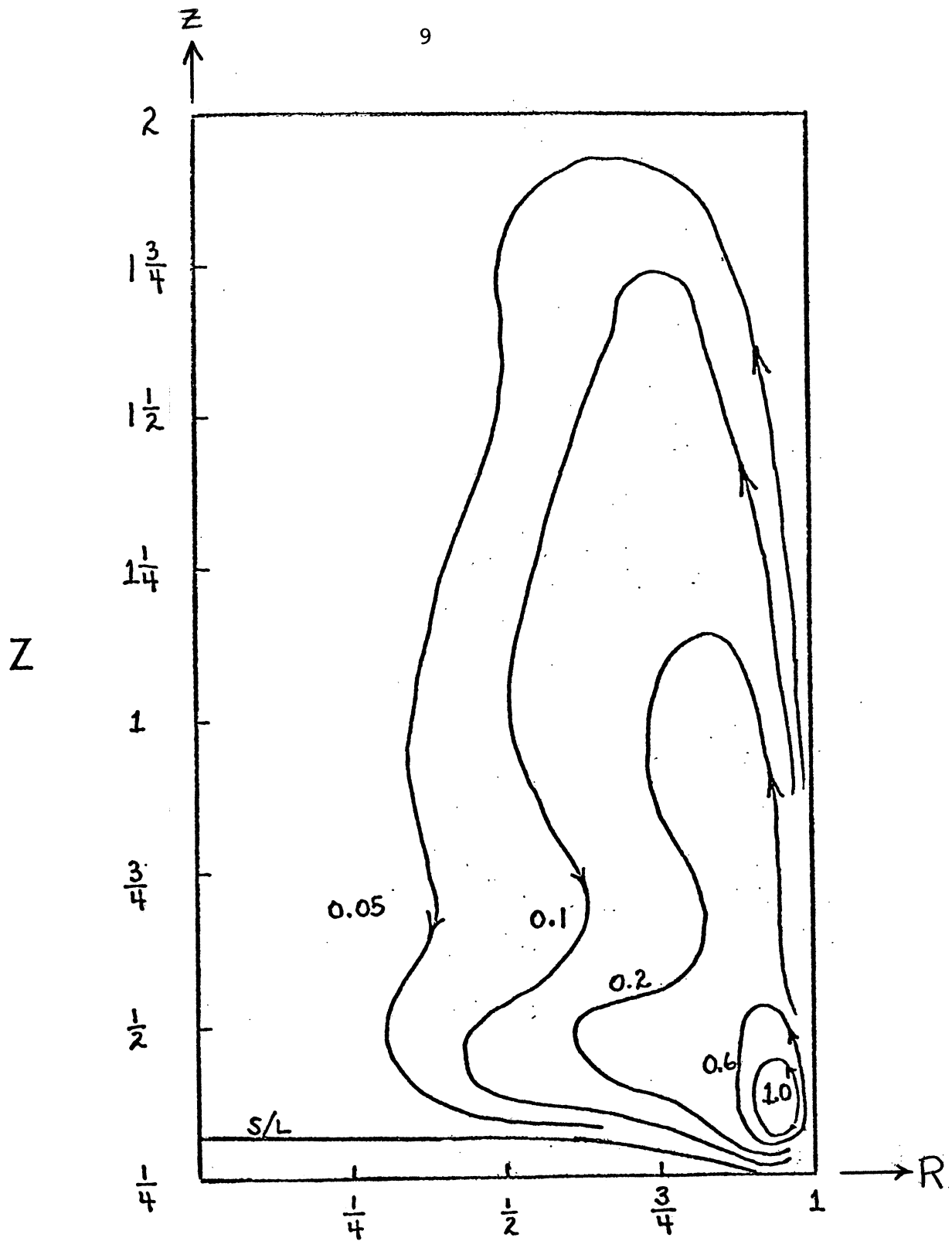


Fig. 4. Contour plot of maximum shear strain.



Axisymmetric heat conduction and thermal convection mesh.
576 Elements.



CONVECTION PATTERNS FOR MESH 2

37 mm ID, STATIONARY, $T_{\text{hot}} = 98^{\circ}\text{C}$

$\text{Gr} = 0.95\text{E}7$

Fig. 7. Convection patterns for molten naphthalene.

In comparing computer costs for this problem with an earlier finite difference solution, it was found that the finite element method was more expensive. The convenience of the finite element method may eventually make it more desirable, even for this type of problem, in which finite difference methods are known to be most competitive.

Finally, on the light side, one can imagine the hero of finite element theory rushing to solve the next problem (Figure 8). After sketching this figure, the author recalled that he had participated in several studies modeling the human, including the human head, knee, artery and heart. A three-dimensional model of the heart is shown in Figure 9. There will be much work in the future along these lines!

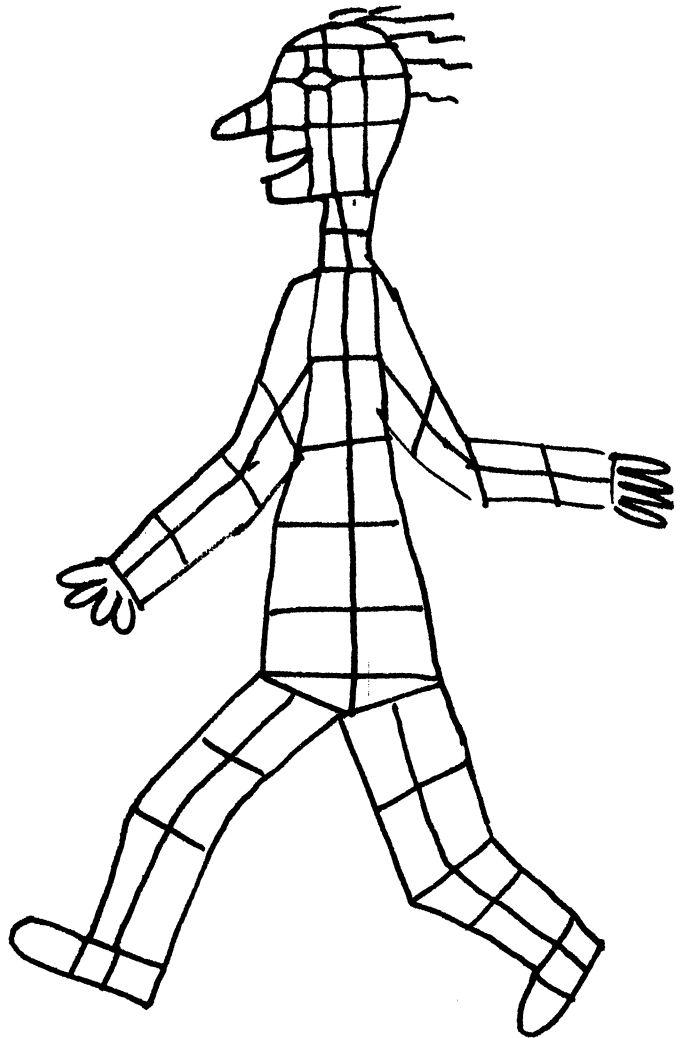


Fig. 8. Finite Element Man to the Rescue!

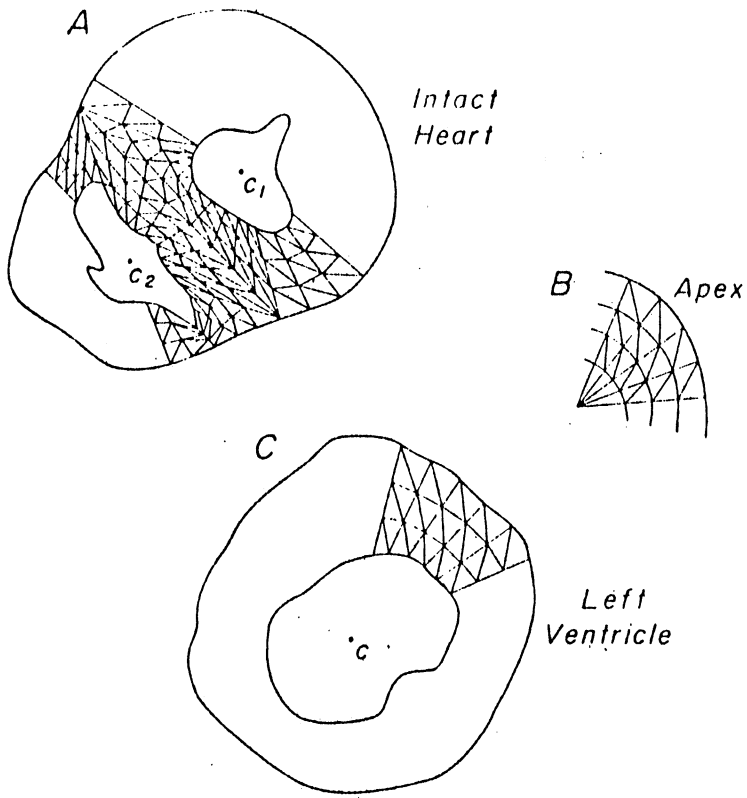


Fig. 1

3-Dimensional Finite Element Analysis

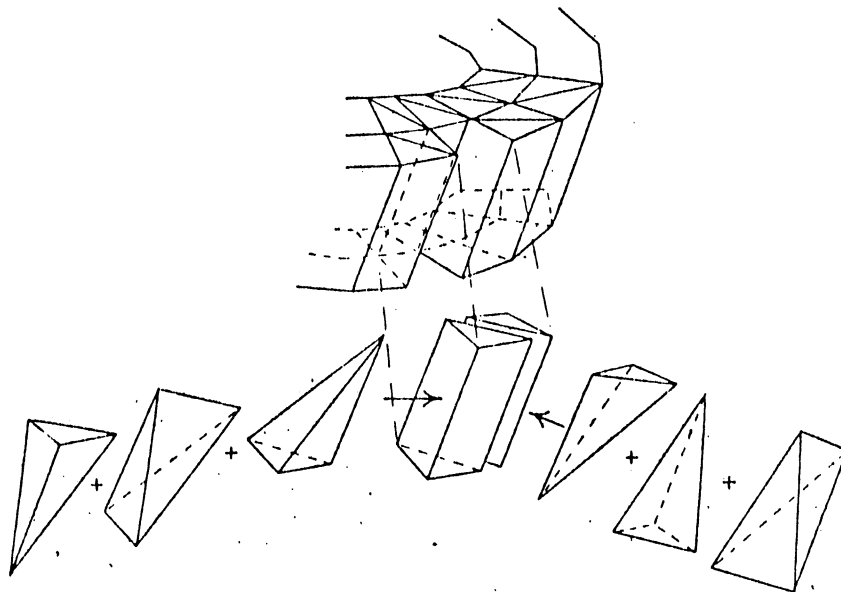


Fig. 9. Finite element model of the heart.
Heethaar, Pao and Ritman, Computer and Biomedical Research, 1977 (approx.).

IV. Notation

A. Vector and Matrix Notation

The notation of Duncan, Frazer and Collar will be used for our matrix work:

column vector $\{x\}$

row vector $[x]$

square or rectangular matrix $[A]$

transpose of a matrix $[A]^T$

diagonal square matrix $[B]$

inverse of a square matrix $[C]^{-1}$.

B. Elasticity (3-D).

$$\text{displacement field } \{u(x,y,z)\} \equiv \begin{Bmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{Bmatrix}$$

$$\text{strain field } \{\epsilon(x,y,z)\} \equiv \begin{Bmatrix} \epsilon_x(x,y,z) \\ \epsilon_y(x,y,z) \\ \epsilon_z(x,y,z) \\ \tau_{xy}(x,y,z) \\ \tau_{yz}(x,y,z) \\ \tau_{zx}(x,y,z) \end{Bmatrix}$$

$$\text{stress field } \{\sigma(x,y,z)\} \equiv \begin{Bmatrix} \sigma_x(x,y,z) \\ \sigma_y(x,y,z) \\ \sigma_z(x,y,z) \\ \tau_{xy}(x,y,z) \\ \tau_{yz}(x,y,z) \\ \tau_{zx}(x,y,z) \end{Bmatrix}$$

C. Finite Element Notation

nodal displacements for a single element:

$$\{q\} \equiv \left\{ \begin{array}{c} q_1 \\ q_2 \\ \vdots \\ q_n \end{array} \right\} .$$

nodal forces for a single element:

$$\{Q\} \equiv \left\{ \begin{array}{c} Q_1 \\ Q_2 \\ \vdots \\ Q_n \end{array} \right\} .$$

nodal displacements for an assembled system:

$$\{r\} \equiv \left\{ \begin{array}{c} r_1 \\ r_2 \\ \vdots \\ r_N \end{array} \right\} .$$

nodal forces for an assembled system:

$$\{R\} \equiv \left\{ \begin{array}{c} R_1 \\ R_2 \\ \vdots \\ R_N \end{array} \right\} .$$

generalized coordinates:

$$\{\alpha\} \equiv \left\{ \begin{array}{c} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{array} \right\} .$$

V. References

1. Zienkiewicz, O. C., "The Finite Element Method," 3rd ed., McGraw-Hill Book Co. (U.K.) Limited, London 1977.
2. Courant, R., "Variational Methods for the Solution of Problems of Equilibrium and Vibration," Bulletin of the American Mathematical Society, vol. 49, 1943 (pp. 1-23).
3. Argyris, J. H., "Energy Theorems and Structural Analysis," Aircraft Engineering, 1954-1955.
4. Turner, M. J., Clough, R. W., Martin, H. C. and Topp, L. J., "Stiffness and Deflection Analysis of Complex Structures," Journal of the Aeronautical Sciences, vol. 23, no. 9, September 1956 (pp. 805-823).

VI. Textbooks

1. Desai & Abel, Introduction to Finite Element Analysis, Van Nostrand-Reinhold (1972). Call # TA 640.2 D48.
2. Zienkiewicz, The Finite Element Method in Engineering Sciences, McGraw-Hill, (1977), 3rd ed. Call # QA 640.2 Z66.
3. Huebner, The Finite Element Method for Engineers, Wiley-Interscience, 1975.
4. Tong & Rosettos, Finite Element Method, MIT Press (1977) . Call # TA 347 F5 T66.
5. Bathe & Wilson, Numerical Methods in Finite Element Methods, Prentice-Hall, 1976. Call #, TA 347 F5 B37
6. Gallagher, Finite Element Analysis Fundamentals, Prentice-Hall, 1975.
7. Martin & Carey, Introduction to Finite Element Analysis, McGraw-Hill, 1973. Call # TA 335 M37.
8. Przemieniecki, Theory of Matrix Structural Analysis, McGraw-Hill, 1968. Call # TA 642 P77.
9. Strang & Fix, An Analysis of the Finite Element Method, Prentice-Hall, 1973.
10. Hamming, R. W., Numerical Methods for Scientists and Engineers, 2nd Ed. , McGraw Hill, 1973.

TABLE 1
 RECOMMENDED TEXTBOOKS FOR FINITE ELEMENT STUDY
 (Books are listed in recommended order for automotive structural design*)

Author	Book Starts at This Level	Title	Publisher	Date	Comment	Cost
Desai & Abel	Introductory	Introduction to the Finite Element Method	Van Nostrand-Reinhold	1972	Good for learning f. e. m. Civil oriented, but relevant	\$ 19.50
Zienkiewicz, O. C.	Intermediate	The Finite Element Method in Engr. Science	McGraw-Hill	1977	Best all around reference. Tough for beginner. Engineering science approach.	\$ 21.00
Huebner, K. H.	Introductory	The Finite Element Method for Engineers	Wiley - Interscience	1975	Good for mechanical engineering, lubrication problems.	\$ 24.75
Cook	Introductory	Concepts & Applications of Finite Element Methods	Wiley	1974	All around text.	
Gallagher, R.	Intermediate	Finite Element Analysis Fundamentals	Prentice Hall	1975	Good for plate and shell sections.	\$ 21.95
Tong & Rossettos	Introductory	Finite Element Method	The MIT Press	1977	Good for energy ideas.	
Bathe & Wilson	Advanced	Numerical Methods in Finite Element Analysis	Prentice-Hall	1976	Very good for numerical methods, dynamics, integration.	\$ 28.95
Oden, J. T.	Advanced	Finite Elements of Nonlinear Continua	McGraw-Hill	1972	Oden is a world leader in nonlinear problems.	\$ 21.50
Desai, C.	Introductory	Elementary Finite Element Method	Prentice-Hall	1979	New elementary text.	\$ 21.95
Martin & Carey	Introductory	Introduction to Finite Element Analysis	McGraw-Hill	1973	Good elementary sections; poorer on advanced work.	\$ 20.50
Przimiencicki, J. S.	Intermediate	Theory of Matrix Structural Analysis	McGraw-Hill	1973	Good as reference for various stiffness matrices. Not good text.	\$ 20.50
Ural	Introductory	Finite Element Method	Intext Educational Publishers	1973		
Strang & Fix	Advanced	An Analysis of the Finite Element Method	Prentice-Hall	1973	Mathematical. Of limited interest to engineers.	\$ 21.95

* Of course, this listing "in order of preference" is very subjective. The experienced finite element user will have his own preferences. The bold step of presenting such a preferred order is done only in the philosophy that a beginning finite element user may be helped, and a certain commonality may be reached, even if the list is incorrect in an absolute sense.

RELATED BOOKS AND NOTES (ALPHABETICAL)

Anderson, W. J.	Introductory	Finite Element Methods in Mechanical Design I, II	Author (313) 764-3397	1978/ 1979	Class notes for University of Michigan two-term course. Uses Desai & Abel notation.	\$ 20.00
Chung, T. J.	Intermediate	Finite Element Analysis in Fluid Dynamics	McGraw-Hill	1978	Fluid mechanics.	
Hamming	Intermediate	Numerical Methods for Scientists & Engineers	McGraw-Hill	1973	Excellent book on numerical analysis.	\$ 14.95
Meirovitch, L.	Intermediate	Element of Vibration Analysis	McGraw-Hill	1975	Dynamics of discrete and continuous systems	\$ 19.50
Pian, T. H. H.	Intermediate	Finite Element Methods in Structural Dynamics	American Institute of Aeronautics & Astronautics	1975	Selected Reprints. Includes original Turner paper.	\$ 7.00
Wilkinson	Introductory	The Algebraic Eigenvalue Problem	Oxford	1965	Excellent book on eigenvalue problem.	\$ 24.00

I. LINE ELEMENT

A two-node, constant area line element is to be created. The element has two nodal degrees of freedom q_1 and q_2 with their associated forces Q_1 and Q_2 . (The general rule is that the product of nodal force and displacement must yield energy.)

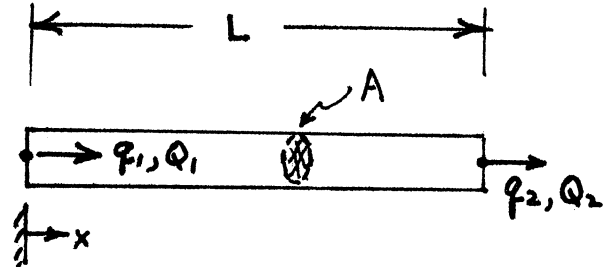


Fig. 1. Two-node line element.

This simple element can illustrate several important features of finite element theory. Since we have not yet developed a general energy approach, we must at this point use equilibrium ideas instead. For instance, it is clear that

$$Q_2 = -Q_1 \tag{1}$$

from equilibrium in the x direction. We also have the one-dimensional stress strain law

$$\sigma_x = E \epsilon_x \quad (\text{materials idea}) \tag{2}$$

and the one-dimensional strain-displacement law

$$\epsilon_x = \frac{d}{dx} u(x) \quad (\text{geometric idea}) \tag{3}$$

where $u(x)$ is the displacement field.

If we assume that strain ϵ_x is constant in the element, we can approximate the strain-displacement law by

$$\begin{aligned} \epsilon_x &\approx \frac{\Delta L}{L} \\ &= \frac{q_2 - q_1}{L} \end{aligned} \tag{4}$$

This turns out to be exact for a constant area element. It also implies (with Eqn. 2) that stress is constant.

From equilibrium at the nodes,

$$Q_1 = -\sigma_x A \tag{5a}$$

$$Q_2 = \sigma_x A \tag{5b}$$

$$= E \epsilon_x A$$

$$= EA \frac{d}{dx} u$$

$$= EA \frac{q_2 - q_1}{L} \tag{17}$$

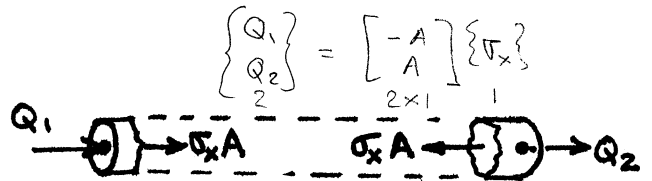


Fig. 2. Equilibrium.

Rewriting this in terms of strain and finally displacement, one has

$$Q_1 = -AE(q_2 - q_1)/L \quad (6a)$$

$$Q_2 = AE(q_2 - q_1)/L \quad (6b)$$

This can be put in matrix form:

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} \quad (7a)$$

$$(7b)$$

This is the fundamental relation in finite element theory, the load-deflection law. The matrix relating load and deflection (including the scalar factor) is defined to be a stiffness $[k]$:

$$[k] = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \text{stiffness matrix} \quad \begin{array}{l} \rightarrow \text{maps from displacements to forces} \\ \text{i.e. apply a displ., measure} \\ \text{resulting forces at each} \\ \text{node.} \end{array} \quad (8)$$

We were fortunate to obtain this answer by equilibrium methods. We were even more fortunate that the answer is exact for the constant area element. Note that the determinant of the stiffness matrix for an element is zero, i.e., the matrix is singular. This is always true.

II. LINE ELEMENT. VARYING AREA.

To push our luck a little, we will attempt to find the stiffness matrix for a two-node line element which has varying area. In this case, we again have

$$Q_2 = -Q_1 \quad (9)$$

from equilibrium.

We again have the stress-strain law

$$\sigma_x = E \epsilon_x \quad (10)$$

and the strain-displacement law

$$\epsilon_x = \frac{du}{dx} \quad (11)$$

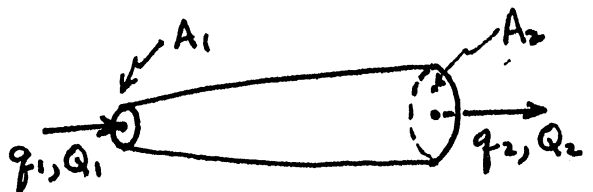


Fig. 3. Varying area line element.

For this element, we avoid assuming constant strain a priori, because we know that it cannot be so. (We have a constant force transmitted through a varying area, which leads to varying stress and then to varying strain.) A general way to attack such a problem, and the reason the method is called the "displacement" method, is to assume a displacement field such as

Using the shape function in Eqn. 13 to represent the internal displacement field,

$$\left. \begin{aligned}
 Q_1 &= -E \left(\frac{q_2 - q_1}{L} \right) A_1 \\
 \text{Likewise} & \\
 Q_2 &= E \left(\frac{q_2 - q_1}{L} \right) A_2
 \end{aligned} \right\} \begin{array}{l}
 (17a) \quad \text{SATISFIES INTERNAL} \\
 \text{EQUILIBRIUM. VIO-} \\
 \text{LATES NODAL (GLOBAL)} \\
 \text{FORCE BALANCE.} \\
 (17b) \quad \text{UNACCEPTABLE!}
 \end{array}$$

An alternate philosophy is to maintain nodal (global) equilibrium and to violate internal equilibrium. For instance, one can take

$$\begin{aligned}
 Q_1 &= -\sigma_x A_{\text{average}} \\
 &= -E \left(\frac{q_2 - q_1}{L} \right) A_{\text{average}} \quad (18a)
 \end{aligned}$$

$$Q_2 = E \left(\frac{q_2 - q_1}{L} \right) A_{\text{average}} \quad (18b)$$

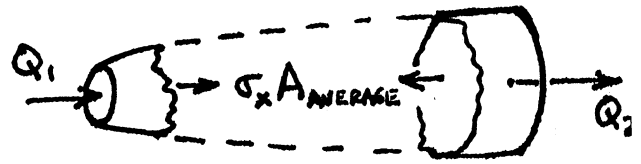


Fig. 5. Satisfaction of nodal equilibrium.

This leads to

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \frac{EA_{\text{average}}}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} \quad (19)$$

Surprisingly, this is a satisfactory model of the varying area line element, even though internal equilibrium of the element is violated. The displacement field leads to a stress field which is nowhere in equilibrium in the interior of the element. This is a common feature of elements to be developed by the displacement finite element method.

QUESTIONS:

1. Write out the matrix form of Equations (17a) and (17b). What is the mathematical problem with [k]?
2. What does A_{average} mean? This question is cleared up for this simple element later by an exact solution using a more complicated (logarithmic) displacement field.
3. Evaluate α_1 and α_2 in Equation (14) in terms of q_1 and q_2 . You must recognize that $u(0) = q_1$ and $u(L) = q_2$ to succeed.

III. ASSEMBLY OF CONSTANT AREA LINE ELEMENTS.

The assembly process for finite element theory is straightforward and powerful. A good way to visualize the process is to look at two disjoint line elements and then to pin them together.

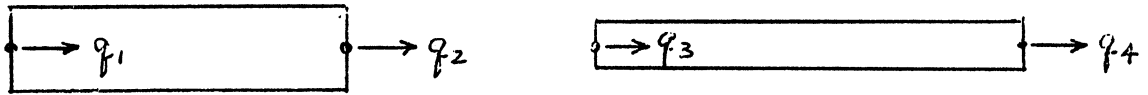


Fig. 6. Two disjoint line elements.

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \frac{E_1 A_1}{L_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}, \quad \begin{Bmatrix} Q_3 \\ Q_4 \end{Bmatrix} = \frac{E_2 A_2}{L_2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} q_3 \\ q_4 \end{Bmatrix} \quad (20)$$

$$= \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

The way to assemble is to create a matrix problem large enough so that each element can be "imbedded" in it. Realizing that upon assembly one has $q_3 = q_2$ and that there will be only 3 independent coordinates, one writes

$$\begin{Bmatrix} \\ \\ \end{Bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_4 \end{Bmatrix} \quad (22)$$

3 3 x 3 3

Equation 22 is a logical format for the assembled system. The equations for the first element can be written, by adding a trivial equation:

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ 0 \end{Bmatrix} = \begin{bmatrix} E_1 A_1 / L_1 & -E_1 A_1 / L_1 & 0 \\ -E_1 A_1 / L_1 & E_1 A_1 / L_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_4 \end{Bmatrix} \quad (23)$$

$Q_1 = k_1 q_1 - k_1 q_2$
 $Q_2 = -k_1 q_1 + k_1 q_2$

Likewise, the second element is described by

$$\begin{Bmatrix} 0 \\ Q_3 \\ Q_4 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & E_2 A_2 / L_2 & -E_2 A_2 / L_2 \\ 0 & -E_2 A_2 / L_2 & E_2 A_2 / L_2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_4 \end{Bmatrix} \quad (24)$$

$Q_3 = k_2 q_2 - k_2 q_4$
 $Q_4 = -k_2 q_2 + k_2 q_4$

Equations (23) and (24) are of the mathematical form

$$\{X\} = [P]\{Z\} \tag{25a}$$

$$\{Y\} = [Q]\{Z\} \tag{25b}$$

and can be added

$$\{X\} + \{Y\} = \underbrace{([P] + [Q])}_{\text{Assembled Stiffness}}\{Z\} \tag{26}$$

This is what people mean by "stiffnesses just add."

In our example:

$$\begin{Bmatrix} Q_1 \\ Q_2+Q_3 \\ Q_4 \end{Bmatrix} = \begin{bmatrix} E_1 A_1 / L_1 & -E_1 A_1 / L_1 & 0 \\ -E_1 A_1 / L_1 & (E_1 A_1 / L_1 + E_2 A_2 / L_2) & -E_2 A_2 / L_2 \\ 0 & -E_2 A_2 / L_2 & E_2 A_2 / L_2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_4 \end{Bmatrix} \tag{27}$$

$Q_1 = k_1 q_1 - k_1 q_2$
 $Q_2 + Q_3 = -k_1 q_1 + (k_1 + k_2) q_2 - k_2 q_4$
 $Q_4 = -k_2 q_2 + k_2 q_4$

\nearrow
force the equality

The stiffness matrix has assembled nicely and the only displacements which appear are the independent degrees of freedom. The force vector can be simplified by imagining a fictitious pin* at the joint. There is in general an external force R on the pin, which satisfies

$$R = Q_2 + Q_3 \tag{28}$$

hence we define the new symbols $[K]$ and $\{r\}$:

$$\begin{Bmatrix} Q_1 \\ R \\ Q_4 \end{Bmatrix} = [K] \{r\} \tag{29}$$

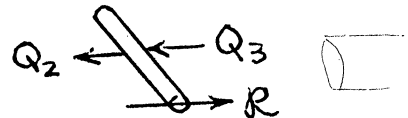


Fig. 7. Fictitious pin. 3/4 view.

\uparrow assembled independent d.o.f.
 \uparrow assembled stiffness

* In most problems, the structure is continuous and this mental process of separating elements and then putting together again with a pin never occurs physically.

TABLE 2

GENERAL PURPOSE F. E. M. PROGRAMS

CODE	COMPANY	CONTACT	COMMENTS
NASTRAN/ MSC	MacNeal-Schwendler Corp. 7442 N. Figueroa Street Los Angeles, CA 90041	J. A. Joseph (213) 254-3456	Linear Stress, Vibration, Nonlinear Material, Nonlinear Geometry, Component Modes, Acoustics, Heat Conduction
ANSYS	Swanson Analysis Systems, Inc. P. O. Box 65 Houston, PA 15342	(412) 746-3304	Linear Stress, Vibration, Nonlinear Material, Geometry
SUPERB	Structural Dynamics Research Corp. 2000 Eastman Drive Milford, OH 45140	(513) 576-2400	Isoparametric Elements, Modal Analysis
NISA	Engineering Mechanics Research Corp. Oak Park, Michigan	(313) 968-3163	Isoparametric Elements, Heat Conduction
MARC	MARC Analysis Research Corp.	H. D. Seamons (415) 326-7511	Good for strong nonlinearities
SAPIV	University of California at Berkeley	Prof. Ed Wilson (415) 642-3743	Cheap! Linear stress and vibration only. 600 copies in use world wide.
SAP6	University of Southern California SAP Users Group, Dept. of Civil Engineering, University Park Los Angeles, CA 90007	Dr. Al Gelman (213) 741-6618	Linear Stress and Vibration .
STARDYNE	Control Data Corporation 21700 Northwestern Highway Southfield, MI 48075	(313) 552-6742	
DANUTA	MCAUTO (McDonnell-Douglas Automation Co.) Southfield, MI	(313) 353-4361	Refined elements for stresses in complex bodies

Fortunately, the force vector consists of only externally imposed loads! This is a wonderful situation, because the finite element method then suppresses the internal forces and the need to know them. (This is comparable to Lagrange's equations in analytical dynamics which are a means of suppressing internal forces of constraint and which allow easier solution.)

One finally defines the external load vector symbol $\{R\}$ and writes:

$$\{R\} = [K]\{r\} \quad (30)$$

This simple matrix equation is the heart of most finite element solutions. The equation still needs to be modified before solution since the assembled stiffness matrix is singular at this stage. This will be discussed later.

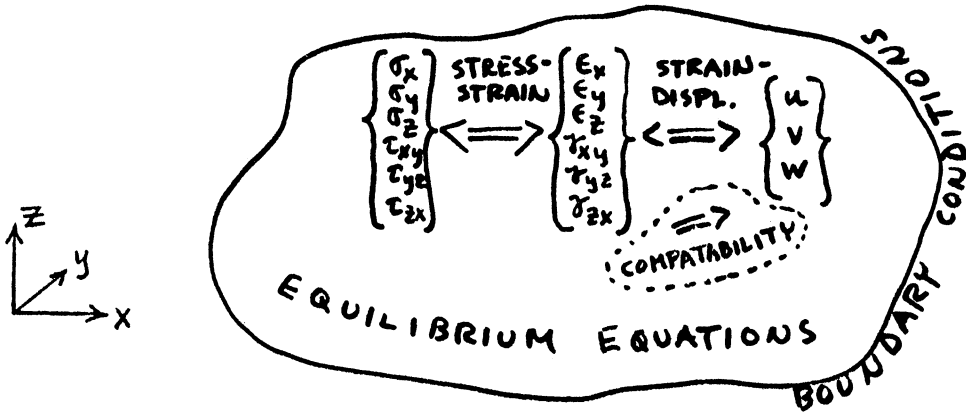
A similar assembly process can be done in electrical and fluid circuits and field problems in many branches of science and engineering. For each class of problem, the law used for assembly is appropriate to the variables, and includes Kirchhoff's current law in electrical circuits and conservation of energy in heat conduction problems.

CONSTANT STRAIN TRIANGLE*

W. J. A.

I. Review of Plane Stress/Plane Strain

The theory of elasticity is concerned with tensor quantities and relations between tensors. For our purposes, we will be happy to deal with vectors and the linear transformations between them. In three-dimensional elasticity, for instance, the sketch below indicates some of the important concepts:



Equilibrium Equations

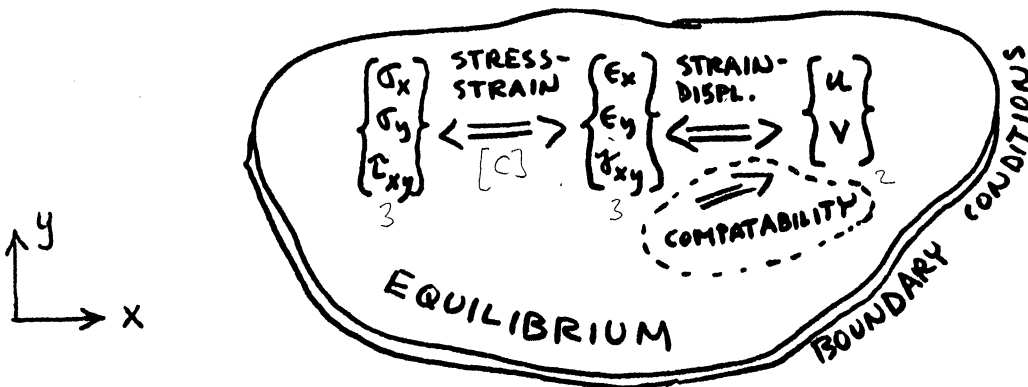
For the more specialized case of plane stress, one has

$$\sigma_z = 0 \tag{1}$$

and

$$\epsilon_z = \text{fcn}(\sigma_x, \sigma_y) \tag{2}$$

so that the significant variables become:



* Reference: Turner, Clough, Martin and Topp, "Stiffness and Deflection Analysis of Complex Structures," Journal of the Aeronautical Sciences, Vol. 23, #9, September 1956, pp. 805-823.

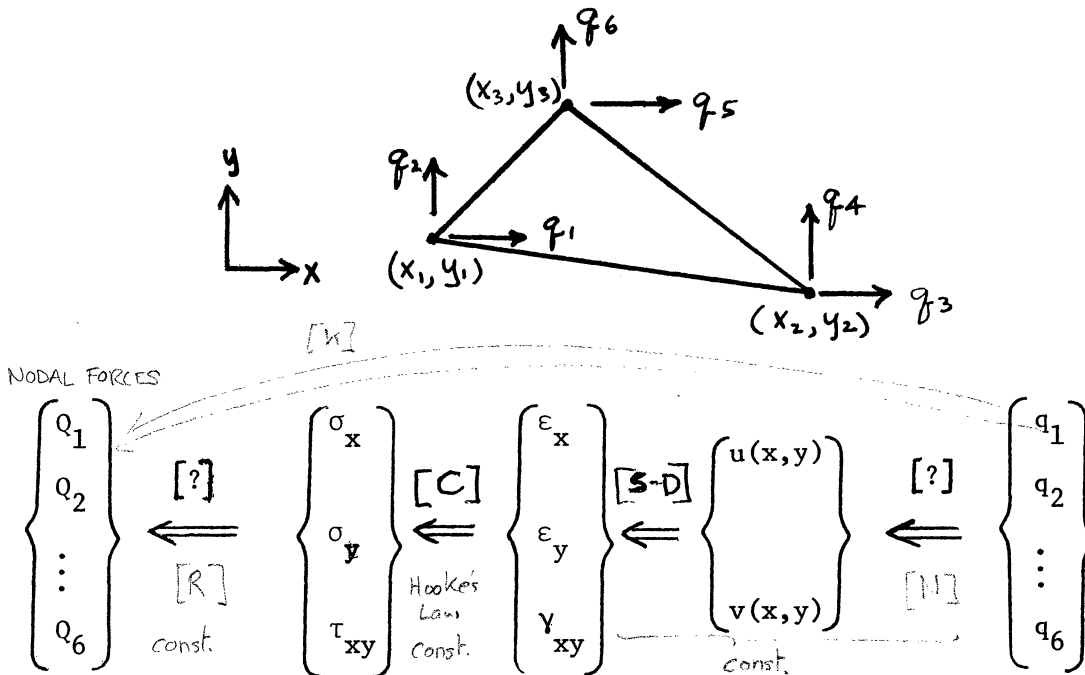
Specifically, the transformations used to relate stresses, strains and displacements, are

$$\begin{matrix}
 \text{HOOKES' LAW - ALGEBRAIC} & & \text{INVOLVES DERIVATIVES} \\
 \left\{ \begin{matrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{matrix} \right\} = \begin{bmatrix} E/1-\nu^2 & E\nu/1-\nu^2 & 0 \\ E\nu/1-\nu^2 & E/1-\nu^2 & 0 \\ 0 & 0 & G \end{bmatrix} \left\{ \begin{matrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{matrix} \right\}, & & \left\{ \begin{matrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{matrix} \right\} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \left\{ \begin{matrix} u \\ v \end{matrix} \right\}
 \end{matrix}$$

Comparable linear transformations exist for plane strain. Note that the stress/strain law involves constants, whereas the strain/displacement law involves derivatives.

II. Development of the Stiffness Matrix for the Constant/Strain Triangle

The case of plane stress discussed in Part A applies to many two-dimensional situations for plate-like bodies. Let us consider the special plane stress problem where a triangular element has been cut from the plate. If we imagine nodal forces and displacements as shown for a general coordinate system, it is then helpful to consider these new quantities in relation to the internal (generic) quantities as follows:



If we remember that, in general, stiffness matrices relate nodal force and displacement vectors, then our job is to find whether linear transformations can be found in the two locations with question marks. One needs only then to proceed leapfrog-fashion with four successive transformations to relate nodal displacements to nodal forces.

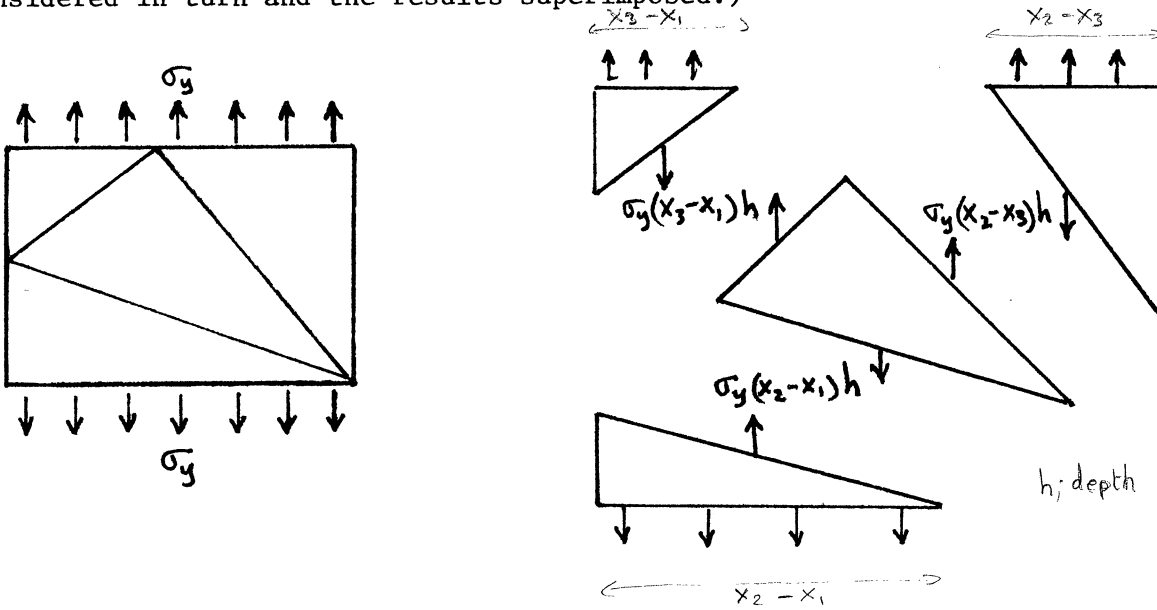
The unknown transformation between nodal forces and internal stresses can be found by simple equilibrium arguments. First, strains are assumed constant.

$$\epsilon_x = a = \epsilon_x(x,y,z) \tag{4a}$$

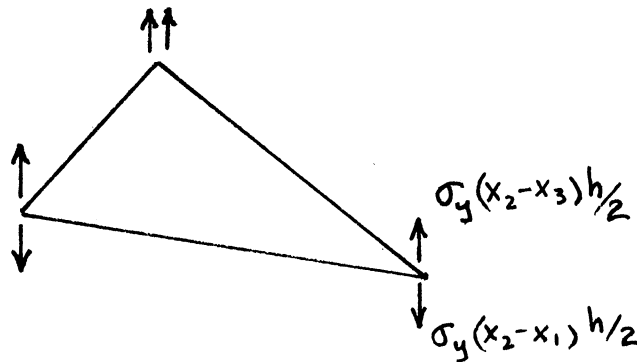
$$\epsilon_y = b \tag{4b}$$

$$\gamma_{xy} = c \tag{4c}$$

Then the triangle is considered to be imbedded in a uniform sheet of material under constant stress in one direction, say y. (The other stresses can be considered in turn and the results superimposed.)



The exploded diagram shows the triangular element with concentrated loads at the midpoint of each side. This is a possible type of concentrated loading that would place the triangular element in equilibrium. A better approach is to break each concentrated load into two equal parts and then to apply half to each nearby node.



$$27 \quad Q_4 = \frac{\sigma_y (x_2 - x_3)h}{2} - \frac{\tau_{xy} (x_2 - x_1)h}{2}$$

It can be seen that

$$Q_4 = \sigma_y (x_1 - x_3)h/2,$$

for instance. One can carry this process out to find the other forces for this loading and then to include σ_x and τ_{xy} stresses. The results is a linear relation:

$$\{Q\} = [\mathcal{E}]\{\sigma\} \quad (5)$$

where the symbol \mathcal{E} has been chosen to stand for "equilibrium."

Now we must solve for the linear relation between generic displacements and nodal displacements. This is more difficult conceptually because an intermediate step develops. First of all, the strain/displacement relations are integrated:

$$u(x,y) = \int \epsilon_x dx \quad v(x,y) = \int \epsilon_y dy \quad (6a)$$

$$= \int a dx \quad = \int b dy \quad (6b)$$

$$= ax + f(y) \quad = by + g(x) \quad (6c)$$

When the unknown functions $f(y)$ and $g(x)$ are substituted into the shear equation

$$\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = c = \tau_{xy} \right] \text{ SHEAR EQN.} \quad (7)$$

one obtains

$$f'(y) + g'(x) = c \quad (8)$$

The derivatives $f'(y)$ and $g'(x)$ must therefore be constants. Arbitrarily choosing $f'(y) \equiv A$ yields $g'(x) = c - A$. One can summarize the results in matrix form:

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} x & 0 & 0 & y & 1 & 0 \\ 0 & y & x & -x & 0 & 1 \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \\ A \\ B \\ C \end{Bmatrix} \quad (9)$$

$$u = ax + Ay + B \\ = \epsilon_x x + f'(y)y + B$$

$$[\phi]\{\alpha\} \quad (10)$$

This doesn't seem to help because we have merely defined a new vector of constant coefficients (three of which are the specified constant strains; the three constants of integration can be shown to represent rigid body modes). It is rather easy, however, to relate these constants to the nodal displacements, since

$$\begin{aligned} q_1 &\equiv u(x_1, y_1) = a x_1 + A y_1 + B \\ q_2 &\equiv v(x_1, y_1) = b y_1 + (c-A)x_1 + C \\ &\vdots \\ q_6 &\equiv v(x_3, y_3) = b y_3 + (c-A)x_3 + C \end{aligned} \quad (11)$$

In matrix form

$$\{q\} = [A]\{\alpha\} \quad \text{or} \quad \{\alpha\} = [A]^{-1}\{q\} \quad (12a,b)$$

Now, put all these results together

$$\begin{aligned} \{Q\} &= [E]\{\sigma\} \\ &= [E] \begin{bmatrix} \text{stress} \\ \text{strain} \end{bmatrix} \{\epsilon\} \\ &= [E][C] \begin{bmatrix} \text{strain} \\ \text{displ.} \end{bmatrix} \{u\} \\ &= [E][C][S-D][\phi]\{\alpha\} \\ \{Q\} &= \underbrace{[E][C][S-D][\phi][A]^{-1}}_{[K]}\{q\} \end{aligned} \quad (13)$$

$$\{Q\} = [K]\{q\} \quad (14)$$

One can also define

$$[B] \equiv [S-D][\phi][A]^{-1}, \quad [B_\alpha] \equiv [S-D][\phi], \quad [N] \equiv [\phi][A]^{-1}$$

Concluding Comments: This equilibrium approach works for this case, but is difficult to extend to more complicated elements. Energy methods must be used instead. Turner, et al., did not use a general cartesian coordinate system (a global system), but rather used a system with the x axis lined up with the base of the triangle (a local system).

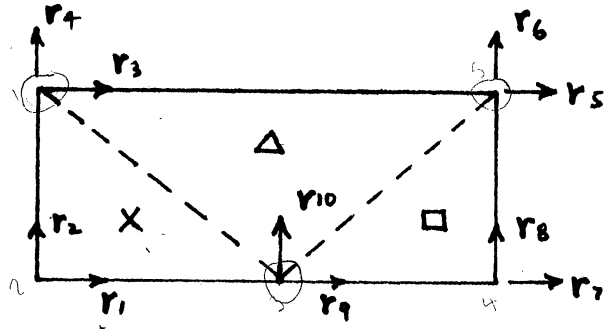
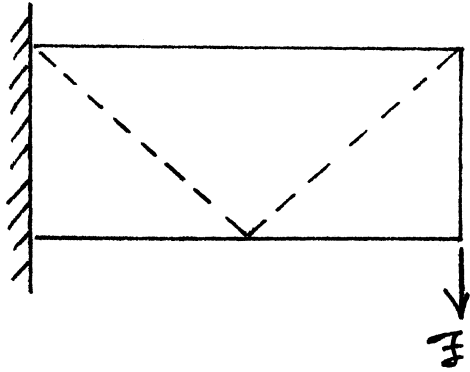
III. Assembly of Triangular Elements

A plate is cantilevered from a wall as shown, with load \bar{F} . Consider a 3-element representation of the plate using the constant-strain triangle. There are 5 nodes and 10 degrees of freedom. For each D.O.F., we must specify a force or a displacement.

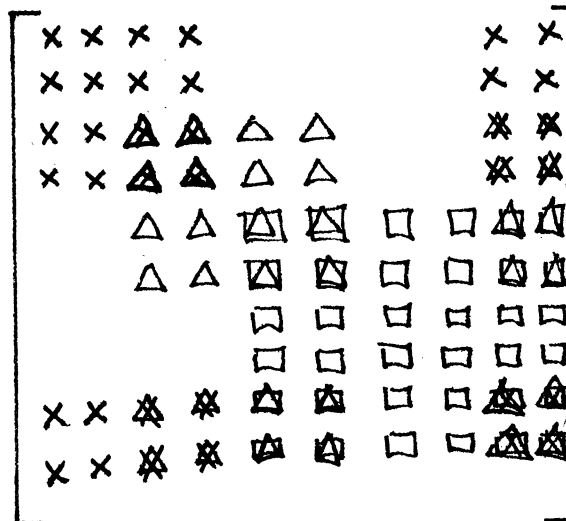
For each element, we have

$$\{Q\} = [k]\{q\}$$

where $[k]$ is a 6×6 matrix. Assemble the stiffness matrix for the structure using symbols for the element stiffnesses as shown.



$$[K] = [k_1] + [k_2] + [k_3] =$$

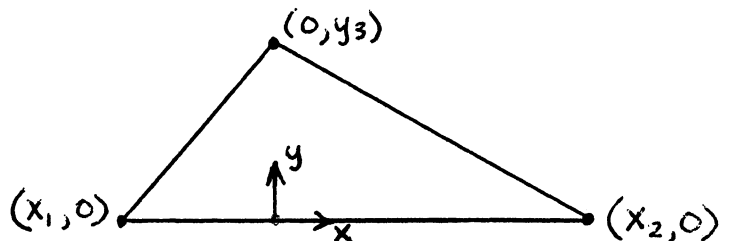


EXAMPLE OF TURNER TRIANGLE

The stiffness matrix for the Turner triangle is found most easily by using a local coordinate system shown. One defines

$$x_{21} \equiv x_2 - x_1$$

and obtains after much calculation:



$$[k] = \frac{Eh}{2(1-\nu^2)}$$

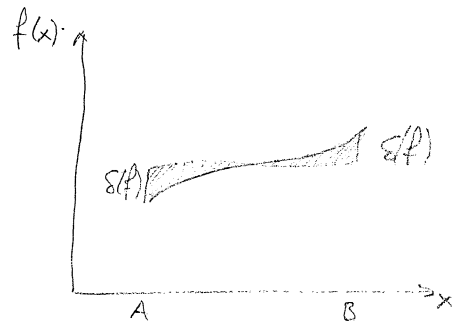
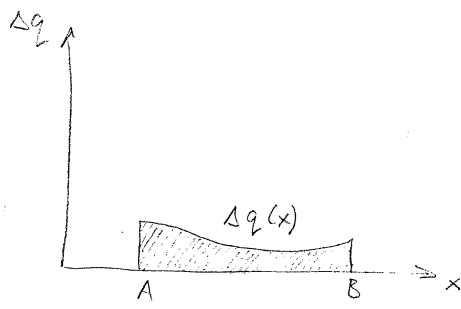
$$\begin{bmatrix} \frac{y_3}{x_{21}} & \frac{\nu x_2}{x_{21}} & -\frac{y_3}{x_{21}} & -\frac{\nu x_1}{x_{21}} & 0 & -\nu \\ \frac{\nu x_2}{x_{21}} & \frac{x_2^2}{y_3 x_{21}} & -\frac{\nu x_2}{x_{21}} & -\frac{x_1 x_2}{y_3 x_{21}} & 0 & -\frac{x_2}{y_3} \\ -\frac{y_3}{x_{21}} & -\frac{\nu x_2}{x_{21}} & \frac{y_3}{x_{21}} & \frac{\nu x_1}{x_{21}} & 0 & \nu \\ -\frac{\nu x_1}{x_{21}} & -\frac{x_1 x_2}{y_3 x_{21}} & \frac{\nu x_1}{x_{21}} & \frac{x_1^2}{y_3 x_{21}} & 0 & \frac{x_1}{y_3} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\nu & -\frac{x_2}{y_3} & \nu & \frac{x_1}{y_3} & 0 & \frac{x_{21}}{y_3} \end{bmatrix}$$

direct stiffness

$$+ \frac{Gh}{2}$$

$$\begin{bmatrix} \frac{x_2^2}{y_3 x_{21}} & \frac{x_2}{x_{21}} & -\frac{x_1 x_2}{y_3 x_{21}} & -\frac{x_2}{x_{21}} & -\frac{x_2}{y_3} & 0 \\ \frac{x_2}{x_{21}} & \frac{y_3}{x_{21}} & -\frac{x_1}{x_{21}} & -\frac{y_3}{x_{21}} & -1 & 0 \\ -\frac{x_1 x_2}{y_3 x_{21}} & -\frac{x_1}{x_{21}} & \frac{x_1^2}{y_3 x_{21}} & \frac{x_1}{x_{21}} & \frac{x_1}{y_3} & 0 \\ -\frac{x_2}{x_{21}} & -\frac{y_3}{x_{21}} & \frac{x_1}{x_{21}} & \frac{y_3}{x_{21}} & 1 & 0 \\ -\frac{x_2}{y_3} & -1 & \frac{x_1}{y_3} & 1 & \frac{x_{21}}{y_3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

shear stiffness



Virtual Displacement

Characteristics

- 1) A small displacement from Π

I. Background

Much of the finite element theory in solid mechanics is concerned with the stress-strain law (constitutive law), the strain-displacement law, and the load-deflection relation. These are symbolically sketched in Figures 1-3, where a scalar version of the vector quantities is given. These sketches are an artifice, but are very useful in sorting out concepts.

Many finite element problems are linear, where each of the relations in Figures 1-3 will be a straight line. Also, most finite element problems have no prestress $\{\sigma_0\}$ or prestrain $\{\epsilon_0\}$ such that the lines in Figures 1-3 would all pass through the origin. (The unloaded, undeflected structure is almost always available as a reference; therefore the load-deflection relation passing through the origin in Figure 3 is the general case.) Let us not assume linearity yet; but, rather, retain the general nonlinear case through the discussion on virtual work.

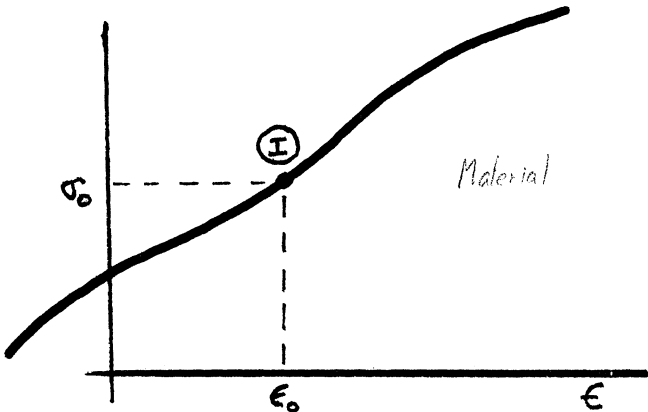


Fig.1. Stress-strain law. (nonlinear elastic).

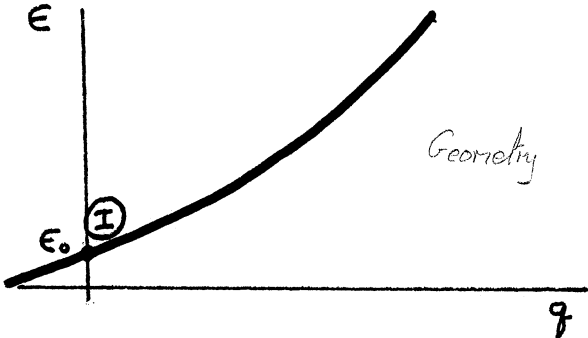


Fig.2. Strain-displacement law. (general case).

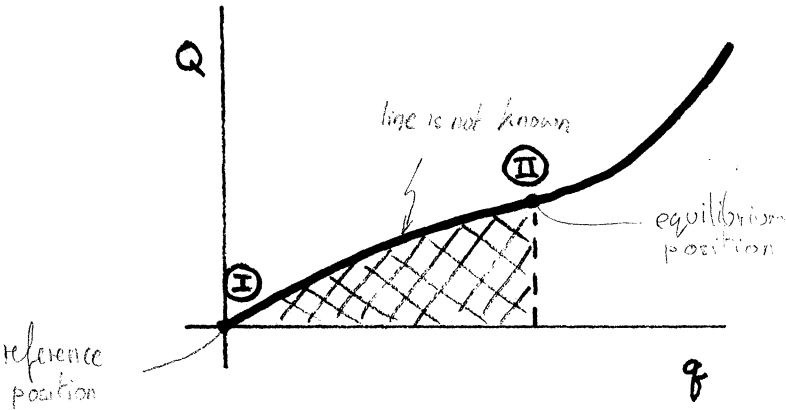
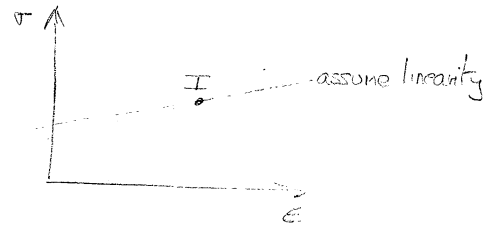


Fig.3. Load-deflection relation. (general case)



A. Stress-Strain Law (Constitutive Law)

The general form of the nonlinear, elastic stress-strain law is of the form

$$\{\sigma\} = \text{function} (\{\epsilon\})$$

A linearized version of this law, valid in the neighborhood of a reference point I (Figure 1) is

$$\{\sigma\} = [C](\{\epsilon\} - \{\epsilon_0\}) + \{\sigma_0\}$$

The standard practice is for this reference point to correspond to the origin in Figure 3; this means that $\{\sigma_0\}$ and $\{\epsilon_0\}$ are prestress and pre-strain present in the unloaded structure. Prestress and prestrain are zero in most structural problems.

B. Strain/Displacement Law

The relation between internal strain and nodal displacements is sketched in Figure 2. This is a geometric relation (kinematic) and is not a function of the material, but does depend on the total strain involved. Most engineering structures remain in the linear range.

C. Equilibrium Position

If a body is loaded with static loads $\{Q\}$, it deforms and reaches a static equilibrium at a point labelled II in Figure 3. The shaded area represents the work done by the external loads on the body, if the loads are slowly applied.

D. Virtual Displacement

A virtual displacement is an infinitesimal displacement from position II, in a manner consistent with geometric constraints. The body moves to position III and undergoes an increment of strain energy shown as the shaded area in Figure 4. The loads during this virtual displacement are assumed to remain constant.

This virtual displacement is called $\{\Delta q\}$. It results in a virtual strain $\{\Delta \epsilon\}$. Because strain energy is approximated by the shaded area in Figure 4, the change in stress going from II to III is negligible. The strain energy due to virtual strain is

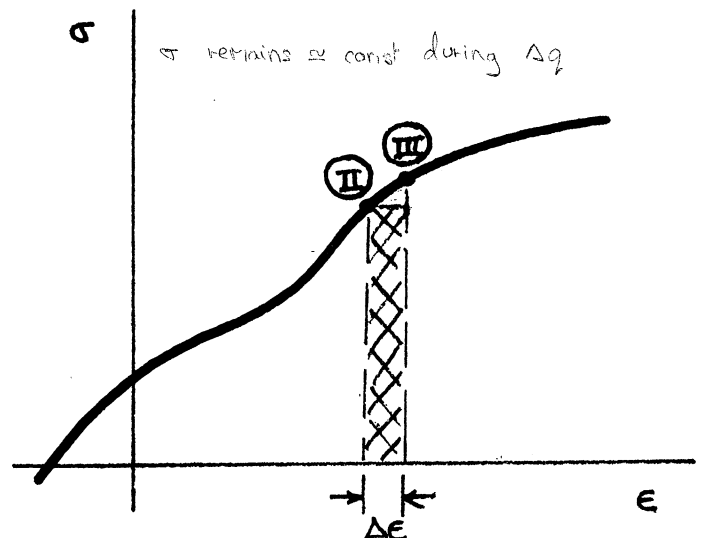


Fig. 4. Strains during a virtual displacement.

$$(3) \Delta U = \int_V (\text{stress}) (\Delta \text{ strain}) dV$$

II II → III

and the amount of work done by the external forces (shaded area in Figure 5) is:

$$(4) \Delta W = \sum_{i=1}^N Q_i \Delta q_i$$

II II → III

The energies in Eqns. 3 and 4 will be equal only when the material is nondissipative, i.e., elastic. The concept of distributed external forces such as gravity has not yet been included.

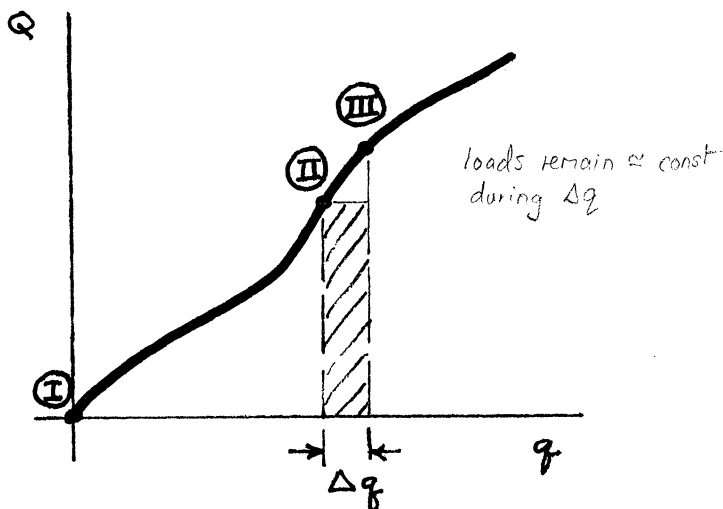


Fig. 5. Virtual displacement.

E. Work

Work is defined most fundamentally in an incremental manner

$$(5) \Delta W \equiv (\text{Force})(\Delta \text{ Displacement})$$

of the body

One must be on guard to never define work as a product of force times displacement and then to take an increment:

Beware!!!

$$\left\{ \begin{array}{l} W = Fd \\ \Delta W = F\Delta d + \Delta Fd \end{array} \right\} \text{ False!!!}$$

Furthermore, the distance involved in this expression is the distance moved by the body on which the force acts, rather than the distance the force moves. This is very important with respect to sliding forces.

F. Virtual Work

Force does not have to move to do work on a body. Displacement of the body counts!

This is the work done, whether by external or internal forces, during a virtual displacement, i.e., in moving from state II to state III.

II. Virtual Work Theorem

In mechanics, there are three fundamental statements of mechanical equilibrium which may be used interchangeably. These are Newton's laws of motion (principally the second law, $F = ma$), Hamilton's principle and the virtual work theorem, all of which can be used for statics and dynamics problems. The virtual work theorem applies to all of mechanics, but we will use a version valid for nonrelativistic, nonthermal problems.

Virtual Work Theorem:

A body is in equilibrium if and only if the sum of all virtual work done during an arbitrary virtual displacement is zero.

$$\begin{array}{l} \text{(increment of work} \\ \text{by internal forces} \\ \text{II} \rightarrow \text{III}) \end{array} + \begin{array}{l} \text{(increment of work} \\ \text{by external forces} \\ \text{II} \rightarrow \text{III}) \end{array} = 0$$

i.e.,

$$(6) \quad \boxed{\begin{array}{l} \Delta W_{\text{internal}} + \Delta W_{\text{external}} = 0 \\ \text{II} \rightarrow \text{III} \quad \text{II} \rightarrow \text{III} \end{array}}$$

General Statement of Virtual Work Theorem for Bodies in Figures 4, 5 and 6.

An important subcase occurs when the material of the body is nondissipative, e.g., nonlinearly elastic. The work done by the internal forces (i.e., stresses and strains) can be shown to be the negative of the strain energy increment by an involved mathematical proof not done here.

$$(7) \quad \begin{array}{l} \Delta W_{\text{internal}} \\ \text{II} \rightarrow \text{III} \end{array} = - \begin{array}{l} \Delta U \\ \text{II} \rightarrow \text{III} \end{array}$$

Inserting Equation 7 into Equation 6, one obtains:

Modified Virtual Work Theorem:

A nonlinearly elastic body is in equilibrium if and only if the increment of work done by external forces equals the change in strain energy during an arbitrary virtual displacement.

$$(8) \quad \boxed{\begin{array}{l} -\Delta U + \Delta W_{\text{external}} = 0 \\ \text{II} \rightarrow \text{III} \quad \text{II} \rightarrow \text{III} \end{array}}$$

strain energy ↗

Virtual Work Theorem for Non-dissipative materials, typically nonlinear elastic with prestress and prestrain.

Comments:

- i) The theorems to this point don't imply the existence of either a function W or a function U which mean anything. It only means one can calculate the increments of energy, i.e., the shaded areas in Sketches 4 and 5, for the case of energy-conserving materials.
- ii) Equation 8 ought to be intuitive to most readers because, for a non-dissipative material, the work done by the external forces during any displacement ought to equal the energy stored in the body. This is really a conservation of energy idea, and some people prefer it to the more general statement of Equation 6 for that reason.

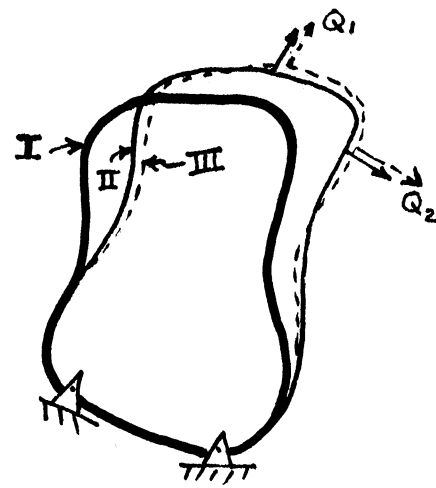


Fig. 6. Configurations of a body under load.

III. Potential Energy Theorem

Finally, the potential energy theorem can be derived. This theorem is less general than virtual work and is restricted to linear elastic systems and to energy conserving external force fields. For the potential energy theorem, we need to use a variational operator δ .

We will change our viewpoint of the Δ symbol used above through the identity

$$(9) \quad \delta\{u\} \equiv \{\Delta u\}$$

II II \rightarrow III

In other words, we pass from an incremental usage of Δ to an operator form δ . The symbol δ will retain the concept of a virtual displacement, i.e., a small change from the II position, with forces and stresses remaining unchanged.

If the external force field is conservative (energy-conserving) and hence can be derived from a potential, i.e.,

$$(10) \quad \vec{F} = -\vec{\nabla} W$$

external forces
work potential
a generalized gradient operator which acts on all displacement-like quantities.

then we can consider W to exist. Putting all of our "work" ideas together:

$$(11) \quad \begin{aligned} \Delta W_{\text{external}} &= (\text{Forces}) \times (\Delta \text{ displacements}) \\ &\quad \text{II} \qquad \qquad \text{II} \rightarrow \text{III} \\ \text{II} \rightarrow \text{III} &= (\text{Forces}) \times \delta(\text{displacements}) \\ &\quad \text{II} \qquad \qquad \text{II} \\ &= \delta(\text{Forces}) \times (\text{displacements}) \\ &\quad \text{II} \qquad \qquad \text{II} \\ &= \delta(-W) \\ &\quad \text{II} \end{aligned}$$

where we now admit the existence of a force potential W and use an operator δ which acts only on displacement-like quantities. Also, since the system is linearly elastic, we can write symbolically

$$\begin{aligned} \Delta U &= \int_V (\text{stress}) (\Delta \text{ strain}) dv \\ \text{II} \rightarrow \text{III} &\quad \text{II} \qquad \text{II} \rightarrow \text{III} \\ &= \int (\text{stiffness}) (\text{strain}) (\Delta \text{ strain}) dv \\ &\quad \text{II} \qquad \text{II} \rightarrow \text{III} \\ &= \int (\text{stiffness}) (\text{strain}) \delta(\text{strain}) dv \\ &\quad \text{II} \qquad \text{II} \\ &= \delta \frac{1}{2} \int (\text{stiffness}) (\text{strain})^2 dv \end{aligned}$$

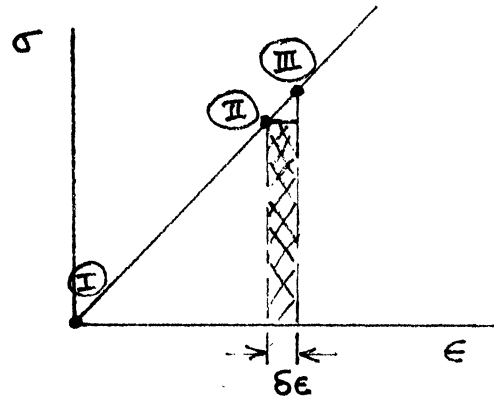


Fig.8. Virtual strain energy. (Increment in strain energy due to virtual strain).

$$(12) \quad \begin{array}{ccc} \Delta U & = & \delta(U) \\ \text{II} \rightarrow \text{III} & & \text{II} \end{array}$$

This, of course, implies a strain energy function U as a function of strains and an operator δ that acts on only strains (which are displacement-like). We now combine results of Equations 8, 11 and 12 to get

$$(13) \quad - \delta \left(\frac{W}{\text{II}} \right) - \delta \left(\frac{U}{\text{II}} \right) = 0$$

$$(14) \quad \delta \left(\frac{U + W}{\text{II}} \right) = 0$$

or

$$(15) \quad \boxed{\delta \left(\frac{\chi}{\text{II}} \right) = 0}$$

Potential Energy Theorem for linearly elastic systems with conservative external forces.

Potential Energy Theorem:

The potential energy of a mechanical system made of linearly elastic elements and exposed to conservative forces is "stationary" (has a zero first variation) at the static equilibrium configuration.

Comment: If one generalizes to systems with even mild nonlinearities, then the question of stability of this equilibrium position becomes important. Then the second variation of the potential energy is needed. This subject is studied in a more advanced finite element course.

DERIVATION OF A FINITE ELEMENT
BY THE VIRTUAL WORK THEOREM

October 10, 1977
W.J.A.

I. DERIVATION FOR CONCENTRATED NODAL LOADS

Consider a single finite element with only concentrated external forces acting on its nodes (Fig. 1). The three configurations are shown. Use the modified form of the virtual work theorem, but use only the lower case δ to indicate both incremental and operational procedures.

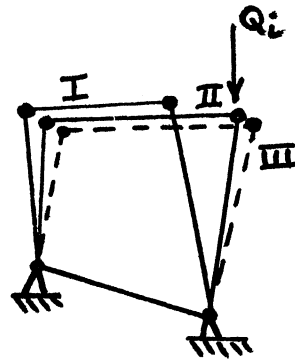
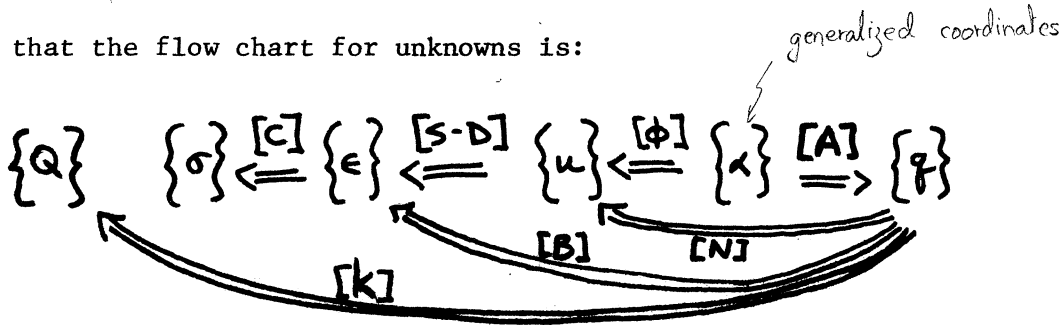


Fig. 1. Typical Finite Element

$$\delta\{u\} \equiv \{\delta u\} \quad \text{etc.}$$

Recall that the flow chart for unknowns is:



where the stress-strain matrix is named [C] and the equilibrium matrix is suppressed. The virtual work theorem becomes

$$\delta W_{II \rightarrow III} - \delta U_{II \rightarrow III} = 0 \quad 1)$$

or

$$\{\delta q\}_{II \rightarrow III}^T \{Q\}_{II} - \int_V \{\delta \epsilon\}_{II \rightarrow III}^T \{\sigma\}_{II} dv = 0 \quad 2)$$

The virtual strain is found from the expression for strain:

$$\{\epsilon\} = [B]\{q\} \quad 3)$$

by operating on both sides with δ :

$$\{\delta \epsilon\} = [B]\{\delta q\} \quad 4)$$

Hence, setting $\{\sigma\} = [C][B]\{q\}$, we have

$$\{\delta q\}^T \{Q\} - \int_V \{\delta q\}^T [B]^T [C] [B] \{q\} dv = 0 \quad 5)$$

Since $\{\delta q\}^T$ does not depend on the integration variables, it can be factored out, as well as $\{q\}$, to get

$$\{\delta q\}^T \left(\{Q\} - \int_V [B]^T [C] [B] dv \{q\} \right) = 0 \quad 6)$$

Since $\{\delta q\}^T$ is an arbitrary vector, the vector enclosed in the large parentheses must be zero. This leads to

$$\{Q\} = \int_V [B]^T [C] [B] dv \{q\} \quad 7)$$

or, upon defining the integral portion to be exactly the stiffness matrix, we have

$$\{Q\} = [k] \{q\}. \quad 8)$$

This approach has defined the stiffness matrix without the need of an equilibrium matrix! Several forms of the stiffness matrix follow, for computational use:

$$[k] = \int_V [B]^T [C] [B] dv \quad 9a)$$

$$= \int_V [N]^T [S-D]^T [C] [S-D] [N] dv \quad 9b)$$

$$= [A]^{-1} \int_V [\phi]^T [S-D]^T [C] [S-D] [\phi] dv [A]^{-1} \quad 9c)$$

The last version is particularly good for computation when displacement functions are given; the next to last can be used when shape functions are given.

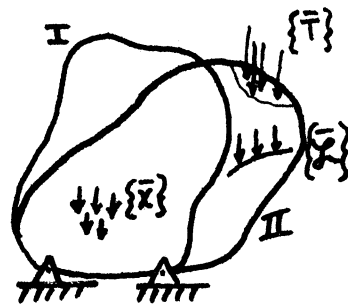
II. DERIVATION INCLUDING EQUIVALENT NODAL LOADS

Now, suppose the element has acting on it several kinds of distributed loads. These can be volumetric (gravity), surface (pressure) or line loads (as on a beam). In each case, a contribution to the virtual work will be made by a force moving the body through an increment of displacement. Although the force and the spatial dimensions vary, the relevant displacement in each case is $\{\delta u\} = [N] \{\delta q\}$. Hence, if

$$\{\bar{X}(x,y,z)\} = \text{volume load}$$

$$\{\bar{T}(x,y,z)\} = \text{surface load}$$

$$\{\bar{L}(x,y,z)\} = \text{line load}$$



Then

$$\begin{array}{ccc} \delta W & - & \delta U = 0 \\ \text{II} \rightarrow \text{III} & & \text{II} \rightarrow \text{III} \end{array}$$

becomes

$$\begin{array}{cccc} \{\delta q\}^T \{Q\} & + \int_{\ell} \{\delta u\}^T \{\bar{L}(x,y,z)\} d\ell & + \int_S \{\delta u\}^T \{T(x,y,z)\} dS & + \int_V \{\delta u\}^T \{\bar{X}(x,y,z)\} dv \\ \text{II} & & \text{II} & \text{II} \\ & & & - \int_V \{\delta \epsilon\}^T \{\sigma\} dv = 0 \\ & & & \text{II} \end{array}$$

This is rewritten

$$\{\delta q\}^T \left(\{Q\} + \int_{\ell} [N]^T \{\bar{L}\} d\ell + \int_S [N]^T \{\bar{T}\} dS + \int_V [N]^T \{\bar{X}\} dv - \int_V [B]^T [C] [B] dv \{q\} \right) = 0$$

The integrals all can be evaluated once the internal displacement fields (and hence [N], [\phi], [A], etc.) have been assigned. Each integral is given a name.

$$\begin{array}{ll} \{Q\}_{\text{e.n.l.}}^{\bar{L}} \equiv \int_{\ell} [N]^T \{\bar{L}\} d\ell & \text{Equivalent nodal load due to line load} \\ \{Q\}_{\text{e.n.l.}}^{\bar{T}} \equiv \int_S [N]^T \{\bar{T}\} dS & \text{Equivalent nodal load due to surface load} \\ \{Q\}_{\text{e.n.l.}}^{\bar{X}} \equiv \int_V [N]^T \{\bar{X}\} dv & \text{Equivalent nodal load due to volume load} \\ [k] \equiv \int_V [B]^T [C] [B] dv & \text{Element stiffness} \end{array}$$

and finally:

$$\{Q\} + \{Q\}_{\text{e.n.l.}}^{\bar{L}} + \{Q\}_{\text{e.n.l.}}^{\bar{T}} + \{Q\}_{\text{e.n.l.}}^{\bar{X}} = [k] \{q\}$$

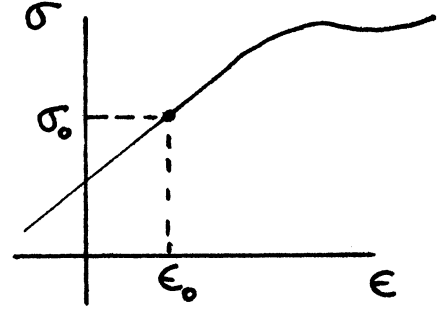
is the general equation of equilibrium.

III. PRESTRAIN AND PRESTRESS

Many physical problems have an initial strain (thermal problems) or initial stress. Also, in nonlinear problems, one often introduces a prestrain or prestress as an artifice to aid solution. In all of these cases, a linear stress-strain law is used:

$$\{\sigma\} = [C](\{\epsilon\} - \{\epsilon_0\}) + \{\sigma_0\}$$

Then the virtual work would give an additional increment of strain energy:



$$\delta U \equiv \int \{\delta\epsilon\}^T \{\sigma\} dv$$

$$= \int ([B]\{\delta q\})^T ([C](\{\epsilon\} - \{\epsilon_0\}) + \{\sigma_0\}) dvol$$

$$= \{\delta q\}^T \int_{vol} [B]^T [C] \{\epsilon\} dvol - \{\delta q\}^T \int_{vol} [B]^T [C] \{\epsilon_0\} dvol + \{\delta q\}^T \int_{vol} [B]^T \{\sigma_0\} dvol$$

After "cancelling" $\{\delta q\}^T$ and combining with the energy balance in the last section:

$$\begin{aligned} \{Q\} + \int_V [N]^T \{\bar{X}\} dv + \int_S [N]^T \{\bar{T}\} dS + \int_\ell [N]^T \{\bar{Z}\} d\ell + \int_V [B]^T [C] \{\epsilon_0\} dvol - \int [B]^T \{\sigma_0\} dvol \\ = \int_V [B]^T [C] [B] dvol \{q\} \end{aligned}$$

$$\begin{aligned} \{Q\} + \{Q\}_{\text{body forces}}^{\text{e.n.l.}} + \{Q\}_{\text{surface forces}}^{\text{e.n.l.}} + \{Q\}_{\text{line loads}}^{\text{e.n.l.}} + \{Q\}_{\text{prestress}}^{\text{e.n.l.}} + \{Q\}_{\text{prestrain}}^{\text{e.n.l.}} = [k] \{q\} \end{aligned}$$

IV. REVIEW OF EQUIVALENT NODAL LOAD CONCEPT

The virtual work theorem has led to

$$\begin{aligned} [k] \{q\} = \{Q\}_{\text{external nodal loads}} + \int_V [N]^T \{\bar{X}\} dv + \int_S [N]^T \{\bar{T}\} dS + \int_\ell [N]^T \{\bar{Z}\} d\ell + \int_V [B]^T [C] \{\epsilon_0\} dv \\ - \int_V [B]^T \{\sigma_0\} dv. \end{aligned}$$

We will define

$$\begin{aligned} \{Q\}_{\text{e.n.l.}} &\equiv \int_{\text{body}} [N]^T \{\bar{X}\} dv \\ &\text{forces} \end{aligned}$$

$$\begin{aligned} \{Q\}_{\text{e.n.l.}} &\equiv \int_{\text{surface}} [N]^T \{\bar{T}\} dS \\ &\text{forces} \end{aligned}$$

$$\begin{aligned} \{Q\}_{\text{e.n.l.}} &\equiv \int_{\text{line}} [N]^T \{\bar{f}\} d\ell \\ &\text{loads} \end{aligned}$$

$$\begin{aligned} \{Q\}_{\text{e.n.l.}} &\equiv \int_{\text{v}} [B]^T [C] \{\epsilon_o\} dv \\ &\text{prestrain} \end{aligned}$$

$$\begin{aligned} \{Q\}_{\text{e.n.l.}} &\equiv - \int_{\text{v}} [B]^T \{\sigma_o\} dv \\ &\text{prestress} \end{aligned}$$

DERIVATION OF LINE ELEMENT
 BY ENERGY EXPRESSION

July 10, 1976
 W.J.A.

I. LINE ELEMENT. ENERGY FORMULATION (called a rod in Msc)

A concrete example is needed to tie down the preceding theory. Consider the constant area, two-node line element. We need to find the operators [C], [S-D], and [N] in the expressions for stiffness:

$$[k] = \int_V [N]^T [S-D]^T [C] [S-D] [N] dV$$

[B]

and for line load:

$$\{Q\}_{e.n.l.} = \int_0^L [N]^T \{ \bar{q}(x) \} dx.$$

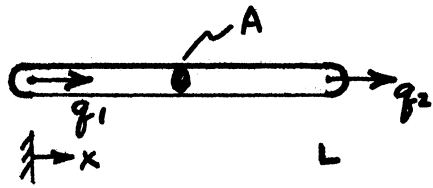


Figure 1. Line Element

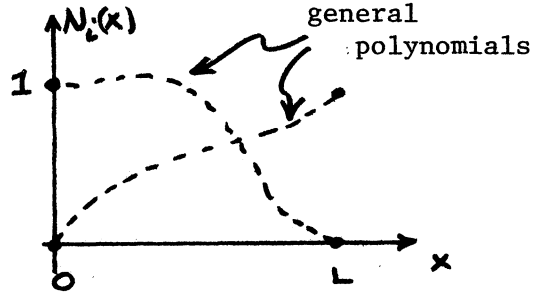


Fig. 2. Shape functions (general).

The shape (interpolation) functions can be found by inspection under the knowledge that

- $N_1(0) = 1$
- $N_1(L) = 0$
- $N_2(0) = 0$
- $N_2(L) = 1$

At this point, the shape functions might have the general shape given by

$$\{u(x)\} = [N_1(x) \quad N_2(x)] \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

and sketched in Figure 2. The shape functions are, in fact, assumed to be linear, however, leading to

$$\{u(x)\} = \underbrace{\begin{bmatrix} 1 - x/L & x/L \end{bmatrix}}_{[N]} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

We found earlier that the stiffness matrix $[C] = [E]$ and the strain displacement matrix

$$[S-D] = \left[\frac{d}{dx} \right]$$

in one-dimensional elasticity.

An intermediate matrix $[B]$ is useful for calculations.

$$\begin{aligned} [B] &\equiv [S-D][N] \\ &= \left[\frac{d}{dx} \right] \begin{bmatrix} 1 - x/L & x/L \end{bmatrix} \\ &= \begin{bmatrix} -1/L & 1/L \end{bmatrix} \end{aligned}$$

The stiffness matrix now becomes

$$\begin{aligned} [k] &= A \int_0^L [B]^T [C] [B] dx \\ &= A \int_0^L \begin{bmatrix} -1/L \\ 1/L \end{bmatrix} [E] \begin{bmatrix} -1/L & 1/L \end{bmatrix} dx \\ &= A \int_0^L \begin{bmatrix} -1/L \\ 1/L \end{bmatrix} \begin{bmatrix} -E/L & E/L \end{bmatrix} dx \\ &= A \int_0^L \begin{bmatrix} E/L^2 & -E/L^2 \\ -E/L^2 & E/L^2 \end{bmatrix} dx \\ &= \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{aligned}$$

$[k] = \int_0^L \begin{bmatrix} 1-x & x \\ -x & x \end{bmatrix}^T \left[\frac{d}{dx} \right]^T [E] \left[\frac{d}{dx} \right] \begin{bmatrix} 1-x & x \\ -x & x \end{bmatrix} A dx$

which is the same result as obtained from equilibrium arguments, and is exact for the constant area case at hand.

The equivalent nodal line load becomes

$$\{Q\}_{e.n.l.} = \int_0^L [N]^T \{\bar{\mathcal{L}}(x)\} dx$$

load per unit length

$$= \int_0^L \begin{Bmatrix} (1 - x/L) \bar{\mathcal{L}}(x) \\ x/L \bar{\mathcal{L}}(x) \end{Bmatrix} dx$$

$$= \begin{Bmatrix} \int_0^L (1 - x/L) \bar{\mathcal{L}}(x) dx \\ \int_0^L x/L \bar{\mathcal{L}}(x) dx \end{Bmatrix}$$

II. EXAMPLE OF EQUIVALENT NODAL LOADS

Choose a specific case, such as $\bar{\mathcal{L}}(x) \equiv \bar{\mathcal{L}}_0 (x/L)^2$. Then

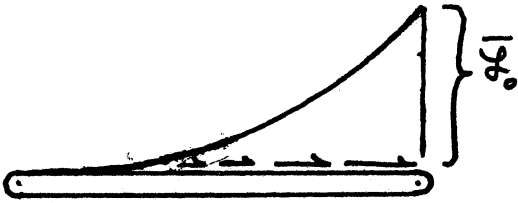


Figure 3. Load Distribution (Line-Load)

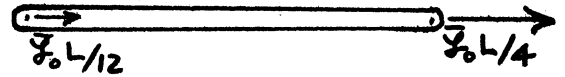


Figure 4. Equivalent Nodal Loads.

$$\{Q\}_{e.n.l.} = \begin{Bmatrix} \bar{\mathcal{L}}_0 \int_0^L (1 - x/L) (x/L)^2 dx \\ \bar{\mathcal{L}}_0 \int_0^L x/L (x/L)^2 dx \end{Bmatrix}$$

$$= \bar{\mathcal{L}}_0 \begin{Bmatrix} \int_0^L [(x/L)^2 - (x/L)^3] dx \\ \int_0^L (x/L)^3 dx \end{Bmatrix}$$

$$= \bar{L}_0 \left\{ \begin{array}{l} \frac{1}{L^2} \left(\frac{x^3}{3} \right) \Big|_0^L - \frac{1}{L^3} \left(\frac{x^4}{4} \right) \Big|_0^L \\ \\ \frac{1}{L^3} \left(\frac{x^4}{4} \right) \Big|_0^L \end{array} \right\}$$

$$= \bar{L}_0 L \left\{ \begin{array}{l} \frac{1}{12} \\ \\ \frac{1}{4} \end{array} \right\}$$

Does this represent faithfully the total running load on the element?

$$\text{Total Load} = \int_0^L \bar{L}_0 (x/L)^2 dx = \bar{L}_0 \frac{1}{L^2} \frac{1}{3} x^3 \Big|_0^L = \bar{L}_0 \frac{L}{3}$$

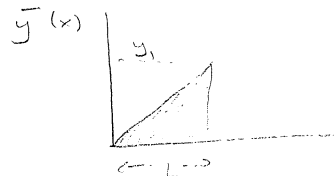
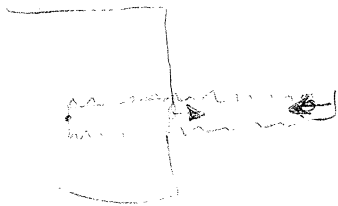
This checks. The equivalent nodal loads are statically equivalent to the original distributed load.

one could have

$$\bar{L}(x) = L_0 \frac{x}{L}$$

↑
const.

$$P = \frac{1}{2} L_0 L$$



I. REVIEW OF CLASSICAL THEORY

Consider a straight, slender elastic beam (Fig. 1). The beam is constrained to move in the x, z plane. Only lateral forces $p(x)$, shearing forces Q_z and bending moments M_y act on the beam. There are no axial forces. The Euler-Bernoulli-Navier approach for this "pure bending" case is to study the motion of the neutral surface (elastica) of the beam under the assumptions:

- 1) linear, elastic material
- 2) small deflections and small slopes
- 3) plane sections initially perpendicular to the beam axis remain perpendicular after deformation
- 4) there is no dependence of stress nor strain on the y coordinate.

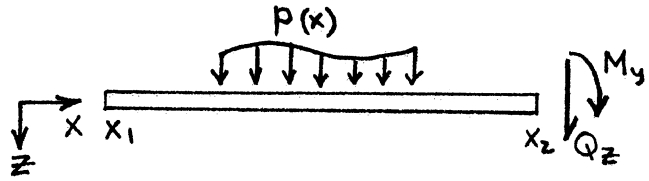


Fig. 1. Straight, slender beam in $x-z$ plane.

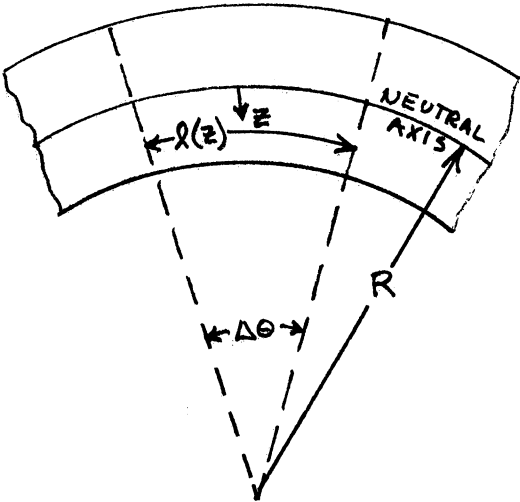


Fig. 2. Beam, curved to circular arc.

The strain-displacement relation for a beam comes immediately from the "plane sections" assumption (Fig. 2). At a distance z from the neutral axis

$$\begin{aligned} \epsilon_x(x, z) &= \frac{l(z) - l(0)}{l(0)} \\ &= \frac{(R-z)\Delta\theta - R\Delta\theta}{R\Delta\theta} \\ &= -\frac{z}{R} \end{aligned}$$

The radius of curvature R is found from differential geometry to be

$$\frac{1}{R} = \frac{\frac{\partial^2 w(x,0)}{\partial x^2}}{\left[1 + \left(\frac{\partial w(x,0)}{\partial x}\right)^2\right]^{3/2}} \quad (2)$$

since the slope $\frac{\partial w}{\partial x}$ is small,

$$\frac{1}{R} \approx \frac{\partial^2 w(x,0)}{\partial x^2}$$

and

$$\epsilon_x = -z \frac{\partial^2 w(x,0)}{\partial x^2} \quad (3)$$

Because of the small lateral dimensions of the beam and because all surface stresses are zero except σ_z on the top surface (Fig. 3), one can argue that

$$\sigma_y(x,z) \cong 0 \quad (4)$$

$$\sigma_z(x,z) \cong 0 \quad (5)$$

$$\tau_{yz}(x,z) \cong 0 \quad (6)$$

Also, the plane sections assumption immediately leads to

$$\gamma_{zx} = 0 \quad (\text{Fig. 4}) \quad (7)$$

$$\gamma_{xy} = 0 \quad (\text{Fig. 5}) \quad (8)$$

This means that the only relevant direct stress in the problem is σ_x , and the stress-strain law of importance is

$$\sigma_x(x,z) = E \epsilon_x(x,z)$$

(The shear stress τ_{zx} does not vanish but the plane sections assumption has made the corresponding shear strain γ_{zx} vanish and therefore no energy is absorbed in shearing action.)

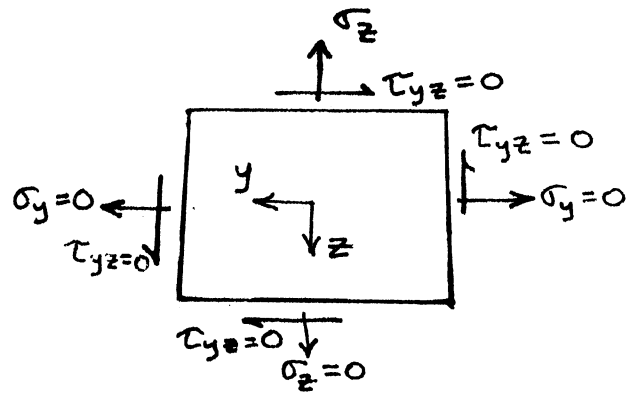


Fig. 3. End view of cut beam.
 $\sigma_y = \sigma_z = \tau_{yz}$ on surface.

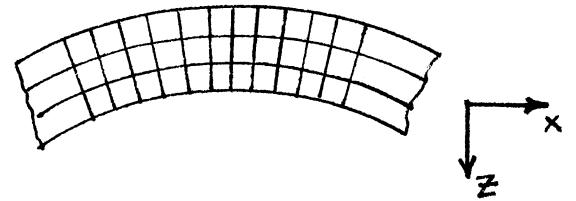


Fig. 4. Side view of plane sections grid. $\gamma_{zx} = 0$

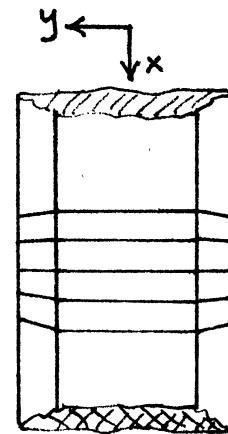


Fig. 5. Top view of beam with plane section grid. $\gamma_{xy} = 0$.

II. MODELING CONCEPTS FOR BEAM

For a beam, which transfers energy to neighboring beams through shearing forces acting through vertical boundary displacements and moments acting through boundary rotations, one must model both deflections and rotations at external nodes. One needs, for a two-node beam element, 4 nodal degrees of freedom:

$$w_1, \theta_1, w_2, \theta_2.$$

Other nodes could be added in the interior for a more refined element.

Consider the two-node, four degree of freedom element. Let us set up the set of finite element vectors and strike out the components known to be zero:

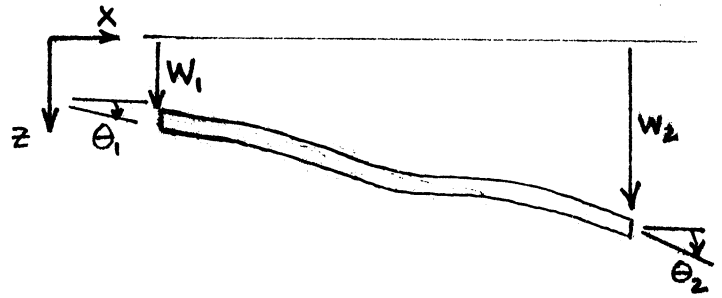


Fig. 6. Two node, four degree of freedom beam element.

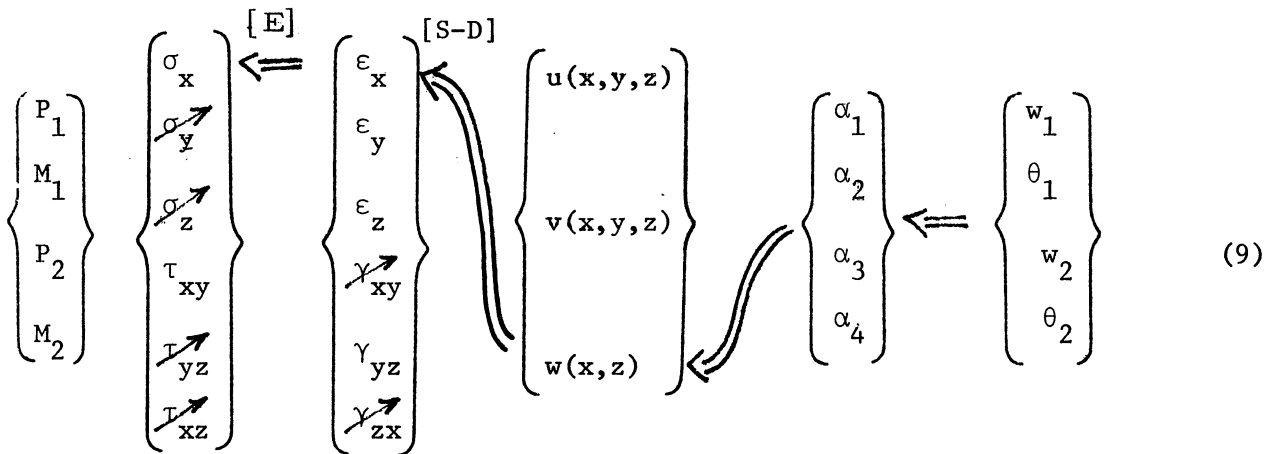


Fig. 7. Chart of vectors appearing in the finite element derivation.

Remembering that the stiffness matrix depends on an integration of strain energy of the form

$$\int_V \{\delta\epsilon\}^T \{\sigma\} dv \tag{10}$$

one sees that only the product of σ_x and ϵ_x survives, i.e.,

$$\delta U = (\delta\epsilon_x)(\sigma_x) \tag{11}$$

and only the mapping of σ_x into ϵ_x (shown in Fig. 7) is important. Note that ϵ_x is completely determined by the neutral surface deformation,

$$\{\epsilon_x\} = \left[-z \frac{d^2}{dx^2} \right] \{w(x,0)\} \quad (12)$$

and this, therefore, provides the only necessary strain-displacement relation. Using the symbol $w(x)$ for the neutral surface deflection, we retain only:

$$\begin{Bmatrix} P_1 \\ M_1 \\ P_2 \\ M_2 \end{Bmatrix} \begin{matrix} \{\sigma_x(x,z)\} \\ \{\epsilon_x(x,z)\} \\ \{w(x,0)\} \end{matrix} \begin{matrix} \xleftarrow{[C]} \\ \xleftarrow{[S-D]} \\ \xleftarrow{[\phi]} \end{matrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} \begin{matrix} \xleftarrow{[A]^{-1}} \\ \xleftarrow{[A]^{-1}} \\ \xleftarrow{[A]^{-1}} \end{matrix} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} \quad (13)$$

III. CONSTRUCTION OF THE BEAM ELEMENT

The only important internal displacement is the vertical displacement of the elastic axis

$$w(x,0) \equiv w(x).$$

It must contain the first 3 generalized coordinates in the displacement function:

$$w(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3$$

in order to meet the convergence requirements, i.e., inclusion of rigid body modes and constant straining mode (constant curvature). The fourth generalized coordinate is needed to provide the fourth nodal D.O.F. mentioned above.

The stress-strain law for uniaxial stress is:

$$\{\sigma_x(x,z)\} = [E]\{\epsilon_x(x,z)\} \quad (14)$$

The strain displacement law is:

$$\epsilon_x(x,z) = \left[-z \frac{d^2}{dx^2} \right] \{w(x)\} \quad (15)$$

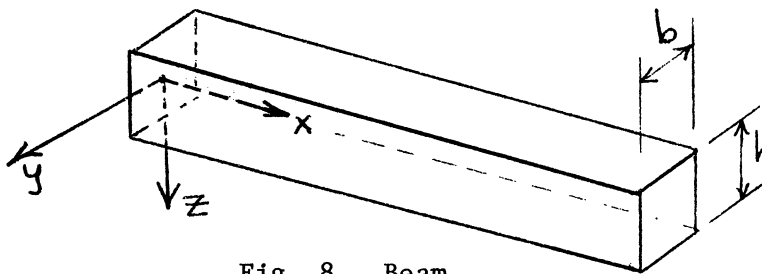


Fig. 8. Beam.

The ϕ matrix is

$$\{w(x)\} = [1 \quad x \quad x^2 \quad x^3] \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} \quad (16)$$

The [A] matrix is found by using the definition

$$\theta(x) \equiv \frac{dw}{dx},$$

and by choosing a local coordinate system $x_1=0$, $x_2=L$

$$\begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & L & L^2 & L^3 \\ 0 & 1 & 2L & 3L^2 \end{bmatrix}}_{[A]} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} \quad (17)$$

Inverting [A] yields:

$$[A]^{-1} = \frac{1}{L^3} \begin{bmatrix} L^3 & 0 & 0 & 0 \\ 0 & L^3 & 0 & 0 \\ -3L & -2L^2 & 3L & -L^2 \\ 2 & L & -2 & L \end{bmatrix} \quad (18)$$

The shape function matrix is

$$\begin{aligned} [N] &= [\phi][A]^{-1} \\ &= \frac{1}{L^3} [L^3 - 3Lx^2 + 2x^3, L^3 x - 2L^2 x^2 + Lx^3, 3Lx^2 - 2x^3, -L^2 x^2 + Lx^3] \end{aligned} \quad (20)$$

The columns in the matrix [N] are "shape functions" and give the internal displacement field corresponding to a unit nodal displacement. Some authors call these "interpolation functions."

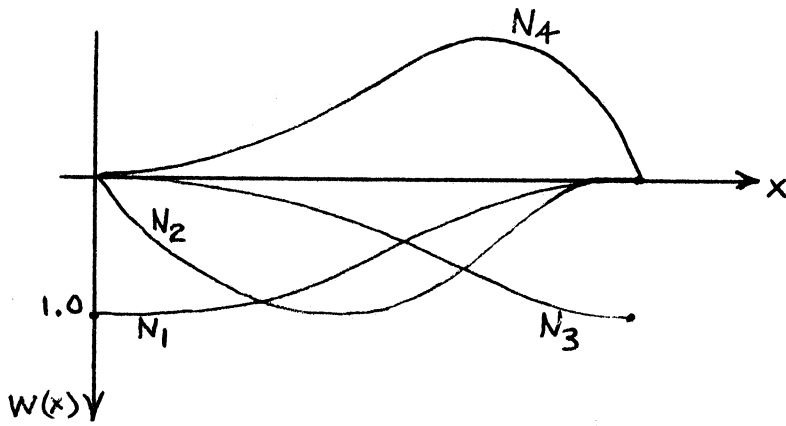


Fig. 9. Shape functions for beam element.

The [B] matrix is

$$\begin{aligned}
 [B] &\equiv [S-D][N] \\
 &= \left[-z \frac{d^2}{dx^2} \right] [N] \\
 &= -\frac{z}{L^3} [-6L+12x, -4L^2+6Lx, 6L-12x, -2L^2+6Lx] \quad (21)
 \end{aligned}$$

The stiffness matrix is

$$[k] \equiv \int_0^L \int_{-b/2}^{b/2} \int_{-h/2}^{h/2} [B]^T [C] [B] dz dy dx \quad (22)$$

After integrating this, one defines

$$I \equiv \frac{1}{12} b h^3$$

and gets

$$[k] = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ \text{(symm)} & & 12 & -6L \\ & & & 4L^2 \end{bmatrix} \quad (23)$$

The equation of equilibrium for the beam is:

$$\{Q\} + \{Q\}_{\text{equiv. nodal loads}} = [k]\{q\} \quad (24)$$

$$\{Q\}_{\text{e.n.l.}} = \int_{\text{space}} [N]^T \{\bar{\mathcal{L}}\} d\text{space} \quad (25)$$

The general laws for equivalent nodal loads worked earlier are valid. These can be evaluated for special cases when desired.

IV. EXAMPLE

Problem: Find the rotation at the ends of a beam with pinned ends when subjected to a line load of 100 lb/in. Use a single element.

$$EI = 10^9 \text{ lb in}^2$$

$$L = 100 \text{ in.}$$

$$\nu = 0.3$$

Solution: The equations of equilibrium will be of the form

$$\{Q\}_{\text{ext. concentrated forces}} + \{Q\}_{\text{equiv. nodal loads}} = [k]\{q\}$$

Find $[k]$. Simply put in the appropriate constants.

$$[k] = 10^3 \begin{bmatrix} 12 & 600 & -12 & 600 \\ & 40000 & -600 & 20000 \\ & & 12 & -600 \\ & & & 40000 \end{bmatrix} \text{ lb/in.}$$

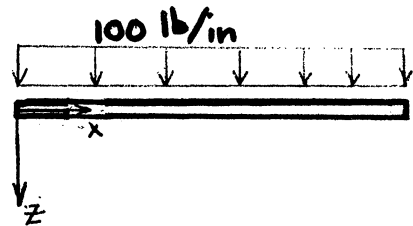


Fig. 10. Beam with constant load distribution.

Find the equivalent nodal loads.

$$\{Q\}_{\text{e.n.l.}} \equiv \int_0^L \frac{1}{L^3} \begin{bmatrix} L^3 - 3Lx^2 + 2x^3 \\ L^3 x - 2L^2 x^2 + Lx^3 \\ 3Lx^2 - 2x^3 \\ -L^2 x^2 + Lx^3 \end{bmatrix} \{100\} dx$$

$$= \frac{100}{L^3} \left\{ \begin{array}{l} L^4 - L^4 + \frac{2}{4} L^4 \\ \frac{L^5}{2} - 2 \frac{L^5}{3} + \frac{L^5}{4} \\ L^4 - \frac{2}{4} L^4 \\ - \frac{L^5}{3} + \frac{L^5}{4} \end{array} \right\}$$

$$= \frac{100}{L^3} \left\{ \begin{array}{l} \frac{L^4}{2} \\ + \frac{L^5}{12} \\ \frac{L^4}{2} \\ - \frac{L^5}{12} \end{array} \right\}$$

$$= \left\{ \begin{array}{l} 5000 \text{ lb.} \\ + 83300. \text{ in. lb.} \\ 5000 \text{ lb.} \\ - 83300. \text{ in. lb.} \end{array} \right\}$$

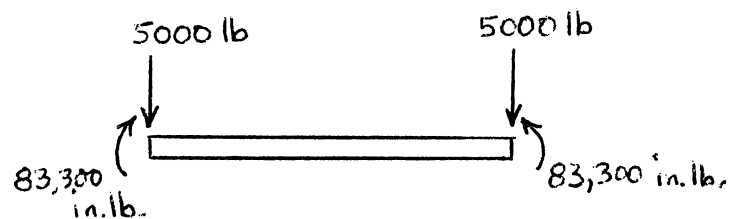


Fig. 11. Equivalent nodal loads.
Sample problem.

Note that substantial moments result. Also note that the vertical equivalent forces vary as L whereas the moments vary as L^2 . As $L \rightarrow 0$, the vertical forces dominate and the rough concept of "lumped" loads (neglecting moments) becomes valid.

Now, proceed with the solution:

$$\begin{Bmatrix} P_1 \\ 0 \\ P_2 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 5,000 \\ 83,300 \\ 5,000 \\ -83,300 \end{Bmatrix} = 10^3 \begin{bmatrix} 12 & 600 & -12 & 600 \\ & 40000 & -600 & 20000 \\ & & 12 & -600 \\ \text{(symm)} & & & 40000 \end{bmatrix} \begin{Bmatrix} 0 \\ \theta_1 \\ 0 \\ \theta_2 \end{Bmatrix}$$

Rather than solving in place, write the first and third equations:

$$\begin{aligned} P_1 + 5000 &= 6 \times 10^5 \theta_1 + 6 \times 10^5 \theta_2 \\ P_2 + 5000 &= -6 \times 10^5 \theta_1 - 6 \times 10^5 \theta_2 \end{aligned}$$

and the second and fourth equations:

$$\begin{aligned} 83,300 &= 4 \times 10^7 \theta_1 + 2 \times 10^7 \theta_2 \\ -83,300 &= 2 \times 10^7 \theta_1 + 4 \times 10^7 \theta_2 \end{aligned}$$

Solving for θ_1 and θ_2 in the latter equations:

$$\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0.00417 \\ -0.00417 \end{Bmatrix} \text{ radians}$$

Then find the external loads P_1 and P_2

$$\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \begin{Bmatrix} -5000 \\ -5000 \end{Bmatrix} \text{ lb.}$$

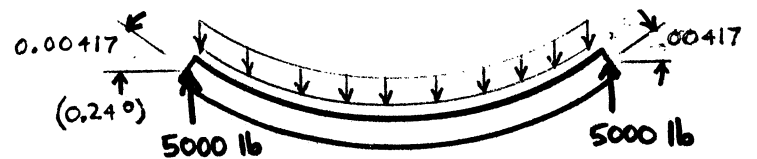


Fig. 12. Beam deflection. Sample problem.

The complete equation of equilibrium is:

$$\begin{Bmatrix} -5000 \\ 0 \\ -5000 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 5,000 \\ 83,300 \\ 5,000 \\ -83,300 \end{Bmatrix} = \begin{bmatrix} k \end{bmatrix} \begin{Bmatrix} 0 \\ 0.00417 \\ 0 \\ -0.00417 \end{Bmatrix}$$

↑ external forces (reactions)
 ↑ equivalent nodal loads
 ↑ stiffness matrix
 ↑ nodal displacements

Q. E. D.

The above answer can be compared to classical Euler-Bernoulli theory and is found to be exact. This means that the cubic shape functions used for the beam are sufficient to exactly model the uniform loading case. They are also exact for concentrated loads at the nodes.

EQUATION SOLVERS

I. STATIC

A. Linear

B. Nonlinear

C. Eigenvalue Problems

II. DYNAMIC

A. Transient Response

B. Periodic Response

C. Eigenvalue Problems

D. Random Response

STATIC - LINEAR EQ'N SOLVERS

GENERAL FORM OF EQUATION:

$$[K]\{r\} = \{R\}$$

↑ ↑ ↖ known
nonsingular unknown

GENERAL CONCEPT FOR SOLUTION:

$$\{r\} = [K]^{-1}\{R\}$$

BUT THIS IS NOT ACTUALLY DONE ON LARGE MATRICES. TOO EXPENSIVE

DIRECT METHODS

ELIMINATION (HIGH SCHOOL)

CRAMER'S RULE (COLLEGE FRESHMAN)

GAUSS ELIMINATION (COLLEGE SENIOR)

* GAUSS-DOLITTLE FACTORIZATION

$$[K] = [L, I][D][L, I]^T \quad \text{SYMM.}$$

ITERATION METHODS

* GAUSS-SEIDEL

SOUTHWELL

ADVANTAGE OF TRIANGULAR FORM :

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{Bmatrix} 7 \\ 8 \\ 9 \end{Bmatrix}$$

$$1x = 7 \quad \boxed{x = 7}$$

$$2x + 3y = 8 \quad \boxed{y = \frac{1}{3}[8 - 2(7)]}$$

ETC.

EXAMPLE OF GAUSS-DOLITTLE DECOMP.
(PROB. 8 IN HOMEWORK)

SOLVE

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 0 \\ 1 & 0 & 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 5 \\ 12 \end{Bmatrix}$$

SOLUTION

Factorize

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 0 \\ 1 & 0 & 10 \end{bmatrix} = [L_1] [D] [L_1]^T$$

$$\begin{bmatrix} 1 & 2 & 1 \\ & & \\ & & \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} d_{11} & d_{11}l_{21} & d_{11}l_{31} \\ 0 & d_{22} & d_{22}l_{32} \\ 0 & 0 & d_{33} \end{bmatrix}$$

Hence

$$\begin{aligned} 1 &= d_{11} & l_{21} &= 2 \\ 2 &= d_{11}l_{21} & l_{31} &= 1 \\ 1 &= d_{11}l_{31} & & \end{aligned}$$

$$5 = l_{21}d_{11}l_{21} + d_{22} \quad d_{22} = 1$$

$$0 = l_{21}d_{11}l_{31} + d_{22}l_{32} \quad l_{22} = -2$$

$$10 = l_{31}d_{11}l_{31} + l_{32}d_{22}l_{32} + d_{33} \quad d_{33} = 5$$

Hence

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 0 \\ 1 & 0 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

Now solve in the forward direction.

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{Bmatrix} r_1^* \\ r_2^* \\ r_3^* \end{Bmatrix} = \begin{Bmatrix} 2 \\ 5 \\ 12 \end{Bmatrix}$$

$$r_1^* = 2$$

$$2r_1^* + r_2^* = 5$$

$$r_2^* = 1$$

$$r_1^* - 2r_2^* + r_3^* = 12$$

$$r_3^* = 12$$

Now back substitute

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 5 \end{bmatrix} \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 1 \\ 12 \end{Bmatrix}$$

$$r_3 = \frac{12}{5}$$

$$r_2 - 2r_3 = 1$$

$$r_2 = \frac{29}{5}$$

$$r_1 + 2r_2 + r_3 = 2$$

$$r_3 = -\frac{60}{5}$$

The answer is

$$\begin{Bmatrix} r_1 \\ r_2 \\ r_3 \end{Bmatrix} = \frac{1}{5} \begin{Bmatrix} -60 \\ 29 \\ 12 \end{Bmatrix}$$

$$= \begin{Bmatrix} -12.0 \\ 5.8 \\ 2.4 \end{Bmatrix}$$

SUBROUTINE "DCOMP"

Application of the finite element method typically leads to sets of linear, algebraic equations. One approach for solution is to decompose the matrix of coefficients and then to solve the equations in a successive fashion. This can be done with the subroutines "DCOMP" and "SOLVE". These follow the modified Gauss decomposition recommended by Melosh in his paper "Manipulation Errors in Finite Element Analyses".

To decompose an NxN real, symmetric, positive definite matrix [A], set

$$[A] = [U]^T [D] [U]$$

where [U] is an upper triangular matrix with unit elements on the main diagonal, and [D] is a diagonal matrix. This decomposition is unique.

In component form, we have

$$\begin{bmatrix} A_{11} & A_{12} & \dots & \\ A_{21} & A_{22} & & \\ \vdots & & & \\ & & & A_{NN} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \dots \\ U_{12} & 1 & 0 & \\ U_{13} & U_{23} & 1 & \\ \cdot & \cdot & \cdot & \cdot \\ & & & 1 \end{bmatrix} \begin{bmatrix} D_{11} & 0 & & \\ 0 & D_{22} & & \\ & & \ddots & \\ & & & D_{NN} \end{bmatrix} \begin{bmatrix} 1 & U_{12} & U_{13} & \dots \\ 0 & 1 & U_{23} & \dots \\ 0 & 0 & 1 & \\ \vdots & \vdots & \vdots & \ddots \\ & & & & U_{NN} \end{bmatrix}$$

Consider the A_{ij} as known and solve for the elements D_{ii} and U_{ij} . Carry enough terms for a 4x4 matrix and try to generalize the results.

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & \\ \vdots & & & \\ & & & A_{44} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ U_{12} & 1 & 0 & 0 \\ U_{13} & U_{23} & 1 & 0 \\ U_{14} & U_{24} & U_{34} & 1 \end{bmatrix} \begin{bmatrix} D_{11} & D_{11}U_{12} & D_{11}U_{13} & D_{11}U_{14} \\ 0 & D_{22} & D_{22}U_{23} & D_{22}U_{24} \\ 0 & 0 & D_{33} & D_{33}U_{34} \\ 0 & 0 & 0 & D_{44} \end{bmatrix}$$

In the first row of A_{ij} :

$$A_{11} = D_{11} \quad \text{hence} \quad D_{11} = A_{11} \quad (1)$$

$$A_{1j} = D_{11}U_{1j} \quad \text{hence} \quad U_{1j} = A_{1j}/D_{11} = A_{1j}/A_{11} \quad (2)$$

For the other rows $i > 1$, we have first the diagonal case where $i = j$:

$$A_{44} = D_{11}(U_{14})^2 + D_{22}(U_{24})^2 + D_{33}(U_{34})^2 + D_{44}$$

and this generalizes to

$$D_{ii} = A_{ii} - \sum_{k=1}^{i-1} D_{kk}(U_{ki})^2 \quad (3)$$

For off-diagonal elements $i > 1$ we have, for instance

$$A_{34} = U_{13}D_{11}U_{14} + U_{23}D_{22}U_{24} + D_{33}U_{34}$$

Solve for the U_{34} element, assuming all other quantities previously found:

$$U_{34} = \frac{A_{34} - \sum_{k=1}^{i-1} U_{k3}D_{kk}U_{k4}}{D_{33}}$$

This can be generalized to

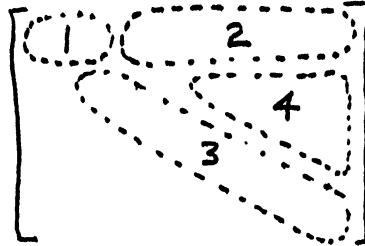
$$U_{ij} = \frac{A_{ij} - \sum_{k=1}^{i-1} U_{ki}D_{kk}U_{kj}}{D_{ii}} \quad (4)$$

The decomposition must proceed in an orderly way. The elements for D and U are calculated and then stored in the same locations reserved for the A matrix. This saves the creation of another NxN matrix, which can be critical if N is of the order of 1000 !

The applicable equations in each region:

$$\underline{A} = \begin{bmatrix} D_{11} & U_{12} & U_{13} & U_{14} \\ & D_{22} & U_{23} & U_{24} \\ & & D_{33} & U_{34} \\ & & & D_{44} \end{bmatrix}$$

undefined



The solution proceeds row by row from the top. It also proceeds from left to right, starting at the diagonal of each row. The newly computed quantities D_{ii} and U_{ij} must replace the original elements A_{ij} as they are calculated. Furthermore, the new elements must also be renamed A_{ij} !

Eqn. #1 Find D_{11}

$$D_{11} = A_{11} \quad \text{Store as } A_{11} \quad A(1,1) = A(1,1)$$

Eqn. #2 Find U_{1j}

$$U_{1j} = A_{1j}/A_{11} \quad A(1,J) = A(1,J)/A(1,1)$$

Eqn. #3 Find D_{ii} $i > 1$

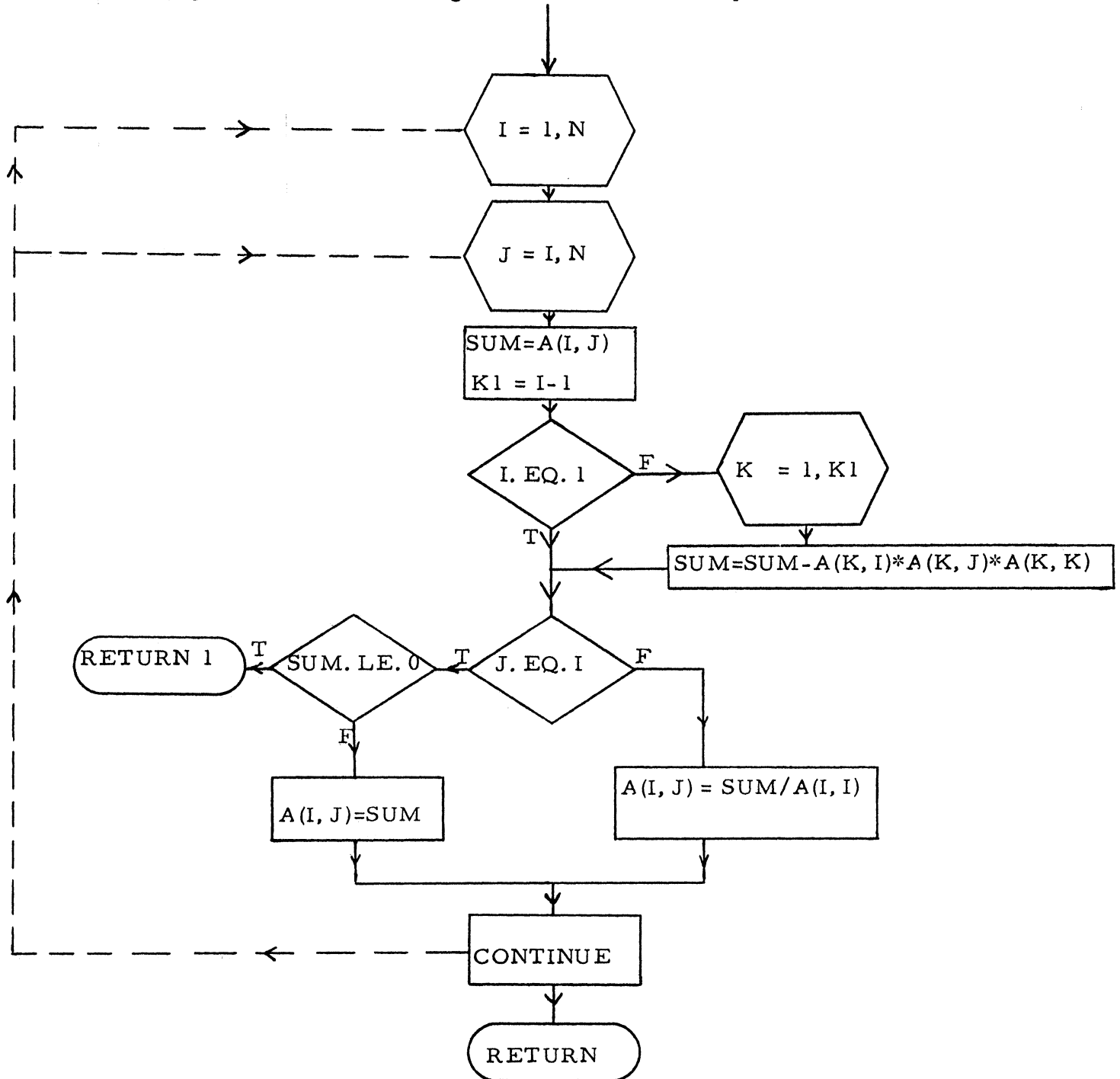
$$D_{ii} = A_{ii} - \sum_{k=1}^{i-1} D_{kk}(U_{ki})^2, \quad A(I,I) = A(I,I) - \sum_{k=1}^{i-1} A(K,K)*A(K,I)**2$$

Eqn. #4 Find U_{ij} for $1 < i < j$

$$U_{ij} = \frac{A_{ij} - \sum_{k=1}^{i-1} U_{ki}D_{kk}U_{kj}}{D_{ii}}, \quad A(I,J) = \frac{A(I,J) - \sum_{k=1}^{i-1} A(K,I)*A(K,K)*A(K,J)}{A(I,I)}$$

SUBROUTINE NAME: DCOMP(N, A, *)

This subroutine decomposes a matrix $[A]$ into a diagonal matrix $[D]$ and an upper triangular matrix $[U]$ such that $[A] = [U]^T[D][U]$. The matrix $[U]$ has unit diagonal elements. The given matrix $[A]$ must be real, $N \times N$, & symmetric, pos. definite. The matrix $[U]$ is stored in the upper right hand of the matrix $[A]$ and the matrix $[D]$ is stored on the diagonal of $[A]$ at the completion of the calculation.



SUBROUTINE "SOLVE"

I. APPROACH

Assume that the matrix $[A]$ in the equation

$$[A]\{X\} = \{B\}$$

has been decomposed, e.g. by the subroutine "DCOMP". Then

$$\underbrace{[U]^T [D] [U]}_{\{X^*\}} \{X\} = \{B\}$$

Define

$$\{X^*\}$$

Such that

$$[U]^T \{X^*\} = \{B\} \quad (1)$$

Solve for $\{X^*\}$ in the forward solution of Eqn 1, and then solve for $\{X\}$ by back substitution in the definition of $\{X^*\}$ in

$$\text{Eqn 2: } [D][U]\{X\} = \{X^*\} \quad (2)$$

II. ALGORITHM

$$\begin{bmatrix} 1 & 0 & 0 & \dots \\ U_{12} & 1 & 0 \\ U_{13} & U_{23} & 1 \\ \vdots & \vdots & \vdots \\ U_{1n} & \cdot & \cdot \end{bmatrix} \begin{Bmatrix} X_1^* \\ X_2^* \\ \vdots \\ X_n^* \end{Bmatrix} = \begin{Bmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{Bmatrix}$$

Hence

$$X_1^* = B_1$$

$$U_{12} X_1^* + X_2^* = B_2$$

$$X_2^* = B_2 - U_{12} X_1^*$$

In i th row

$$U_{1i} X_1^* + U_{2i} X_2^* + \dots + U_{i-1,i} X_{i-1}^* + X_i^* = B_i$$

General rule:

$$X_i^* = B_i - \sum_{k=1}^{i-1} U_{ki} X_k^* \quad (i=2,3,\dots,n)$$

Case $i=1$ is degenerate because of rules for computer summation (always takes one term in sum).

Finally

$$\begin{bmatrix} D_{11} & D_{11}U_{12} & D_{11}U_{13} & D_{11}U_{14} \\ 0 & D_{22} & D_{22}U_{23} & D_{22}U_{24} \\ 0 & 0 & D_{33} & D_{33}U_{34} \\ 0 & 0 & 0 & D_{44} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{Bmatrix} = \begin{Bmatrix} X_1^* \\ X_2^* \\ X_3^* \\ X_4^* \end{Bmatrix}$$

For last eqn,

$$D_{44} X_4 = X_4^* \Rightarrow X_n = \frac{X_n^*}{D_{nn}}$$

and

$$D_{33} X_3 + D_{33} U_{34} X_4 = X_3^* \Rightarrow X_{n-1} = \frac{X_{n-1}^* - D_{n-1,n-1} U_{n-1,n} X_n}{D_{n-1,n-1}}$$

The general term is for $i < n$

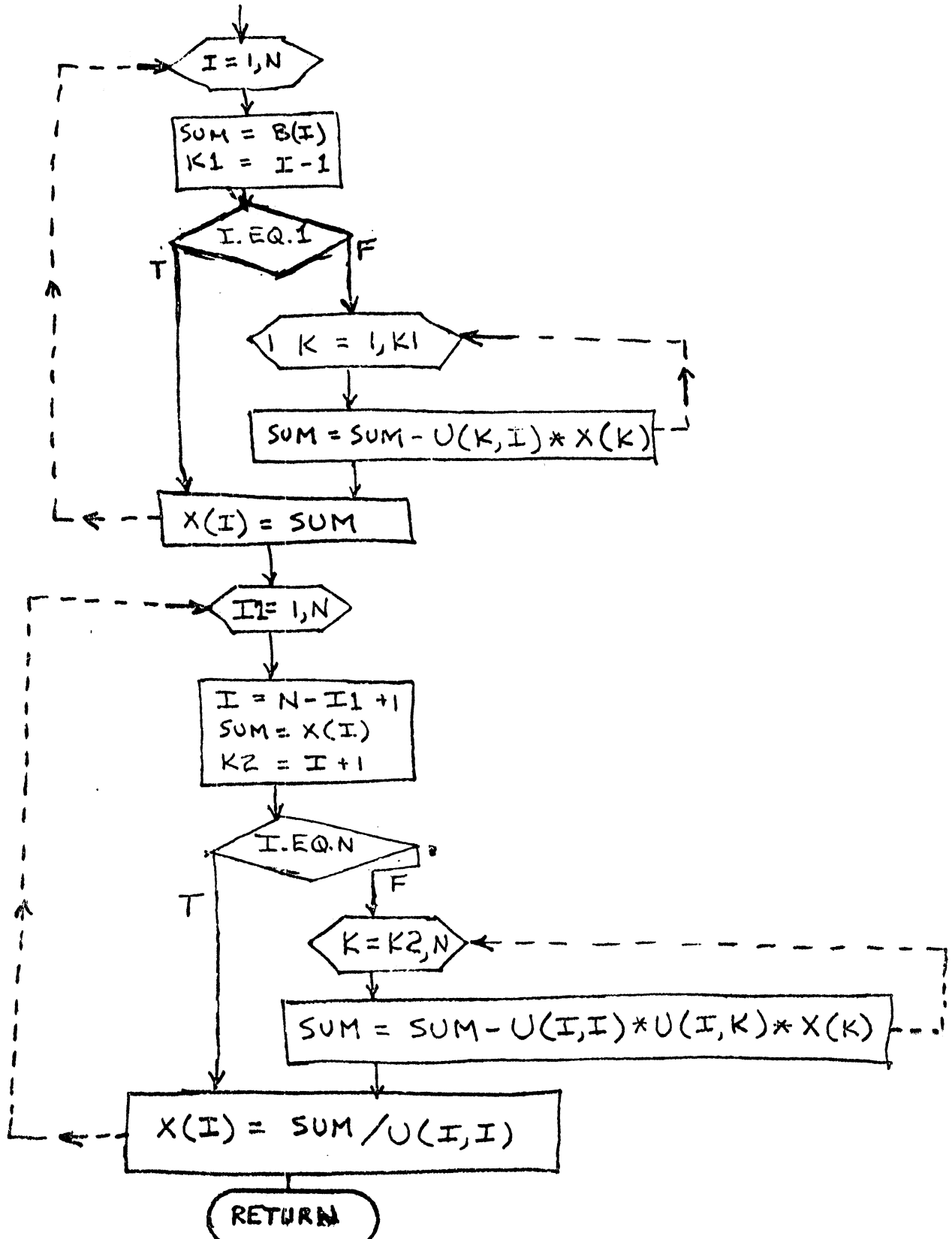
$$X_i = \frac{X_i^* - \sum_{k=i+1}^{n-1} D_{ii} U_{ik} X_k}{D_{ii}}$$

$$X_i = \frac{X_i^* - D_{ii} \sum_{k=i+1}^n U_{ik} X_k}{D_{ii}}$$

where again the $i = n$ case is degenerate,

SUBROUTINE NAME: SOLVE(N,U,B,X)

This subroutine completes the solution of the matrix equation $[A]\{X\} = \{B\}$. The matrix $[A]$ is real, symmetric and positive definite. It has been decomposed and stored in the matrix $[U]$. Both $[A]$ and $[U]$ are $N \times N$.




```

1      SUBROUTINE DCOMP(N,A,*)
2      C      THIS SUBROUTINE DECOMPOSES AN N BY N SYMMETRIC MATRIX INTO THE
3      C      PRODUCT OF AN UPPER TRIANGULAR MATRIX PREMULTIPLIED BY ITS
4      C      TRANSPOSE AND A DIAGONAL MATRIX.      A = U' D U
5      C      A = ORIGINAL SYMMETRIC MATRIX
6      C      U = UPPER TRIANGULAR MATRIX WITH UNITY ON MAIN DIAGONAL.
7      C      D = DIAGONAL MATRIX
8      C      THE MATRIX U IS RETURNED TO THE ORIGINAL PROGRAM IN THE UPPER
9      C      RIGHT PORTION OF A(I,J).
10     C      THE MATRIX D IS RETURNED TO THE MAIN PROGRAM ON THE DIAGONAL OF

```

```

12     DIMENSION A(10,10)

```

```

13     DO 4 I=1,N
14     DO 4 J=I,N
15     SUM=A(I,J)
16     K1=I-1
17     IF (I.EQ.1) GO TO 2
18     DO 1 K=1,K1
19     1 SUM=SUM-A(K,I)*A(K,J)*A(K,K)
20     2 IF (J.NE.I) GO TO 3
21     IF (SUM.LE.0.0) RETURN 1
22     A(I,J)=SUM
23     GO TO 4
24     3 A(I,J)=SUM/A(I,I)
25     4 CONTINUE
26     RETURN
27     END

```

END OF FILE

```

1      SUBROUTINE SOLVE(N,U,B,X)

```

```

2      C      THIS SUBROUTINE SOLVES THE LINEAR ALGEBRAIC SET OF EQUATIONS AX
3      C      A IS AN NX N , REAL, SYMETERIC MATRIX.
4      C      X AND A ARE VECTORS WITH N ELEMENTS.
5      C      THIS SUBROUTINE MAKES USE OF THE DECOMPOSITION OF A INTO A
6      C      PRODUCT OF TWO TRIANGULAR AND ONE DIAGONAL MATRIX. SEE SUBROUT
7      C      DCOMP FOR DETAILS.
8      C      INTEGER K,K1,I,K2,N
9      C      REAL X(10), B(10),U(10,10)
10     DO 2 I=1,N
11     SUM=B(I)
12     K1=I-1
13     IF(I.EQ.1) GO TO 2
14     DO 1 K=1,K1
15     1 SUM=SUM-U(K,I)*X(K)
16     2 X(I)=SUM
17     DO 4 I1=1,N
18     I=N-I1+1
19     SUM=X(I)
20     K2=I+1
21     IF(I.EQ.N) GO TO 4
22     DO 3 K=K2,N
23     3 SUM=SUM-U(I,K)*X(K)*U(I,I)

```

```

24     4 X(I)=SUM/U(I,I)
25     RETURN
26     END

```

END OF FILE

Modification of Equilibrium Equations
Prior to Solution

W. J. Anderson

I. PROBLEM STATEMENT

Assuming that a finite element problem has been properly posed and the element equilibrium equations have been successfully assembled, one has a set of equations

$$\begin{matrix} \text{assembled vector of} & \nearrow & \{R\} & = & [K]\{r\} & \longleftarrow & \text{degrees of freedom} \\ \text{external loads, plus} & & & & \uparrow & & \\ \text{equivalent nodal loads} & & & & \text{assembled} & & \\ & & & & \text{stiffness} & & \\ & & & & \text{matrix} & & \end{matrix} \quad (1)$$

Mathematically, 3 possibilities exist:

- (1) All components of $\{r\}$ are known and $\{R\}$ is to be found.
- (2) All components of $\{R\}$ are known and $\{r\}$ is to be found.
- (3) A mixture of components of $\{r\}$ and $\{R\}$ are known, and the remainder of each are to be found.

The first case is easy to solve; the $\{R\}$ vector is recovered by direct multiplication. This case almost never occurs in practice! The second case is the standard form desired for most equation solvers but is not obtained as a well-posed problem upon first assembly. Rigid body modes must be constrained by setting certain displacements zero, and this casts the second case into the third case. The only assembly of finite element equations of any practical interest and of any difficulty, is case 3. This is called a mixed boundary-value problem.

If we concentrate on the mixed boundary value problem, case 3, we need to modify the equations in some way to recover the form of case 2 suitable for equation solvers. There are several methods available which reduce the set of equations in size or substitute a dummy equation for the equations with specified displacement.

II. PARTITIONING

Partitioning is a method of reducing the number of equations studied to only those where displacements are unknown. It proceeds more easily when there is a fortuitous or pre-planned numbering of degrees of freedom so that all specified displacements are left to the last (say):

$$\begin{matrix} \text{known} \rightarrow & \left\{ \begin{matrix} R_1 \\ \dots \\ R_2 \end{matrix} \right\} & = & \begin{bmatrix} K_{11} & \dots & K_{12} \\ \dots & \dots & \dots \\ K_{21} & \dots & K_{22} \end{bmatrix} & \begin{matrix} \left\{ \begin{matrix} r_1 \\ \dots \\ r_2 \end{matrix} \right\} \leftarrow \text{unknown} \\ \leftarrow \text{known} \end{matrix} \end{matrix} \quad (2)$$

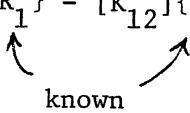
The equations are then partitioned, where each submatrix follows the normal laws for matrix operations:

$$\{R_1\} = [K_{11}]\{r_1\} + [K_{12}]\{r_2\} \quad (3)$$

$$\{R_2\} = [K_{21}]\{r_1\} + [K_{22}]\{r_2\} \quad (4)$$

Equation 3 can be solved for the unknown displacements $\{r_1\}$:

$$\{r_1\} = [K_{11}]^{-1}(\{R_1\} - [K_{12}]\{r_2\}) \quad (5)$$



If the reactions at the supports (degrees of freedom where displacement is specified) are desired, then the final step is to use equation 4 to directly solve for $\{R_2\}$, since the R.H.S. is known.

III. SOLVING "IN-PLACE" (PAYNE & IRONS)

One can solve the equations for a mixed boundary value problem, without regard to the order in which they appear, by an artifice due to Payne & Irons.* The set of equations is modified by converting the known displacements into unknowns, and then forcing them to the desired value by choosing a very large value on the diagonal as shown:

ORIGINAL PROBLEM:

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \begin{Bmatrix} r_1 \\ r_2 \\ 2.7 \end{Bmatrix} = \begin{Bmatrix} 4000. \\ 7000. \\ R_3 \end{Bmatrix} \quad (6)$$

MODIFIED PROBLEM:

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & 10^{12} \times K_{33} \end{bmatrix} \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \end{Bmatrix} = \begin{Bmatrix} 4000. \\ 7000. \\ 10^{12} \times 2.7 \times K_{33} \end{Bmatrix} \quad (7)$$

The modified equations are then solved in the usual way by versions of Gaussian elimination or by iteration.

To see the effect of the large constant which is used, separate out the last equation from (7):

$$K_{31}r_1 + K_{32}r_2 + K_{33} \times 10^{12}r_3 = 10^{12} \times 2.7 \times K_{33} \quad (8)$$

* For engineers and mathematicians who have worked with the Simplex method of linear programming, this approach is similar to the "big M" method of driving a variable to its constrained value.

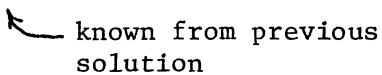
Since stiffness matrices are diagonally dominant, i.e., K_{33} is of the same order as K_{31} and K_{32} ,

$$r_3 \approx 2.7 \quad (9)$$

The value of r_3 can be made as close to 2.7 as wanted by increasing the constant $10^{12} r_3$ even further.

In solving "in place", one inserts some dummy equations in the process and therefore must lose (at least temporarily) some information from the original set of equations. The information lost is the reaction force at each node, which can be recovered by a final step using direct summation, e.g.

$$R_3 = \sum_{j=1}^3 K_{3j} r_j \quad (10)$$



IV. SOLVING IN PLACE (WEAVER)

An alternate method of solving in place is to extensively modify the stiffness matrix as shown, using the same original matrix in the last section:

$$\begin{bmatrix} K_{11} & K_{12} & \overline{0} \\ K_{21} & K_{22} & 0 \\ 0 & 0 & \underline{1} \end{bmatrix} \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \end{Bmatrix} = \begin{Bmatrix} 4000. - K_{13}(2.7) \\ 7000. - K_{23}(2.7) \\ 2.7 \end{Bmatrix} \quad (11)$$

This method is exact. It involves more manipulation than the previous method, however. It is practical, and has been used in some smaller programs such as Weaver's "PS2."

V. STATIC CONDENSATION

Every finite element solution must use a modification procedure similar to those in sections II, III and IV to put the equations in the standard form for solution. There is a further way to eliminate a portion of the problem and reduce the number of equations further. This is called static condensation. It really is Gaussian elimination in a preferred order. A method described here is based on partitioning.

Suppose we are dealing with a set of equations in which we are not interested in several degrees of freedom, i.e., we are willing to suppress them, or "condense them out." Let $\{r_2\}$ represent the undesired degrees of freedom, which have been numbered as the last degrees of freedom.

$$\begin{bmatrix} K_{11} & \dots & K_{12} \\ \dots & \dots & \dots \\ K_{21} & \dots & K_{22} \end{bmatrix} \begin{Bmatrix} r_1 \\ \dots \\ r_2 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ \dots \\ R_2 \end{Bmatrix} \quad (12)$$

all forces assumed known

undesired d.o.f.

Partitioning the equations gives

$$[K_{11}]\{r_1\} + [K_{12}]\{r_2\} = \{R_1\} \quad (13)$$

$$[K_{21}]\{r_1\} + [K_{22}]\{r_2\} = \{R_2\} \quad (14)$$

One solves for $\{r_2\}$ from (14) and inserts it in (13), thereby eliminating $\{r_2\}$, as shown below:

$$\{r_2\} = [K_{22}]^{-1}(\{R_2\} - [K_{21}]\{r_1\})$$

Hence (13) becomes

$$[K_{11}]\{r_1\} + [K_{12}][K_{22}]^{-1}(\{R_2\} - [K_{21}]\{r_1\}) = \{R_1\}$$

Collecting terms, one can define new stiffnesses and forces:

$$\underbrace{([K_{11}] - [K_{12}][K_{22}]^{-1}[K_{21}])}_{[\bar{K}]} \{r_1\} = \underbrace{\{R_1\} - [K_{12}][K_{22}]^{-1}\{R_2\}}_{\{\bar{R}\}}$$

with the result that a new problem involving fewer degrees of freedom, but more complicated stiffness and force system, is defined:

$$[\bar{K}]\{r_1\} = \{\bar{R}\} .$$

Comments on static condensation:

- A. Partitioning is usually not possible because of the scrambling of desired and undesired degrees of freedom, but other methods are used.
- B. One can use static condensation to allow a better elastic representation of a body, particularly its interior, without putting external loads at those suppressed d.o.f., i.e. $\{R_2\} = 0$. The practical reason for this is that the user does not want output data at these d.o.f. and likewise does not want to have to prescribe input data there either.
- C. Static condensation can be used for single elements to condense out unwanted nodes in the manner of comment B.

VI. SUBSTRUCTURING

This is a method of dividing large structures into substructures for easier solution. It is philosophically similar to static condensation except that the force vector $\{\bar{R}_2\}$ is not usually zero.

VII. FINAL COMMENT

The partitioning method for static condensation, substructuring, Gauss-Dolittle decomposition, Cholesky decomposition and high school elimination methods are all just variations on Gauss elimination using a preferred order for eliminating variables.

REMOVAL OF RIGID BODY MODES

I. DEFINITION OF RIGID-BODY MODES

Many elastic bodies can translate freely or rotate freely because there are not enough constraints applied to fix the body to the earth. The body of a typical desk telephone often is free to translate in two directions and to rotate about a vertical axis.

Each such distinct type of motion is called a rigid body mode.

Mathematically, one says that a displacement field is possible which causes no strain energy, i. e.,

$$\frac{1}{2} \{ r \}_{I \rightarrow II} [K] \{ r \}_{I \rightarrow II} = 0 \quad (1)$$

This equation implies that $\det[K] = 0$.

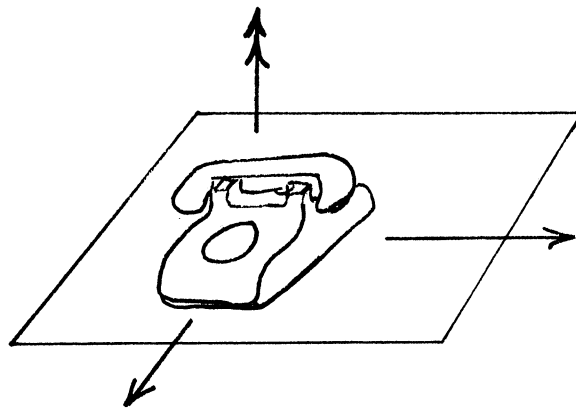


Fig. 1. Desk telephone.

Rigid body modes are legitimate in many analyses, particularly dynamic problems where the overall motion of a body is important. Their presence causes trouble in static stress analysis, however, because they prevent the solution of the equilibrium equations. Every student in finite element theory will sooner or later attempt to solve a stress problem

$$[K]\{r\} = \{R\} \quad (2)$$

without constraining the rigid body modes and will get warning messages from the finite element program. The reason is that $[K]$ is singular and cannot be triangularly factorized.

The requirement for removal of rigid body modes is a nuisance in many stress problems which could reasonably be stated as if the body were in mid-air. For instance, stress in an inflated tire results primarily from

the internal pressure and vehicle weight whereas it should not depend on which direction the car is traveling. Nevertheless, for a unique solution, one must specify the orientation of the time in space to not allow rigid body modes to occur.

In the theory of elasticity, there are:

6 rigid body modes in 3 dimensional space

3 rigid body modes in 2 dimensional space

1 rigid body mode in 1 dimensional space.

For structural elements such as beams and plates, there are 6 rigid body modes in 3 dimensions. When imbedded in 2-D, however, one must decide each case separately.

The opposite of a rigid body mode is an elastic mode, i. e., a displacement field which causes strain energy.

II. HOW TO REMOVE RIGID BODY MODES

In many finite element programs, the user must artificially constrain degrees of freedom to prevent rigid body translations and rotations. Other computer programs such as MSC/NASTRAN have automated procedures for this. Nevertheless, every user should understand the logic required to remove these modes. Generally, one must artificially constrain enough displacements to prevent rigid body translation or rotation but not enough to constrain elastic deformation between any nodes. Application of such an artificial constraint at a node should not introduce any external force at the node.

Every three-dimensional body should have at least six displacement components specified (typically zero) to remove six rigid body modes. Likewise, two-dimensional bodies require three displacements specified, and one-dimensional bodies require one displacement specified. One rigid body mode is removed by constraint of each properly chosen displacement.

The original physical problem often has displacement constraints that fortuitously remove some rigid body modes. The remaining rigid body modes must be removed by the analyst.

III. EXAMPLE

Let us look at the stress around the circular hole in a sheet under uniform tension (Fig. 2).

Note that the problem says nothing about orientation of the sheet because it has no effect on stresses.

The engineer must ensure that the system does not rotate or translate, however.

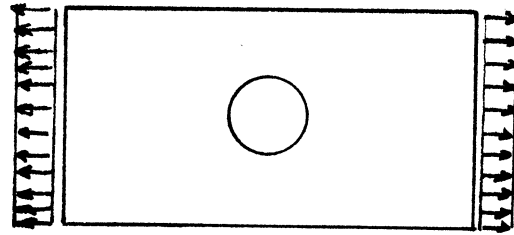


Fig. 2. Rectangular sheet with circular hole, under tension. Two-dimensional problem.

If the full problem is to be solved (neglecting symmetry for now), one could use the extremely crude grid in Fig. 3. To constrain the three rigid modes, one could set

to zero:

- (a) horizontal displacement at node 4
- (b) vertical displacement at node 4
- (c) vertical displacement at node 8

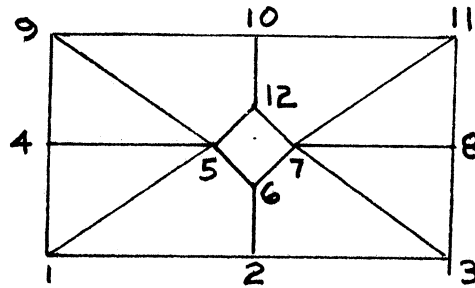


Fig. 3. Node pattern.

These constraints do not affect the elastic modes, whereas the following set does and is therefore unacceptable:

- (a) horizontal displacement at node 4
- (b) vertical displacement at node 4
- (c) horizontal displacement at node 8.

The latter set of constraints does not allow any cumulative elastic strain in the x direction between nodes 4 and 8. It also does not prevent small rotations of the body about node point 4 and is therefore a complete failure.

Finally, if symmetry had been used, one would solve the problem in Fig. 3. In this case, however, symmetry sets to zero:

- (a) horizontal displacement at node 10
- (b) horizontal displacement at node 12
- (c) vertical displacement at node 7
- (d) vertical displacement at node 8.

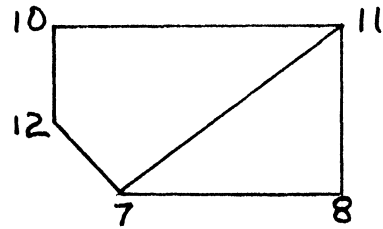


Fig. 4. Fortuitous removal of rigid body modes by symmetry considerations.

This is sufficient to remove rigid body modes, i. e., the physical problem contains none and therefore further artificial constraints are not needed.[†]

[†] Symmetry also implies that the vertical forces on 10 and 12 are zero, and horizontal force on 7 is zero. Node 8 is unique because it is a loaded node on a plane of symmetry. The horizontal load due to the neighboring mirror image acting on 8 is zero by symmetry, so the only load on 8 is the equivalent nodal load resulting from the edge stress.

IV. MATHEMATICAL NOTE

Consider the strain energy in a structure at equilibrium, state II.

If the structure were at zero stress and strain at state I, then the increase in energy absorbed equals the work done:

$$\Delta_{I \rightarrow II} U = \Delta_{I \rightarrow II} W$$

$$\int \frac{1}{2} \{\epsilon\}_{II}^T \{\sigma_{II}\} dV = \frac{1}{2} \{r\}_{II}^T \{R\}_{II}$$

$$= \frac{1}{2} \{r\}_{II}^T [K]_{II} \{r\}_{II}$$

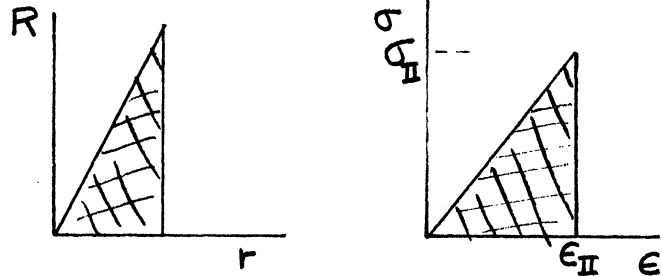


Fig. 5. Work and energy.

Generally speaking, the strain energy must be greater than or equal to zero for any displacement $\{r\}$. Hence,

$$\frac{1}{2} \{r\}^T [K] \{r\} \geq 0.$$

If, however, a nonzero set of displacements $\{\hat{r}\}$ exists for which the strain energy is identically zero:

$$\frac{1}{2} \{\hat{r}\}^T [K] \{\hat{r}\} = 0$$

then the displacement $\{\hat{r}\}$ is a rigid body displacement. Furthermore, the matrix $[K]$ is then called positive semi-definite.

Reference: Desai & Abel: pp. 19-21; pp. 162-163.

The computational difficulty in solving large sets of linear, algebraic equations depends on the way the equations are mathematically coupled to each other. Three concepts are important: "bandwidth," "wavefront," and "pointer" methods. Equation solvers usually require that one or the other of these be minimized in order to reduce computer CPU time. Equation solvers of the banded (SAP4, SAP 6) or wavefront (SUPERB) types require careful ordering of nodes or elements, respectively. Equation solvers using pointers (NASTRAN) use sparse matrix methods that only store nonzero stiffness terms.

I. BANDWIDTH

Bandwidth refers to the width of the diagonal band made up of nonzero terms in a banded, sparse matrix. Typically, by numbering the nodes properly, the F.E. method leads to a stiffness matrix which has nonzero terms near the main diagonal and zero elsewhere. The bandwidth is given as a dimensionless number indicating how many terms wide this band is; no nonzero terms may fall outside the band. Bandwidth can be given in terms of compact or detailed notation. The semi-band width, B, is more important in symmetric matrices and is used almost exclusively.*

$$\text{SEMI-BANDWIDTH} = \frac{1}{2} (\text{BANDWIDTH} + 1)$$

This means that the semi-bandwidth includes the main diagonal term.

Consider a physical problem with nodes numbered in two different ways. In each case, it can be shown that

$$B = (D + 1)f$$

where D is the greatest difference in node number within an element, and f is the number of D.O.F. per node. In the first case,

$$\begin{aligned} B &= (5 + 1)(2) \\ &= 12 \end{aligned}$$

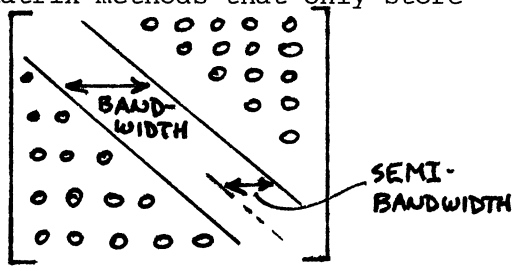


Fig. 1. Bandwidth and semi-bandwidth.

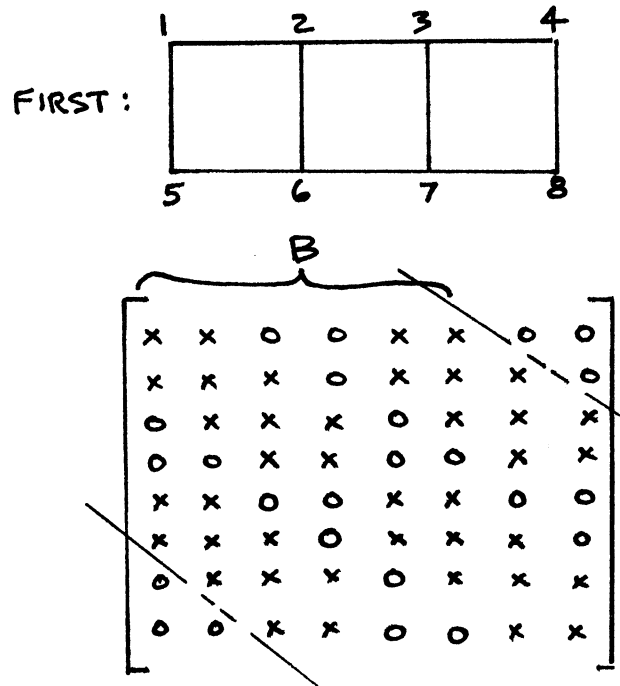


Fig. 2 . First nodal numbering scheme. (compact notation)

* In fact, for most structural problems, authors will use the term bandwidth when semi-bandwidth is actually meant. This seems to cause little trouble in practice.

In the second case

$$B = (3+1)(2) \\ = 8$$

Hence, the bandwidth depends on the way the nodes are numbered.

For Gaussian elimination types of solution, the CPU time required is proportional to the size of the matrix times the semi-bandwidth squared:

$$\left(\frac{\text{CPU}}{\text{TIME}}\right) \propto (N)(B)^2$$

In the example discussed, the running time for the second case would be

$$\begin{aligned} (\text{CPU})_2 &= \frac{N_2}{N_1} \frac{B_2^2}{B_1^2} (\text{CPU})_1 \\ &= \frac{8^2}{12^2} (\text{CPU})_1 \\ &= 0.44 (\text{CPU})_1 \end{aligned}$$

and the CPU time is cut by more than half by renumbering the nodes!

SAPIV and SAP6, as well as most other public domain finite element codes are dependent on bandwidth ideas. Note that the numbering scheme for elements is unimportant for bandwidth ideas.

II. WAVEFRONT

Another type of equation solver, also based on Gaussian elimination, is the "wavefront" approach. This was originally developed in England by Bruce Irons. In this approach, the element numbering is important, rather than the node numbering. The goal is to hold as few equations in high speed core as possible, putting the remaining terms on disk or other lower cost storage. The solution proceeds element by element. The equations of equilibrium for a node are activated when an element is first formed which contains that node. The equations of equilibrium for that node remain active until the last element containing the node is processed. That node is then deactivated. The solution is merely a Gaussian elimination in a preferred order.

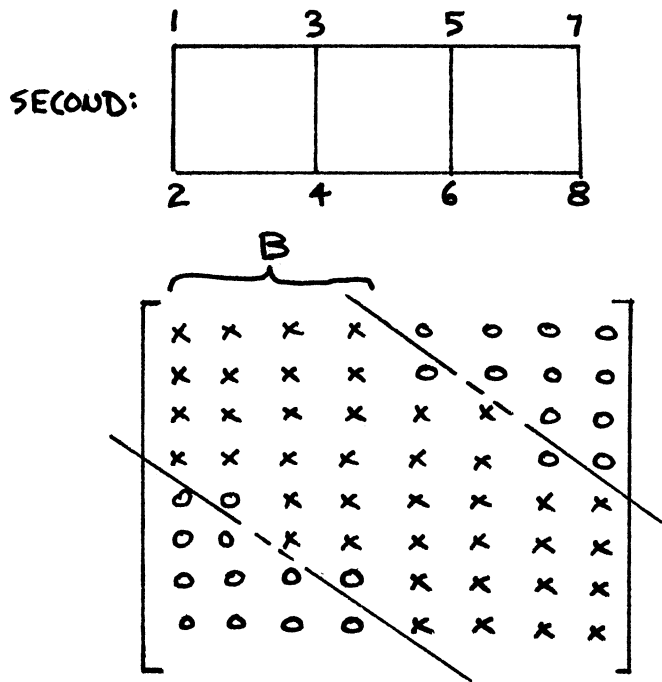


Fig. 3. Second nodal numbering scheme. (compact notation)

An example is now given where the same problem is solved with two different element number schemes.

FIRST CASE

Element	Active Node	# of Active Nodes	Nodes Dropped
1	1,2,3,8	4	1,8
2	2,3,4,5,6,7	6	4,6
3	2,3,5,7	4	2,3,5,7

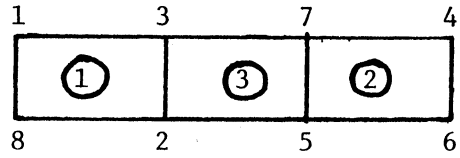


Fig. 4. First element numbering scheme for wavefront solver.

SECOND CASE

Element	Active Node	# of Active Nodes	Nodes Dropped
1	1,2,3,8	4	1,8
2	2,3,5,7	4	2,3
3	4,5,6,7	4	4,5,6,7

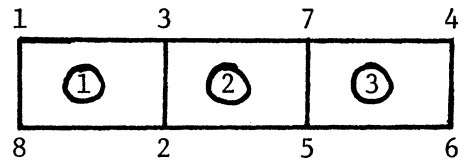


Fig. 5. Second element numbering scheme for wavefront solver.

The second case has a smaller wavefront. Note that the numbering scheme or the nodes is unimportant for wavefront ideas.

The proprietary program "SUPERB" has a wavefront solver. NASTRAN did not originally have a wavefront solver, but the recent proprietary version (MSC) does.

III. "POINTER" METHODS FOR SPARSE MATRICES

IV. DIFFICULTIES WITH CPU TIME FOR 3-D SOLUTIONS USING SOLID ELEMENTS

Reference: Zienkiewicz, Edition 3, pp. 135-136.

The run time for 3-D problems can be prohibitive if care is not taken. An example which is rather scary is presented in Zienkiewicz's book, where comparable sized 1-D, 2-D, and 3-D problems are discussed. Suppose each problem has 20 elements in each direction as shown. It is known that CPU time T for the equation solver varies as:

$$T \propto \left(\frac{\text{Number of Equations}}{\text{Semi-Bandwidth}} \right)^2$$

Note that the number of degrees of freedom per node increases from 1-D to 3-D. The number of nodes and the bandwidth also increase strongly. Indeed:

$$T_{1-D} = \text{const}(21)(2)^2 = \text{const}(84)$$

$$T_{2-D} = \text{const}(882)(46)^2 = \text{const}(1,866,312)$$

$$T_{3-D} = \text{const}(27,783)(1392)^2 \\ = \text{const}(5.83 \times 10^{10})$$

Even comparing the 2-D and the 3-D solution, we see that the 3-D solution requires 24,600 times the run time as the 2-D. This is a fantastic increase in complexity as dimensions increase.

IV. CONCLUSIONS & COMMENTS

Minimizing bandwidth (or wavefront) saves on CPU time and on core storage requirements.

Core storage times typically represent 2/3 of the total cost of running F.E. programs. At present, then, one should attempt to reduce core storage as the highest priority.

Three-dimensional solutions (solid elements) are extremely expensive if care is not taken. Effort in the direction of using fewer but more sophisticated elements, is the proper approach. Isoparametric elements appear to be the answer here.

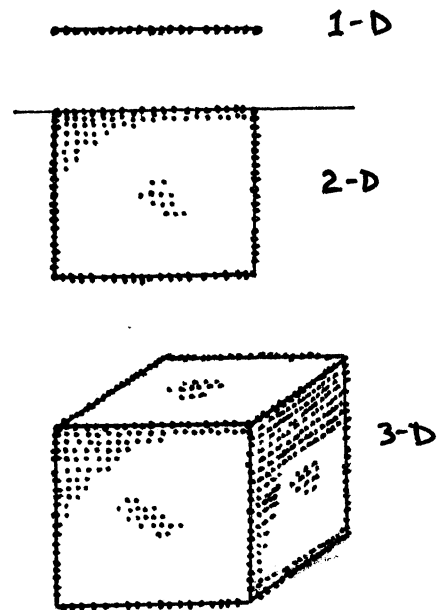
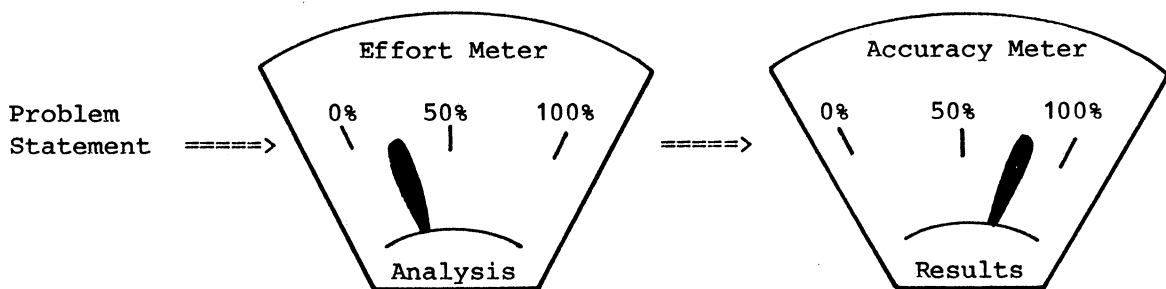


Fig. 6. Comparison of 1-, 2-, and 3-dimensional spaces

Convergence Concepts in Finite Element Analysis

by
Robert E. Sandstrom, Ph.D.
University of Michigan, 1981

The goal in any computational analysis is obtain solutions for a given problem as efficiently and accurately as possible. The purpose of this lecture is to expose the utility of convergence concepts as a tool to assist you in reaching this goal. Understanding these concepts will help you become a "smart" finite element user.



Convergence in finite element analysis generally includes those factors which prevent us from computing the exact answers to our "real world" problem. Some factors which effect the rate of convergence include:

1. Real world complications such as nonlinearities and ill defined or unknown parameters.
2. Modeling concepts, practices and accuracy.
3. Discretization of Continuum.
4. Solution algorithm.
5. Roundoff error.

This lecture will deal with item 3. Despite the fact that items 1 and 2 are probably the most important factors affecting the accuracy and validity of the analysis; they will not be discussed in this lecture. They reflect broad consideration which is not restricted to finite element application. Items 4 and 5 will not be discussed here because most modern computer codes employ accurate procedures and which generally are not within the control of the user. On the other hand, the finite element user is directly involved with item 3, discretization of the continuum. Users must make decisions with regard to element type and mesh size. These decisions can drastically effect the quality and cost of the analysis.

In this presentation the mathematical aspects of convergence will be reserved for the text books. Those concepts which will enhance our "smartness" as finite element users will be presented in an intuitive manner.

Concepts

Remember that the finite element method is basically a Rayleigh-Ritz procedure. To refresh your memory the Rayleigh-Ritz technique is used to solve structural problems in the following manner. First, we select (or invent) a set of admissible displacement fields. They may be continuous or piecewise continuous and they must satisfy the "geometric" boundary conditions on displacement and slope. We then tune the displacement fields in our set so that the potential energy of the structural system is a minimum. The "best" answer will produce the lowest value of potential energy.

A basic characteristic of the Rayleigh-Ritz method is that it will:

1. Underestimate displacements.
2. Underestimate strains.
3. Underestimate stresses.
4. Overestimate stiffness.

Furthermore, it is generally recognized that the Rayleigh-Ritz method may produce displacements of acceptable accuracy; however, the stresses and strains computed through differentiation of the admissible displacement fields may yield poor results. Users of the finite element method should be aware that stresses and strains computed from a finite element analysis may be very poor even though the displacements are accurate.

The accuracy of a Rayleigh-Ritz analysis can be improved by doing any of the following:

1. Increase the number of admissible displacement fields.
2. Increase the order of the displacement fields.

In the finite element analysis the accuracy can be improved by refining the mesh and/or using higher order elements. Refining the mesh is equivalent to increasing the number of admissible piecewise displacement fields and using higher order elements is equivalent to increasing the order of the displacement fields. Both improvements may increase the cost of the analysis. One still may ask 'What is the cost of a poor result?' Failure?

Element Convergence Requirements Taken from Reference [1]

Is it possible to converge to the "exact" result? The answer to this question is yes, provided that the following conditions are satisfied.

1. The displacement functions chosen should be such that it does not permit straining of an element to occur when the nodal displacements are caused by a rigid body displacement.
2. The displacement function has to be of such a form that if nodal displacements are compatible with a constant strain condition such a constant strain will in fact be obtained.
3. The displacement functions should be so chosen that the strains at the interface between elements are finite.

When these conditions are satisfied, the finite element discretization will approach the exact answer in the limit as the mesh size is reduced. This idea is consistent with concepts used in the study of calculus.

The point here is that we can achieve acceptable accuracy, but again it may be expensive.

Convergence Rates

How far must one go to obtain acceptable results?

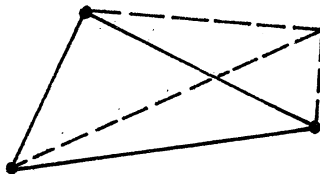
Zienkiewicz gives the following guidelines regarding solution errors and convergence rates.

1. The exact solution will be obtained if the displacement function for an element exactly fits the correct solution.
2. The solution error for displacements is given by $O(h^{p+1})$ where h refers to the meshsize and p equals the order of the displacement function used in the element.
3. The solution error for stresses and strains is given by $O(h^{p+1-m})$ where m equals the m^{th} derivative of the displacement field used to compute the stresses and strains.

As an example the convergence rates for plane strain triangles are given below.

Constant Strain Triangle:

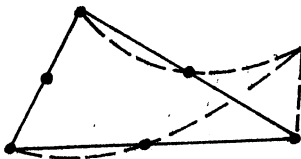
Shape function: Linear



Convergence Rates:
 Displacements: $O(h^2)$
 Stress/strain: $O(h)$

Linear Strain Triangle:

Shape function: Quadratic



Convergence Rates:
 Displacements: $O(h^3)$
 Stress/strain: $O(h^2)$

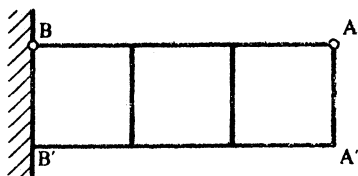
It can be seen from the convergence rates given above that if the mesh size is refined by a factor of two that the error in the stress results will be reduced by an order of $O(h/2)$ for the CST element and $O(h^2/4)$ for the LST element. It should be noted that the concept of convergence rate is relative. It will not measure the absolute error reduction; it will only indicate the relative error reduction one can obtain through mesh refinement and the use of higher order elements.

Mesh Refinement verses Higher Order Elements

Mesh refinement and the use of higher order elements should be governed by:

1. Desired output (displacement and/or stress strain data).
2. Stress-strain distribution throughout the region of interest.

A simple point to remember is that constant strain elements (linear shape functions) will represent the displacement field by a patchwork of discrete linear interpolation functions and the stress-strain field will appear as a histogram type distribution. This histogram analogy should be kept in mind when selecting element type and mesh size. Disregard for this concept may lead to serious modeling problems. Linear strain elements are more forgiving in this regard, relaxing (but not deleting) the requirements on mesh size. Linear strain elements will model the stress-strain distribution by a patchwork of linear interpolation fields and the mesh size should be selected with this mind. These arguments could be extended for higher order elements.



A cantilever beam analyzed by several plane stress elements. Note accuracy improvement with higher order elements.

(REF. 1)

Type of element	Vertical Load of A		Couple at AA'	
	Max. defl. at AA'	Max. stress BB'	Max. defl. at AA'	Max. stress BB'
	0.26	0.19	0.22	0.22
	0.65	0.56	0.67	0.67
	0.53	0.51	0.52	0.55
	0.99	0.99	1.00	1.00
	1.00	1.00	1.00	1.00
EXACT	1.00	1.00	1.00	1.00

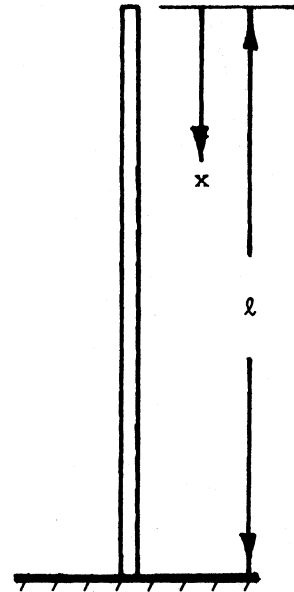
A second point which should be kept in mind is that constant strain elements may produce acceptable displacement fields (provided that a proper mesh size is used); however, stress and strains may not be accurate for reasons given earlier. Higher order elements may not always significantly improve the displacement results, but they usually will improve the stress-strain results.

Example

The example shown of the next few pages demonstrates the improved stress-strain data obtainable from linear strain elements over constant strain elements.

Suppose we want to compute the stresses in a bar due to its own weight.

Let E = Young's modulus
 A = Cross sectional area
 w_0 = weight per unit length



Exact Solution

Differential Equation: $EA \frac{d^2u}{dx^2} = -w_0$

Boundary Conditions: Displacement $u(x=l) = 0$

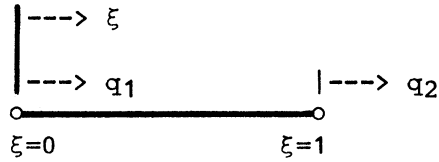
Strain $\epsilon_x = \frac{du(x=0)}{dx} = 0$

Displacements: $u(x) = \frac{w_0 l^2}{2EA} \left[1 - \left(\frac{x}{l}\right)^2 \right]$

Stresses: $\sigma_x = \frac{-w_0 l}{A}$

Finite element solution using two constant strain truss elements.

Element Data



Shape functions:

$$N_1 = 1 - \xi$$

$$N_2 = \xi$$

Stiffness Matrix:

$$\frac{EA}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Global Solution

Equivalent Nodal loads:

$$R_1 = \frac{w_0 l}{4}$$

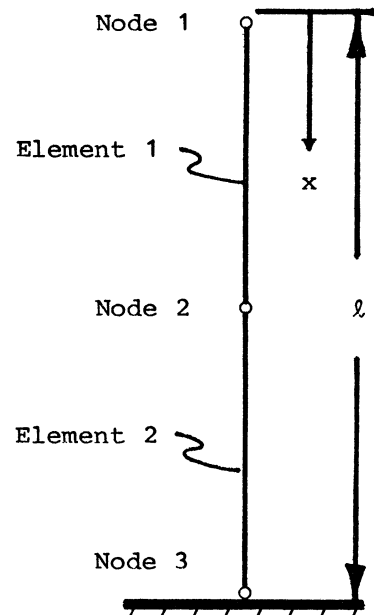
$$R_2 = \frac{w_0 l}{2}$$

Boundary Conditions:

$$r_3 = 0$$

Global Equations:

$$\begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \frac{2EA}{l} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix}$$



Nodal Displacements:

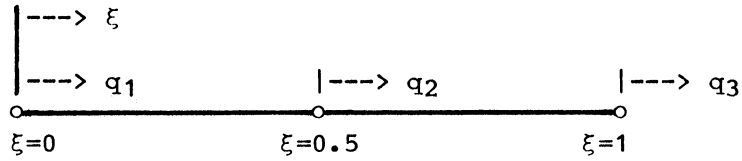
$$r_1 = \frac{1}{2} \frac{w_0 l^2}{EA} \quad r_2 = \frac{3}{8} \frac{w_0 l^2}{EA} \quad \leftarrow \text{These displacements are EXACT!}$$

Stress at node 3 of element 2:

$$\sigma_x = E \epsilon_x = E \left[\frac{dN_1}{d\xi} r_2 + \frac{dN_2}{d\xi} r_3 \right]_{\xi=1} = - \frac{3}{4} \frac{w_0 l}{A} \quad \leftarrow \text{25\% ERROR!}$$

Finite element solution using one linear strain truss element.

Element Data



Shape functions:

$$N_1 = 2\xi^2 - 3\xi + 1$$

$$N_2 = 4\xi - 4\xi^2$$

$$N_3 = 2\xi^2 - \xi$$

Stiffness Matrix:

$$\frac{EA}{3l} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}$$

Global Solution

Equivalent Nodal loads:

$$R_1 = \frac{w_0 l}{6}$$

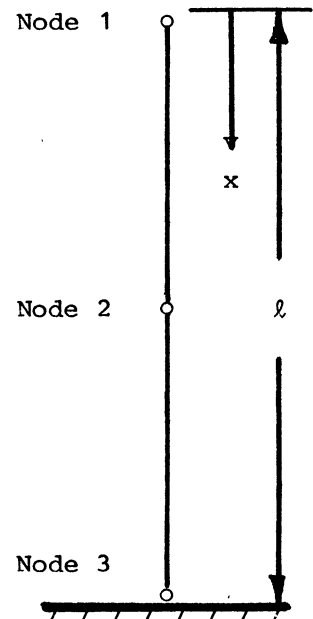
$$R_2 = \frac{2w_0 l}{3}$$

Boundary Conditions:

$$r_3 = 0$$

Global Equations:

$$\begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \frac{EA}{3l} \begin{bmatrix} 7 & -8 \\ -8 & 16 \end{bmatrix} \begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix}$$



Nodal Displacements:

$$r_1 = \frac{1}{2} \frac{w_0 l^2}{EA} \quad r_2 = \frac{3}{8} \frac{w_0 l^2}{EA} \quad \leftarrow \text{These displacements are EXACT!}$$

Stress at x within the element:

$$\sigma_x = E\epsilon_x = E \left[\frac{dN_1}{d\xi} r_1 + \frac{dN_2}{d\xi} r_2 + \frac{dN_3}{d\xi} r_3 \right]_{\xi=x} = \frac{-w_0 x}{A} \quad \leftarrow \text{EXACT!}$$

References

1. Zienkiewicz, O. C., The Finite Element Method, Third Edition, McGraw-Hill, 1977, Chapter 2.
2. Desai, C. S. and Abel, J. F., Introduction to the Finite Element Method, Van Nostrand, 1972, Chapter 5.

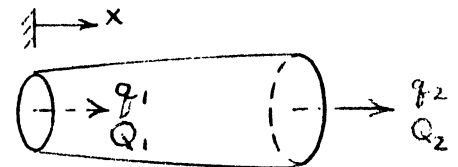
A SPECIAL LINE ELEMENT
DERIVED USING EQUILIBRIUM

Equilibrium ideas will be used to derive the stiffness matrix for a special element: a line element with varying cross-sectional area

$$A(x) = A_0 (1 + Bx/L).$$

The reference area A_0 has dimensions of L^2 and the dimensionless constant B is positive for an element with area increasing from left to right and negative otherwise.

An exact solution can be found. This element will be useful for comparing with later approximate solutions to the varying area element.



Procedure

We wish to fill in the missing operators in the general scheme:

$$\left\{ \begin{matrix} Q_1 \\ Q_2 \end{matrix} \right\} \begin{matrix} [?] \\ \\ \\ \\ [?] \end{matrix} \leftarrow \{ \sigma_x \} \begin{matrix} [E] \\ \\ \\ \\ [?] \end{matrix} \leftarrow \{ \epsilon_x \} \begin{matrix} [d/dx] \\ \\ \\ \\ [?] \end{matrix} \leftarrow \{ u(x) \} \begin{matrix} [?] \\ \\ \\ \\ [?] \end{matrix} \leftarrow \{ \alpha \} \begin{matrix} [?] \\ \\ \\ \\ [?] \end{matrix} \leftarrow \left\{ \begin{matrix} q_1 \\ q_2 \end{matrix} \right\}$$

The equilibrium matrix relating nodal loads and stresses is found by a trick:

$$\begin{aligned} \sigma_x(x) &= Q_2/A \\ &= \frac{Q_2}{A_0(1+Bx/L)} \end{aligned}$$

or

$$Q_2 = A_0(1+Bx/L)\sigma(x).$$

Also

$$Q_1 = -Q_2 = -A_0(1+Bx/L)\sigma(x).$$

Putting this in matrix form, one has

$$\left\{ \begin{matrix} Q_1 \\ Q_2 \end{matrix} \right\} = \begin{bmatrix} -A_0(1+Bx/L) \\ A_0(1+Bx/L) \end{bmatrix} \sigma_x(x)$$

The relation between nodal displacements and $u(x)$ is found from our knowledge of stress in the body. (This is similar to Turner's procedure of starting with known strains and then working out displacements.)

$$\epsilon_x(x) = \frac{\sigma_x(x)}{E}$$

or

$$\begin{aligned} u(x) &= \int \epsilon_x(x) dx + C \quad \leftarrow \text{integration constant} \\ &= \int \frac{\sigma_x(x)}{E} dx + C \\ &= \int \frac{Q_2 dx}{E A_o (1+Bx/L)} + C \\ &= \underbrace{\frac{Q_2}{EA_o} \frac{L}{B}}_{\alpha_1} \ln(1+Bx/L) + \underbrace{C}_{\alpha_2} \end{aligned}$$

If one identifies the constants shown as generalized coordinates α_i , we can proceed.

$$\begin{aligned} \{u(x)\} &= \underbrace{[\ln(1+Bx/L) \quad 1]}_{[\phi]} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} \\ \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} &\equiv \begin{Bmatrix} u(0) \\ u(L) \end{Bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ \ln(1+B) & 1 \end{bmatrix}}_{[A]} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} \end{aligned}$$

The [A] matrix is easily inverted to get:

$$[A]^{-1} = \begin{bmatrix} -\frac{1}{\ln(1+B)} & \frac{1}{\ln(1+B)} \\ 1 & 0 \end{bmatrix}$$

We now have all the ingredients for the stiffness matrix, using the sequence of operators from nodal displacement to nodal forces:

$$\begin{aligned}
[k] &= [\xi][S-S][S-D][\phi][A]^{-1} \\
&= \begin{bmatrix} -A_o(1+Bx/L) \\ A_o(1+Bx/L) \end{bmatrix} [E] \left[\frac{d}{dx} \right] \begin{bmatrix} \ln(1+Bx/L) & 1 \\ 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} -A_o(1+Bx/L) \\ A_o(1+Bx/L) \end{bmatrix} [E] \left[\frac{d}{dx} \right] \underbrace{\begin{bmatrix} 1 - \frac{\ln(1+Bx/L)}{\ln(1+B)} & \frac{\ln(1+Bx/L)}{\ln(1+B)} \\ 1 & 0 \end{bmatrix}}_{[N]}
\end{aligned}$$

The shape function matrix has been calculated as an intermediate step in the equation above. It is a set of interpolation functions, of course, with unit value at the appropriate node. Carrying out the derivative

$$[k] = \begin{bmatrix} -A_o(1+Bx/L) \\ A_o(1+Bx/L) \end{bmatrix} [E] \begin{bmatrix} -\frac{B/L}{(1+Bx/L)\ln(1+B)} & \frac{B/L}{(1+Bx/L)\ln(1+B)} \\ 1 & 0 \end{bmatrix}$$

and, finally,

$$[k] = \frac{E A_o B}{L \ln(1+B)} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \leftarrow \text{stiffness for varying-area line element}$$

Comments:

- 1.) Note that the "effective area" of the line element is $\frac{A_o B}{\ln(1+B)}$. If $B = 1.0$, for example, the effective area is $1.44 A_o$, which is perhaps smaller than one would expect. The more flexible portion of the element dominates in this case.
- 2.) Other solutions for this element, which use approximate shape functions, will yield "stiffer" results, i.e., the stiffness matrices will have larger terms in general. This is because approximate solutions constrain displacements.

COMPARISON OF VARIOUS LINE
ELEMENTS

November 9, 1977
W.J.A.

From several lectures and homework problems, a sequence of line elements has been generated which can be used to model the varying area link shown:

In increasing order of accuracy, we can use

- (A) A single, two node constant area line element. This is a constant strain element. One would choose the reference area, say at $x/L = 1/2$ to get

$$[k] = \frac{EA_o (1+B/2)}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

- (B) A single, two-node area line element. This is again a constant strain element, in spite of the violation of internal equilibrium which results. One has

$$[k] = \frac{EA_o}{L} (1+B/2) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

- (C) A single, three-node, varying area line element. This is a linear strain element.

$$[k] = \frac{EA_o}{L} \frac{(6+6B+B^2)}{3(2+B)} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

- (D) The exact solution, yielding a logarithmic displacement function:

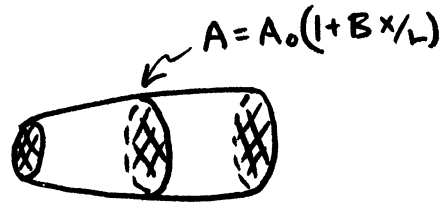


Fig. 1. Physical Case



Fig. 2. Model A. Two nodes.



Fig. 3. Model B. Two nodes.

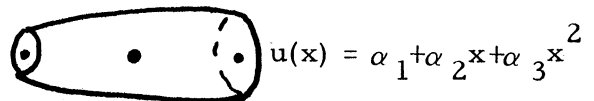


Fig. 4. Model C. Three nodes.

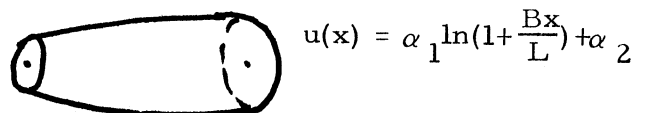


Fig. 5. Model D. Two nodes, logarithmic displacement.

$$[k] = \frac{EA_o B}{L \ln(1+B)} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

One can now compare the multiplication factors, using $B=1$, say, where

$$[k] = k_f \frac{EA_o}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

- (A) $k_f = 1.5$ (4.0% high)
- (B) $k_f = 1.5$ (4.0% high)
- (C) $k_f = 1.445$ (0.1% high)
- (D) $k_f = 1.443$ (exact)

It will be left as a challenge to the student to do a model with two constant-area, two-node elements to see if the answer is better than model C, above. The question is whether one parabola or two straight line segments more accurately model the true logarithmic curve. The question raised here is fundamental: Do you in general get more accuracy by using a larger number of simple elements, or by using a smaller number of more refined elements? Current belief is that fewer, but more refined elements are the "way to go." See the attached page for a review of a paper by I. Fried, in which this conclusion is reached in regard to round-off error in dynamics problems (This is different from the above case concerning modelling error in a static problem, but is the only definitive work on the general question at the moment.)

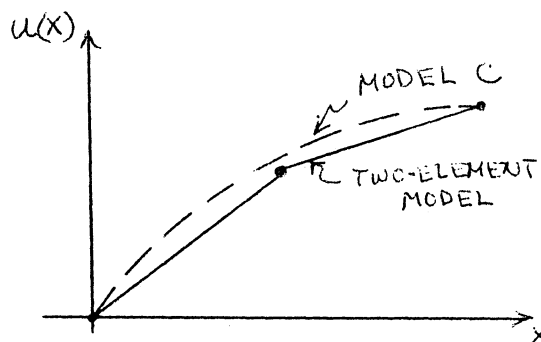


Fig. 6. Comparison of Model C and proposed 2 element solution.

BOUNDS ON THE EXTREMAL EIGEN-
VALUES OF THE FINITE ELEMENT STIFF-
NESS AND MASS MATRIXES AND THEIR
SPECTRAL CONDITION NUMBER

Fried I.

J. Sound and Vib. 22(4), 407-418
(June 22, 1972) 14 refs

Refer to Abstract No. 72-1592

This paper deals with the conditioning of matrixes in finite element approximations to eigenvalue problems. Much of the approach has to do with the spectral condition number (the ratio of the maximum to minimum eigenvalue) of the mass and stiffness matrixes. It has been previously established that roundoff error in computer solutions of eigenvalue problems depends on this spectral condition number.

The eigenvalues of the assembled mass and stiffness matrixes are first bounded using straightforward arguments such as Rayleigh's principle. These bounds depend on properties of the individual elements and the maximum number of elements connected at a single node. Several theorems result, yielding upper and nontrivial lower bounds for the eigenvalues. The latter is important for the case of element stiffness because some zero eigenvalues are present (corresponding to rigid body modes).

The most important theorem (Theorem 4) shows dependence of the spectral condition number on the element length scale h and on the order of original field equation. Specifically, for a differential equation of order $2m$, the spectral condition number varies as h^{-2m} for the stiffness matrix and is independent of h for the mass matrix. Hence, as element size vanishes, the stiffness matrix becomes increasingly ill-conditioned.

A number of examples commonly known to be ill-conditioned are studied and estimates for spectral condition number given. These cases include nearly incompressible solids, nearly inextensional rings and bending with shear.

It is concluded that one obtains better total accuracy in a problem by using a smaller number of higher order elements (those with higher degree polynomials as shape functions) than using a larger number of simpler elements.

The paper is well written and is recommended for both researchers in finite element methods as well as users of finite element programs. One small typographical error is the omission of the symbol λ_1 from R. H. S. of Eq. 36.

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Reprint from THE SHOCK AND VIBRATION DIGEST, Vol.5, No.9

- Reference: (1) Argyris, J. H., "Recent Advances in Matrix Methods," vol. 4, Progress in Aeronautical Sciences, Pergamon Press, Oxford, 1964.
- (2) Weaver, William W., "Outline of Notes on Supplementary Topics," Unpublished Notes, 1969.

I. BACKGROUND

The natural mode method used in finite elements is attributed to Argyris (Ref. 1), but is closely related to the modal methods used in dynamics for many years. The general idea is to identify physical types (modes) of deformation according to whether the motion is of a rigid body type or a straining type. We know that rigid body modes are present in an element; the present discussion takes advantage of these modes by including them explicitly in a description of the displacement field. The "unknowns" in the problem then become the generalized coordinates rather than nodal coordinates.

The end result of using natural modes is to have much simpler stiffness matrices when expressed in the natural form. One can also gain physical intuition from their use.

Do not confuse natural modes with natural coordinates! Natural modes have to do with the field variables whereas the natural coordinates have to do with the underlying coordinate system.

II. NATURAL MODES

The generalized coordinates $\{\alpha\}$ in the expression

$$\{u\} = [\phi] \{\alpha\}$$

will represent certain specific modes of displacement if the ϕ functions are carefully chosen. For the line elements, if we choose:

$$\{u\} = \begin{bmatrix} 1 & (-1 + 2 \frac{x}{l}) \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix}$$

↑

rigid
body
mode

↑

straining
mode

Then α_1 is the magnitude of the rigid body translation, and α_2 is the magnitude of the straining mode. This can also be shown in the solution

$$\{q\} = [A] \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix}$$

where

$$\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix}$$

Note that post- multiplication by α_1 and α_2 brings out the "column-wise" character of the matrix $[A]$, as can be seen by rewriting:

$$\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \alpha_1 \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \alpha_2 \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

↑ rigid body mode ↑ straining mode

The choice of natural modes is not unique because they all have arbitrary amplitude and because any amount of rigid body mode can be introduced into a straining mode with the results being still a straining mode. One must choose as many rigid body modes as required for the space in which the element is imbedded. One also must include all forms of constant strain. It is best to choose rigid body and straining modes which are as dissimilar (orthogonal) as possible.

III. CASTING THE ENTIRE F. E. DERIVATION INTO NATURAL MODES

Since the generalized coordinates $\{\alpha\}$ correspond to a displacement, we need a generalized force $\{Q\}$ to correspond. [See Fig. 1.] We would like to cast the problem in terms of generalized forces and displacements.

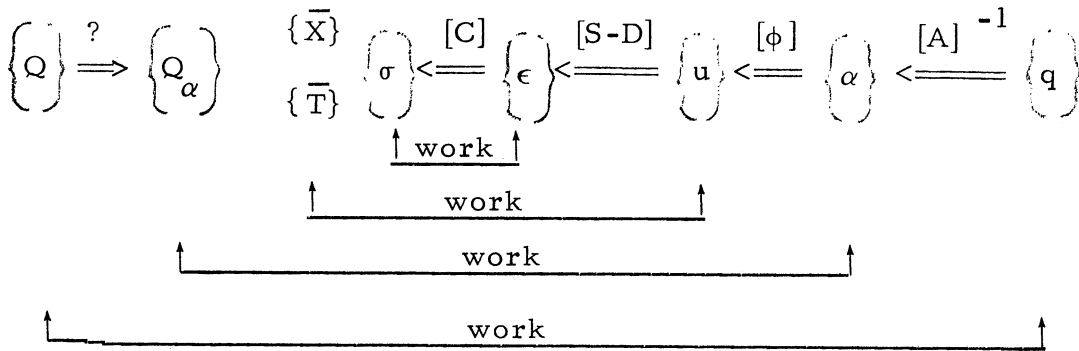


Fig. 1. Vector transformations.

Let us define $\{Q\}$ by requiring that the amount of external work done during a virtual displacement should be the same whether described by physical or generalized coordinates:*

$$\begin{aligned}
 \{d\alpha\}^T \{Q_\alpha\} &= \{dq\}^T \{Q\} \\
 &= ([A] \{d\alpha\})^T \{Q\} \\
 &= \{d\alpha\}^T [A]^T \{Q\} \\
 &= \{d\alpha\}^T [A]^T \{Q\}
 \end{aligned}$$

Since $\{d\alpha\}^T$ is arbitrary, we must have

$$\{Q_\alpha\} = [A]^T \{Q\} \quad \leftarrow \quad \text{generalized forces}$$

This defines the generalized forces. Now look at the equilibrium equation to see how the generalized forces and displacements are related. In the original form

$$\{Q\} = [k] \{q\}$$

Replace the physical variables with generalized variables

$$[A]^T \{Q_\alpha\} = [k] [A] \{q\}$$

* This is not trivial because in general the increment of work done does depend on the coordinate system used, and differs for coordinate systems moving with respect to each other. Our two coordinate systems will be "at rest" with respect to each other.

Premultiplying both sides by $[A]^T$

$$\{Q_\alpha\} = [A]^T [k] [A] \{\alpha\}$$

This is exactly the relation needed for a generalized stiffness,

$$[k_\alpha] \equiv [A]^T [k] [A] \quad \leftarrow \text{generalized stiffness}$$

This generalized stiffness is actually much easier to calculate than the stiffness because

$$\begin{aligned} [k_\alpha] &\equiv [A]^T \int_V [A]^{-1} [\phi]^T [S-D]^T [C] [S-D] [\phi] [A]^{-1} dV [A] \\ &= \int_V [\phi]^T [S-D]^T [C] [S-D] [\phi] dV \end{aligned}$$

Generalized equivalent nodal loads are related to equivalent nodal loads the same way that ordinary nodal loads are.

$$\begin{aligned} \{Q_\alpha\}_{e.n.l.} &= [A]^T \{Q\}_{e.n.l.} \\ &= [A]^T \int_{\text{space}} [N]^T \{f\} d\text{Space} \\ &= [A]^T \int_{\text{space}} ([\phi] [A]^{-1})^T \{f\} d\text{Space} \\ &= [A]^T \int [A]^{-1} [\phi]^T \{f\} d\text{Space} \\ &= \int_{\text{space}} [\phi]^T \{f\} d\text{Space} \quad \leftarrow \text{generalized e.n.l.} \end{aligned}$$

It is possible to solve a problem in terms of these generalized coordinates; the procedure is of primary value for academic insight.

IV. EXAMPLE USING LINE ELEMENT

For the line element,

$$[\phi] = \begin{bmatrix} 1 & -1 + 2 \frac{x}{l} \end{bmatrix}$$

$$[S-D] \equiv \left[\frac{\partial}{\partial x} \right]$$

$$[C] = [E]$$

$$[k_{\alpha}] \equiv \int_V [\phi]^T [S-D]^T [C] [S-D] [\phi] dV$$

$$= \int_0^l \begin{bmatrix} 1 \\ -1 + 2 \frac{x}{l} \end{bmatrix} \left[\frac{\partial}{\partial x} \right] [E] \left[\frac{\partial}{\partial x} \right] \begin{bmatrix} 1 & -1 + 2 \frac{x}{l} \end{bmatrix} A dx$$

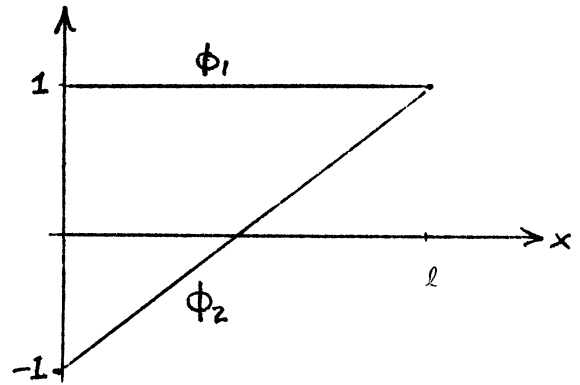


Fig. 2. Natural modes ϕ_1, ϕ_2 .

Carry out the matrix multiplication involving $[\phi]$ and $[S-D]$ matrices

$$= A \int_0^l \begin{bmatrix} 0 \\ \frac{2}{l} \end{bmatrix} [E] \begin{bmatrix} 0 & \frac{2}{l} \end{bmatrix} dx$$

$$[k_{\alpha}] = A \int_0^l \begin{bmatrix} 0 & 0 \\ 0 & \frac{4E}{l^2} \end{bmatrix} dx$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & \frac{4AE}{l} \end{bmatrix}$$

Note that the presence of a rigid body mode has made this stiffness matrix "sparse" (with many zeroes).

Because this stiffness matrix isolates the stiffnesses causing strain energy, it ought to be easy to calculate strain energy using $[k_\alpha]$. Strain energy in a multiple degree-of-freedom, discrete system is α identified

$$U = \frac{1}{2} \{\alpha\}^T [k_\alpha] \{\alpha\}$$

Students: Can you prove this relation from

$$U = \frac{1}{2} \{q\}^T [k] \{q\} ?$$

$$= \frac{1}{2} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix}^T \begin{bmatrix} 0 & 0 \\ 0 & \frac{4AE}{l} \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix}$$

$$= \frac{1}{2} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix}^T \begin{Bmatrix} 0 \\ \frac{4AE}{l} \alpha_2 \end{Bmatrix}$$

$$= \frac{1}{2} \left(\frac{4AE}{l} \right) \alpha_2^2$$

and, as expected, strain energy depends only on the generalized coordinate corresponding to the straining mode.

INTERPRETATION OF
RIGID BODY MODES USING
DISPLACEMENT FUNCTIONS

Oct. 27, 1978

W. J. A.

The presence of rigid body modes in a single element can best be seen from displacement functions, rather than shape functions. For instance, Turner's original triangle element has

$$\begin{Bmatrix} u(x, y) \\ v(x, y) \end{Bmatrix} = \begin{bmatrix} x & 0 & 0 & y & 1 & 0 \\ 0 & y & x & -x & 0 & 1 \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \\ A \\ B \\ C \end{Bmatrix}$$

straining modes
rigid body modes

generalized coordinates

where a, b and c are the constant strains and A, B and C cause rigid body rotation. In more detail:

- a ~ constant direct strain in x direction
- b ~ constant direct strain in y direction
- c ~ constant shear strain
- A ~ rigid body clockwise rotation about the x, y origin
- B ~ rigid body translation in x direction
- C ~ rigid body translation in y direction

One concludes that the generalized coordinates and displacement functions used in this Turner triangle are equivalent to Argyris' natural modes.

Another way to see the rigid body modes inherent in an element formulation is by the [A] matrix, if it is available. A convenient coordinate system such as one at the centroid (Fig. 1) is helpful but not necessary.

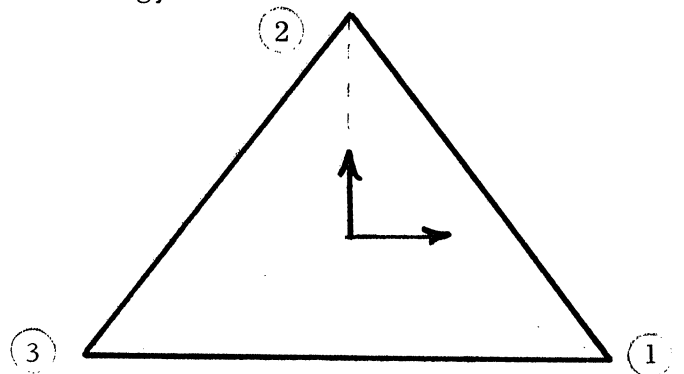
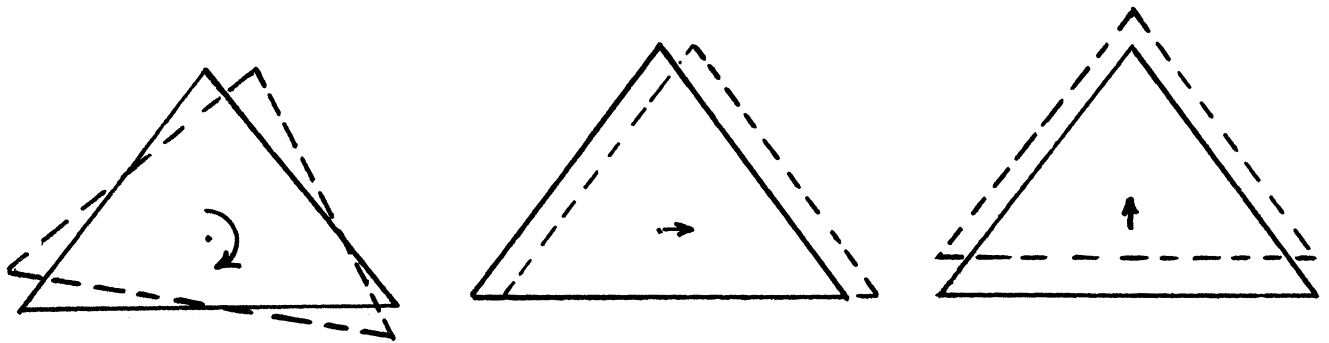


Fig. 1. Two-dimensional triangle.

$$[A] = \begin{bmatrix} \phi(x_1, y_1) \\ \phi(x_2, y_2) \\ \phi(x_3, y_3) \end{bmatrix} = \begin{bmatrix} x_1 & 0 & 0 & y_1 & 1 & 0 \\ 0 & y_1 & x_1 & -x_1 & 0 & 1 \\ x_2 & 0 & 0 & y_2 & 1 & 0 \\ 0 & y_2 & x_2 & -x_2 & 0 & 1 \\ x_3 & 0 & 0 & y_3 & 1 & 0 \\ 0 & y_3 & x_3 & -x_3 & 0 & 1 \end{bmatrix}$$

rigid body modes

The rotation is clockwise around the origin in this case.



(a) Rotation about centroid (b) Horizontal translation (c) Vertical translation.

Fig. 2. Rigid body modes for Turner triangle.

A square two-dimensional element is considered next. An elastic body in 2-D must have 3 rigid body modes as seen here:

$$\begin{Bmatrix} u(x, y) \\ v(x, y) \end{Bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & -y & \dots & \dots & \dots \\ 0 & 1 & x & \dots & \dots & \dots \end{bmatrix}}_{\text{rigid body modes}} \underbrace{\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_q \end{bmatrix}}_{\text{straining modes}}$$

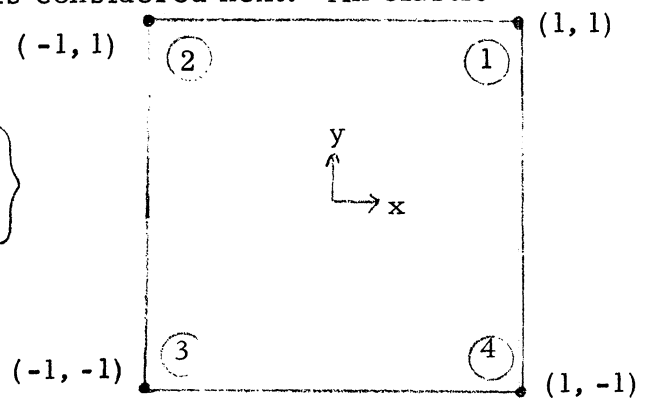


Fig. 3. Plane stress square.

The [A] matrix may be formed to yield

$$[A] \equiv \begin{bmatrix} \phi(x_1, y_1) \\ \text{-----} \\ \phi(x_2, y_2) \\ \text{-----} \\ \phi(x_3, y_3) \\ \text{-----} \\ \phi(x_4, y_4) \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & -1 & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & -1 & \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & 1 & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & -1 & \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & 1 & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & 1 & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

$\underbrace{\hspace{10em}}$
 rigid
body
modes

$\underbrace{\hspace{10em}}$
 straining
modes

Note that the choice of modes is not unique. One could choose rigid body translations in two arbitrary, different directions. The rotation could occur about any other axis in addition to the origin. The magnitude of the modes is also arbitrary, since the generalized coordinates α_j adjust to whatever displacement functions are used.

Reference: Desai and Abel, Section 6.5, 6.6

I. LOCAL VS. GLOBAL COORDINATES

Thus far, we have not distinguished between "local" and "global" coordinate systems, and indeed, they are often identical. There are times, however, when certain elements are located in a structure in skewed positions such that a single coordinate system for the entire system is impractical. One then prefers to have a separate "local" coordinate system for each element and a "global" coordinate system for the assembled structure. In Fig. 1, the 3 line elements shown have been imbedded in a 2-D space at various angles. Each line element is well understood in its own local system and might be easier to handle in a local system before assembling the stiffnesses into a global system.

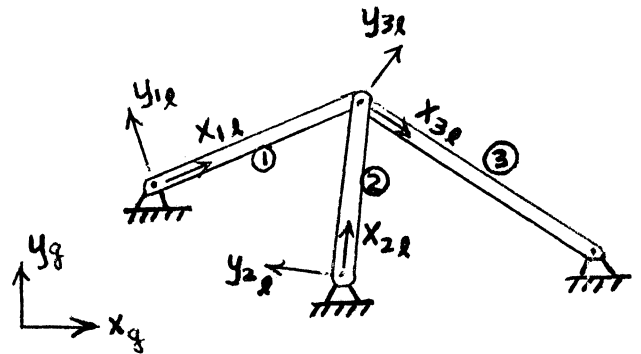


Fig. 1. Several line elements in a 3-D structure.

II. IMBEDDING A LINE ELEMENT INTO 2-D AND 3-D SPACE

To generalize the earlier solution of a line element, let us imbed the line element in a 2-D space. It then becomes a 2-D truss member.

$$\begin{Bmatrix} Q_{1l} \\ Q_{2l} \\ Q_{3l} \\ Q_{4l} \end{Bmatrix} = \begin{bmatrix} EA/L & 0 & -EA/L & 0 \\ 0 & 0 & 0 & 0 \\ -EA/L & 0 & EA/L & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} q_{1l} \\ q_{2l} \\ q_{3l} \\ q_{4l} \end{Bmatrix}$$

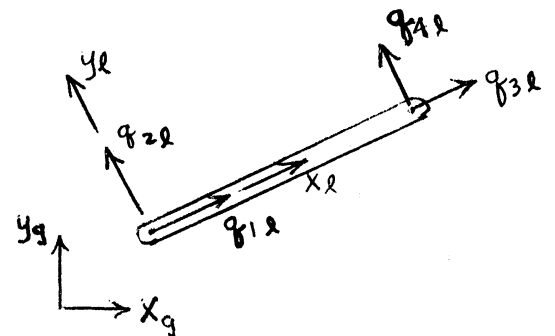


Fig. 2. Line element embedded in 2-D.

This step is particularly easy to do with an x_l axis lined up with the element axis - i.e., a local coordinate system.

One can also imbed the 1-D line element into a 3-D local coordinate system:

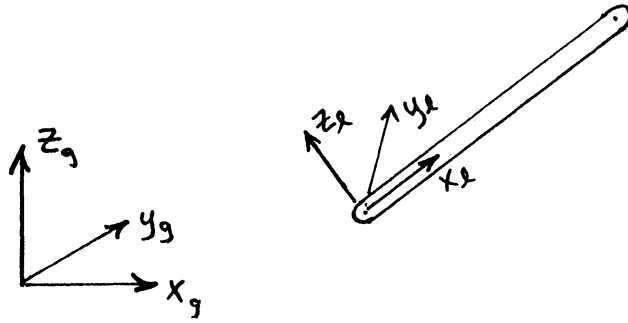


Fig. 3. Line element imbedded in 3-D.

$$\begin{Bmatrix} Q_{1l} \\ Q_{2l} \\ Q_{3l} \\ Q_{4l} \\ Q_{5l} \\ Q_{6l} \end{Bmatrix} = \begin{bmatrix} EA/L & 0 & 0 & -EA/L & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -EA/L & 0 & 0 & EA/L & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} q_{1l} \\ q_{2l} \\ q_{3l} \\ q_{4l} \\ q_{5l} \\ q_{6l} \end{Bmatrix} \quad (2)$$

One concludes that local coordinates systems can be very helpful in developing element stiffness matrices.

III. ABORTED ATTEMPT TO ASSEMBLE, USING LOCAL COORDINATES

If one tries to assemble a stiffness matrix for the two truss elements shown using local coordinates, the result is a disaster. The trouble is at the center node, where the stiffnesses lead to force components in 4 different directions. One can't sum stiffnesses easily. This can be solved by developing a transformation of coordinates for each element before attempting assembly.

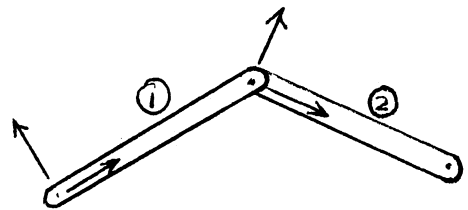


Fig. 4. Attempt to assemble in local coordinates.

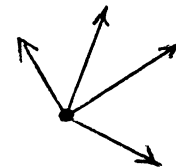


Fig. 5. Difficulty with coordinates at center node.

IV. DISPLACEMENT TRANSFORMATION AT A NODE. 2-D.

The approach to use in Fig. 4. is not to relate the displacements of element ① to those of element ②, but rather to relate both local systems to a single global one. Furthermore, the problem is to concentrate on coordinate rotations, since translations of coordinate systems don't affect stiffnesses, displacement or forces.

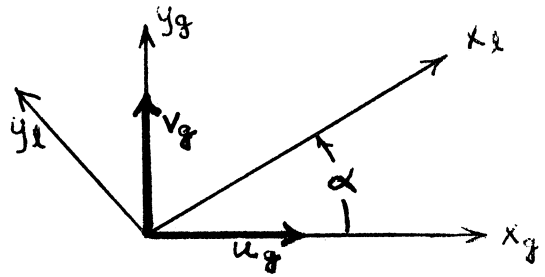


Fig. 6. Coordinate transformation.

Consider a centered displacement vector $\begin{Bmatrix} u \\ v \end{Bmatrix}$ with its description $\begin{Bmatrix} u_l \\ v_l \end{Bmatrix}$ in the local system and $\begin{Bmatrix} u_g \\ v_g \end{Bmatrix}$ in the global system. Imagine that the vector $\begin{Bmatrix} u_l \\ v_l \end{Bmatrix}$ has been found. By projecting these components on the local coordinate system

$$u_l = u_g \cos \alpha + v_g \sin \alpha$$

$$v_l = -u_g \sin \alpha + v_g \cos \alpha$$

one has:

$$\begin{Bmatrix} u_l \\ v_l \end{Bmatrix} = \underbrace{\begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix}}_{[t]} \begin{Bmatrix} u_g \\ v_g \end{Bmatrix}$$

where $[t]$ is a transformation matrix for the node in question. If this same process is repeated at each node in the element, one obtains:

$$\{q_l\} \equiv \begin{Bmatrix} u_{1l} \\ v_{1l} \\ \dots \\ u_{2l} \\ v_{2l} \\ \dots \\ \vdots \end{Bmatrix} = \underbrace{\begin{bmatrix} [t] & [0] & [0] & \cdot \\ \dots & \dots & \dots & \dots \\ [0] & [t] & [0] & \cdot \\ \dots & \dots & \dots & \dots \\ [0] & [0] & [t] & \cdot \\ \dots & \dots & \dots & \dots \end{bmatrix}}_{[T]} \begin{Bmatrix} q_g \end{Bmatrix}$$

i. e. ,

$$\{q_\ell\} = [T] \{q_g\}$$

Forces likewise transform:

$$\{Q_\ell\} = [T] \{Q_g\}$$

The equilibrium equation for a single element has already been derived in a form that holds either for local or global coordinates. In local coordinate form:

$$\{Q_\ell\} = [k_\ell] \{q_\ell\}$$

Then using the transformations above,

$$\begin{aligned} [T] \{Q_g\} &= [k_\ell] [T] \{q_g\} \\ \{Q_g\} &= [T]^{-1} [k_\ell] [T] \{q_g\} \end{aligned}$$

A curious feature of the rotation matrix $[T]$ is that $[T]^{-1} = [T]^T$. (you can check this out on $[T]$ for yourself.) This matrix $[T]$ is said to be "orthogonal." Hence

$$\{Q_g\} = [T] \underbrace{[k_\ell]}_{[k_g]} [T] \{q_g\}$$

Hence, at this point, we know how to transform displacements, forces and stiffnesses from local to global coordinates. Equivalent nodal loads will transform exactly as do the above concentrated nodal loads.

A common user procedure is to lay out a problem in terms of a global coordinate system. The typical computer program will, however, create stiffnesses for elements such as beams and membranes in local coordinate systems and then transform those stiffnesses to global coordinates, as shown above. Assembly is done in the global system.

V. SKEWED BOUNDARY CONDITIONS

An exception to the above procedure is for a skewed boundary condition, such as an inclined roller sketched here. Most general purpose computer programs will separate out the assembled equations for that node and will write them in terms of a local coordinate system aligned with the skewed support. The

boundary conditions are then uncoupled in these nodal degrees of freedom, and the assembled equations can be solved. Such skewed boundary conditions arise often because of symmetry in structures, such as in the pie-shaped wedge removed from the cartridge chamber sketched in Fig. 7.

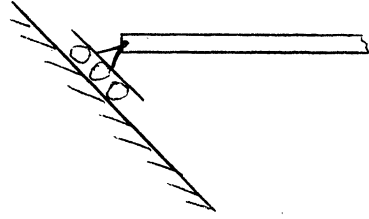


Fig. 6. Inclined support.

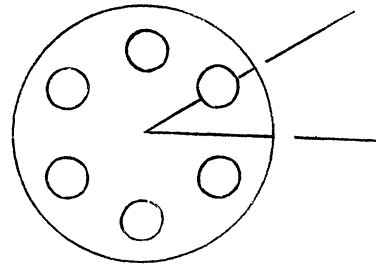


Fig. 7. Cartridge chamber.

VI. FINAL COMMENTS

Many students, upon working out their first sample problem on coordinate transformations, do it in the wrong order. In the common local-to-global transformation, you should develop a $[T]$ matrix for elements one at a time and find the global stiffness for the elements one at a time. Assembly is then carried out. For the case of the inclined support or a plane of symmetry one can work with the assembled global stiffness matrix and use global transformation matrices of the form:

$$[T_{\text{global}}] = \begin{bmatrix} [I] & 0 & 0 & 0 \\ 0 & [I] & 0 & 0 \\ 0 & 0 & [t] & 0 \\ 0 & 0 & 0 & [I] \end{bmatrix}$$

transformation for the particular node with skewed boundary conditions.

where the $[t]$ matrix in this case rotates the global coordinates at a node back to a preferred local system.

In general, then, use care as to whether transformations for a node, an element or assembled structure are required.

THERMAL STRESS

7 Nov. 1980

W. J. A.

- References: 1) Zienkiewicz, 3rd edition, pp. 96-98
2) Desai & Abel, pp. 216-217

I. Equivalent Nodal Load

We have shown in previous lectures that a prestrain $\{\epsilon_0\}$ causes an equivalent nodal load

$$\{Q\}_{e.n.l.} = \int_{vol} [B]^T [C] \{\epsilon_0\} dV . \quad (1)$$

This came as a result of the stress-strain law

$$\{\sigma\} = [C](\{\epsilon\} - \{\epsilon_0\}) + \{\sigma_0\} . \quad (2)$$

Let us concentrate on thermal strain. We will set $\{\sigma_0\} = 0$ and use the law

$$\{\sigma\} = [C](\{\epsilon\} - \{\epsilon_0\}) . \quad (3)$$

To develop feeling for the prestrain $\{\epsilon_0\}$, use a scalar example of a line element with left node fixed. Suppose the element is to be both loaded and heated. If the line element is initially unstressed at $q_2=0$, then the load-deflection and the stress-strain curve for zero temperature (reference level) would pass through the origin, as shown for $T=0$ in each figure. Once heated, however, a new load-deflection curve shown as $T = +100^\circ$ is used. One could interpret the new stress-strain curve as a prestress effect,

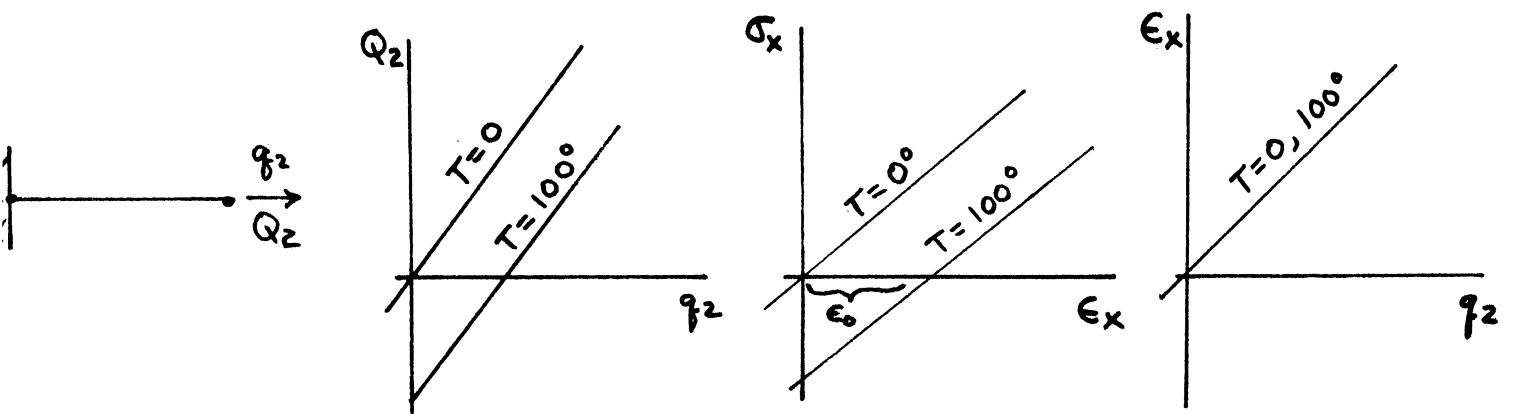


Fig. 1 Line Element

2a, b, c. Heated line element under load.

namely as the stress needed to hold the heated line element in place with zero strain, but we have decided not to use prestress. (Note that under that interpretation, a negative stress is needed to hold the element in undeformed shape.) One does, however, identify ϵ_0 as the amount of strain caused by the free thermal expansion with no stress applied. I personally like this approach because one can keep signs straight - the thermal strain is positive when associated with a positive temperature.

II. Thermal Strain in Various Situations

In a 3-D solid, if one has local coordinates aligned with principal material directions

$$\{\epsilon_0\} \equiv \begin{Bmatrix} \epsilon_{x_0} \\ \epsilon_{y_0} \\ \epsilon_{z_0} \\ \gamma_{xy_0} \\ \gamma_{yz_0} \\ \gamma_{zx_0} \end{Bmatrix} = \begin{Bmatrix} \alpha_1 \Delta T \\ \alpha_2 \Delta T \\ \alpha_3 \Delta T \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad (4)$$

where α_1 , α_2 and α_3 are coefficients of thermal expansion. These α_i may be functions of temperature and should account for all strain between the current temperature and the reference temperature (here taken as zero for convenience).

In a 2-D, plane stress case

$$\{\epsilon_0\} \equiv \begin{Bmatrix} \epsilon_{x_0} \\ \epsilon_{y_0} \\ \gamma_{xy_0} \end{Bmatrix} = \begin{Bmatrix} \alpha_1 \Delta T \\ \alpha_2 \Delta T \\ 0 \end{Bmatrix} \quad (5)$$

In a 2-D, plane strain case

$$\{\epsilon_0\} \equiv \begin{Bmatrix} \epsilon_{x_0} \\ \epsilon_{y_0} \\ \gamma_{xy_0} \end{Bmatrix} = (1+\nu) \begin{Bmatrix} \alpha_1 \Delta T \\ \alpha_2 \Delta T \\ 0 \end{Bmatrix} \quad (6)$$

If an element has orthotropic material properties, but the coordinate system used, say (\bar{x}, \bar{y}) , does not lie in principal directions, one has for plane stress:

$$\{\epsilon_o\} \equiv \begin{Bmatrix} \epsilon_{\bar{x}} \\ \epsilon_{\bar{y}} \\ \gamma_{\bar{x}\bar{y}} \end{Bmatrix} = \begin{Bmatrix} \bar{\alpha}_1 \Delta T \\ \bar{\alpha}_2 \Delta T \\ \bar{\alpha}_3 \Delta T \end{Bmatrix} \quad (7)$$

If a material has a temperature dependent coefficient of expansion, then, in each of the expressions above, one replaces the simple $\alpha \Delta T$ with

$$\int_{T_o}^T \alpha_i(T) dT,$$

where T_o is a reference temperature.

III. Example: Two-Node Line Element

Consider an aluminum link 10" long. Model the link as a two-node line element. Properties are

$$E = 10^7 \text{ psi}$$

$$\nu = 0.3$$

$$\alpha = 1.23 \times 10^{-5} \frac{\text{in}}{\text{in.}^\circ\text{F.}}$$

$$A = 1 \text{ in}^2$$

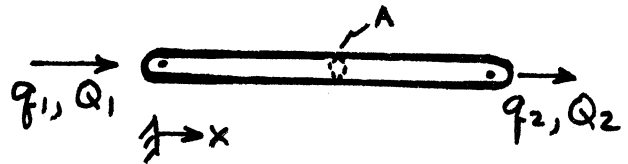


Fig. 3. Aluminum link.

The link is to be heated at 100 °F. and then subjected to various boundary conditions and loading situations.

The stress-strain law becomes

$$\begin{aligned} \{\sigma_x\} &= [E](\{\epsilon\} - \{\epsilon_o\}) \\ (8) \quad &= [10^7 \text{ psi}](\{\epsilon\} - \{1.23 \times 10^{-3}\}) \end{aligned}$$

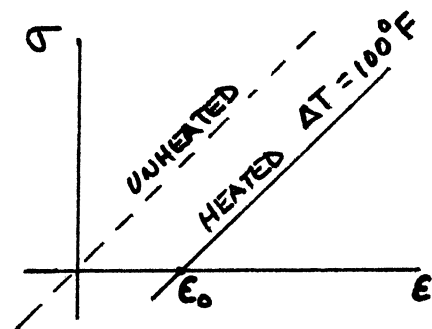


Fig. 4. Stress-strain curves for heated and unheated case.

The equations of equilibrium are

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} + \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix}_{\substack{\text{e. n. l.} \\ \epsilon_o}} = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} \quad (9)$$

where

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix}_{\substack{\text{e. n. l.} \\ \epsilon_o}} = \int_{\text{vol}} [B]^T [C] \{\epsilon_o\} dV \quad (10)$$

$$= \int_0^L \begin{bmatrix} -1/L \\ 1/L \end{bmatrix} [E] \{\alpha \Delta T\} A dx$$

$$= \begin{Bmatrix} -EA\alpha \Delta T \\ EA\alpha \Delta T \end{Bmatrix}$$

$$= \begin{Bmatrix} -12,300 \text{ lb} \\ 12,300 \text{ lb} \end{Bmatrix} \quad (11)$$

We have, prior to specifying loads and boundary conditions,

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} + \begin{Bmatrix} -12,300 \\ 12,300 \end{Bmatrix} = (10^6 \frac{\text{lb}}{\text{in}}) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} \quad (12)$$

The sketch of load vs. deflection is given. The initial loads (equivalent nodal loads due to initial strain) are absorbed into $\{Q\}_{TOTAL}$.

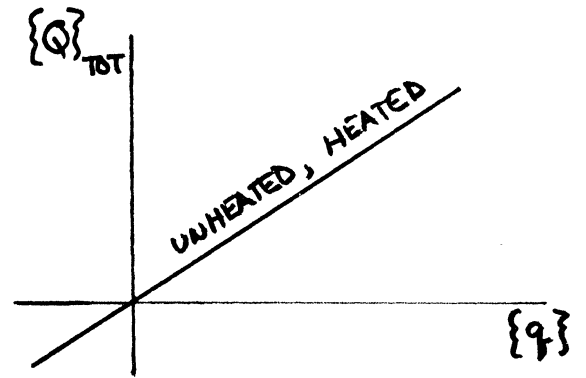


Fig. 5. Combined loading.

A. Constrained Element

If the nodes are constrained, the element is held in place by the external concentrated forces.

Solve Eqn. 12.

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} + \begin{Bmatrix} -12,300 \\ 12,300 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Hence

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \begin{Bmatrix} 12,300 \\ -12,300 \end{Bmatrix} \text{ lb.}$$

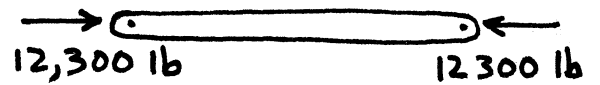


Fig. 6. Loads on constrained line element.

The points on the stress strain diagram and the load deflection diagram are given. Point I on the load deflection diagram is unmotivated and is probably irrelevant.

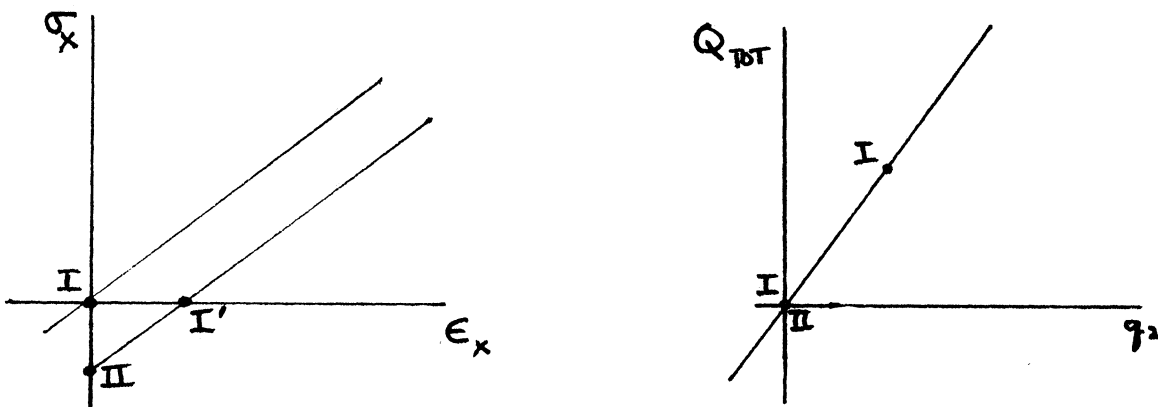


Fig. 7. Stress-Strain and Load-Displacement for Constrained Line Element.

B. Freely Expanding Element

If the same line element as above is heated 100°F . and allowed to expand freely, find the resulting displacements. The external nodal forces are zero. Set $q_1 = 0$ to remove the rigid body mode. Eqn. 12 gives

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} -12,300 \\ +12,300 \end{Bmatrix} \text{ lb.} = (10^6 \text{ psi}) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

From the second equation,

$$q_2 = \underline{0.0123 \text{ in.}}$$

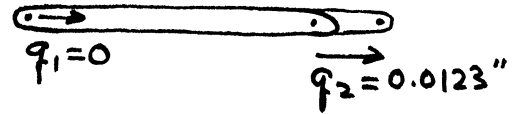


Fig. 8. Free expansion of aluminum link.

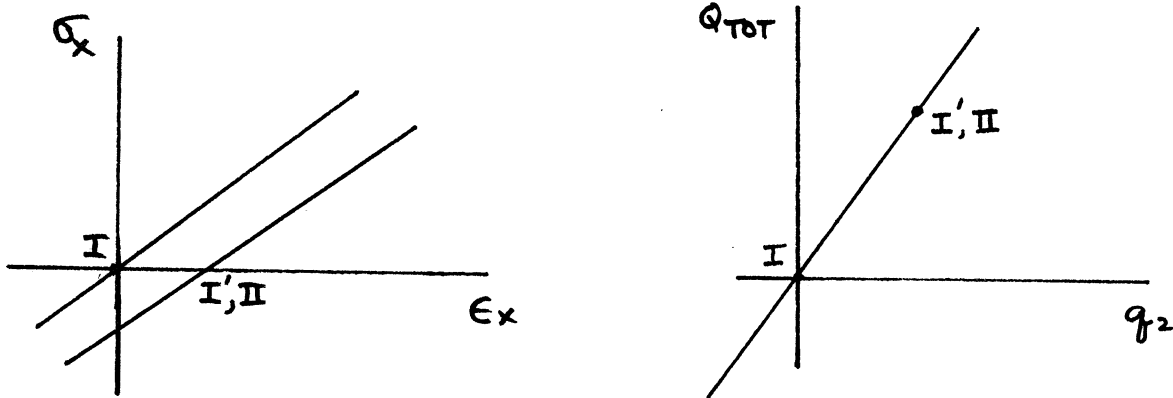


Fig. 9 Stress-Strain and Load-Displacement for Freely Expanding Line Element

ELECTRICAL & FLUID NETWORKS

- References: 1) Zienkiewicz, "The Finite Element Method", Ed 2, 1971, Section 1.5.
2) Murphy, Glenn, "Mechanics of Fluids", Ed, Int. Textbook Co., 1952.

I. Electrical Network

Consider a network of resistors subjected to a.c. or d.c. voltage. (We will not include capacitors or inductors because that requires complex arithmetic.) There is an analogy between the electrical and the mechanical cases:

MECHANICAL	ELECTRICAL
force at a node due to a certain member	= current
displacement at a node, common to all members	= voltage
stiffness of a member	= conductance ($1/R$)

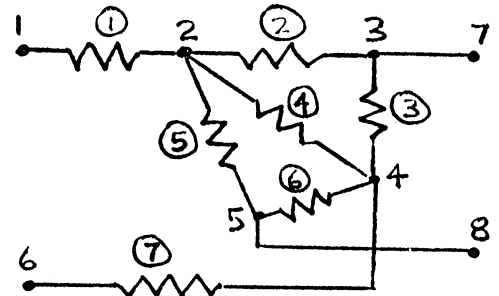


Fig. 1. Electrical network

For the example shown, nodes are 1, 2, ... 8 and elements are 1, 2, ... 7.

Consider a single element with nodes i and j . Define voltage and current at a node, with current positive for flow into the element. (Note a fundamental difference between current in a resistor and force in a 1-D structure, in regard to sign convention.)

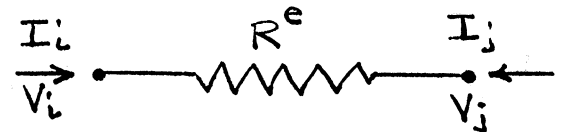


Fig. 2. Resistive element

By continuity:

$$I_j = -I_i \tag{1}$$

By Ohm's law:

$$I_i = \frac{V_i - V_j}{R^e} \tag{2}$$

$$I_j = \frac{V_j - V_i}{R^e} \tag{3}$$

In matrix form:

$$\begin{Bmatrix} I_i \\ I_j \end{Bmatrix} = \frac{1}{R^e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} V_i \\ V_j \end{Bmatrix} \tag{4}$$

In summation form:

$$I_p^e = \sum_{q=i,j} k_{pq}^e V_q \quad (p=i,j) \tag{5}$$

↖
admittance matrix term

Assembly is done by looking at a node and using Kirchhoff's current law, (current at a node) = 0. We do a slight departure by labeling the external current as positive into the node and the internal element current positive away from the node (into the element):

$$\mathcal{I}_i = \sum_e \sum_q \underbrace{k_{ij}^e}_{\substack{\text{current away from node } i \\ \text{toward node } j \text{ through} \\ \text{element } e}} V_q \tag{6}$$

↖ external current into node "i"

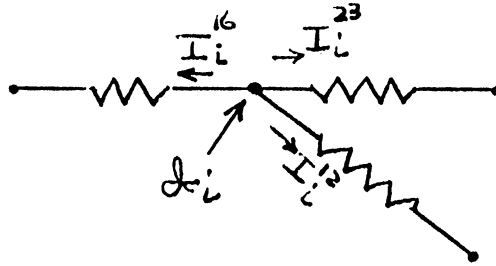


Fig. 3 Kirchhoff's current law

This law is just like the case of a truss assembly. The assembled form is easy to see because of the scalar nature of current flow.

Consider the example shown in Fig. 4. One can do this either from Eq. (6) or by inspection using the topology of the network.

$$\begin{Bmatrix} \mathcal{I}_1 \\ \mathcal{I}_2 \\ \mathcal{I}_3 \\ \mathcal{I}_4 \end{Bmatrix} = \begin{bmatrix} 1/R^1 & -1/R^1 & 0 & 0 \\ -1/R^1 & (1/R^1 + 1/R^2 + 1/R^3) & -1/R^2 & -1/R^3 \\ 0 & -1/R^2 & 1/R^2 & 0 \\ 0 & -1/R^3 & 0 & 1/R^3 \end{bmatrix} \begin{Bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{Bmatrix} \tag{7}$$

The admittances "add" into the stiffness matrix in the same way as stiffnesses do.

The \mathcal{I}_i are the external currents fed in at each node. The internal currents do not appear in this matrix formulation, although they can be recovered, if desired.

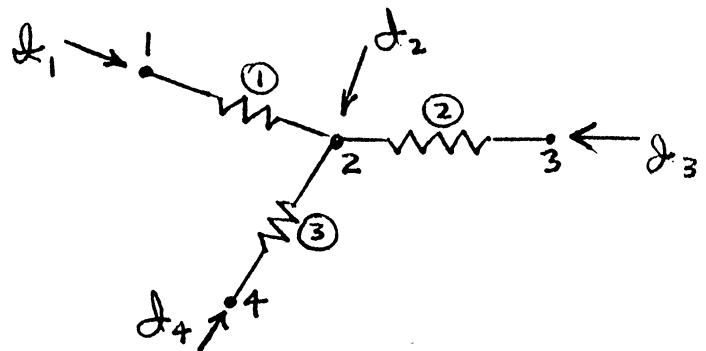


Fig. 4. Example assembly

II. Fluid Network - Pipe Flow

Consider the pipe element in Fig. 5. Assume volume flow is proportional to pressure drop across the pipe, and assume volume flow is inversely proportional to a friction coefficient f .

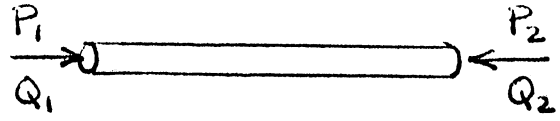


Fig. 5 Pipe element

$$Q_1 = \frac{P_1 - P_2}{f^e} \quad (8)$$

\uparrow
 friction coeff.

Also

$$Q_2 = \frac{P_2 - P_1}{f^e} \quad (9)$$

In matrix form:

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \frac{1}{f^e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} \quad (10)$$

The sign convention on pressure and volume flow agree. Both tend to cause flow into the pipe. Again, as in the electrical network, this problem has only limited vector character. It is inherently scalar (1-D) and is not a degenerate 2-D or 3-D problem. The pipe can be bent as in Fig. 6 or even tied in a knot!

In general, the friction coefficient f is a function of volume flow, but we will at first consider the case where f is constant (slow, viscous flow).



Fig. 6 Bent pipe

Assembly

Use an assembly process based on imbedding each element equation into an expanded matrix the full size of the assembly. In Fig. 7, by continuity, at the junction:

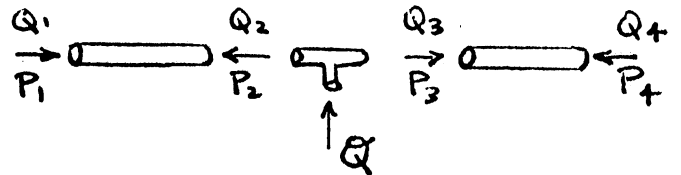


Fig. 7 Pipe assembly

$$Q = Q_2 + Q_3$$

(11)

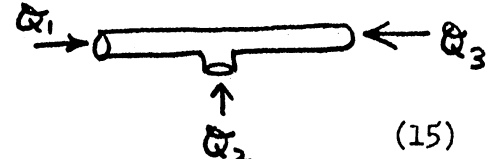
For the separate pipes:

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \begin{bmatrix} 1/f' & -1/f' \\ -1/f' & 1/f' \end{bmatrix} \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} \quad \begin{Bmatrix} Q_3 \\ Q_4 \end{Bmatrix} = \begin{bmatrix} 1/f^2 & -1/f^2 \\ -1/f^2 & 1/f^2 \end{bmatrix} \begin{Bmatrix} P_3 \\ P_4 \end{Bmatrix} \quad (12,13)$$

Imbedding these equations in a 3x3 matrix form, and setting $P_3 = P_2$:

$$\begin{Bmatrix} Q_1 \\ Q_2 + Q_3 \\ Q_4 \end{Bmatrix} = \begin{bmatrix} 1/f' & -1/f' & 0 \\ -1/f' & (1/f' + 1/f^2) & -1/f^2 \\ 0 & -1/f^2 & 1/f^2 \end{bmatrix} \begin{Bmatrix} P_1 \\ P_2 \\ P_4 \end{Bmatrix} \quad (14)$$

We can now identify each of the volume flow terms on the left as the external flow into the node

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Bmatrix} = \begin{bmatrix} \text{(as above)} \end{bmatrix} \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix} \quad (15)$$


external fluid sources only assembled "flow" matrix nodal pressures

Fig. 8. Assembled system. (Nodes renumbered)

The script notation is intended to signify an assembled system, in contrast to capital letters used for single elements.

III. Nonlinear Systems

Actual pipe flow problems are nonlinear in the sense that the pressure drop is approximately proportional to the square of the discharge in turbulent flow, rather than proportional to the first power. The expression for head loss H_f due to friction is

$$H_f = f \left(\frac{l}{d} \right) \frac{V^2}{2g} \quad (16)$$

where f is a friction coefficient, l/d is the pipe length to diameter ratio, V is the flow velocity and g is the acceleration of gravity. The friction coefficient has a mild dependence on flow velocity, as shown in standard Stanton diagrams (Ref. 2). The result is that pressure drop is proportional to velocity slightly less than squared, perhaps to the 1.8 power.

In finite element theory, one casts nonlinear problems into quasi-linear ones, such as by absorbing the nonlinearity into the "stiffness" matrix. A typical relation between pressure drop and discharge might be

$$\Delta P = \underbrace{\bar{f}}_{\text{constant}} Q^r \quad (17)$$

for instance. One would factor out Q as shown

$$\Delta P = (\bar{f} |Q|^{r-1}) Q \quad (18)$$

The previous expression for flow in a pipe becomes

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \begin{bmatrix} \frac{1}{\bar{f}^e |Q^e|^{r-1}} & -\frac{1}{\bar{f}^e |Q^e|^{r-1}} \\ -\frac{1}{\bar{f}^e |Q^e|^{r-1}} & \frac{1}{\bar{f}^e |Q^e|^{r-1}} \end{bmatrix} \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} \quad (19)$$

One now views the flow matrix as a function of volume flow, i.e.

$$[k^e] = [k^e(Q^e)] \quad (20)$$

Because the flow matrix for an element depends on the volume flow through the element, the problem is nonlinear. For an assembled system:

$$\{Q\} = [K(Q)] \{P\} \quad (21)$$

For such nonlinear problems, one typically uses an iteration solution. One type of iteration requires an initial guess for flow rates. The system flow matrix is then calculated, and the set of coupled, linear algebraic equations is solved for unknown Q_i and P_i . The process is repeated until convergence to proper flow rates and pressures is achieved. This is called a secant stiffness approach and is one of the oldest ways to solve nonlinear problems. The method is sometimes useful for smaller problems, but often does not converge.

IV. Example of Nonlinear System

Suppose pipe flow in the system of two pipes in Fig. 7, Section II follows a law $\Delta P = \bar{F} Q^{1.8}$. One can show after some discussion that the assembled equations should be:


```

1      C          PIPE
2      C          THIS PROGRAM HAS BEEN CREATED FOR AERO 510
3      C
4      C          IT IS MEANT TO ILLUSTRATE A VARIABLE STIFFNESS
5      C          METHOD, USING FLOW THROUGH PIPES AS AN EXAMPLE.
6      C
7          DIMENSION Q(3,100)
8          REAL K(3,3)
9          READ (5,1) F1,F2,P1,P2,P3,E,ERROR,Q1,Q2,Q3,ITER
10         1 FORMAT (8F10.0/2F10.0,I5)
11         WRITE (6,4) F1,F2,P1,P2,P3,E,ERROR,Q1,Q2,Q3,ITER
12         4 FORMAT (10(F14.6/),I5)
13         I=0
14         2 I=I+1
15         K(1,1)=1./F1/(ABS(Q1)**E)
16         K(3,3)=1./F2/(ABS(Q3)**E)
17         K(1,2)=-K(1,1)
18         K(1,3)=0.
19         K(2,1)=K(1,2)
20         K(2,2)=K(1,1)+K(3,3)
21         K(2,3)=-K(3,3)
22         K(3,1)=0.
23         K(3,2)=-K(3,3)
24         Q1=K(1,1)*P1+K(1,2)*P2
25         Q2=K(2,1)*P1+K(2,2)*P2+K(2,3)*P3
26         Q3=K(3,2)*P2+K(3,3)*P3
27         Q(1,I)=Q1
28         Q(2,I)=Q2
29         Q(3,I)=Q3
30         WRITE (6,3) I,Q1,Q2,Q3
31         3 FORMAT (' THE',I5,' TH ITERATION GIVES A FLOW VOLUMEVECTOR'/
32         1 F14.4/F14.4/F14.4)
33         IF (I.EQ.1) GO TO 999
34         ILAST=I-1
35         ERR=ABS(Q(1,I)-Q(1,ILAST))+ABS(Q(2,I)-Q(2,ILAST))+ABS(
36         1 Q(3,I)-Q(3,ILAST))
37         IF (ERR.LE.ERROR) GO TO 1000
38         999 IF (I.GE.ITER) GO TO 1000
39         GO TO 2
40     1000 CONTINUE
41         END

```


DYNAMICS

I. Reversed Effective Forces

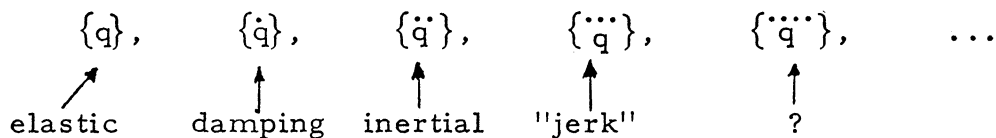
Newton said: $\{Q\} = [m]\{\ddot{q}\}$ and got great fame.

D' Alembert said: $\{Q\} - [m]\{\dot{q}\} = 0$ and also got fame, fortune, etc.

Moral: Look at things in a different way! You can treat dynamics problems as if they were static by including inertial forces as reversed effective forces. For lumped systems where finite dimensions are involved in equilibrium processes, reversed effective moments are also generated and must be applied about centers of gravity.

II. Consistent Mass and Damping Matrices

Generally speaking, one can imagine internal forces being generated in a structure due to increasing orders of derivatives of the internal displacement field



We have discussed elastic forces extensively up to this point. Damping and inertial forces are also familiar. No significant forces are generated by higher derivatives of the displacement field, although the third derivative causes physiological discomfort and is a factor in ride quality of vehicles.

Let us develop equivalent nodal loads for damping and inertial forces in the identical way as for other body forces:

$$\{\bar{X}\} = \{\bar{X}\}_{\text{externally caused}} + \{\bar{X}\}_{\text{damping}} + \{\bar{X}\}_{\text{inertia}}$$

Then

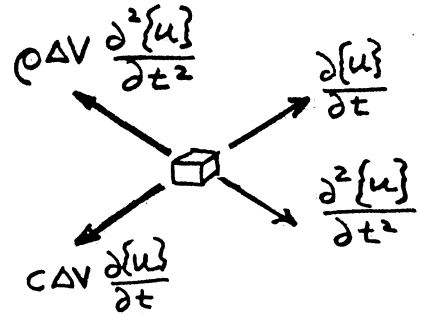
$$\{Q\}_{\text{e.n.l.}} = \int_{\text{Vol}} [N]^T \{\bar{X}\} d\text{Vol}$$

A commonly used, but often artificial, type of damping is "viscous" damping in which a body force tends to oppose the instantaneous velocity of the particles:

$$\{\bar{X}\}_{\text{damping}} = -c \frac{\partial}{\partial t} \{u\}$$

The inertial force is universally acclaimed as:

$$\{\bar{X}\}_{\text{inertial}} = -\rho \frac{\partial^2}{\partial t^2} \{u\}$$



Equivalent nodal loads for these effects are:

$$\{Q\}_{\text{e.n.l.}} = \int_V [N]^T \left(-c \frac{\partial}{\partial t} \{u\} \right) dV + \int_V [N]^T \left(-\rho \frac{\partial^2}{\partial t^2} \{u\} \right) dV$$

Discretization with shape functions

$$\{u\} = [N]\{q\} \equiv [N(x, y, z)]\{q(t)\}$$

where

$$\frac{\partial}{\partial t} \{u\} = [N]\{\dot{q}\}$$

$$\frac{\partial^2}{\partial t^2} \{u\} = [N]\{\ddot{q}\}$$

leads to

$$\{Q\}_{\text{e.n.l.}} = - \underbrace{\int_V c [N]^T [N] dV}_{[C]} \{\dot{q}\} - \underbrace{\int_V \rho [N]^T [N] dV}_{[m]} \{\ddot{q}\}$$

Hence the equation of motion becomes:

$$[K]\{q\} = \{Q\} - [C]\{\dot{q}\} - [m]\{\ddot{q}\}$$

In standard form, one writes:

$$[m]\{\ddot{q}\} + [C]\{\dot{q}\} + [K]\{q\} = \{Q\}$$

The finite element theory is an excellent method for discretizing a field problem and reducing it to this standard form. Methods for solving this equation of motion are universal in the sense that other discretization methods such as lumped mass methods and finite differencing also produce this standard form. In this respect, the methods to follow, such as the normal mode method, apply equally well to problems generated in other ways.

The method of developing mass and damping matrices above is based on virtual work. It gives matrices which depend upon the type of motion assumed through the shape functions, and hence the matrices are consistent with the displacement field. These are called "consistent" damping and mass matrices.

III. Example of a Consistent Mass Matrix --- Beam

Evaluation of the consistent mass matrix for a beam is not trivial, because the shape function is intricate. From earlier notes (29 Sept. 1975), the shape function for a 2-node, Euler-Bernoulli beam is:

$$[N] = \frac{1}{L^3} [L^3 - 3Lx^2 + 2x^3, L^3x - 2L^2x^2 + Lx^3, 3Lx^2 - 2x^3, -L^2x^2 + Lx^3]$$

If the mass density of the beam is constant over its volume, the consistent mass matrix is found after much manipulation to be:

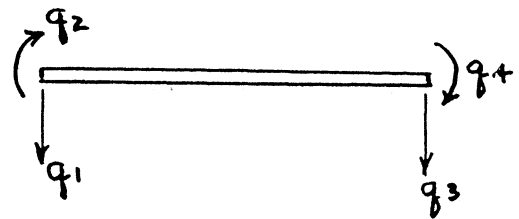
$$[m] = \frac{\rho AL}{420} \begin{bmatrix} 156 & 22L & 54 & -13L \\ & 4L^2 & 13L & -3L^2 \\ & & 156 & -22L \\ \text{(Symm.)} & & & 4L^2 \end{bmatrix}$$

(See Przemieniecki, Theory of Matrix Structural Analysis, page 297)

IV. Lumped Mass Matrix

An old-fashioned but effective way to calculate the mass matrix in a discretized problem is to divide the structure into subdomains surrounding each degree-of-freedom (d. o. f.) and arbitrarily assign a portion (lump) of mass to that degree of freedom. This makes sense with regard to translational degrees of freedom but is far less intuitive with rotations.

A good example of a lumped mass matrix is for the Euler-Bernoulli beam, where half the mass of the beam is concentrated at each translational node and there is assigned no rotational inertia at all. The latter assumption has some basis in the fact that rotational inertia is already neglected on a local basis in this beam theory, but is really not justified for finite length beams.



The result is:

$$[m] = \frac{\rho AL}{2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

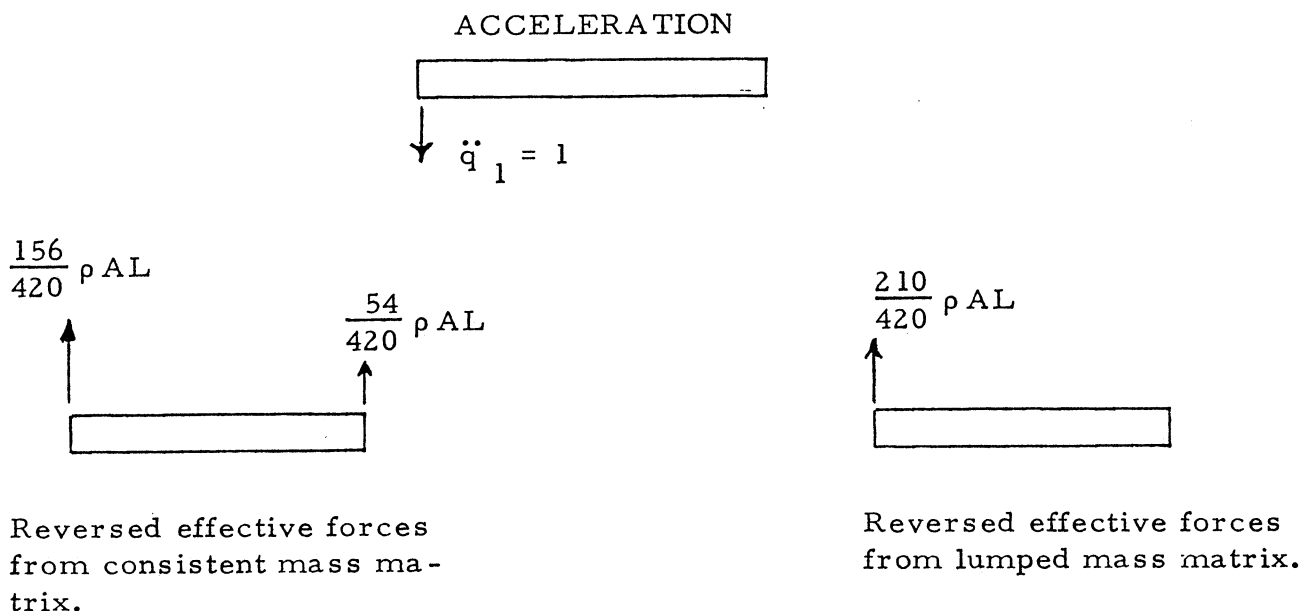
V. Comparison of Consistent and Lumped Mass Matrices

The consistent mass matrix is a much richer source of inertial effects than the lumped mass matrix. Each nodal displacement and rotation causes inertial effects (reactions) at all other degrees of freedom when the consistent mass matrix is used. One limiting case of comparison is useful: $L \rightarrow 0$. One might suspect that as the length of the beam element becomes infinitesimal, the distinction between the consistent and lumped masses might disappear. This is not so. Taking the consistent mass matrix, and letting $L \rightarrow 0$, and discarding terms of

order (L^2) compared to those of order (L):

$$\lim_{L \rightarrow 0} [m]_{\text{consistent}} = \frac{\rho AL}{420} \begin{bmatrix} 156 & 0 & 54 & 0 \\ 0 & 0 & 0 & 0 \\ 54 & 0 & 156 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This shows the coupling that remains between displacements at the two ends of the beam, even in the limit of small length. As a specific example of this behavior, consider a positive, unit acceleration in the first nodal coordinate. The reversed effective forces, as converted to equivalent nodal loads, are shown:



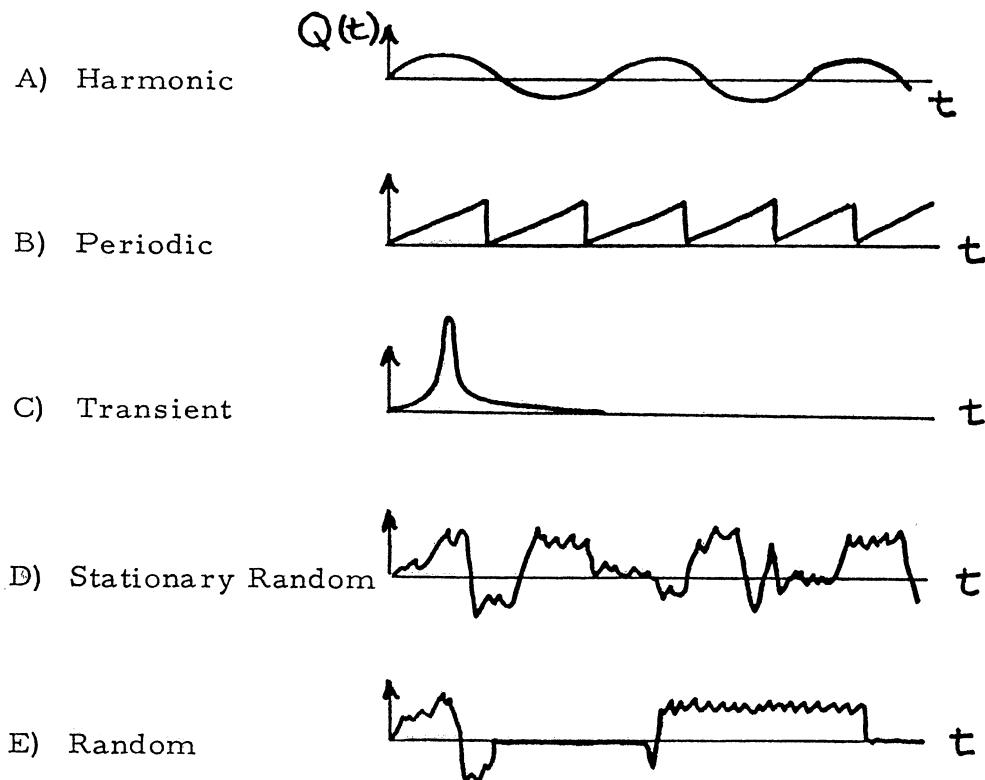
VI. Types of Vibration Problems

Vibration problems are divided into two major classes, free and forced.

1. Free Vibration. No external forces act on the body. The motion of the elastic body is due only to a balance of internal forces, including elastic, damping and inertial effects. Mathematically, this is an eigenvalue problem, and reveals much of the inner character of an elastic system. If the system has no damping, or if it is neglected, one often speaks of

"natural vibration" in the so-called natural frequencies and modes. These quantities are often used as building blocks for understanding the next category (forced vibration) and so studies of this idealized system are very important.

2. Forced Vibration. In a linear system, the character of the force input largely determines the character of the response. Proceeding from the simplest to the most difficult:



We will spend most of our time on free vibration and the response of linear systems to harmonic or general forcing. Often the goal of the forced vibration solution is to reduce the problem to one of solving uncoupled ordinary differential equations with constant coefficients.

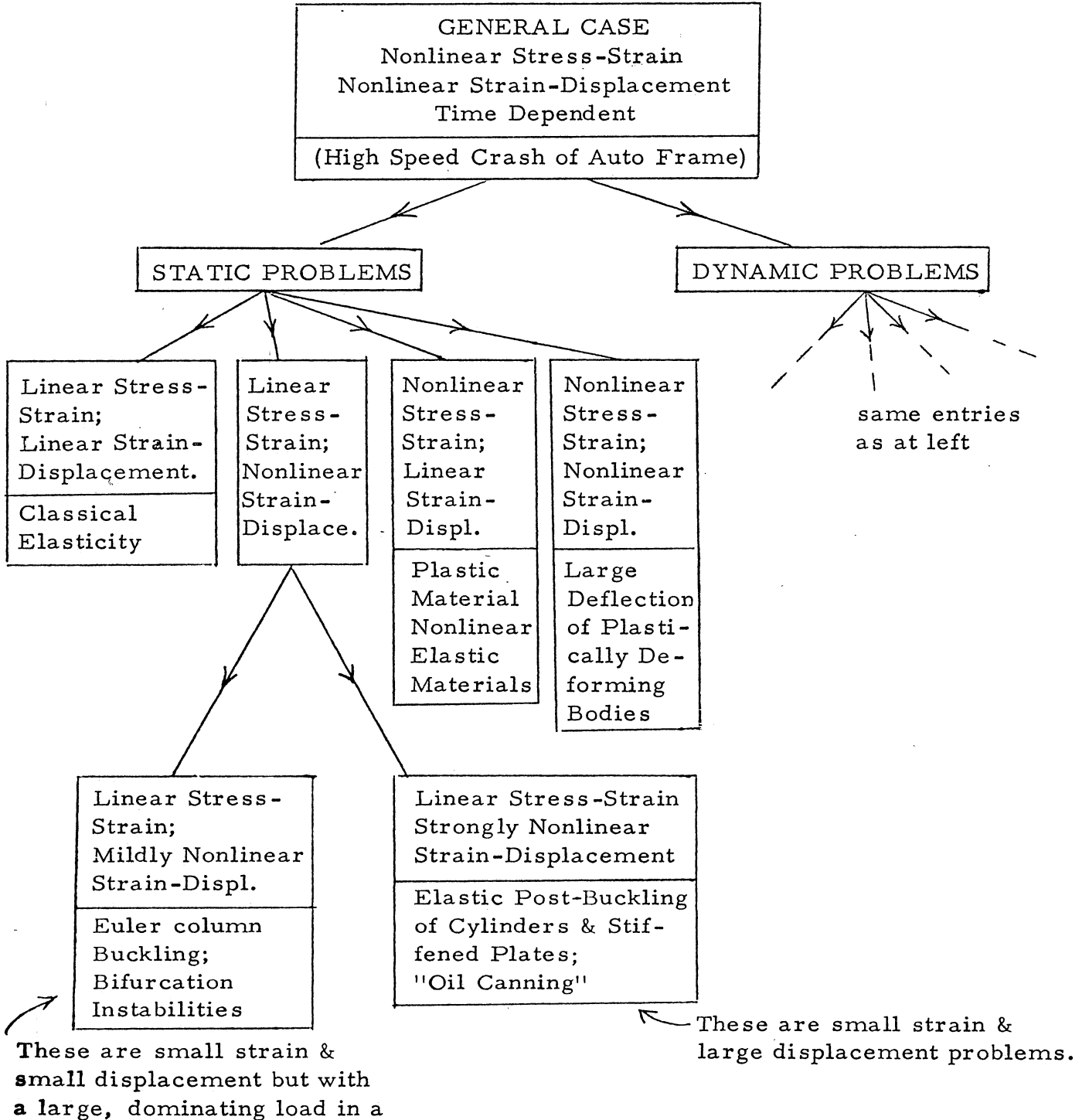
W. J. A.

Categorization of Nonlinear Problems

11 January 1977

References: Section 7.3 in Desai & Abel

Nonlinearities enter solid mechanics through the constitutive law (stress-strain) and through the strain-displacement law. Nonlinearities destroy the uniqueness of solutions in elasticity and allow multiple solutions, bifurcations and jump instabilities. Graphically,



Reference: Desai & Abel, "Introduction to the Finite Element Method,"
Sections 5.4, 5.5

I. INTERPOLATION

Reference: Hamming, "Numerical Methods for Scientists and Engineers,"
pp. 227-250.

Interpolation is the basic feature of many engineering methods, including finite element methods. Let us concentrate on a scalar function of one independent variable, $f(x)$, as sketched. A linear interpolation of $f(x)$ for points lying between x_1 and x_2 can be based on an average slope

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

to get

$$f(x) = f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1} (x - x_1)$$

By rearranging terms, we get

$$f(x) = f(x_1) \left[1 - \frac{x - x_1}{x_2 - x_1} \right] + f(x_2) \left(\frac{x - x_1}{x_2 - x_1} \right)$$

and

$$f(x) = f(x_1) \left(\frac{x_2 - x}{x_2 - x_1} \right) + f(x_2) \left(\frac{x - x_1}{x_2 - x_1} \right)$$

This particular formula samples the function at two points (the endpoints) and interpolates by multiplying the weighting functions which account for "how close" x is to x_1 and x_2 . We have already done this sort of linear interpolation in our constant strain line element and the Turner (constant strain) triangle, where for the line element

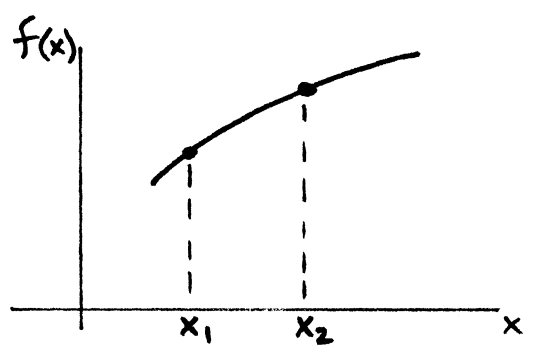


Fig. 1. General function.

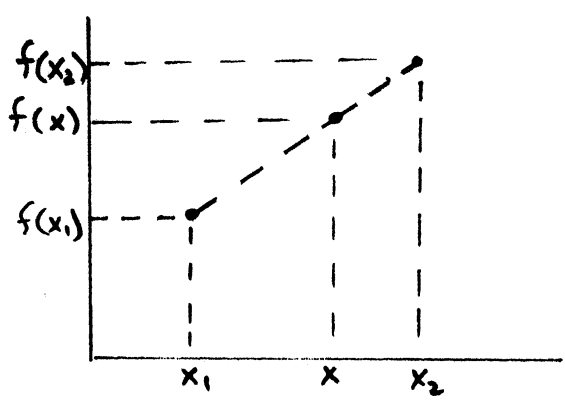


Fig. 2. Linear interpolation using function values only.

$$u(x) = q_1(1 - x/L) + q_2(x/L) \quad (4)$$

in local coordinates, for example.

Hamming suggests that interpolation uses information about the function $f(x)$ for interpolation in the form of:

- 1) function values only
- 2) values of derivatives
- 3) differences of function values
- 4) arbitrarily placed samples.

Where functional values only are considered, the finite element expression previously used for the displacement field is adequate:

$$\begin{aligned} u(x) &= \sum N_i(x)q_i \\ \{u\} &= [N(x)]\{q\} \\ 1 & \quad 1 \times n \quad n \end{aligned} \quad (5)$$

and can be used for higher order approximation, such as with quadratic shape functions (interpolation functions).

II. NATURAL COORDINATE SYSTEMS: LINE ELEMENT.

Reference: Huebner, "The Finite Element Method for Engineers," pp. 134-150.

For a line element of length ℓ , lying on (x_1, x_2) , define

$$\ell_1 = x_2 - x \quad L_1 = \frac{x_2 - x}{x_2 - x_1} \quad (6)$$

$$\ell_2 = x - x_1 \quad L_2 = \frac{x - x_1}{x_2 - x_1}$$

The normalized coordinates L_1 and L_2 satisfy:

$$L_1 + L_2 = 1 \quad (7)$$

Note that L_1 is unity when x is at the left node ($x=x_1$) and L_2 is unity when x is at the right node $x=x_2$. This means that the "natural coordinates" L_1 and L_2 are themselves interpolation functions (like $N_1(x)$ and $N_2(x)$ used previously) and can be used to express the internal displacement field:

$$u(x) = q_1 L_1 + q_2 L_2 \quad (8)$$

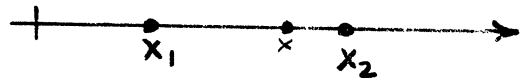


Fig. 3. One dimensional domain.

On the other hand, L_1 and L_2 define the Cartesian coordinate x in a similar interpolation:

$$x = x_1 L_1 + x_2 L_2 \quad (9)$$

This can be proven:

$$\begin{aligned} x &\stackrel{?}{=} x_1 \left(\frac{x_2 - x}{x_2 - x_1} \right) + x_2 \left(\frac{x - x_1}{x_2 - x_1} \right) \\ &\stackrel{?}{=} \frac{x_1 x_2 - x_1 x + x_2 x - x_2 x_1}{x_2 - x_1} \\ &\stackrel{?}{=} \frac{x(x_2 - x_1)}{(x_2 - x_1)} \\ &= x \quad \text{This checks!} \end{aligned}$$

The fact that the natural coordinates L_i can be used to interpolate both the dependent and the independent physical variables is important. In higher dimensional spaces, this allows curved boundary elements (and not just straight) to go with higher order displacement polynomials (not just linear) in isoparametric element theory.

Now consider the mapping between x and the L_i . Put the relations

$$x = \text{fcn}(L_1, L_2) \quad \text{and} \quad L_1, L_2 = \text{fcns}(x)$$

into matrix form. Equations 6 yields, directly from the definition of the natural coordinates:

$$\begin{Bmatrix} L_1 \\ L_2 \end{Bmatrix} = \begin{bmatrix} x_2/\ell & -1/\ell \\ -x_1/\ell & 1/\ell \end{bmatrix} \begin{Bmatrix} 1 \\ x \end{Bmatrix} \quad (10)$$

and Equations 8 and 9 give:

$$\begin{Bmatrix} 1 \\ x \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ x_1 & x_2 \end{bmatrix} \begin{Bmatrix} L_1 \\ L_2 \end{Bmatrix} \quad (11)$$

Equations 10 and 11 form an unlikely transformation "pair." There is only one Cartesian coordinate x and really only one independent L_i . The relations could be simplified to single equations if desired, but that is not helpful for computations.

The chain rule for differentiation is:

$$\frac{d}{dx} = \frac{\partial}{\partial L_1} \frac{dL_1}{dx} + \frac{\partial}{\partial L_2} \frac{dL_2}{dx} \quad (12)$$

and using Equation 7 for the derivatives, w.r.t. x :

$$\frac{d}{dx} = -\frac{1}{\ell} \frac{\partial}{\partial L_1} + \frac{1}{\ell} \frac{\partial}{\partial L_2} \quad (13)$$

Integration in the natural coordinate system for arbitrary polynomial terms becomes

$$\int_{x_1}^{x_1+\ell} L_1^p L_2^q dx = \frac{p! q! \ell}{(p+q+1)!} \quad (14)$$

Example 1: Two node, constant area line element.

Assume constant strain in the element, due to a linear displacement field:

$$\{u(x)\} = [L_1 \quad L_2] \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

Then

$$\begin{aligned} [K] &= \int_{x_1}^{x_1+\ell} [N]^T \left[\frac{d}{dx} \right]^T [E] \left[\frac{d}{dx} \right] [N] dx \\ &= \int_{x_1}^{x_1+\ell} \begin{bmatrix} \frac{dL_1}{dx} \\ \frac{dL_2}{dx} \end{bmatrix} [E] \begin{bmatrix} \frac{dL_1}{dx} & \frac{dL_2}{dx} \end{bmatrix} dx \\ &= \int_{x_1}^{x_1+\ell} \begin{bmatrix} -\frac{1}{\ell} \\ \frac{1}{\ell} \end{bmatrix} \begin{bmatrix} -\frac{E}{\ell} & \frac{E}{\ell} \end{bmatrix} dx \\ &= \frac{EA}{\ell} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{aligned}$$

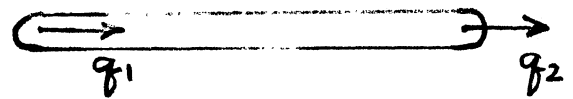


Fig. 4. Two node, constant area line element.

This checks our earlier solution with Cartesian coordinates.

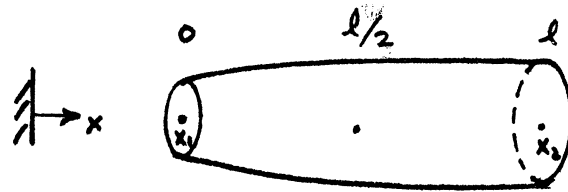
Example 2: Three node, variable area line element.

Given:

$$x = x_1 L_1 + x_2 L_2$$

we will choose:

$$u(x) = q_1 2L_1 \left(L_1 - \frac{1}{2}\right) + q_2 4L_1 L_2 + q_3 2L_2 \left(L_2 - \frac{1}{2}\right)$$



$$A = A_0 \left(1 + B \frac{x-x_1}{l}\right) = A_0 (1 + BL_2)$$

In matrix form this becomes:

$$\{u(x)\} = \left[2L_1 \left(L_1 - \frac{1}{2}\right) \quad 4L_1 L_2 \quad 2L_2 \left(L_2 - \frac{1}{2}\right) \right] \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}$$

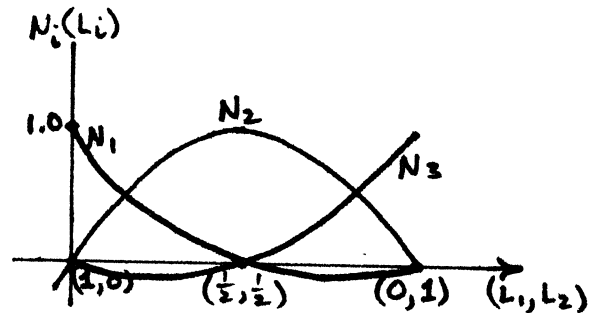


Fig. 5. Three-node, variable area line element.

$$[K] = \int_{x_1}^{x_1+l} [N]^T \left[\frac{d}{dx} \right]^T [E] \left[\frac{d}{dx} \right] [N] A_0 (1+BL_2) dx$$

$$= \int_{x_1}^{x_1+l} \begin{bmatrix} \frac{d}{dx} (2L_1^2 - L_1) \\ \frac{d}{dx} (4L_1 L_2) \\ \frac{d}{dx} (2L_2^2 - L_2) \end{bmatrix} [E] \left[\frac{d}{dx} (2L_1^2 - L_1), \frac{d}{dx} (4L_1 L_2), \frac{d}{dx} (2L_2^2 - L_2) \right] A_0 (1+BL_2) dx$$

$$= EA_0 \int_{x_1}^{x_1+l} \begin{bmatrix} -\frac{1}{l} (4L_1 - 1) \\ \frac{1}{l} 4L_2 + \frac{1}{l} 4L_1 \\ -\frac{1}{l} (4L_2 - 1) \end{bmatrix} \left[-\frac{1}{l} (4L_1 - 1), -\frac{4}{l} (L_2 - L_1), \frac{1}{l} (4L_2 - 1) \right] A_0 (1+BL_2) dx$$

$$= EA_0 \int \begin{bmatrix} \frac{1}{l^2} (4L_1 - 1)^2 & \frac{4}{l^2} (4L_1 - 1) (L_2 - L_1) & -\frac{1}{l^2} (4L_1 - 1) (4L_2 - 1) \\ \frac{4}{l} (4L_1 - 1) (L_2 - L_1) & \frac{16}{l^2} (L_2 - L_1)^2 & -\frac{4}{l^2} (L_2 - L_1) (4L_2 - 1) \\ -\frac{1}{l^2} (4L_2 - 1) (4L_1 - 1) & -\frac{4}{l^2} (4L_2 - 1) (L_2 - L_1) & \frac{1}{l^2} (4L_2 - 1)^2 \end{bmatrix} (1+BL_2) dx$$

Admittedly, for this relatively simple problem, the integration above is equally as difficult as a Cartesian integration would be. The advantage of the method lies in higher dimensioned domains. Typical calculations are:

$$\begin{aligned} k_{11} &= EA_o \int \frac{1}{\ell^2} (16L_1^2 - 8L_1 + 1)(1 + BL_2) dx \\ &= \frac{EA_o}{\ell^2} \int_{x_1}^{x_2} [16L_1^2 + 16BL_1^2L_2 - 8L_1 - 8BL_1L_2 + 1 + BL_2] dx \end{aligned}$$

Now,

$$\int_{\ell} L_1^2 d\ell = \frac{2! 0!}{3!} \ell = \frac{2\ell}{3 \cdot 2} = \frac{1}{3} \ell$$

$$\int_{\ell} L_1^2 L_2 d\ell = \frac{2! 1!}{4!} \ell = \frac{2\ell}{4 \cdot 3 \cdot 2} = \frac{1}{12} \ell$$

$$\int_{\ell} L_1 d\ell = \frac{1! 0!}{2!} \ell = \frac{1}{2} \ell$$

$$\int_{\ell} L_1 L_2 d\ell = \frac{1! 1!}{3!} \ell = \frac{1}{6} \ell$$

$$\int_{\ell} 1 d\ell = \ell$$

$$\int_{\ell} L_2 d\ell = \frac{1}{2} \ell$$

$$\begin{aligned} k_{11} &= \frac{EA_o}{\ell^2} \left[16 \left(\frac{1}{3} \ell \right) - 16B \frac{1}{12} \ell - 8 \left(\frac{1}{2} \ell \right) - 8B \frac{\ell}{6} + 1\ell + B \frac{\ell}{2} \right] \\ &= \frac{EA_o}{\ell} \left[\left(\frac{16}{3} - 4 + 1 \right) + B \left(+ \frac{16}{12} - \frac{8}{6} + \frac{1}{2} \right) \right] \\ &= \frac{EA_o}{\ell} \left[\frac{7}{3} + \frac{1}{2} B \right]. \end{aligned}$$

The total stiffness becomes:

$$[K] = \frac{EA_o}{6\ell} \begin{bmatrix} 14 + 3B & -16 - 4B & 2 + B \\ -16 - 4B & 32 + 16B & -16 - 12B \\ 2 + B & -16 - 12B & 14 + 11B \end{bmatrix}$$

In this element, the independent variable x was given a linear representation in L_1, L_2 whereas the displacement field is given a quadratic interpolation. Because the independent variable is interpolated with a lower power polynomial than the dependent (field) variable, the element is said to be subparametric. There are also superparametric elements where the converse is true, and isoparametric when both are interpolated to the same order. Isoparametric elements are most common of the three.

III. NATURAL COORDINATE SYSTEM: 2-D, AREA COORDINATES.

For a triangular element, one can interpolate

$$\begin{aligned} x &= x_1 L_1 + x_2 L_2 + x_3 L_3 \\ y &= y_1 L_1 + y_2 L_2 + y_3 L_3 \\ 1 &= L_1 + L_2 + L_3 \end{aligned} \quad (15)$$

The quantities L_i can be found from inverting this relation to obtain

$$\begin{aligned} L_1(x,y) &= \frac{1}{2\Delta} [a_1 + b_1 x + c_1 y] \\ L_2(x,y) &= \frac{1}{2\Delta} [a_2 + b_2 x + c_2 y] \\ L_3(x,y) &= \frac{1}{2\Delta} [a_3 + b_3 x + c_3 y] \end{aligned} \quad (16)$$

where $\Delta \equiv$ area of triangle and

$$\begin{aligned} a_1 &= x_2 y_3 - x_3 y_2, \\ b_1 &= y_2 - y_3, \\ c_1 &= x_2 - x_3. \end{aligned} \quad (17)$$

The pair of matrix equations 16 and 17 define the L_i once given the linear interpolation model, or one can start with the linear surface such as $L_1(x,y)$ sketched and then prove the interpolation. The integration for the area coordinates can be shown to be

$$\int L_1^p L_2^q L_3^r dA = \frac{p! q! r!}{(p+q+r+2)!} 2A$$

IV. NATURAL COORDINATE SYSTEM: 3-D, VOLUME COORDINATES.

The volume coordinates for a tetrahedron are shown in TABLE I. One cannot plot the domain of the (L_1, L_2, L_3, L_4) quadruple because it is a surface in a four-dimensional space. Again, one is able to invert the relation between x, y, z and the L_i to obtain

$$L_i = \text{function}(x, y, z) \quad i=1, 2, 3$$

so that the chain rule for differentiation can be applied.

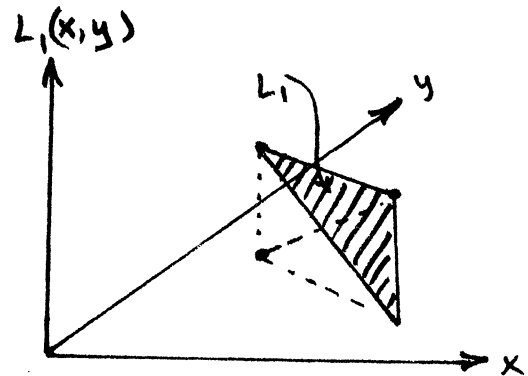
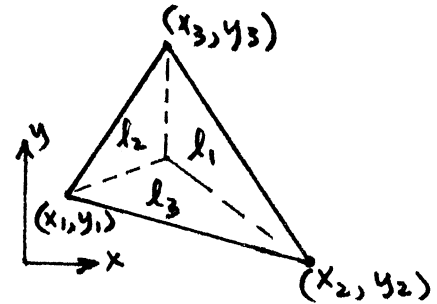


Fig. 6. Area coordinates for a triangle.

V. "MAPPING" COORDINATES. THE SERENDIPITY FAMILY.

TABLE I shows a system of coordinates in which the N_i shape functions are related to a ξ, η, ζ system. The mapping is done through the use of the "serendipity" interpolation formulae; e.g., for a hexahedron in 3-D:

$$N_i = \frac{1}{8} \sum_{i=1}^8 (1 + \xi \xi_i)(1 + \eta \eta_i)(1 + \zeta \zeta_i)$$

and where (ξ, η, ζ) are coordinates with nodes (ξ_i, η_i, ζ_i) lying at the eight corners of a cube. The cube has sides two units long each, extending from $(-1, 1)$ on each axis. This means that the original body is mapped into a cube.

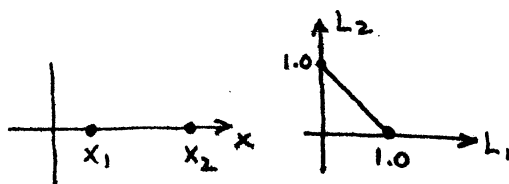
TABLE I

TABLE OF NATURAL COORDINATES

Length Coordinate:

$$x = L_1 x_1 + L_2 x_2$$

$$1 = L_1 + L_2$$



$$\int_{x_1}^{x_2} L_1^p L_2^q dx = \frac{p!q!}{(p+q+1)!}$$

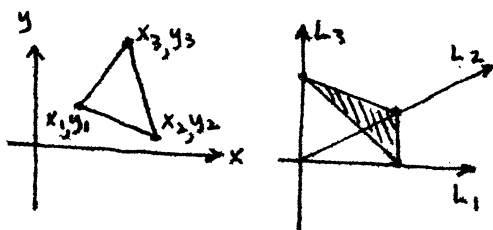
Area Coordinates:

(suitable for triangle)

$$x = L_1 x_1 + L_2 x_2 + L_3 x_3$$

$$y = L_1 y_1 + L_2 y_2 + L_3 y_3$$

$$1 = L_1 + L_2 + L_3$$



$$\int_A L_1^p L_2^q L_3^r dA = \frac{p!q!r!2A}{(p+q+r+2)!}$$

Volume Coordinates:

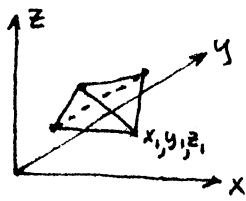
(suitable for tetrahedra)

$$x = L_1 x_1 + L_2 x_2 + L_3 x_3 + L_4 x_4$$

$$y = L_1 y_1 + L_2 y_2 + L_3 y_3 + L_4 y_4$$

$$z = L_1 z_1 + L_2 z_2 + L_3 z_3 + L_4 z_4$$

$$1 = L_1 + L_2 + L_3 + L_4$$



(cant plot)
(SURFACE IN 4-D)

$$\int_V L_1^p L_2^q L_3^r L_4^s dV = \frac{p!q!r!s!6V}{(p+q+r+s+3)!}$$

TABLE II

SERENDIPITY, LINEAR "MAPPING" COORDINATES.

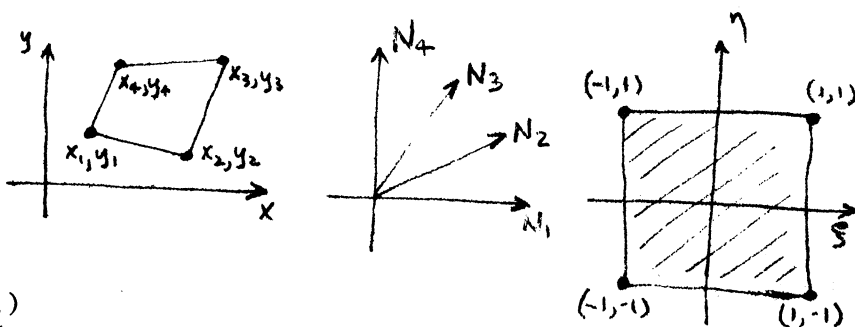
2-D: Quadrilateral Coordinates:

$$x = \frac{1}{4} [(1-\xi)(1-\eta)x_1 + (1+\xi)(1-\eta)x_2 + (1+\xi)(1+\eta)x_3 + (1-\xi)(1+\eta)x_4]$$

$$= \frac{1}{4} \sum_1^4 (1+\xi\xi_i)(1+\eta\eta_i)x_i$$

$$= \sum N_i x_i \text{ where } N_i = \frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i)$$

$$y = \sum N_i y_i$$

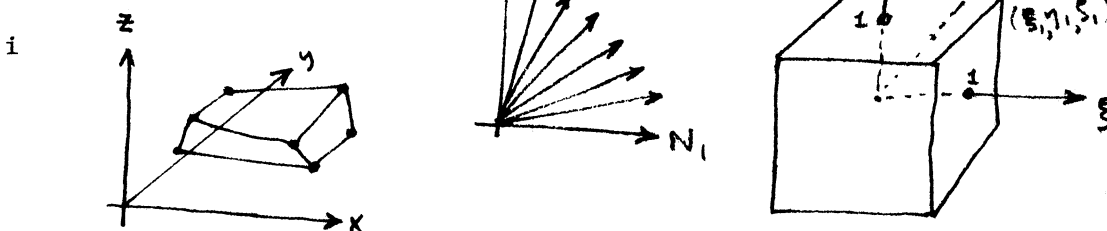


3-D: Hexahedral Coordinates:

$$x = \sum N_i x_i \quad y = \sum N_i y_i \quad z = \sum N_i z_i$$

where

$$N_i = \frac{1}{8}(1 + \xi\xi_i)(1 + \eta\eta_i)(1 + \zeta\zeta_i)$$



I. POLYNOMIAL APPROXIMATION - CLASSICAL THEORY

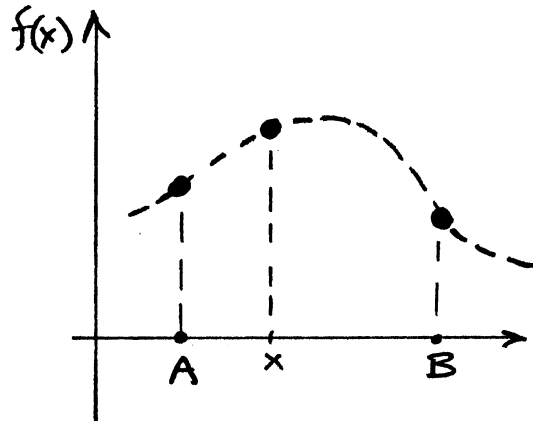
Reference: Hamming - "Numerical Methods for Scientists and Engineers," Chapters 14-19.

A. Motivation

- We often need to integrate a scalar function (such as strain energy) over a volume.
- Can develop the theory in terms of scalar function of one variable, e.g., strain energy in a line element.
- The function being integrated is often not available in closed form. In the case of isoparametric elements, the Jacobian is very complicated - possibly the determinant of a 32 x 32 matrix.

B. Interpolation

Hamming gives four basic ways of finding a functional value $f(x)$ at a general point x when the function is not known everywhere analytically:



- 1) Methods which use tabulated data from points near $f(x)$, say $f(A)$ and $f(B)$ and to estimate a value between, as one does in log tables.
- 2) Methods using differences of functional values, e.g., the straight line approximation:

$$f(x) = f(A) + \left(\frac{x-A}{B-A}\right)(f(B) - f(A))$$

- 3) Methods involving derivatives e.g., Newton's method,

$$f(x) = f(A) + \left.\frac{df}{dx}\right|_A (x-A).$$

- 4) Gaussian quadrature using arbitrary points on the x axis.

II. GAUSSIAN QUADRATURE

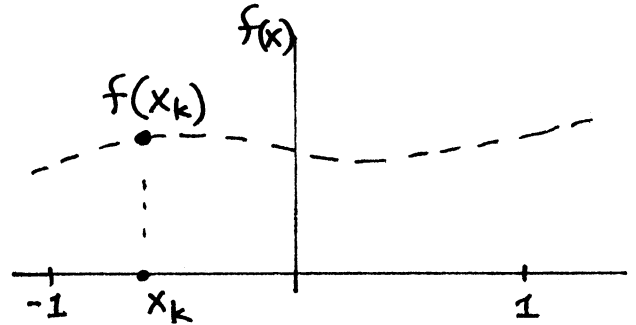
Reference: Zienkiewicz, The Finite Element Method, vol. 2, pages 144-147.

A.

We will discuss the symmetric form of Gaussian quadrature. The goal is to integrate the area under a curve for $-1 \leq x \leq 1$, where the function can be sampled at any point. One proposes a form of solutions:

$$\int_{-1}^1 f(x) dx = \sum_{k=1}^N w_k f(x_k)$$

In other words, one replaces the integral with a finite sum of terms, each consisting of the product of a "weight factor" w_k and the function evaluated at an "abscissa" x_k . This is a remarkably bold step inasmuch as not only the weight factors, but the abscissa must be found during the process itself.



B. Defining Equations

Reference: Hamming, Chapter 19

Suppose we wish to use a two-term approximation to the symmetric integration:

$$\int_{-1}^1 f(x) dx = w_1 f(x_1) + w_2 f(x_2).$$

This means there are four unknowns, w_1, w_2, x_1 and x_2 . We assume $f(x)$ is available in some form for numerical sampling, that is, we can pop a value of x into $f(x)$ and get out a number, even though we can't write out $f(x)$ analytically ($f(x)$ is a "black box").

We can impose four conditions on the problems, in equation form, and hopefully have four equations for the unknowns. One way to proceed is to imagine that $f(x)$ is a polynomial (it often is). If $f(x)$ were cubic,

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3,$$

we could insure that our integration process would be exact, by demanding that it separately be exact for a constant, a linear term, a parabolic and a cubic.

If $f(x) = 1$, then the integration formula becomes

$$\int_{-1}^1 (1) dx = w_1(1) + w_2(1)$$

or

$$\boxed{2 = w_1 + w_2} \quad \leftarrow \text{1st defining equation}$$

If $f(x) = x$,

$$\int_{-1}^1 x dx = w_1 x_1 + w_2 x_2$$

or

$$\boxed{0 = w_1 x_1 + w_2 x_2} \quad \leftarrow \text{2nd defining equation}$$

If $f(x) = x^2$

$$\int_{-1}^1 x^2 dx = w_1 x_1^2 + w_2 x_2^2$$

$$\boxed{\frac{2}{3} = w_1 x_1^2 + w_2 x_2^2} \quad \leftarrow \text{3rd defining equation}$$

and if $f(x) = x^3$

$$\int_{-1}^1 x^3 dx = w_1 x_1^3 + w_2 x_2^3$$

$$\boxed{0 = w_1 x_1^3 + w_2 x_2^3} \quad \leftarrow \text{4th defining equation}$$

We have created 4 defining equations and can solve for all w_k and x_k .
The expression

$$\int_{-1}^1 f(x) dx = w_1 f(x_1) + w_2 f(x_2)$$

can be used to integrate an arbitrary function $f(x)$; furthermore, it will exactly integrate polynomials up to cubics. The latter idea is confirmed by writing out the special case

$$\begin{aligned}\int_{-1}^1 f(x) dx &\equiv \int_{-1}^1 (a_0 + a_1 x + a_2 x^2 + a_3 x^3) dx \\ &= a_0 \int_{-1}^1 (1) dx + a_1 \int_{-1}^1 x dx + a_2 \int_{-1}^1 x^2 dx + a_3 \int_{-1}^1 x^3 dx\end{aligned}$$

and our defining equations assured us that each of these integrals would be exact. This is linearity of one type. The equations for the w_k and x_k are certainly not linear, however!

III. EXAMPLE OF GAUSSIAN INTEGRATION IN ONE DIMENSION

Reference: Desai and Abel, page 147, Ex. 5-11.

This example is taken from a textbook but recast into symmetric form. Let $f(x) = 0.5 - 0.5x$ and use the 3 point Gaussian integration

$$\int_{-1}^1 f(x) dx = \sum_{i=1}^3 w_i f(x_i).$$

Assume someone else has already solved the defining equations to give

$$\begin{array}{ll} x_1 = -0.775 & w_1 = 0.556 \\ x_2 = 0 & w_2 = 0.889 \\ x_3 = 0.775 & w_3 = 0.556 \end{array}$$

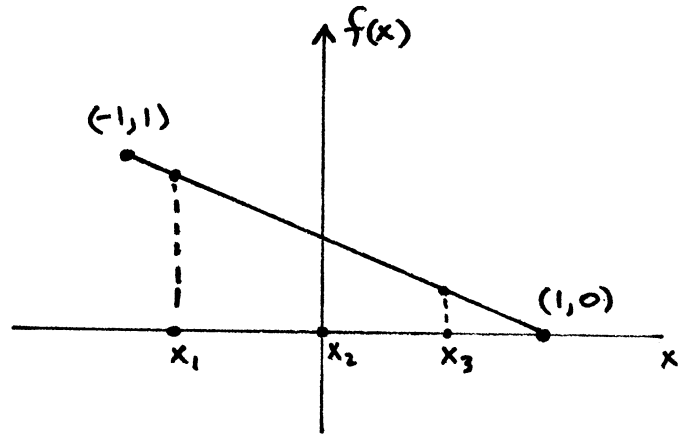
one has:

$$\begin{aligned}\int_{-1}^1 f(x) dx &= (.556)f(-.775) + (.889)f(0) + (.556)f(.775) \\ &= (.556)(.8875) + (.889)(.500) + (.556)(.1125) \\ &= 1.0005\end{aligned}$$

This is an approximate check for what one expects for the area, but it should be exact! (Because $f(x)$ is a quintic or less). The error is due to round-off of the w_i and x_i given by Desai and Abel.

Redoing this problem with more exact values from the "Handbook of Math. Functions," by Abramowitz and Stegun, one has

$$\begin{aligned}\int_{-1}^1 f(x) dx &= (0.55555 \dots)f(-0.7745966692) + (.88888 \dots)f(0) + (.55 \dots)f(0.7745966692) \\ &= 1.000000000 \quad (\text{accurate to 10 significant figures}).\end{aligned}$$



NONLINEAR PROBLEMS

I. General

We have some numerical methods that require solution of

$$[k(\{q\}, \{Q\})] \{q\} = \{Q\} \quad (1)$$

and in which the secant stiffness is used (by definition). Other problems are of the form

$$[k(\{q\}, \{Q\})] \{\Delta q\} = \{\Delta Q\} \quad (2)$$

in which the tangent stiffness is used for the most accurate solution, but in which the elastic stiffness or the secant stiffness can be used as approximations. Finally, because the stiffnesses in (1) and (2) are generally very complicated to recalculate, we might want to get the nonlinear effect by replacing (1) by

$$[k_e] \{q\} = \{Q\} + \{Q\}_{\epsilon_o, \text{e.n.l.}} + \{Q\}_{\sigma_o, \text{e.n.l.}} \quad (3)$$

or by replacing (2) by

$$[k_e] \{\Delta q\} = \{\Delta Q\} + \{\Delta Q\}_{\epsilon_o, \text{e.n.l.}} + \{\Delta Q\}_{\sigma_o, \text{e.n.l.}} \quad (4)$$

(Usually only one of initial stress or initial strain is used at a time.)

II. Virtual Work

It greatly helps understanding of the different approaches if one returns to the virtual work formulation and includes the nonlinearity. Suppose that, indeed, both the stress-strain law

$$\{\sigma\} = [C(\{\epsilon\}, \{\sigma\})] \{\epsilon\} \quad (5)$$

and the strain-displacement law are nonlinear:

$$\{\epsilon\} = [B(\{q, Q\})] \{q\} \quad (6)$$

Note that the nonlinear quantity has been factored in a way to remove the linear part, i. e., the vectors $\{\epsilon\}$ and $\{q\}$ appear to the first power on the right hand side.

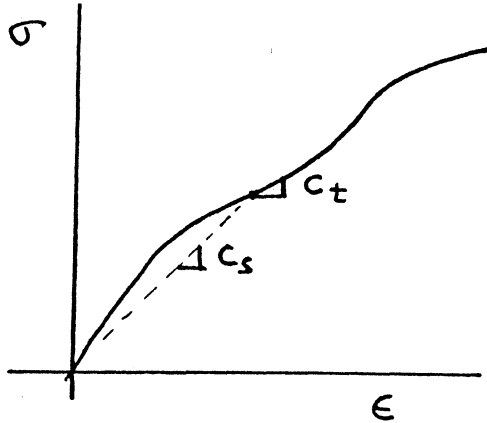


Fig. 1. Nonlinear stress-strain law.

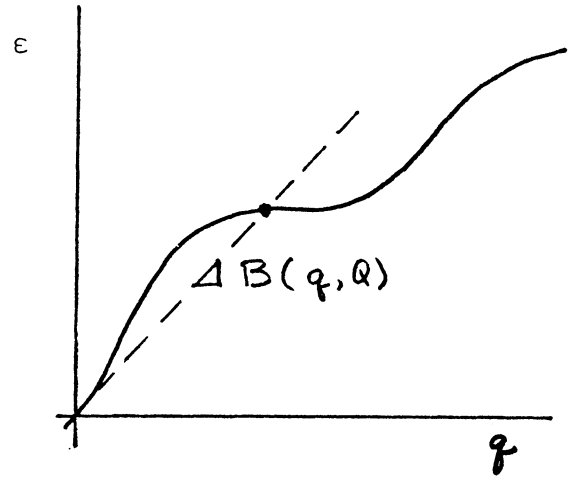


Fig. 2. Nonlinear strain-displacement law.

The modified virtual work expression, for total work done during a virtual displacement is:

$$-(\delta U) + (\delta w_{\text{external}}) = 0 \quad (7)$$

$$-\int_v \{\delta \epsilon\}^T \{\sigma\} dv + \{\delta q\}^T \{Q\} + \int_v \{\delta u\}^T \{\bar{X}\} dv + \int_s \{\delta u\}^T \{\bar{T}\} ds = 0 \quad (8)$$

strain energy	conc. loads	body forces	surface forces
------------------	----------------	----------------	-------------------

This expression is valid for nonlinear elastic material cases and many nonlinear geometry cases*. We are interested in cases with no physical prestress or pre-strain.

A. Material Nonlinearities

Let us consider the easiest nonlinearity, material, alone:

$$\{\sigma\} = [C_s(\{\epsilon\}, \{\sigma\})] \{\epsilon\} \quad (9)$$

$$\{\epsilon\} = [B] \{q\} \quad (10)$$

↑
constant

* For strong nonlinear geometry, one needs to adjust the coordinate systems somewhat.

V. RADAU AND LOBATTO INTEGRATION

Reference: Section 19.7 in Hamming

A minor variation in Gaussian quadrature is to take one or two sample points to be fixed, typically at the endpoints of the region (nodes of an element). If you choose one sample point a priori to be at the end of the region, you have Radau integration whereas choosing two such points gives Labotto integration. Specification of a sample point provides more accurate information at that point; however, it also eliminates a defining equation and reduces the overall accuracy of the method. There is clearly a trade-off involved in the choice of methods.

The example

$$\int_{-1}^1 f(x) dx \equiv \int_{-1}^1 (0.5 - 0.5x) = \sum_{i=1}^3 w_i f(x_i)$$

has been integrated exactly since 6 defining equations were used. The 3-point Gaussian quadrature can integrate a 5th-degree polynomial exactly.

IV. COMMENTS ABOUT GAUSS' QUADRATURE

- (a) If you know the degree of a polynomial to be integrated, it does not pay to use more integration points than necessary for an exact answer. Typical examples are in plate and shell elements. Even Zienkiewicz had some trouble with this several years back.
- (b) Abramowitz and Stegun give tables for "abscissas" and weight factors for Gaussian integration for up to 96 sample points! They truncate the values to 15 decimal places.
- (c) The solution of the defining equations in general involves finding zeroes of a polynomial called the Legendre polynomial.
- (d) There are multiple solutions for the abscissa and weight factors, which is reasonable because of the nonlinearity of the defining equations. There are 3 sets for cases with less than 13 sample points ($n < 13$) and 4 sets for $16 \leq n \leq 96$. There may be additional solutions, but we discard them because we desire real numbers and desire the abscissae lie in the integration region.
- (e) Desai and Abel show a nonsymmetric form of Gaussian quadrature:

$$\int_a^b f(y) dy = \left(\frac{b-a}{2}\right) \sum_{i=1}^n w_i f(y_i)$$

where

$$y_i = \left(\frac{b-a}{2}\right)x_i + \left(\frac{b+a}{2}\right)$$

This allows one to convert from the symmetric, normalized data pairs (w_i, x_i) given in math tables to apply to a given physical problem.

The virtual work expression becomes:

$$\underbrace{-\{\delta q\}^T \int_v [B]^T [C_s(\{\epsilon\}, \{\sigma\})] [B] dv}_{\text{strain}} \underbrace{\{Q\} + \{\delta q\}^T \int_v [N]^T \{\bar{x}\} dv}_{\text{stress}} + \{\delta q\}^T \int_s [N]^T \{\bar{T}\} ds = 0 \quad (11)$$

Note that the δ operator acts only on displacement-like quantities, and not on the $[C]$ matrix.

There is a common, nonzero leading vector $\{\delta q\}^T$:

$$\{\delta q\}^T \{ \} = 0 \quad (12)$$

therefore the second vector must be identically zero, leading to

$$\int_v [B]^T [C(\{q\}, \{Q\})] [B] dv \{q\} = \{Q\} + \int_v [N]^T \{\bar{x}\} dv + \int_s [N]^T \{\bar{T}\} ds \quad (13)$$

or

$$[k_s] \{q\} = \{Q\} + \underbrace{\{Q\}}_{\text{body forces e.n.l.}} + \underbrace{\{Q\}}_{\text{surface forces e.n.l.}} \quad (13')$$

We have arrived at the happy conclusion that

$$[k_s] = \int_v [B]^T [C_s(\{q\}, \{Q\})] [B] dv \quad (5.45a)$$

Desai & Abel

which would be an intuitive guess, but is not a trival result. This allows us to solve equation (1) by the variable stiffness approach.

We now need to know how to solve the incremental equation (2) for nonlinear material. We can do this by a variational process in which we have the equation

$$[k_s] \{q\} - \{Q\} = 0 \quad (14)$$

but wish to study small displacements away from the current equilibrium solution, without violating equilibrium. Using the Δ symbol as an operator (not the same as for a virtual displacement), it is claimed that the tangent stiffness is defined by

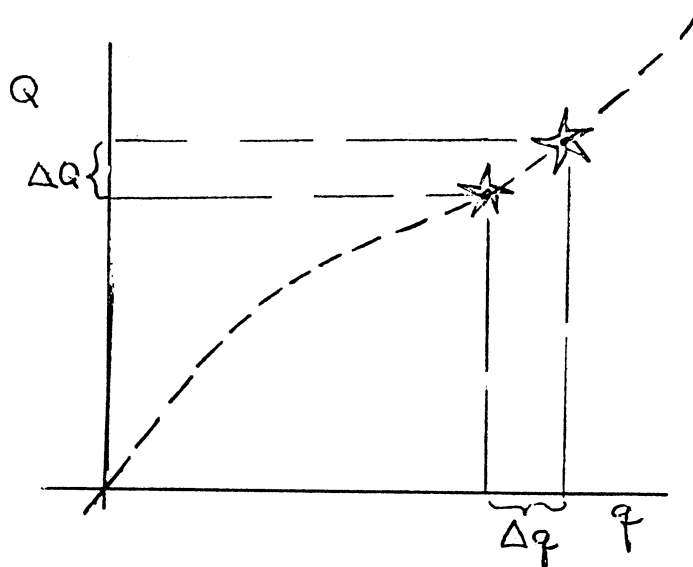


Fig. 3. Increment taken along true load-deflection curve.

$$\{\Delta Q\} \equiv [k_T] \{\Delta q\} \quad (15)$$

Data used to find $[k_T]$ comes from material tests in the incremental form:

$$\{\Delta \sigma\} \equiv [C_T(\epsilon, \sigma)] \{\Delta \epsilon\} = [C_T][B] \{\Delta q\} \quad (16)$$

(In other words, material tangent moduli will be used to calculate structural tangent stiffness.)

Rewriting our old expression for load-deflection:

$$\begin{aligned} \{Q\} &= [k_s] \{q\} \\ &= \int_v [B]^T [C(q, Q)] [B] dv \{q\} \\ &= \int_v [B]^T \{\sigma\} dv \end{aligned} \quad (17)$$

\swarrow nonlinear stress
 \nwarrow constant

One can take the variation

$$\begin{aligned}
\{\Delta Q\} &= \int_{\mathbf{v}} [B]^T \{\Delta \sigma\} d\mathbf{v} \\
&= \int_{\mathbf{v}} [B]^T [C_T] \{\Delta \epsilon\} d\mathbf{v} \\
&= \int_{\mathbf{v}} [B]^T [C_T] [B] d\mathbf{v} \{\Delta q\}
\end{aligned} \tag{18}$$

or

$$\{\Delta Q\} = [k_T] \{\Delta q\} \tag{19}$$

where $[k_T]$ is the tangent stiffness, and is defined:

$$[k_T] \equiv \int [B]^T [C_T(\epsilon, \sigma)] [B] d\mathbf{v} \tag{20}$$

This could be true for any nonlinear elastic material, or for elastic-plastic behavior as given in equations 7.19 and 7.20 in Desai and Abel.

We have now found secant and tangent stiffnesses for nonlinear material problems.

B. Geometrical Nonlinearity (Zienkiewicz, pp. 413-417)

Now suppose that

$$\{\sigma\} = [C] \{\epsilon\} \quad \{u\} = [N] \{q\} \tag{21, 22}$$

$$\begin{array}{c} \nearrow \text{constant} \\ \{\delta u\} = [N] \{\delta q\} \end{array} \tag{23}$$

$$\{\epsilon\} = [B(\{q, Q\})] \{q\} \tag{24}$$

Furthermore, if $[\bar{B}]$ is defined so that

$$\{\delta \epsilon\} \equiv [\bar{B}(\{q, Q\})] \{\delta q\} \tag{25}$$

we are set to study the nonlinear geometry problem.

First of all, for large strain, the virtual work expression, equation (8) must have proper interpretation. The stresses involved must be Kirchhoff-Piola stress and the volume and surface integrals must be in the deformed configuration (State II). We are interested in both $[k_s]$ and in the tangent stiffness form $[k_T]$:

$$\{\Delta Q\} \equiv [k_T] \{\Delta q\} \quad (\text{repeated}) \tag{26}$$

both of which we can get from equation (8) after much manipulation.

$$\begin{aligned}
-\{\delta q\}^T \int [\bar{B}]^T \{\sigma\} dv + \{\delta q\}^T \{Q\} + \{\delta q\}^T \int_{\underline{v}} [N]^T \{\bar{X}\} dv \\
+ \{\delta q\}^T \int_{\underline{s}} [N]^T \{\bar{T}\} ds = 0
\end{aligned} \tag{27}$$

Again, factoring out $\{\delta q\}^T$ and realizing that it is arbitrary,

$$\int_{\underline{v}} [\bar{B}]^T [C][B] dv \{q\} = \{Q\} + \{Q\}_{\text{e.n.l.}}^{\bar{X}} + \{Q\}_{\text{e.n.l.}}^{\bar{T}} \tag{28}$$

From here, we see that the secant stiffness is

$$[k_s] \equiv \int_{\underline{v}} [\bar{B}]^T [C][B] dv \tag{29}$$

Note that both $[B]$ and $[\bar{B}]$ enter this expression.

Again, we wish to vary equation (28) so as to move along the true load deflection path. Combine all loads into a single $\{Q\}$ vector for this argument and again use the Δ operator.

$$\Delta \left(\int_{\underline{v}} [\bar{B}]^T \{\sigma\} dv \right) = \Delta \{Q\} \tag{30}$$

$$\int_{\underline{v}} [\Delta \bar{B}]^T \{\sigma\} dv + \int_{\underline{v}} [\bar{B}]^T \{\Delta \sigma\} dv = \{\Delta Q\}$$

$$\int_{\underline{v}} [\Delta \bar{B}]^T \{\sigma\} dv + \int_{\underline{v}} [\bar{B}]^T [C] \{\Delta \epsilon\} dv = \{\Delta Q\}$$

$$\int_{\underline{v}} [\Delta \bar{B}]^T \{\sigma\} dv + \int_{\underline{v}} [\bar{B}]^T [C] [\bar{B}] dv \{\Delta q\} = \{\Delta Q\} \tag{31}$$

Now, $[\bar{B}]$ can always be written

$$[\bar{B}] = [B_o] + [B_L(q)] \tag{32}$$

constant

where $[B_L(q)]$ is a linear function of $\{q\}$. Equation (31) then becomes

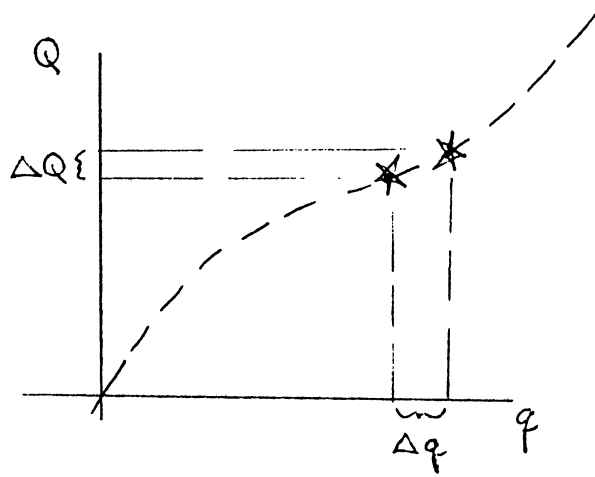


Fig. 4. Increment taken along true load-deflection curve. (As in Fig. 3.)

$$\begin{aligned}
 \int_{\mathbf{v}} [\Delta \bar{\mathbf{B}}]^T \{\sigma\} d\mathbf{v} + & \left[\int_{\mathbf{v}} [\mathbf{B}_o]^T [\mathbf{C}] [\mathbf{B}_o] d\mathbf{v} + \int_{\mathbf{v}} [\mathbf{B}_L]^T [\mathbf{C}] [\mathbf{B}_o] d\mathbf{v} \right. \\
 & + \int_{\mathbf{v}} [\mathbf{B}_o]^T [\mathbf{C}] [\mathbf{B}_L] d\mathbf{v} \\
 & \left. + \int_{\mathbf{v}} [\mathbf{B}_L]^T [\mathbf{C}] [\mathbf{B}_L] d\mathbf{v} \right] \{\Delta q\} = \{\Delta Q\} \quad (33)
 \end{aligned}$$

The first term is rewritten, as it always can be done (need to work on specific cases to see):

$$\int_{\mathbf{v}} [\Delta \bar{\mathbf{B}}]^T \{\sigma\} d\mathbf{v} \equiv [\mathbf{k}_\sigma] \{\Delta q\} \quad (34)$$

\swarrow geometric (mild) and not a function of $\{q\}$.

and the other terms are defined

$$\int_{\mathbf{v}} [\mathbf{B}_o]^T [\mathbf{C}] [\mathbf{B}_o] d\mathbf{v} \equiv [\mathbf{k}_e] \quad (35)$$

$$\int_{\mathbf{v}} [\mathbf{B}_L]^T [\mathbf{C}] [\mathbf{B}_o] d\mathbf{v} + \int_{\mathbf{v}} [\mathbf{B}_o]^T [\mathbf{C}] [\mathbf{B}_L] d\mathbf{v} + \int_{\mathbf{v}} [\mathbf{B}_L]^T [\mathbf{C}] [\mathbf{B}_L] d\mathbf{v} \equiv [\mathbf{k}_L] \quad (36)$$

so that equation (33) becomes

$$([k_{\sigma}] + [k_e] + [k_L])\{\Delta q\} = \{\Delta Q\} \quad (37)$$

and by definition, the tangent stiffness is

$$[k_T] = [k_e] + [k_{\sigma}] + [k_L]$$

↖ elastic
↖ geometric matrix
(mildly nonlinear)
↖ large displacement matrix
(strong nonlinearities)

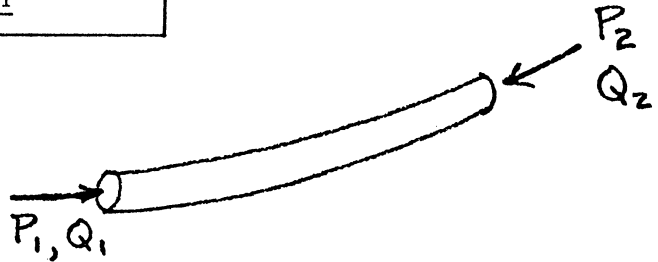
This is the tangent stiffness used in the Newton-Raphson iteration process.

Q. E. D.

Example of a Nonlinear Problem

Turbulent Pipe Flow

We previously developed (Lecture 25, Aero 510) equations for pipe flow. Assuming a uniform, cylindrical pipe as shown, the pressure drop is related to the volume flow as



$$Q_1 = \frac{P_1 - P_2}{f^e(Q^e)}$$

It was pointed out that the constant f depends on the flow rate. For turbulent flow, one might take

$$f^e(Q^e) = \bar{f}^e Q^{0.8},$$

where the superscript "e" on the volume flow Q has been dropped for simplicity. This relation leads to:

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \begin{bmatrix} \frac{1}{\bar{f}^e Q^{0.8}} & -\frac{1}{\bar{f}^e Q^{0.8}} \\ -\frac{1}{\bar{f}^e Q^{0.8}} & \frac{1}{\bar{f}^e Q^{0.8}} \end{bmatrix} \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}$$

The difficulty here is the dependence of the 2×2 flow matrix upon the flow rate itself.

Numerical Example of Turbulent Pipe Flow

Suppose

$$\bar{f}^e = 10 \frac{\text{psi-sec}^2}{\text{gallon}^2}$$

$$P_1 = 5 \text{ psi}$$

$$P_2 = 3 \text{ psi}$$

Then

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \begin{bmatrix} \frac{0.1}{Q^{0.8}} & -\frac{0.1}{Q^{0.8}} \\ -\frac{0.1}{Q^{0.8}} & \frac{0.1}{Q^{0.8}} \end{bmatrix} \begin{Bmatrix} 5 \\ 3 \end{Bmatrix}$$

One can solve for Q_1 and Q_2 by making an initial guess for Q calculating the values of Q_1 and Q_2 , and iterating. This is a 'variable stiffness' approach and is classified as an iterative method using a secant stiffness. More often, the stiffness matrix is a function of the vector on the R. H. S.

STEADY STATE
HEAT CONDUCTION

Nov. 24, 26, 1976

W.J.A.

- References: 1) Desai & Abel, Chapter 12.
2) Zienkiewicz, Edition 2, Chapter 15.

Many field problems involve the Poisson equation, an elliptic, second-order P. D. E. with nonhomogeneous terms:

$$\frac{\partial}{\partial x} \left(k_x \frac{\partial \Psi}{\partial x} \right) + \frac{\partial}{\partial y} \left(k_y \frac{\partial \Psi}{\partial y} \right) + \frac{\partial}{\partial z} \left(k_z \frac{\partial \Psi}{\partial z} \right) + \bar{Q}(x, y, z) = 0 \quad (1)$$

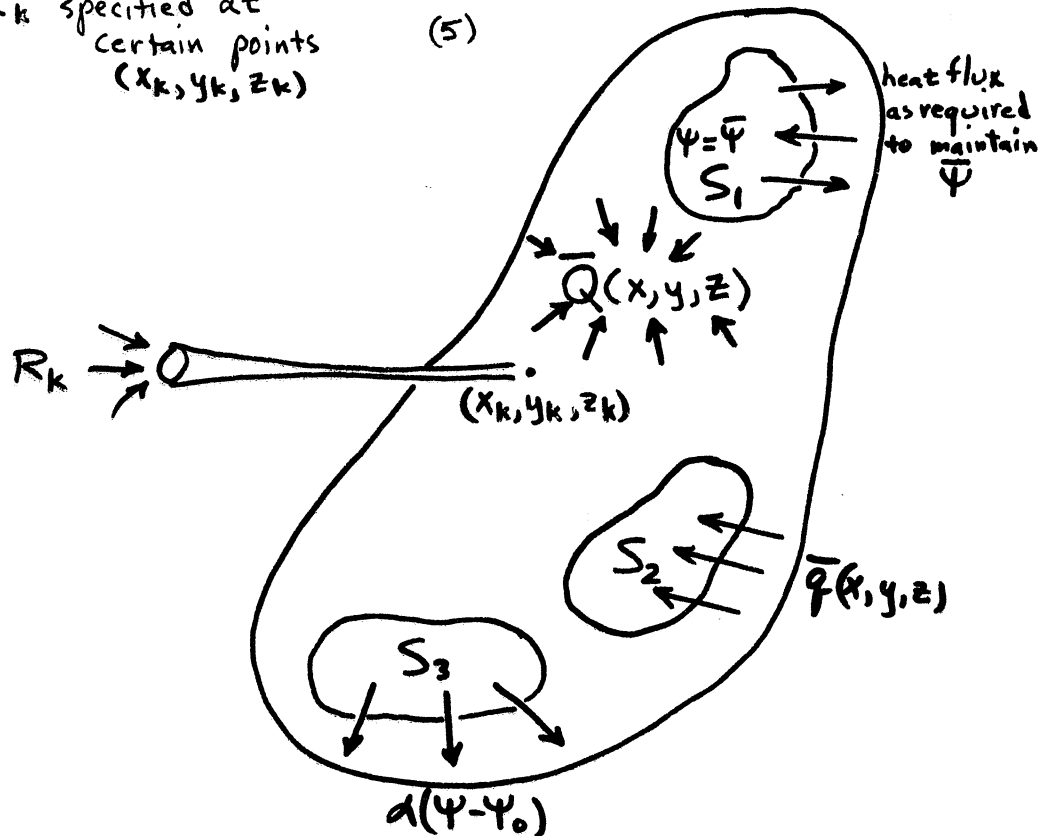
In the heat conduction problem, ψ is the temperature, k_x , k_y , and k_z are thermal conductivities in those directions and \bar{Q} is a heat flux per unit volume.* The k_x , k_y , k_z and \bar{Q} are all functions of spatial coordinates. Boundary conditions for Equation 1 are

$$\Psi = \bar{\Psi}(x, y, z) \quad \text{on } S_1 \quad (2)$$

$$k_x \frac{\partial \Psi}{\partial x} l_x + k_y \frac{\partial \Psi}{\partial y} l_y + k_z \frac{\partial \Psi}{\partial z} l_z - \bar{q}(x, y, z) = 0 \quad \text{on } S_2 \quad (3)$$

$$k_x \frac{\partial \Psi}{\partial x} l_x + k_y \frac{\partial \Psi}{\partial y} l_y + k_z \frac{\partial \Psi}{\partial z} l_z + \alpha(\Psi - \Psi_0) = 0 \quad \text{on } S_3 \quad (4)$$

R_k specified at certain points (x_k, y_k, z_k) (5)



* \bar{Q} is a heat source.

where l_x, l_y, l_z are direction cosines of the normal to the surface, directed outward, $\bar{\psi}_1(x, y, z)$ is a prescribed temperature on the surface S_1 , $\bar{q}(x, y, z)$ is a prescribed flux into the body on surface S_2 and $\alpha(\psi(x, y, z) - \psi_0(x, y, z))$ is a flux due to convection on surface S_3 , (positive outward). The heat transfer coefficient α is a function of space. The concentrated heat flux R_k is needed for later discretization work and needs to be included as a concept at this stage.

Other types of boundary conditions due to radiation and conduction do not fit easily into the linear form needed here and will not be discussed.

Let us restate the field problem as a global conservation of thermal energy for the body. This can be done by minimizing the following functional with respect to the temperature field $\psi(x, y, z)$.

$$A(\psi) = \int_{vol} \left\{ \frac{1}{2} \left[k_x \left(\frac{\partial \psi}{\partial x} \right)^2 + k_y \left(\frac{\partial \psi}{\partial y} \right)^2 + k_z \left(\frac{\partial \psi}{\partial z} \right)^2 \right] - \bar{Q} \psi \right\} dV - \int_{S_2} \bar{q} \psi dS + \frac{1}{2} \int_{S_3} \alpha (\psi - \psi_0)^2 dS - \sum_{k=1}^K R_k \psi(x_k, y_k, z_k) \quad (6)$$

where there are a total of K concentrated heat fluxes. We could recover the differential equation and boundary conditions by carrying out a general variation $\delta A(\psi) = 0$, but this is not our goal. Rather, we wish to discretize the system.

To discretize, lay out a grid of N nodes and E elements. Include all previously assigned concentrated flux locations in the larger, nodal pattern. Consider shape functions within each element N^e to be sufficiently smooth, and the elements to be conforming so that no heat energy is lost in the cracks between elements.* One can then subdivide the functional:

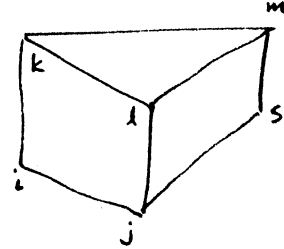
$$A(\psi) = \sum_{e=1}^E A(\psi^e) \quad (7)$$

(Assign the concentrated flux R_k to only one element so that it does not appear repeatedly.)

* This requires that the temperature field be continuous across element boundaries but allows discontinuity of flux (slope) across boundaries. We have seen this type of discontinuity in the Turner triangle.

For each element, use a shape function N^e . Drop the superscript under the condition that different shape functions can still be used if necessary.

$$\begin{aligned}\psi^e(x, y, z) &= [N] \begin{Bmatrix} \psi_i \\ \psi_j \\ \vdots \\ \psi_s \end{Bmatrix} \\ &= \sum_{r=1}^s N_r \psi_r\end{aligned}$$



The functional depends on all N nodal temperatures, so the variation can be rewritten:

$$\delta A(\psi) \equiv \frac{\partial A}{\partial \psi_1} \delta \psi_1 + \frac{\partial A}{\partial \psi_2} \delta \psi_2 + \dots + \frac{\partial A}{\partial \psi_N} \delta \psi_N = 0 \quad (9)$$

This statement must hold true for all values of $\delta \psi_p$, hence the derivatives must be separately zero:

$$\frac{\partial A}{\partial \psi_p} = 0 \quad (p=1, 2, \dots, N) \quad (10)$$

But by partitioning A ,

$$\frac{\partial \sum A^e}{\partial \psi_p} = 0 \quad (p=1, 2, \dots, N) \quad (11)$$

Interchanging differentiation and summation:

$$\sum_{e=1}^E \frac{\partial A^e}{\partial \psi_p} = 0 \quad (p=1, 2, \dots, N) \quad (12)$$

We now need to form A^e and determine $\frac{\partial A^e}{\partial \psi_p}$. This is the longest and most difficult step of the process. Once the individual derivatives are

found, we must assemble them in the form of Equation 12 to get the matrix equation of equilibrium.

The functional is discretized:

$$\begin{aligned}
 A^e \equiv & \int_{\text{Vol}^e} \left\{ \frac{1}{2} \left[k_x \left(\frac{\partial}{\partial x} \sum_r N_r \Psi_r \right)^2 + k_y \left(\frac{\partial}{\partial y} \sum_r N_r \Psi_r \right)^2 + k_z \left(\frac{\partial}{\partial z} \sum_r N_r \Psi_r \right)^2 \right] \right. \\
 & \left. - \bar{Q} \sum_r N_r \Psi_r \right\} dV - \int_{S_2^e} \bar{q} \sum_r N_r \Psi_r dS \\
 & + \int_{S_3^e} \frac{1}{2} \alpha \left(\sum_r N_r \Psi_r - \Psi_0 \right)^2 dS - \sum_{r=i,j,\dots}^s R_r^e \Psi_r \quad (\neq 0) \quad (13)
 \end{aligned}$$

where the concentrated fluxes R_k^e are acting on element e and are assigned to only one element (i.e., not counted twice).

Differentiation yields:

$$\begin{aligned}
 \frac{\partial A^e}{\partial \Psi_p} \equiv & \int_{\text{Vol}} \left\{ \left[k_x \left(\frac{\partial}{\partial x} \sum_r N_r \Psi_r \right) \frac{\partial N_p}{\partial x} + k_y \left(\frac{\partial}{\partial y} \sum_r N_r \Psi_r \right) \frac{\partial N_p}{\partial y} \right. \right. \\
 & \left. \left. + k_z \left(\frac{\partial}{\partial z} \sum_r N_r \Psi_r \right) \frac{\partial N_p}{\partial z} \right] - \bar{Q} N_p \right\} dV - \int_{S_2} \bar{q} N_p dS \\
 & + \int_{S_3} \alpha \left(\sum_r N_r \Psi_r - \Psi_0 \right) N_p dS - R_p^e \quad (\neq 0) \quad (14)
 \end{aligned}$$

$r = i, j, \dots, s$ dummy index

$p = 1, 2, \dots, N$ free index

$e = 1, 2, \dots, E$ free index

Note that there are actually $N \times E$ such quantities defined; the functional within each element is differentiated by each nodal temperature. Now interchange \sum and \int and factor ψ_r out of the integral:

$$\begin{aligned} \frac{\partial A^e}{\partial \psi_p} = & \sum_{r=1}^s \left\{ \int_{Vol} \left[k_x \frac{\partial N_p}{\partial x} \frac{\partial N_r}{\partial x} + k_y \frac{\partial N_p}{\partial y} \frac{\partial N_r}{\partial y} + k_z \frac{\partial N_p}{\partial z} \frac{\partial N_r}{\partial z} \right] dV \right\} \psi_r \\ & - \int_{Vol} \bar{Q} N_p dV \quad - \int_{S_2} \bar{q} N_p dS \\ & + \sum_{r=1}^s \left\{ \int_{S_3} \alpha N_p N_r dS \right\} \psi_r - \int_{S_3} \alpha \psi_0 N_p dS \\ & - R_p \quad (\neq 0) \end{aligned}$$

(15)

r is a dummy index summed up to s

$p = 1, 2, \dots, N$

$e = 1, 2, \dots, E$

In matrix form,

$$\frac{\partial A^e}{\partial \{\psi\}} \equiv [h^e] \{\psi\} + [\bar{h}^e] \{\psi\} + \{\eta^e\} \quad (16)$$

internal
energy

convection
flux,
outflow

nodal flux,
outflow

where typical terms are written in summation form

$$h_{pr}^e \equiv \int_{Vol} \left(k_x \frac{\partial N_p}{\partial x} \frac{\partial N_r}{\partial x} + k_y \frac{\partial N_p}{\partial y} \frac{\partial N_r}{\partial y} + k_z \frac{\partial N_p}{\partial z} \frac{\partial N_r}{\partial z} \right) dV \quad (17)$$

$$\bar{h}_{pr}^e \equiv \int_{S_3} \alpha N_p N_r dS \quad (18)$$

$$\eta_p^e \equiv - \int_{Vol} \bar{Q} N_p dV - \int_{S_2} \bar{q} N_p dS - \int_{S_3} \alpha \psi_0 N_p dS - R_p \quad (19)$$

The vector fluxes are rather easily written in vector notation.

$$\{\eta^e\} \equiv - \int_{Vol} \underline{L} N \underline{Q} dV - \int_{S_2} \underline{L} N \underline{q} dS - \int_{S_3} \alpha \underline{L} N \psi_0 dS - \{R\} \quad (20)$$

The heat conduction and convection matrices are given in D and A as Equation 12.4c.

For assembly, one should view Equation 16 as a relatively sparse set of equations imbedded in an $N \times N$ matrix format. If each element is so developed, and all equations summed properly, the assembled system becomes

$$\frac{\partial A}{\partial \{\psi\}} \equiv [H]\{\psi\} + [\bar{H}]\{\psi\} + \{f\} = 0 \quad \begin{array}{l} \text{first variation} \\ \text{of energy} \\ \text{functional} \end{array} \quad (21)$$

In this case of thermal equilibrium, there is some question as to which side of the equation the various nodal fluxes and equivalent nodal fluxes should appear. Separate out the prescribed nodal fluxes $\{R\}$ and label the rest:

$$[H]\{\psi\} + [\bar{H}]\{\psi\} = \{f\} + \{f\} + \{f\} + \{R\} \quad (22)$$

equiv. nodal flux, inflow from volume sources
e.n.f. inflow, due to \bar{q} on S_2
e.n.f., inflow due to ψ_0 on S_3
prescribed nodal flux, inflow

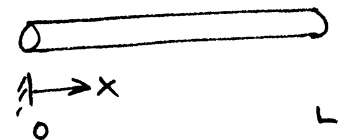
internal energy
convection flux outflow due to ψ on S_3

The $\{R\}$ vector is defined at every node and plays the same role as the concentrated external force in the elasticity problem. The three $\{f\}_{e.n.f.}$ terms are then assembled versions of the first three terms on the R.H.S. of Equation 20, but with minus signs dropped.

Equation 22 requires discussion. Basically, it is a sum of heat fluxes. The first term represents the thermal state of the material in the body, and all the remaining terms are heat flow into or out of the body. Some students would prefer to see all the flux terms on the R.H.S., in which case, the thermal state of the body is in response to all heat fluxes into the body. The homogeneous convection terms $[\bar{H}] \{\psi\}$ will always be lumped into the "internal energy" terms for solution, however.

II. LINE ELEMENT FOR HEAT CONDUCTION

Given a rod of constant cross section A , length L , and constant thermal conductivity k , develop the heat conduction matrix $[h]$ for a 2-node element.



Assume

$$\psi(x) = \begin{matrix} 1-x/L & x/L \end{matrix} \begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix}$$

$$h_{pr}^e = Ak \int_0^L \frac{\partial N_p}{\partial x} \frac{\partial N_r}{\partial x} dx$$

$$h_{11} = Ak \int_0^L (-1/L)^2 dx = \frac{Ak}{L}$$

$$h_{12} = h_{21} = Ak \int_0^L (-1/L)(1/L) dx = -\frac{Ak}{L}$$

$$h_{22} = Ak \int_0^L (1/L)^2 dx = \frac{Ak}{L}$$

$$[h] = \frac{Ak}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

EXAMPLE OF A HEAT CONDUCTION PROBLEM USING LINE ELEMENTS

A pair of pliers is often used as a heat sink when soldering semiconductor diodes into a circuit. By grasping the diode wire between the diode and the heated joint, a secondary heat path is given and the diode hopefully does not receive as much heat. This is in fact a transient heat conduction problem, but we will look at the heat flow before the diode and pliers change temperature, i.e., a short time, quasi static approach. Study the idealized model shown and find the difference in heat flow into the diode (at node 4) with and without the heat sink. Neglect the heat capacity of the wires.

Physical parameters:

The pliers and diode are at 72°F. The iron is at 500°F. Jaws are assumed 1/4" D and 1 1/2" long. Wire segments are 0.015" D. and 1" long.

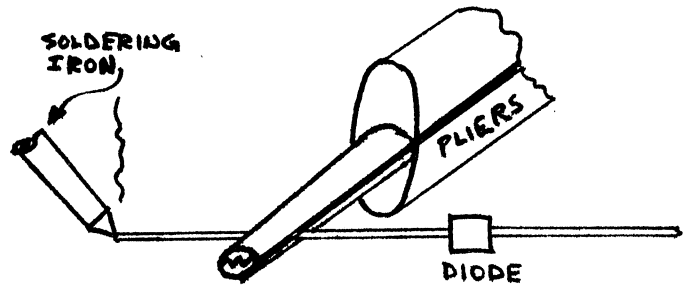


Fig. 1. Physical problem.

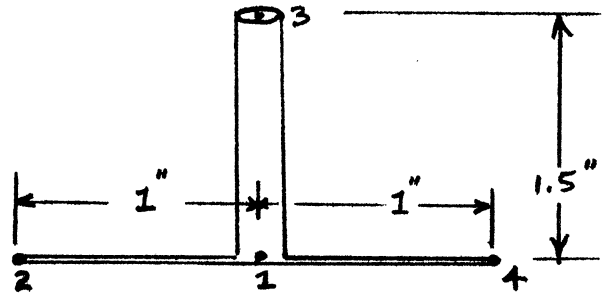


Fig. 2. Idealized model.

$$k_{\text{pliers}} = 6.00 \times 10^{-4} \frac{\text{BTU}}{\text{sec.}^\circ\text{F. in}} \quad (\text{heat conductivity for steel})$$

$$k_{\text{wire}} = 5.05 \times 10^{-3} \frac{\text{BTU}}{\text{sec.}^\circ\text{F. in}} \quad (\text{heat conductivity for copper})$$

The element heat conduction matrices are:

$$[h^P] = \frac{Ak}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{(\pi/4)(.25)^2 6 \times 10^{-4}}{1.5} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = 1.96 \times 10^{-5} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[h^W] = \frac{(\pi/4)(.015)^2 (5.05 \times 10^{-3})}{1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 8.92 \times 10^{-7} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The steel jaws clearly conduct heat better, primarily because of their larger size.

The matrix equations will first be found, including the pliers.

The assembled heat conduction matrix, by inspection, is

$$[H] \equiv \begin{bmatrix} 2h^w + h^p & -h^w & -h^p & -h^w \\ -h^w & h^w & 0 & 0 \\ -h^p & 0 & h^p & 0 \\ -h^w & 0 & 0 & h^w \end{bmatrix}$$

$$= 10^{-6} \begin{bmatrix} 21.4 & -.892 & -19.6 & -.892 \\ -.892 & +.892 & 0 & 0 \\ -19.6 & 0 & 19.6 & 0 \\ -.892 & 0 & 0 & +.892 \end{bmatrix}$$

Solve for the required heat flux, R_4 , defined positive into the circuit at node 4.

$$10^{-6} \begin{bmatrix} 21.4 & -.892 & -19.6 & -.892 \end{bmatrix} \cdot \begin{Bmatrix} \psi_1 \\ 500 \\ 72 \\ 72 \end{Bmatrix} = \begin{Bmatrix} 0 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix}$$

Unknown temperature (s) are found first. In this case, only one temperature is unknown.

$$21.4\psi_1 - .892(500) - 19.6(72) - .892(72) = 0$$

$$\psi_1 = \underline{\underline{89.8^\circ\text{F}}} \quad \longleftarrow \text{TEMP at junction of pliers and lead.}$$

Now get R_2 .

$$R_4 = 10^{-6} [(-.892)(89.8) + (.892)(72)]$$

$$= \underline{\underline{-1.59 \times 10^{-5} \frac{\text{BTU}}{\text{sec}}}} \quad \longleftarrow \text{HEAT flow out of diode at } 72^\circ\text{F.}$$

For the case without a heat sink, merely set $k^P = 0$, and resolve.

$$2k^W \psi_1 - k^W(500) - k^W(72) = 0$$

$$\psi_1 = \frac{500 + 72}{2}$$

$$\psi_1 = 286^\circ.$$

TEMP at midpoint of wire without pliers

Solve for R_4 .

$$R_4 = -.892 \times 10^{-6}(286) + (.892 \times 10^{-6})(72)$$

$$= \underline{\underline{-1.02 \times 10^{-4} \frac{\text{BTU}}{\text{sec}}}}$$

HEAT flow out of diode at 72°F .

Knowing the heat capacity of the diode, one could find the rate of temperature rise in the diode. Q. E. D.

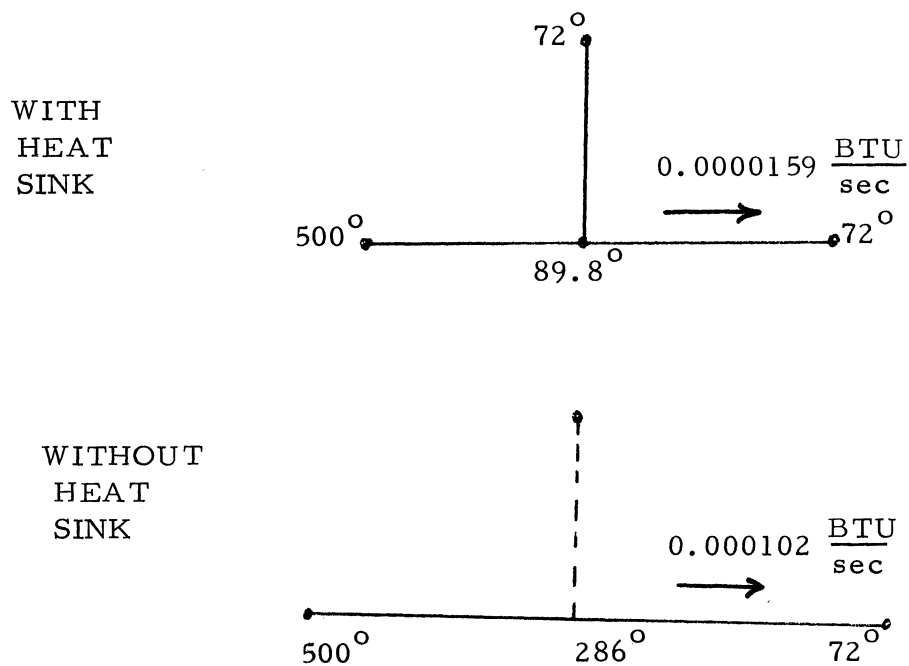


Fig. 3. Schematic of temperatures with and without heat sink.

12 April 1978

VARIATIONAL APPROACH TO
FIELD PROBLEMS

W. J. A.

- References: (1) Zienkiewicz, O. C., "The Finite Element Method," vol. 2, McGraw-Hill, 1971
(2) Desai, C. S. & Abel, J. F., "Introduction to the Finite Element Method," Van Nostrand-Reinhold, Chapter 12, 1972.

I. INTRODUCTION

Variational methods are important to finite element methods; indeed, we have already seen virtual work and potential energy. The current lecture attempts to back off a bit and look at the variational process itself in more detail. The examples chosen are energy conserving structures and the function is the potential energy of such a structure, but the applicability is broader - to steady state field problems in torsion, heat conduction, electric fields and magnetic fields. These all represent energy conserving systems which possess an underlying variational statement.

II. ALTERNATIVE USES OF VARIATIONAL METHODS

There are two basic variational approaches, general and direct. Both require a functional such as potential energy.

- A. General Approach - This method leaves the dependent field variables in functional form (such as $\Phi(x,y,z)$). A variation of the dependent variables is then performed. This leads to a differential equation (Euler equation) and associated boundary conditions.
- B. Direct Variation - A solution is assumed in series form before the variation is taken. A variation with respect to discrete variables (generalized coordinates) is then done. In the application to finite elements, the assumed series do not satisfy boundary conditions; rather, the boundary conditions are applied as a last step to an assembled, discrete system.

III. GENERAL VARIATIONAL APPROACH

Many field problems have an energy conservation law which states that a certain functional $\pi(\Phi)$ has a minimum at the solution $\Phi(x,y,z) = \Phi^*$. This means that $\delta\pi(\Phi) = 0$ at $\Phi = \Phi^*$. A typical functional, which leads to Poisson's equation, is:

$$\begin{aligned} \Pi = \int_{\text{vol}} \left\{ \frac{1}{2} \left[k_x \left(\frac{\partial \Phi}{\partial x} \right)^2 + k_y \left(\frac{\partial \Phi}{\partial y} \right)^2 + k_z \left(\frac{\partial \Phi}{\partial z} \right)^2 \right] - \bar{Q}\Phi \right\} d\text{Vol} \\ - \int_{\text{surf}} \left(\bar{q} \Phi - \frac{1}{2} \alpha \Phi^2 \right) d\text{Surf} \end{aligned} \quad (1)$$

The dependent field variable $\Phi(x,y,z)$ might represent a displacement or potential function. (For structural problems, Φ is a displacement and π is potential energy). The variation with respect to (w.r.t.) Φ yields

$$\delta\pi = \int_{\text{vol}} \left\{ \frac{1}{2} \left[2k_x \frac{\partial\Phi}{\partial x} \frac{\partial}{\partial x} (\delta\Phi) + 2k_y \frac{\partial\Phi}{\partial y} \frac{\partial}{\partial y} (\delta\Phi) + 2k_z \frac{\partial\Phi}{\partial z} \frac{\partial}{\partial z} (\delta\Phi) \right] - \bar{Q}\delta\Phi \right\} d\text{Vol} \\ - \int_{\text{surf}} (\bar{q}\delta\Phi - \alpha\Phi \delta\Phi) d\text{Surf} = 0 \quad (2)$$

The terms involving derivatives of $\delta\Phi$ must be transformed. Use a version of the unsymmetric Green's formula* to obtain

$$\delta\pi = - \int_{\text{vol}} \left\{ \left[\frac{\partial}{\partial x} k_x \frac{\partial\Phi}{\partial x} + \frac{\partial}{\partial y} k_y \frac{\partial\Phi}{\partial y} + \frac{\partial}{\partial z} k_z \frac{\partial\Phi}{\partial z} \right] - \bar{Q} \right\} \delta\Phi d\text{Vol} \\ + \int_{\text{surf}} \left[k_x \frac{\partial\Phi}{\partial x} + k_y \frac{\partial\Phi}{\partial y} + k_z \frac{\partial\Phi}{\partial z} - \bar{q} + \alpha\Phi \right] \delta\Phi d\text{Surf} = 0 \quad (3)$$

Since $\delta\Phi$ is an arbitrary variation of Φ , we will argue that the integrands in the volume and the surface integrals must separately vanish. If the curly-bracketed quantity in the volume integral did not vanish identically at some point in space, then the arbitrary variation $\delta\Phi$ could be chosen as a delta function centered at that point and the law would be violated. This leads to the differential equation (Euler equation)

$$\frac{\partial}{\partial x} k_x \frac{\partial\Phi}{\partial x} + \frac{\partial}{\partial y} k_y \frac{\partial\Phi}{\partial y} + \frac{\partial}{\partial z} k_z \frac{\partial\Phi}{\partial z} - \bar{Q}(x,y,z) = 0 \quad (4)$$

Likewise, the surface integral must vanish, but this can be accomplished in two ways, by the square-bracketed quantity vanishing on a part of the surface, say S_1 :

$$k_x \frac{\partial\Phi}{\partial x} l_x + k_y \frac{\partial\Phi}{\partial y} l_y + k_z \frac{\partial\Phi}{\partial z} l_z - \bar{q}(x,y,z) + \alpha\Phi = 0 \quad (\text{on } S_1) \quad (5)$$

and by the variation vanishing on the remaining surface, say S_2 :

$$\delta\Phi(x,y,z) = 0 \quad \text{or} \quad \Phi(x,y,z) = \Phi_B \quad (\text{on } S_2) \quad (6)$$

This completes the theory, in a sketchy way, for a typical "general" variation.



Fig. 1. Surface regions

*The standard form of Green's formula is given in Wiley, C. R., "Advanced Engineering Mathematics," 4th Ed., McGraw-Hill, 1975, p. 676, Eq. 8.

IV. EXAMPLE. LINE ELEMENT WITH ELASTIC SPRINGS. GENERAL VARIATION.

Consider a line element with springs at each end connected to fixed supports. The coordinate system is adjusted so that the springs are unstretched at $\Phi(0)$ and $\Phi(L)$. Concentrated loads act at $x=0$ and $x=L$. A distributed line load $\bar{\mathcal{L}}(x)$ acts along the length of the element. The element has varying area and modulus of elasticity. The springs have constants α_1 and α_2 . The situation is intended to represent a field problem for $\Phi(x)$ in the region $0 \leq x \leq L$. The general variational approach will be used to develop the differential equation and the boundary conditions.

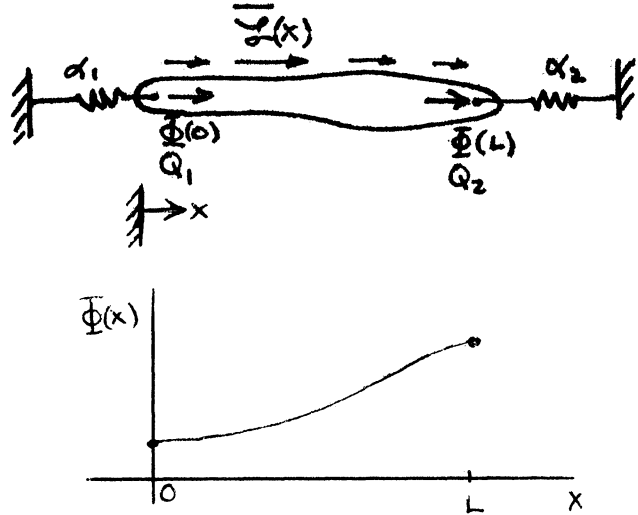


Fig. 2. Simple field problem. Line element.

We have the familiar line element cast in the role of a field problem, with the inclusion of springs. The springs are treated in the common potential energy manner as elastic elements within the system rather than as external forces.

Potential energy for the system (wall to wall) is

$$\pi(\Phi) \equiv (\text{STORED STRAIN ENERGY}) + (\text{WORK POTENTIAL OF EXTERNAL FORCES}) \tag{7}$$

Remember that the work potential \bar{W} is given a sign convention from the classical physics notation where $\{Q\} = -\{\nabla\} \bar{W}$ and where $\{\nabla\}$ is a generalized gradient operator. (In this case

$$\text{Force} = -\frac{d}{d\Phi} \bar{W}(\Phi),$$

for instance). Hence:

$$\pi(\Phi) = \underbrace{\frac{1}{2} \int_0^L A(x) \{\epsilon_x\}^T [C] \{\epsilon_x\} dx + \frac{1}{2} \{\Phi\}^T \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix} \{\Phi\}}_{\text{STRAIN ENERGY}} + \underbrace{\left[- \int_0^L \bar{\mathcal{L}}(x) \Phi(x) dx - \{Q\}^T \{\Phi\} \right]}_{\text{WORK POTENTIAL}} \tag{8}$$

where

$$\{\Phi\} \equiv \begin{Bmatrix} \Phi(0) \\ \Phi(L) \end{Bmatrix} \quad (9)$$

$$\{Q\} \equiv \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} \quad (10)$$

$$\{\epsilon_x\} = \begin{Bmatrix} \frac{d\Phi}{dx} \end{Bmatrix} \quad (11)$$

$$[C] = [E] \quad (12)$$

The variation with respect to Φ is carried out:

$$\begin{aligned} \delta\pi(\Phi) &= \int_0^L A(x)E(x) \left(\frac{d}{dx} \Phi(x)\right) \left(\frac{d}{dx} \delta\Phi(x)\right) dx + \alpha_1 \Phi(0) \delta\Phi(0) + \alpha_2 \Phi(L) \delta\Phi(L) \\ &\quad - \int_0^L \bar{\mathcal{L}}(x) \delta\Phi(x) dx - \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix}^T \begin{Bmatrix} \delta\Phi(0) \\ \delta\Phi(L) \end{Bmatrix} \end{aligned} \quad (13)$$

Integrating the first term on the right hand side (R.H.S.) by parts and collecting terms, one has

$$\begin{aligned} \pi(\Phi) &= - \int_0^L \left[\frac{d}{dx} AE \frac{d\Phi}{dx} + \bar{\mathcal{L}}(x) \right] \delta\Phi(x) dx \\ &\quad + [AE \frac{d}{dx} \Phi(L) + \alpha_2 \Phi(L) - Q_2] \delta\Phi(L) + [-AE \frac{d\Phi(0)}{dx} + \alpha_1 \Phi(0) - Q_1] \delta\Phi(0) \end{aligned} \quad (14)$$

We want this to vanish. The integral term is easy, because we want $\delta\Phi(x)$ to be arbitrary in the interior of the link. Hence, the Euler equation is

$$\frac{d}{dx} \left(AE \frac{d\Phi(x)}{dx} \right) + \bar{\mathcal{L}}(x) = 0 \quad \text{Equation of Equilibrium} \quad (15)$$

The boundary conditions are tougher. At the left end, we can either specify a natural (force) or a geometric boundary condition:

$$- AE \frac{d\Phi(0)}{dx} + \alpha_1 \Phi(0) - Q_1 = 0 \quad (16a)$$

or

$$\delta\Phi(0) = 0 \quad (16b)$$

respectively.

Likewise at the right end of the link:

$$AE \frac{d}{dx} \Phi(L) + \alpha_2 \Phi(L) - Q_2 = 0 \quad (16c)$$

or

$$\delta\Phi(L) = 0 \quad (16d)$$

The force condition corresponds to an equilibration of forces on the pin.

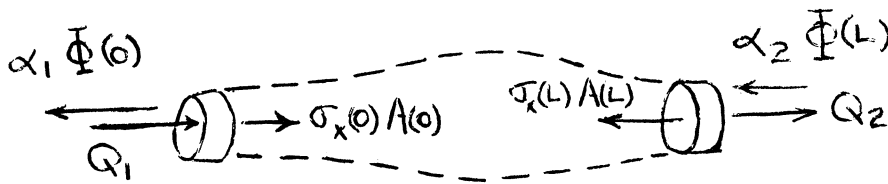


Fig. 3. Interpretation of boundary conditions.

Hence, at each end, we may either set the pin in equilibrium or we may hold the pin at a prescribed displacement. The latter causes $\delta\Phi$ to be zero during the variation.

The general variation yields a differential equation and consistent boundary conditions. Perhaps this problem is sufficiently intricate to show that the boundary conditions are not easy to see from equilibrium. (Note that we did not use equilibrium to get the sign sense of the elastic forces - they come from the variation.)

V. DIRECT VARIATIONAL APPROACH

The functional in Eqn. (1) is to be minimized. We have previously shown a general variation w.r.t. the scalar function $\phi(x)$. We can now take a different route, to discretize the function $\phi(x)$ and to take a variation w.r.t. the discrete values. In other words, $\phi(x)$ is sampled at a certain number of x points (nodes) and those values of $\phi(x_i)$ used as new variables. This is the Rayleigh-Ritz approach.

Let us conceive "global shape functions," which are nonzero within only one element. For a scalar field variable $\phi(x)$, and for a total of M nodes, one has

$$\Phi(x) = \sum_{i=1}^M N_i(x) \phi_i \quad (17)$$

field
variable

$$= [N_1(x), N_2(x), \dots, N_M(x)] \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_M \end{Bmatrix} \quad (18)$$

nodal values
of $\Phi(x)$

When x lies in an element e , the shape functions $N_i(x)$ switch to a set of shape functions zero everywhere except in that element.* Instead of a scalar variation

$$\delta\pi \equiv \frac{\partial\pi}{\partial\Phi} \delta\Phi,$$

we now need a vector variation:

$$\delta\pi = \frac{\partial\pi}{\partial\phi_1} \delta\phi_1 + \frac{\partial\pi}{\partial\phi_2} \delta\phi_2 + \dots + \frac{\partial\pi}{\partial\phi_M} \delta\phi_M \quad (19)$$

In summation notation, we have

$$\delta\pi = \sum_{i=1}^M \frac{\partial\pi}{\partial\phi_i} \delta\phi_i \quad (20)$$

In vector form, using Zienkiewicz's shorthand notation for the gradient vector

$$\delta\pi = \left\{ \frac{\partial\pi}{\partial\{\Phi\}} \right\}^T \{\delta\Phi\} \quad (21)$$

This, of course, is the process on which the entire finite element theory is based.

One can view Eqn. (20) as an approximation to the full Taylor series expansion:

* This is a difficult concept for engineers but seems easy for mathematicians to visualize. In any event, the generation of such global shape functions means the assembly is accomplished automatically, and, in fact, is not needed. The shape functions must still allow for inter-element compatibility of derivatives on Φ up to order one less than the highest derivative appearing in the function (Eq. (1)). See Zienkiewicz, Ed. 2, pp. 24-26 for details.

$$\Delta\pi \equiv \pi(\{\Phi\} + \{\Delta\Phi\}) - \pi(\{\Phi\})$$

$$\Delta\pi = \underbrace{\sum_{i=1}^M \left. \frac{\partial\pi}{\partial\phi_i} \right|_{\{\Phi\}} \delta\phi_i}_{\text{"Differential"}} + \frac{1}{2} \sum_{i=1}^M \sum_{j=1}^M \left. \frac{\partial^2\pi}{\partial\phi_i\partial\phi_j} \right|_{\{\Phi\}} \delta\phi_i \delta\phi_j + \dots$$

"Differential"

CONCLUSION

Aug. 25, 1980

W. J. A.

We have seen the fundamentals of a new approach to problem solving, the finite element method. In teaching finite elements for ten years at the University of Michigan, the author has developed an increasing enthusiasm for the method. Master's level students have entered a year-long program with no knowledge of the method. After studying for two terms, they have been able to start work immediately on doctoral dissertations. There is possibly no other training tool which has increased intuition and knowledge of mechanics as much as the finite element method.

Upon reviewing the various branches of mechanics and physics which are studied in undergraduate engineering courses, one sees that many courses could be redone using finite element methods. At the first blush of enthusiasm, one might wish to recast all undergraduate teaching into a finite element mold. One such attempt was made in solid mechanics at the Aerospace Department of the University of Texas. It did not work and the department eventually backed away from this. One can conclude that although finite element methods are powerful and can solve many problems, they cannot replace classical methods because classical methods are required to create individual elements. Indeed, the interpolation scheme used in the displacement method is mathematically equivalent to the Rayleigh-Ritz method. Galerkin's method is often used to create elements, particularly fluid flow and other non-conservative elements. It appears that the finite element method must be a strong partner with the classical methods of analysis rather than supplanting them.

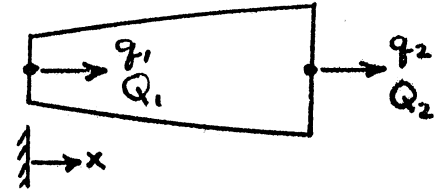
HOMEWORK

HOMEWORK
PROBLEMS

June 23, 1978
W. J. A.

Problem 1

Find the exact stiffness matrix $[k]$ for a two-node line element with varying area $A = A_0(1+Bx/L)$. Use an equilibrium method, relying on the fact that the line element is statically determinate.



Problem 2

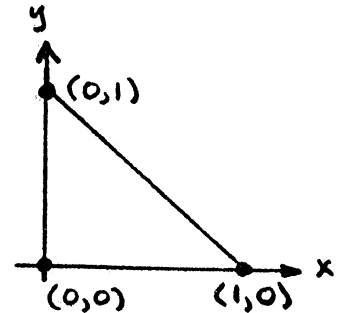
Find an approximate stiffness matrix $[k]$ for the two-node line element in Problem 1 using the energy approach. Assume constant strain.

Problem 3

Find the stiffness matrix for a plane stress, constant strain triangle with local coordinates shown. Use the equilibrium approach with assumed displacement function:

$$u(x, y) = \alpha_1 + \alpha_2 x + \alpha_3 y$$

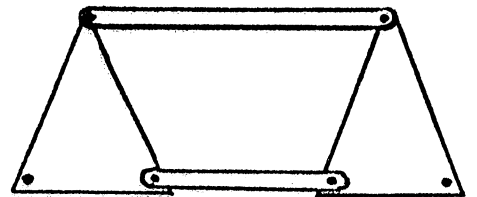
$$v(x, y) = \alpha_4 + \alpha_5 x + \alpha_6 y$$



The fact that the vertices have unit coordinate values will simplify the problem greatly. Use the expression for $[k]$ involving $[A]^{-1}$ outside the integral.

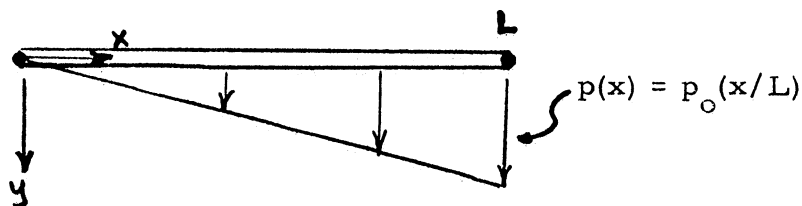
Problem 4

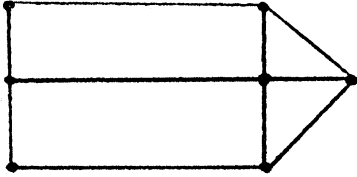
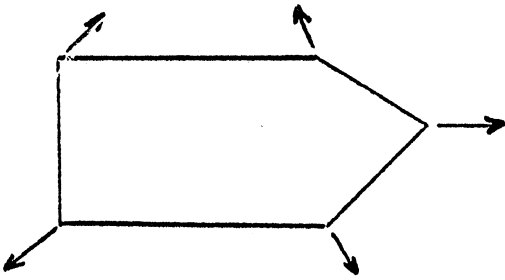
Assemble the stiffness matrix for the following structure using symbols Δ and \ominus to represent the stiffness terms. Number nodes as you wish. Use the compact notation.



Problem 5

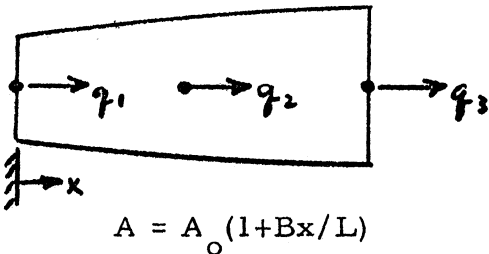
For the beam element based on Euler-Bernoulli theory and with displacement function $w(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3$, develop the equivalent nodal loads for the loading shown.



Problem 6

A thin plate sketched in Figure 1 is to be subjected to various inplane loads. A finite element study is to be made in which two rectangular and two triangular elements will be used. The rectangles each have eight nodal degrees of freedom and the triangles each have six d. o. f.

- Number the nodal degrees of freedom in such a way as to minimize the bandwidth of the assembled stiffness matrix.
- Which of the elements in the assembled stiffness matrix are zero?
- Explain in words how a distributed loading along one edge of an element can be handled. What are the principles involved?

Problem 7

$$A = A_0(1+Bx/L)$$

- A line element has changing cross-sectional area as shown. Develop the stiffness matrix for a 3-node element, choosing the interior midpoint as a node. Use a quadratic displacement function.
- Reduce the element's stiffness to a 2×2 stiffness matrix by "condensing out" the middle node. Present this matrix in its simplest form.
- Compare the 2×2 stiffness matrix with that obtained earlier for a uniform prismatic link and with that obtained in Problems 1 & 2.

Problem 8

- A) Solve the following set of equations with the use of a Gauss-Dolittle decomposition:

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 0 \\ 1 & 0 & 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 5 \\ 12 \end{Bmatrix}$$

- B) Solve the equations by a Choleski decomposition method.

Problem 9

Solve the set of equations given in Problem 8 by using a Gauss-Seidel iteration. The method converges slowly; a programmable calculator may be needed to get satisfactory convergence. Start with an initial guess:

$$\begin{Bmatrix} x \end{Bmatrix}^{(0)} = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

Problem 10

Write out the equations necessary to program in FORTRAN the solution by Gauss-Dolittle factorization of the standard equation:

$$[K] \{r\} = \{R\}$$

Assume the matrix $[K]$ is an $n \times n$, real, symmetric, positive-definite matrix. The vector $\{R\}$ is known and $\{r\}$ is unknown. Assume $[K]$ is a full matrix and not banded. Use the lower triangular factorization.

Problem 11

Develop a FORTRAN subroutine for the Gauss-Dolittle decomposition required in Problem 10. Name the subroutine DCOMP(N, K, *), and write the decomposed portions $[L_1]$ and $[D]$ back on the lower triangular part of $[K]$, in order to save space. Use the * location for an error return.

Problem 12

Develop a FORTRAN subroutine for the factorization solution of the equation studied in Problem 10.

$$[L_1] [D] [L_1]^T \{r\} = \{R\}$$

(CONTINUED)

Problem 12 Continued

Assuming the $[L_1]$ and $[D]$ are available in the lower triangular portion and diagonal portion respectively of $[D]$. Name the subroutine SOLVE(N, K, R, D) where $R(I)$ is the reaction vector and $D(I)$ is the displacement vector. Carry out both the forward solution and the back substitution.

Problem 13

This question is intended to test your knowledge of the stiffness matrix, its definition and physical interpretation.

A linearly elastic coil spring is to be treated as a single finite element. Nodal degrees of freedom are the axial displacement u_1, u_2 and the rotations v_1, v_2 . The spring is 12" long and 2" outside diameter. It is made of steel wire of 0.050" diameter.

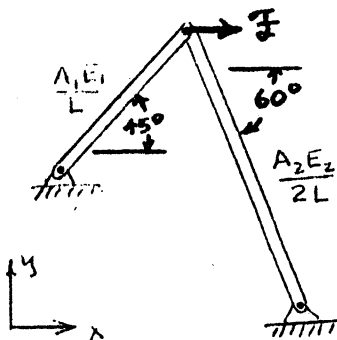


Nodal Displacements

Two experiments have been carried out to provide stiffness data. The left end of the spring is clamped firmly. A displacement u_2 of 1" causes forces $U_1 = -100$ lb and $V_1 = -50$ in. lb. ($v_2 = 0$). A rotation v_2 of 1 radian causes forces $U_1 = -50$ lb. and $V_1 = -100$ in. lb. ($u_2 = 0$). (Assume the spring remains within the linear range during this rotation).

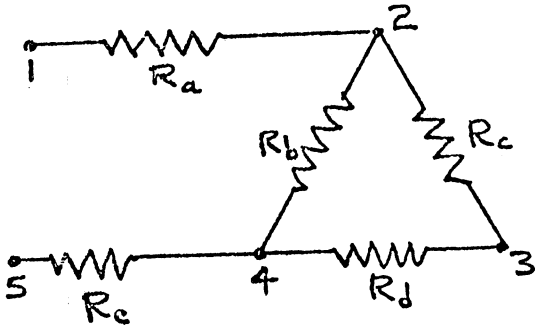
- Using equilibrium concepts, construct the stiffness matrix for this spring.
- If two such identical spring elements are joined in series, find the stiffness matrix for the assembled system. Use letters to represent any stiffness components not found in (a).

Problem 14



A truss is made of two elements as shown. A horizontal force F is applied at the center node (the vertex). Find the displacements at this node in terms of the length L , areas A_1 , moduli E_1 and force F . Use a global system aligned with the horizontal to express your results.

Problem 15



An electrical network is shown.

- (a) Set up the equation for a single element with nodes i and j.
- (b) In analogy with the structural stiffness problem, does the voltage at node V_i correspond to force or displacement? To what does the current I_i correspond?
- (c) Give the assembled equations in matrix form for the network. Do by inspection.

Problem 16

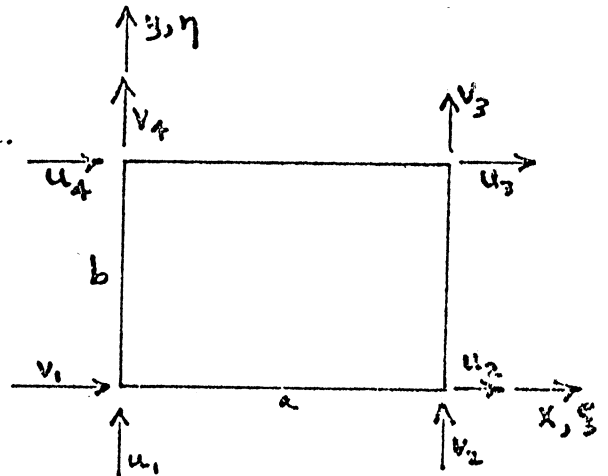
(Weaver)

Consider the rectangular element for plane stress and plane strain with generic and nodal displacements as shown. Assume the displacement functions for $u(\xi, \eta)$ and $v(\xi, \eta)$ to be

$$u(\xi, \eta) = \alpha_1 + \alpha_2 \xi + \alpha_3 \eta + \alpha_4 \xi \eta$$

$$v(\xi, \eta) = \alpha_5 + \alpha_6 \xi + \alpha_7 \eta + \alpha_8 \xi \eta$$

in which $\xi = x/a$ and $\eta = y/b$. Also let $\beta = b/a$.



The stress-strain matrix is

$$[D] = \frac{E}{(1+\nu)e_2} \begin{bmatrix} e_1 & \nu & 0 \\ \nu & e_1 & 0 \\ 0 & 0 & e_3 \end{bmatrix}$$

where plane stress: $e_1 = 1$, $e_2 = 1-\nu$, $e_3 = e_2/2$

plane strain: $e_1 = 1-\nu$, $e_2 = 1-2\nu$, $e_3 = e_2/2$

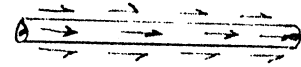
Derive the following items for plane stress and plane strain:

- a. Stiffness matrix $[k]$ of order 8×8 .
- b. Equivalent nodal loads $\{F\}_{e.n.l.}$ for a linearly varying force normal to edge 1-2 and of intensity p_1 at node 1 and intensity p_2 at node 2.
- c. Equivalent nodal loads for uniform gravity loading p_g (per unit volume), acting in the $-y$ direction.

Problem 17

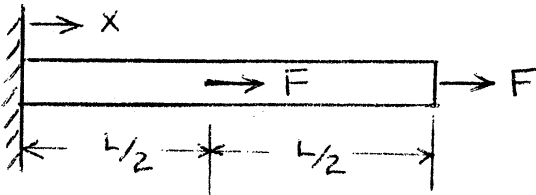
What are the equivalent nodal loads for a 10" long cylindrical bar due to a surface traction

$$\{q(x_s, y_s, z_s)\} = \{100 \text{ lb/in}^2\}$$



if the bar is one inch in diameter? A two-node line element is used.

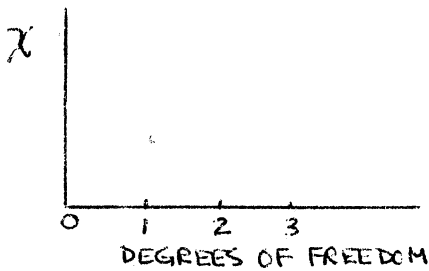
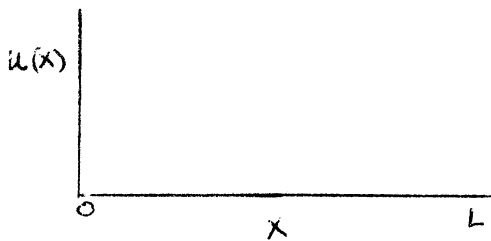
Problem 18



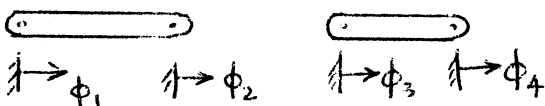
A linear elastic rod is loaded as shown. It is of uniform cross section and mechanical properties.

(a) Carry out a one-element and a two-element solution for displacement of the rod using the 2 D. O. F. line element.

(b) Make comparisons of strain energies, potential energies and displacements for the two cases. Present this in graphical form as displacement field $u(x)$ versus x and energy at state II (equilibrium) vs. number of degrees of freedom.



Problem 19

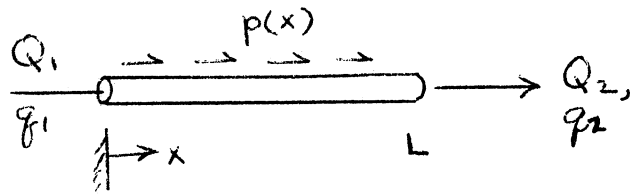


Use Lagrange multiplier s to impose the constraint of assembly $\Phi_3 = \Phi_2$ on the two prismatic links shown. The result is the matrix equation of equilibrium with 5 unknowns.

Problem 20

Find the equations of equilibrium for a finite element for a prismatic link, using Galerkin's method.

Diff. Eqn. :



$$EA \frac{d^2 u(x)}{dx^2} + p(x) = 0$$

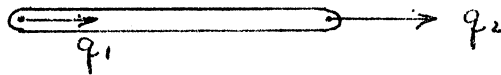
The relation between nodal loads and strains at nodes is:

$$Q_1 = -EA \left. \frac{du}{dx} \right|_0$$

$$Q_2 = EA \left. \frac{du}{dx} \right|_L$$

Problem 21

Cast the entire prismatic link development into the natural mode approach. Assume a uniform cross section and constant mechanical properties.



- Define the two modes needed, one a straining and one a rigid body mode.
- Develop the relation between nodal displacement and generalized coordinates

$$\{q\} = [A] \{\alpha\}$$

- Develop the relation between the generalized forces and the given nodal loads

$$\{Q_\alpha\} = [A]^T \{Q\}$$

- Develop the stiffness matrix $[k_\alpha]$ which relates the generalized coordinates and the nodal action vector.

$$\{Q_\alpha\}^e = [k_\alpha] \{\alpha\}^e$$

Problem 22

Zienkiewicz claims in Section 3.5 of "The Finite Element Method," that minimizing the functional

$$\chi = \int_{\text{surf}} \left[\frac{1}{2} \left(\frac{\partial \phi}{\partial x} \right)^2 + \frac{1}{2} \left(\frac{\partial \phi}{\partial y} \right)^2 - C\phi \right] d(\text{surf})$$

$$+ \int_{\text{boundary}} \left(q\phi + \frac{1}{2} \alpha \phi^2 \right) d(\text{boundary})$$

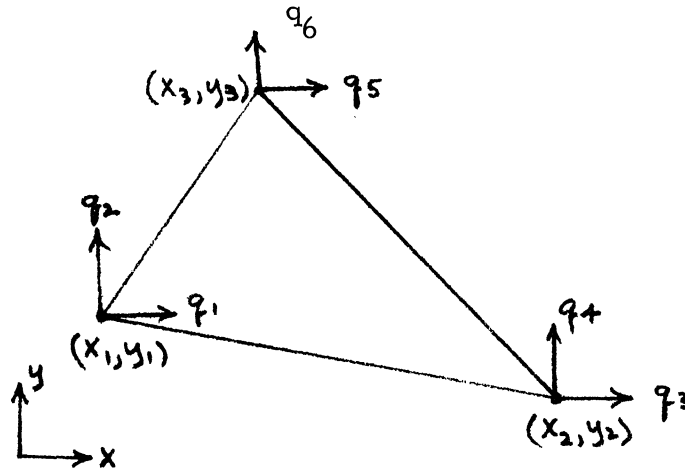
is equivalent to solving the differential equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + C = 0$$

Prove this by carrying out a general variation of χ and show the resulting Euler equation and consistent boundary conditions.

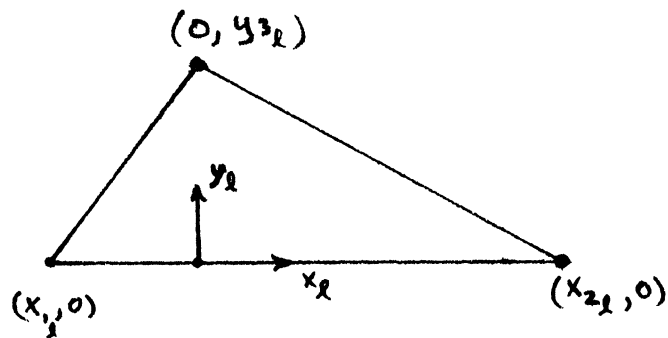
Problem 23. (Estimated time: 1 1/2 hour)

Develop the matrix which relates internal stresses to nodal forces for the constant-strain, plane-stress triangle. This is the $[C]$ matrix mentioned in the Lecture 4. Use a global coordinate system.

Problem 24.

Develop the stiffness matrix for the constant-strain, plane-stress triangle in the local coordinates shown. You may drop the subscript l on the coordinates for the calculation, since it is obvious that a local system is implied. Use the direct method, with the equilibrium matrix developed in Problem 23.

Comment: The toughest part of this problem is the inversion of the $[A]$ matrix. Work with another student on this to avoid error. Also, it is easier to use the 3rd expression for $[k]$ on page 38, involving $[A]^{-T} \int (\quad) dv [A]^{-1}$.



Problem 25. Fluid Flow in Pipes

A pipe assembly is shown. If

$$\Delta P^e = \bar{f} Q^{1.8}$$

Q in gallons/sec

ΔP in psi

and if

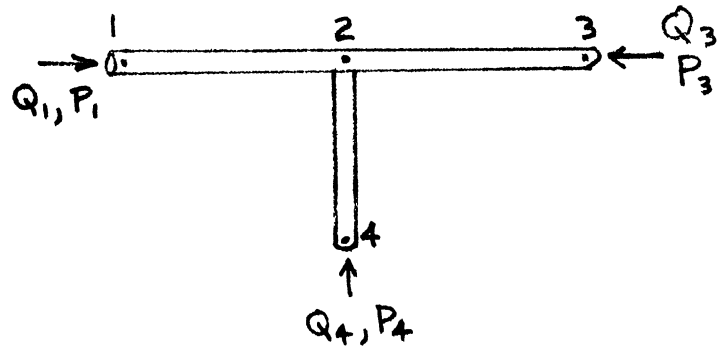
$$Q_{i_1} = 2 \text{ gallons/sec}$$

$$Q_3 = -5 \text{ gallons/sec}$$

$$Q_4 = 3 \text{ gallons/sec}$$

$$\bar{f} = 0.5 \text{ lb. sec}^{1.8} / (\text{in}^2 \cdot \text{gal}^{1.8}),$$

find the fluid pressure at each node.

Problem 26.

$$\text{If } \tilde{f}(y) = 1 + y^2 - y^3,$$

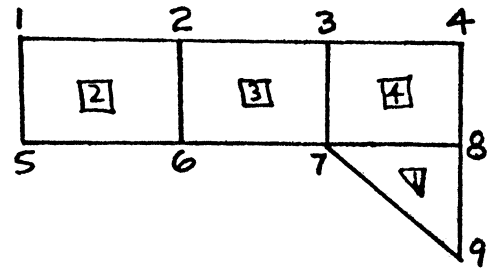
evaluate

$$\int_1^5 \tilde{f}(y) dy$$

using Gaussian quadrature. Use the minimum number of sample points.

Problem 27

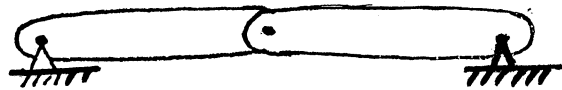
A sheet of metal has been modelled with an assembly of plane strain quadrilaterals and triangles as shown. Using "compact" notation,



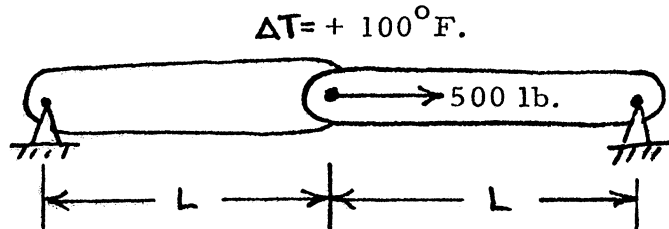
- the half bandwidth is _____.
 - the maximum wavefront is _____.
 - the number of zero terms in the assembled stiffness matrix is _____.
- If an optimum number of nodes had been done, the half bandwidth could have been reduced to _____.
- If an optimum numbering of elements had been done, the maximum wavefront would have been _____.

Problem 28

Two identical line elements are joined at the center and fixed at the ends as shown, with zero initial strain. A concentrated load of 500 lbs. is applied to the right at the connection, and the bodies are heated 100°F . Using finite element theory and notation, find the displacement at the center connection and the reactions at the fixed ends.



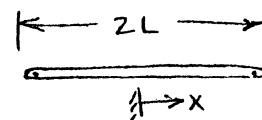
$E = 30 \times 10^6 \text{ psi}$
 $A = 1 \text{ in.}^2$
 $L = 10 \text{ in.}$
 $\alpha = 7 \times 10^{-6} \text{ in/in. } ^\circ\text{F.}$



Problem 29

Consider a coordinate system with origin located at the center of a line element. Use a displacement function

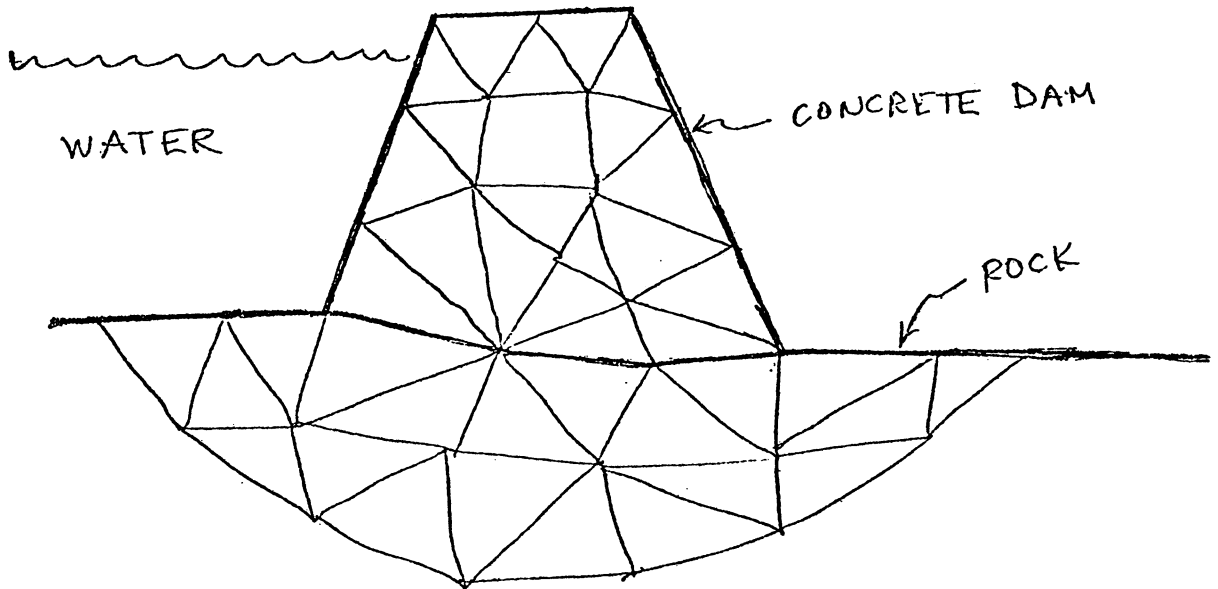
$$u(x) = \alpha_1 + \alpha_2 x/L$$



Find the equivalent nodal reactions for a distributed load $p(x) = p_0(x/L)^2$ where p_0 has dimensions of force per unit length.

Problem 30

A concrete dam is to be constructed on rock as shown. The heavy lines indicate boundaries between materials (rock-concrete, concrete-water, etc). A plane-strain solution using triangular elements is to be found. The rock will be modelled only as far from the dam as felt necessary to give accurate stresses in the dam.



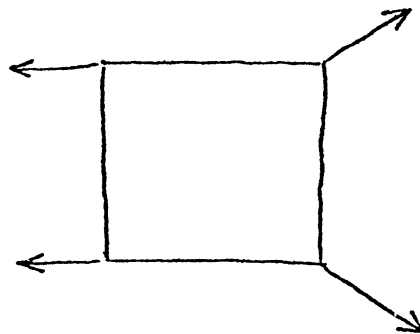
- (1) How many nodes will have nonzero equivalent nodal loads due to water pressure?
- (a) 4 (b) 5 (c) 6 (d) 11
- (2) What boundary conditions should be applied at the bottom most nodes in the rock?
- (a) zero forces (b) zero displacements
(c) zero strains (d) finite forces
- (3) Which of the following would be impossible to model with today's state of the art?
- (a) thermal prestrain due to heat in concrete
(b) earthquake effects
(c) water height which causes crack initiation at base of dam.
(d) natural modes of vibration of dam
(e) none of the above are beyond the state of the art

(4) What do you see as the major weakness of the given approach?

- (a) (a) There are not enough small elements at the base of the dam where cracking in the rock might initiate.
- (b) A rectangular element would be better for rock.
- (c) Dam problems are really plane stress.
- (d) It is difficult to include equivalent nodal loads for air pressure.

Problem 31

A plane-stress problem has been posed. A thin sheet of metal is to be acted upon by four loads as shown. In addition, gravity acts on the plate. As presented, no constraints on displacement are given. Assume that the external forces place body in force equilibrium.

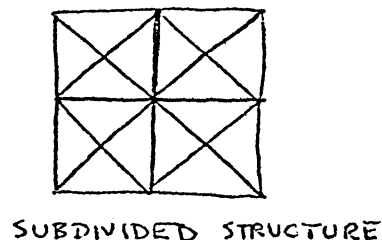


(1) How many elements does the subdivided structure have?

- 8
- 16
- 24

(2) How many degrees of freedom are there in the assembled structure if the Turner Triangle is used?

- 13
- 26
- 36

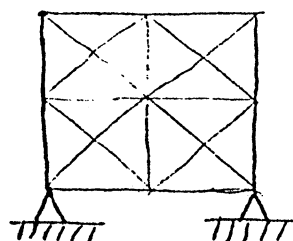


(3) How many degrees of freedom must be constrained in order to eliminate rigid body modes?

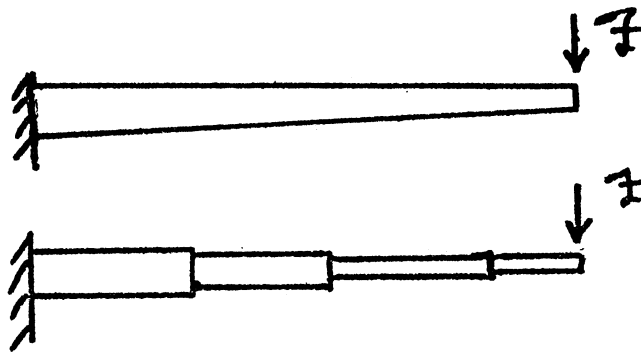
- 3
- 4
- 6

(4) Would the following be an acceptable way to eliminate rigid body modes without introducing additional stresses?

- yes
- no



Problem 32

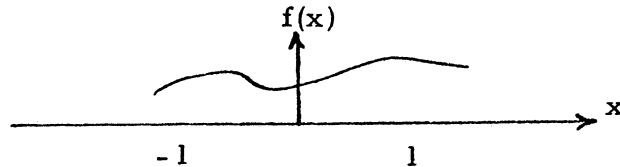


A cantilevered pole is modelled by Euler-Bernoulli beam elements as shown. Each element is of constant cross-sectional properties and has four nodal degrees of freedom. Including the node at the support,

- 1) How many nodes are there in the assembled structure? _____
 - 2) How many D. O. F. (detailed notation) in the assembled structure? _____
 - 3) How many equations of equilibrium in assembled form before applying boundary conditions (detailed notation)? _____
 - 4) What is the minimum half bandwidth of the assembled stiffness matrix? _____
- 5) Is it possible to solve for the unknown displacements in this problem without finding the unknown forces? Why?
- | | Yes | No |
|--|--------------------------|--------------------------|
| | <input type="checkbox"/> | <input type="checkbox"/> |
- 6) Suppose a more refined element were used which involved $w(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3 + \alpha_5 x^4 + \alpha_6 x^5$ but still assumed constant area. The refined element would lead to a solution with smaller or greater deflections under the load P ?
- | | smaller | greater |
|--|--------------------------|--------------------------|
| | <input type="checkbox"/> | <input type="checkbox"/> |

Problem 33

The integrand for a certain isoparametric element integration involves quartic polynomials. The numerical integration is to be done with Gaussian quadrature. Write out the necessary defining equations used to generate the weighting factors and sample points, using the minimum number of sample points for an exact integration. Use the symmetric form of integration. You don't need to solve the equations.

Problem 34

For the quartic above, if one used Radau integration where one fixes one sample point (at $x = -1$, say), how many sample points would be needed for an exact integration, including the fixed point?

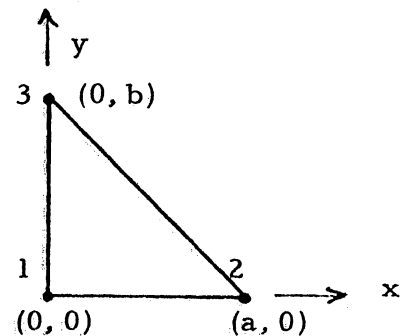
Problem 35

A constant temperature gradient triangle for use in steady-state heat conduction is to be developed. It will be the equivalent of a Turner triangle for the heat problem. A right triangle with a local coordinate system will be considered (see sketch).

The nodes are numbered as shown.

The first shape function is given:

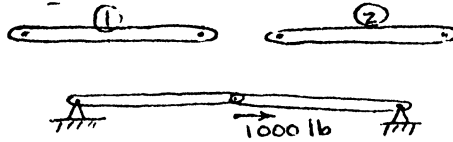
$$N_1 = 1 - x/a - y/b.$$



1. Find N_2 and N_3 .
2. Calculate the heat conduction term h_{22} .
3. Set up the expression for equivalent nodal flux due to a surface flux source $\bar{q}(x, y) = \bar{q}_0 x^2 y^2$. You need not carry out the integral.

Problem 36

Two identical steel bars have been purchased and holes have been drilled at each end for pins. The stiffness EA/L is 1000 lb/in. for each bar. The bars are cooled ten degrees Fahrenheit from room temperature and then assembled in such a way as to stretch them to their original room tempera-

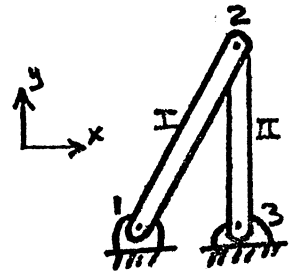


ture length. This results in a tension of 100 lb in each bar. A load of 1000 lb. to the right is then applied at the center pin. Use an initial strain concept to find:

$$\{Q\}_{\epsilon_0}^1, \{Q\}_{\epsilon_0}^2, \{r\}, \{R\}$$

Problem 37

A truss structure is assembled as shown. The nodes are numbered 1-3. The elements are numbered with Roman numerals. Suppose the structure has been assembled:



$$\begin{Bmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \\ U_3 \\ V_3 \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} \\ & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} \\ & & & \cdot & & \\ & & & \cdot & & \\ & & & \cdot & & \\ & & & & & \\ & & & & & k_{66} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

(symmetry)

1) What sign does k_{43} have? Why?

2) What sign does $(k_{33}k_{44} - k_{34}^2)$ have? Why?

SAMPLE EXAMS

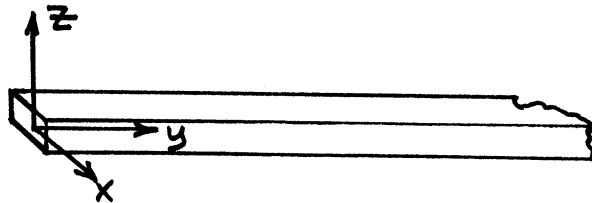
FROM

"FINITE ELEMENTS IN MECHANICS, I"

(Aero 510, M.E. 557, AM/ES 510)

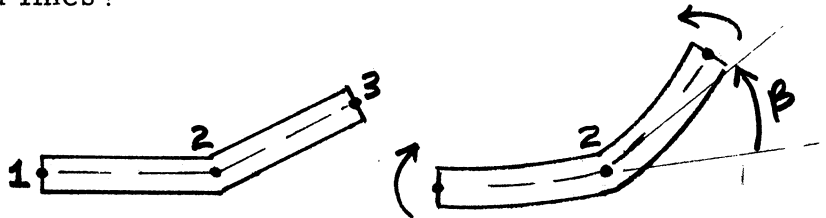
- 1) An assembly has a plane of symmetry at $y = 0$. A beam passes through this plane as shown, with axis in the y direction. Using global coordinates, and standard MSC/NASTRAN notation, which of the following degrees of freedom should be constrained at the plane of symmetry:

- (a) 2, 4, 6
- (b) 1, 3, 5
- (c) 2, 5, 6
- (d) 1, 2, 3
- (e) 3, 5, 6



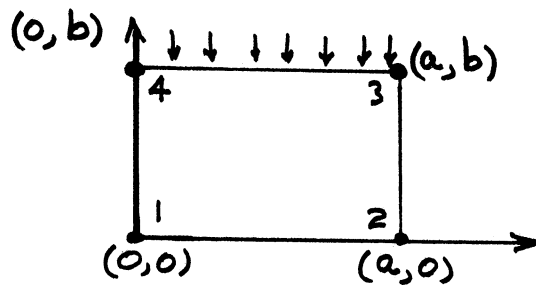
- 2) Two straight beams are welded together at an angle of 30° as shown. A scribed line is drawn along the neutral axis. The body is then loaded. An analysis predicts that a rotation of 10° should occur at node 2. After loading, then, what is the local angle β shown in the deformed body, as indicated by the scribed lines?

- (a) 20°
- (b) 30°
- (c) 40°
- (d) None of the above
- (e) There is not enough information to tell.



- 3, 4, 5) A heat conduction rectangle has shape function:

$$[N] = [(1 - x/a)(1 - y/b) \quad x/a(1-y/b) \quad \frac{xy}{ab} \quad (1-x/a)y/b]$$



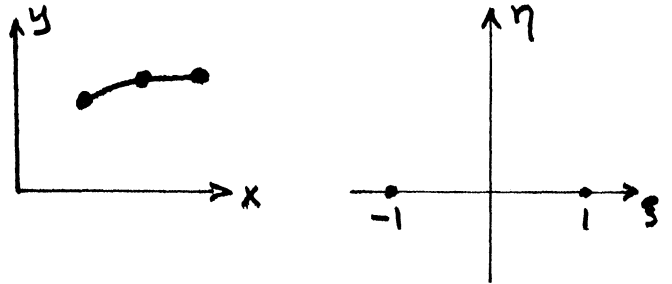
What is the equivalent nodal flux into node 3 due to an input of f $N \cdot \text{mm}/\text{mm}/\text{sec}$ along the top edge? The element has thickness h and heat conductivity

- (a) $(1/3)fa$
- (b) $(1/4)fa^2$
- (c) $(1/2)kfa$
- (d) ab
- (e) $(1/2)fa$

- 6) A slightly curved line element lying in the x, y plane is to be mapped onto a double unit line segment in an attempt to create an isoparametric element. How many Gauss points are needed to exactly integrate the physical length of the element given by the formula

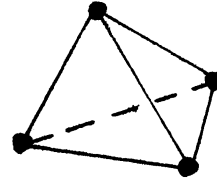
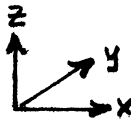
$$L = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

- (a) 1
 (b) 2
 (c) 3
 (d) 4
 (e) Transcendental functions can't be exactly integrated.



- 7) A solid, tetrahedron element is to be generated with displacement functions. How many rigid-body modes does it have?

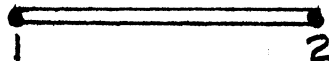
- (a) 1
 (b) 2
 (c) 3
 (d) 4
 (e) 6



- 8) How many straining modes does it have?
 (a) 4 (b) 6 (c) 10 (d) 12 (e) 18

- 9, 10) A uniform line element has length L and cross-sectional area A . What should the consistent mass terms m_{11} and m_{22} be?

- (a) $A \rho L/2$
 (b) $A \rho/2$
 (c) $\rho AL/3$
 (d) $\rho AL/4$
 (e) $\rho AL/6$



- 11) What are the lumped mass terms m_{11} and m_{22} for the same uniform line element of the previous problem?

- (a) $\rho AL/6$
 (b) $\rho AL/2$
 (c) $\rho AL/4$
 (d) $\rho AL/8$
 (e) None of the above.

- 12) Suppose you were asked to develop a Gauss integration procedure for integration in a triangular region in the x, y plane. The Gauss formula would look like

$$\int_A f(x, y) dA = \sum_{i=1}^3 w_i f(x_i, y_i).$$

(continued on page 3)

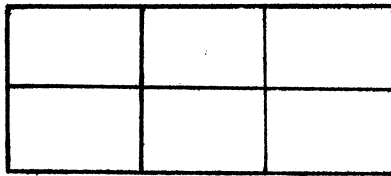
12) (continued)

What functional values would you choose to develop the required defining equations?

- (a) $1, x, y, x^2, xy, y^2$
- (b) $1, x, y, x + y$
- (c) $1, x, y$
- (d) x, y, z
- (e) $1, x, y, z, x^2, y^2$

13, 14) A plate problem has been solved using 6 QUAD4 elements as shown. It is desired to redo the problem with the same number of QUAD8 elements. If a Gauss elimination solution is used, by what factor will the computer CPU time increase for the equation solving portion?

- (a) 2
- (b) 3
- (c) 5
- (d) 10
- (e) 20



15, 16) A hypothetical structural element with stiffness $[k]$ has a set of loads applied which give an equilibrium displacement $\{q^*\}$. A subsequent virtual displacement $\{\Delta q\}$ is given. How much virtual work is done by the external forces? Units are Newtons and mm.

$$[k] = 10^2 \begin{bmatrix} 3 & 1 & 2 \\ 1 & 4 & 1 \\ 2 & 1 & 5 \end{bmatrix}$$

$$\{q^*\} = \begin{Bmatrix} 1 \\ 1 \\ 2 \end{Bmatrix}$$

$$\{\Delta q\} = \begin{Bmatrix} 0 \\ 0 \\ .01 \end{Bmatrix}$$

- (a) - 5 N mm
- (b) 0 N mm
- (c) 10 N mm
- (d) 50 N mm
- (e) 100 N mm

17) Design sensitivity parameters $\frac{\partial \{r^*\}}{\partial x_i}$ can be as easily found in a static stress problem as an additional

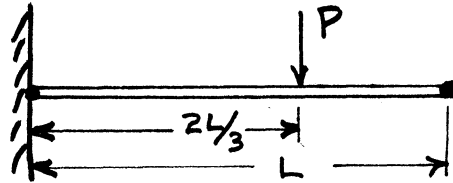
- (a) set of boundary conditions
- (b) material property
- (c) generation of element stiffnesses
- (d) natural frequency
- (e) none of the above.

18) Electrical and fluid pipe networks have a sign convention for electrical and fluid flow that is:

- (a) the same as for trusses in 2-D
- (b) based on flow into a node
- (c) depends on the underlying coordinate system
- (d) not valid for curved pipes
- (e) none of the above.

- 19) DMAP procedures in MSC/NASTRAN can be used to
- exit early in a solution sequence
 - insert additional calculations
 - resequence nodes
 - print out additional information
 - all of the above.
- 20) A beam has a concentrated load P at a point $2L/3$ from its left end. What fraction of this load should be applied to the right end of the beam as an equivalent nodal load, if only one beam element is used?

- 0.75
- 0.71
- 0.66
- 0.60
- 0.50



Note: If the previous question is too difficult mathematically for you, it has the same answer as this question (which is viewed as an alternate question, only — not as extra credit):

The Direct Matrix Insertion feature of MSC/NASTRAN allows a person to

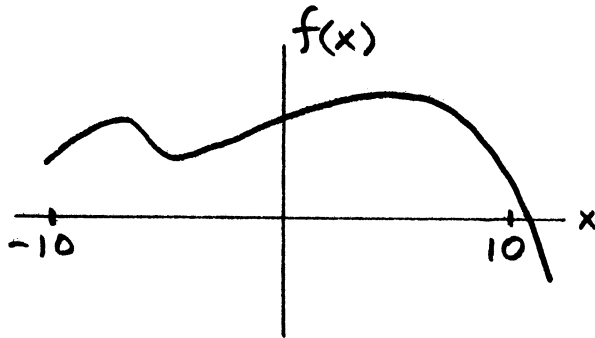
- Insert real and complex matrices and vectors
 - Allows only insertion of real matrices
 - Allows insertion of only real and complex vectors
 - Insert data through the executive deck portion of the deck
 - None of the above.
- 21) The Turner (constant strain) triangle) has linear interpolation functions. This means
- no stresses can be recovered
 - the element is more accurate than 6-noded triangles
 - the σ_x stress is the same throughout the element
 - the σ_x and σ_y stresses equal each other throughout the element
 - none of the above.
- 22) A line element with 3 nodes
- requires quadratic shape functions
 - has 3 generalized coordinates
 - has one rigid body mode
 - all of the above.
- 23) In any linear, elastic structure, $K_{ij} = K_{ji}$ because
- forces in the x direction cancel those in the y direction
 - positive strain energy must be generated by any displacement
 - the system can create energy
 - angles are small
 - all of the above.

24) Equivalent nodal loads

- (a) are always smaller than lumped loads
- (b) depend on the shape functions
- (c) are found by differentiation
- (d) are easier to calculate than lumped loads
- (e) none of the above.

25) Gauss integration is to be used to integrate a function on a domain $[-10, 10]$. The function is believed to be a quintic (fifth degree polynomial). How few sample points are required for an exact integration?

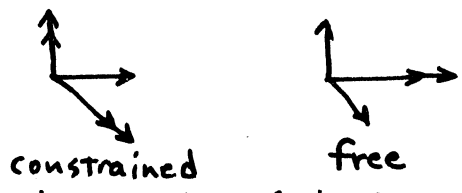
- (a) 1
- (b) 2
- (c) 3
- (d) 4
- (e) 5



AERO SIO / AMES SIO / ME 557
FINAL EXAM - INSTRUCTORS SOLUTION.

24 Dec 198
Wja

PROB. 1. (a). 2, 4, 6



PROB. 2. (b) 30°. There is no relative rotation (hinging effect).

PROB. 3, 4, 5. (e)

$$\{f\} = \int_0^a \begin{bmatrix} : \\ xy/ab \end{bmatrix} \{f\} dx \quad (\text{a line integral})$$

$$f_3 = \int_0^a \frac{x}{a} f dx = \frac{f}{a} \frac{1}{2} x^2 \Big|_0^a = \frac{1}{2} f a$$

PROB. 6. (e)

PROB. 7. (e)

PROB. 8. (b)

PROB. 9, 10. (c)

$$\begin{aligned} m_{11} &= \int_0^L N_1 N_1 \rho A dx \\ &= \rho A \int_0^L (1 - x/L)^2 dx \\ &= \rho A \int_0^L (1 - 2x/L + x^2/L^2) dx \\ &= \rho A \left[x - x^2/L + \frac{1}{3} x^3/L^2 \right]_0^L \\ &= \frac{1}{3} \rho A L \end{aligned}$$

PROB. 11. (b)

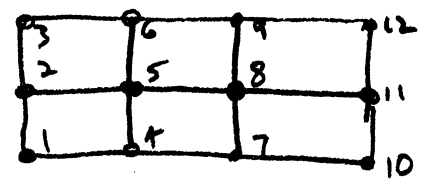
PROB. 12. (a)

PROB. 13, 14 (d)

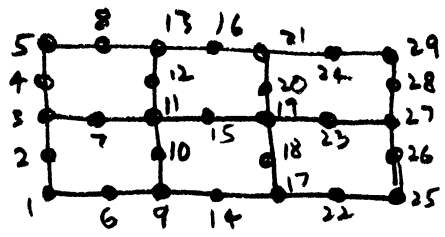
$$\begin{aligned} CPU_1 &= k 60(25)^2 \\ &= k 37,500 \end{aligned}$$

$$\begin{aligned} CPU_2 &= k 145(55)^2 \\ &= k 438,625 \end{aligned}$$

$$\frac{CPU_2}{CPU_1} = 11.69$$



$$\begin{aligned} N &= 12 \times 5 = 60 \\ B &= (5-1+1)5 = 25 \end{aligned}$$



$$\begin{aligned} N &= 29 \times 5 = 145 \\ B &= (11-1+1)5 \\ &= 55 \end{aligned}$$

PROB. 15, 16. (c)

$$\{q^*\} = \begin{Bmatrix} 1 \\ 1 \\ 2 \end{Bmatrix}$$

$$\{Q^*\} = 10^2 \begin{bmatrix} 3 & 1 & 2 \\ 1 & 4 & 1 \\ 2 & 1 & 5 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 2 \end{Bmatrix} = \begin{Bmatrix} 800 \\ 700 \\ 1300 \end{Bmatrix}$$

$$\begin{aligned} \Delta W_{EXT} &= \{Aq\}^T \{Q^*\} \\ &= \begin{Bmatrix} 0 \\ 0 \\ .01 \end{Bmatrix}^T \begin{Bmatrix} 800 \\ 700 \\ 1300 \end{Bmatrix} \\ &= \underline{\underline{13.0 \text{ N mm}}} \end{aligned}$$

PROB. 17. (e)

PROB. 18. (b)

PROB. 19. (e)

PROB. 20 (a)

$$\begin{aligned} \{Q\}_{e.n.l.} &= \int_0^L [N]^T \{p(x)\} dx \\ &= \int_0^L \begin{bmatrix} : \\ (3Lx^2 - 2x^3)/L^3 \end{bmatrix} P \delta(x - \frac{2L}{3}) dx \\ &\quad \uparrow \text{DIRAC DELTA FUNCTION} \\ Q_3 &= P \left[\frac{3L}{L^3} \left(\frac{2L}{3}\right)^2 - \frac{2}{L^3} \left(\frac{2L}{3}\right)^3 \right] \\ &= P \left[\frac{4}{3} - \frac{16}{27} \right] \\ &= \underline{\underline{0.741 P}} \end{aligned}$$

PROB. 21. (c)

PROB. 22 (d)

PROB. 23 (b)

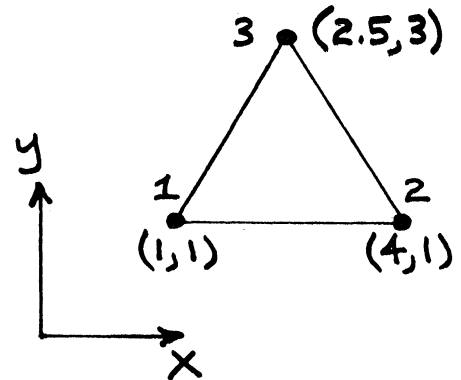
PROB. 24 (b)

PROB. 25 (c)

Problem 1. (10 points)

For the constant strain (Turner) triangle shown, find the shape function $N_3(x, y)$. Note: The form of the 2-D shape function is:

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} q_1 \\ \vdots \\ q_6 \end{Bmatrix}$$



Problem 2. (30 points)

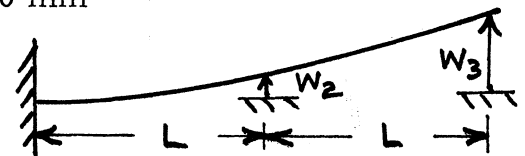
Two identical beam elements are joined as shown to form a cantilever beam. If the free end of the beam is deflected 1 mm upward and rotated 0.2 rad counterclockwise, and the midpoint is deflected 0.4 mm upward and 0.1 rad clockwise, what is the vertical (shear) force on the left end of the beam?



$$EI = 10^8 \text{ N mm}^2$$

$$L = 100 \text{ mm}$$

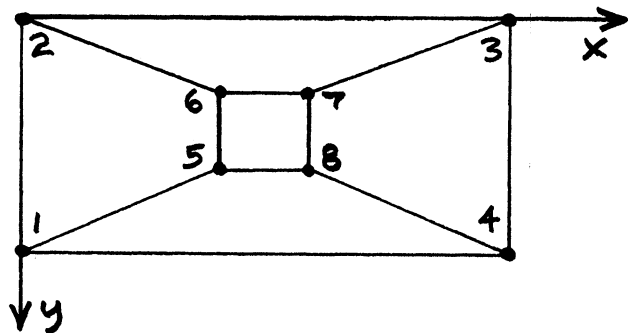
Comments: Use finite element concepts. Do not derive any stiffness matrices or equivalent nodal load vectors from first principles.



Problem 3. (20 points)

A rectangular sheet of metal has a square hole cut from its center as shown. The sheet is clamped at its left and upper edges and is free on its right and lower edges. A 100 N force is applied downward at its lower right corner as shown.

Write down a physical description of all boundary conditions that need to be explicitly entered into a general purpose code such as MSC/NASTRAN.



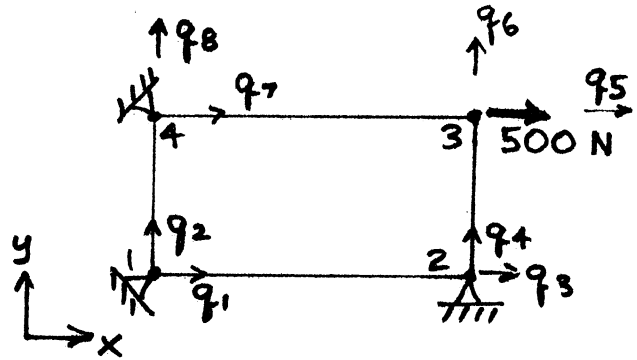
A hypothetical example is given in tabular form:

Node	Displ.	Force
27	1, 4, 6	$F_z = 250 \text{ N}$
38	-----	$F_y = -100 \text{ N}$

Problem 4. (10 points)

An elastic plane-stress quadrilateral is shown. The active load is a 500 N force at node 3.

The stiffness $k_{1,5}$ is 1000 N/mm (detailed notation) and the sheet is 0.5 mm thick. How much virtual work is done by the external forces if node 3 undergoes a 1 mm virtual displacement?

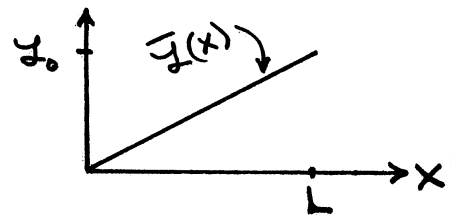


Problem 5. (20 points)

A 3-noded line element has a linearly-varying line load acting:

$$\bar{q}(x) = q_0 x/L$$

The total value of the line load is $q_0 L/2$. If the nodes are equally spaced, how much of this line load should be applied to node 2 as an equivalent nodal load?



$$N_1(x) = (1 - 2x/L)(1 - x/L)$$

$$N_2(x) = \frac{4x}{L}(1 - x/L)$$

$$N_3(x) = \frac{2x}{L}(1 - 2x/L)$$

Problem 6. (10 points)

Carry out a Gauss-Dolittle factorization of the following matrix:

$$\begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix}$$

2 Hours
 Open Book
 Honor Code

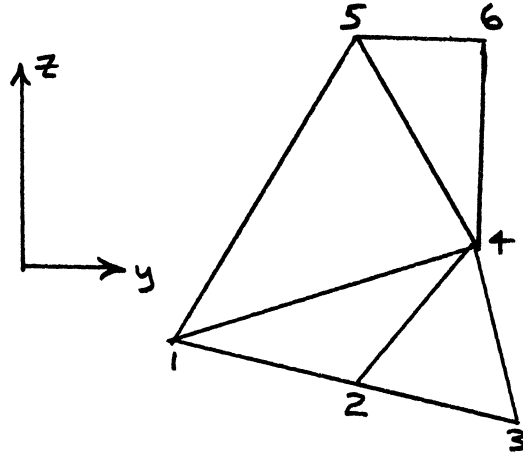
Aero 510/M.E. 557/A.M. 510
 FINAL EXAM

W. J. Anderson
 19 December 1980

There are 9 problems totaling 200 points.
 Please put the solutions in order in your
 bluebook or paper.

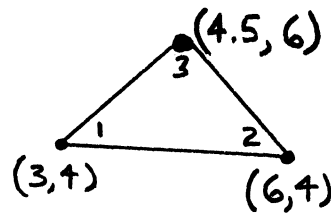
Problem 1. (10 points)

Find the semi-bandwidth, in compact notation, for the given axisymmetric body.



Problem 2. (15 points)

Find the shape function $N_3(x,y)$ for the given constant strain triangle.

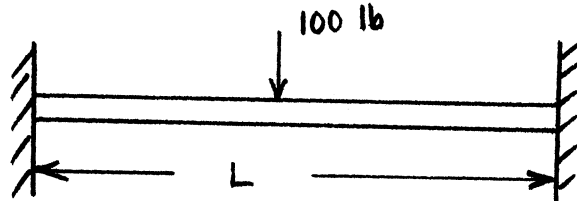


Problem 3. (40 points)

A clamped-clamped beam is loaded at its center with a 100 lb. load as shown. Calculate the stored strain energy and the potential energy at equilibrium for the following constants:

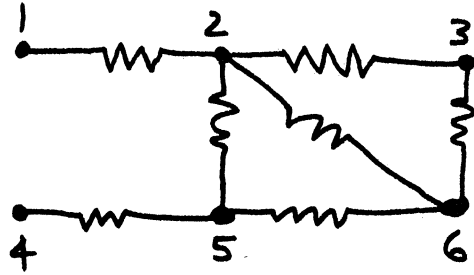
$$EI = 10^7 \text{ lb.in}^2$$

$$L = 50 \text{ in.}$$

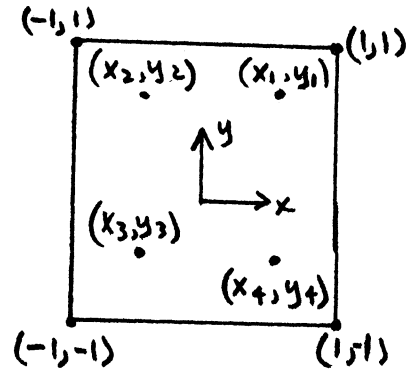


Problem 4. (20 points)

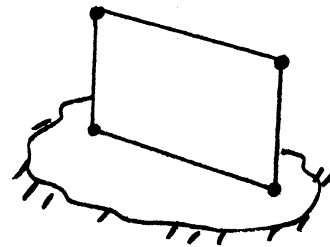
An electrical circuit is shown. It consists of 7 equal resistors. Using the notation $C \equiv 1/R$ (conductance), write out the assembled conductance matrix for the system.

**Problem 5.** (20 points)

Assume that Gaussian integration is to be developed in two dimensions for quadrilateral elements. Write out the first three (lower degree polynomial) defining equations. A total of four sample points are to be used, as shown.

**Problem 6.** (30 points)

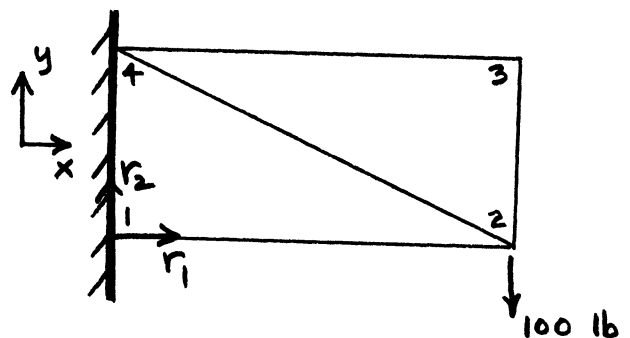
- How many rigid body modes does an axisymmetric element have?
- A certain line element problem has 10 elements connected end-to-end in a straight line. By what factor will the CPU time for the equation solver (Gaussian elimination) be increased if the problem is re-solved with 30 elements?
- A single plate element is clamped along one edge as shown. How many degrees of freedom will be unconstrained in the solution of the problem, either with SAP6 or with NASTRAN's QUAD4 element?
- Is Euler buckling a geometric or a material nonlinearity?
- Near equilibrium, a virtual displacement causes no virtual work. Does potential energy change much with the same virtual displacement?

**Problem 7.** (20 points)

Write down the appropriate nodal force and displacement vectors to be used in a plane stress solution of a cantilever beam, as shown.

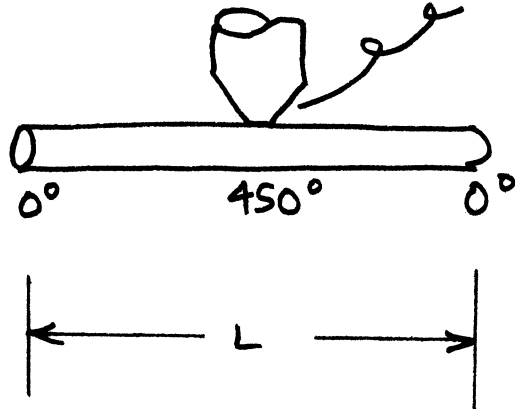
$$\{R\} = ?$$

$$\{r\} = ?$$



Problem 8. (35 points)

A copper wire is modelled with a single quadratic element. The ends of the element are held at 0 degrees F. and a soldering iron at 450 degrees F. is firmly held at the center of the wire as shown. Using a quadratic element, how much heat flux into the wire occurs at the soldering iron contact point?

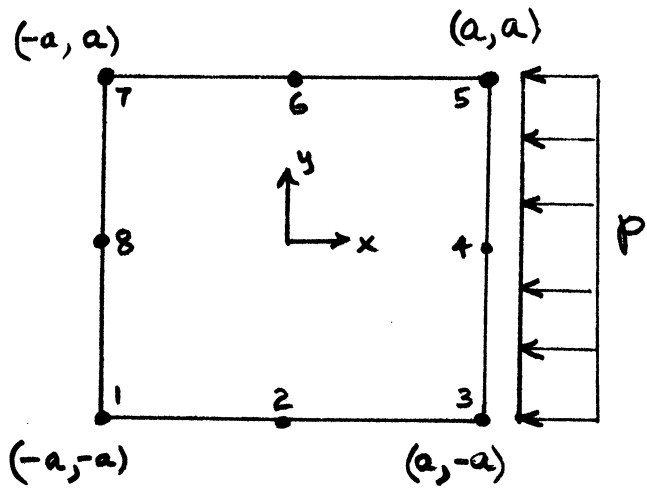


- A = 0.001 in²
- k = 5.05 x 10⁻³ BTU/(in.sec.°F.)
- L = 4 in.

Problem 9. (30 points)

An 8-noded square element is given as shown. A constant pressure p acts on the right face. Calculate the equivalent horizontal nodal load at node 4 due to this pressure. The shape functions evaluated on the line x = a must be parabolic. The element has thickness h.

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1(x,y) & 0 & N_2(x,y) & 0 & \cdot & \cdot & \cdot \\ 0 & N_1(x,y) & 0 & N_2(x,y) & \cdot & \cdot & \cdot \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ \cdot \\ \cdot \\ q_{16} \end{Bmatrix}$$



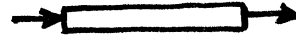
Place all answers on the computer-graded form provided. Your final score will be the total number of right answers. Wrong answers do not count against you, so you should guess answers to any unsolved problems.

Choose the answer supplied which is closest to the correct answer. It is not intended that the exact answer is always provided.

Each answer will count 4 points. Some of the questions are meant to carry 8 or 12 points credit and require repeating the answer on several lines on the answer sheet.

Please print your name and sign the honor code on the answer sheet.
 Make no marks on the R.H. margin of the answer sheet. You should not hand in the exam sheets, only the answer sheet.

1. Which approach would yield the most accurate model of a 6" long, constant-area, line element with only end loads?



- (a) one two-noded element with constant area.
- (b) one three-noded element with constant area.
- (c) two two-noded elements with varying area.
- (d) two three-noded elements with constant area.
- (e) all of the above.

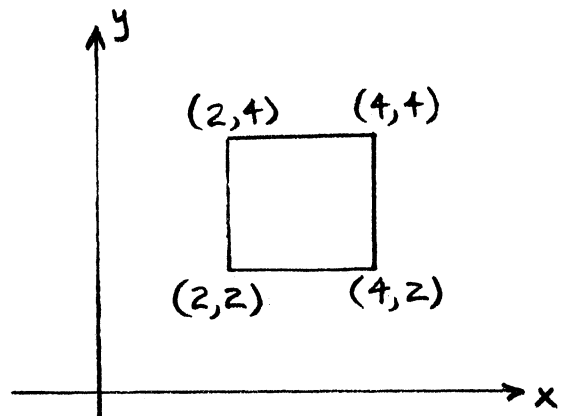
2. Generalized coordinates

- (a) must have the same dimensions as physical coordinates.
- (b) must be the same in number as physical coordinates.
- (c) must not be used for beams.
- (d) are used in shape functions.
- (e) cannot cause rigid body motion.

3. The plane stress rectangle shown has a $[\phi]$ matrix with a rotational mode in it. Which column corresponds to a rotation?

$$[\phi] = \begin{bmatrix} 1 & 0 & x & 0 & x & -y & x^2 & xy \\ 0 & 1 & 0 & y & y & x & y^2 & xy \end{bmatrix}$$

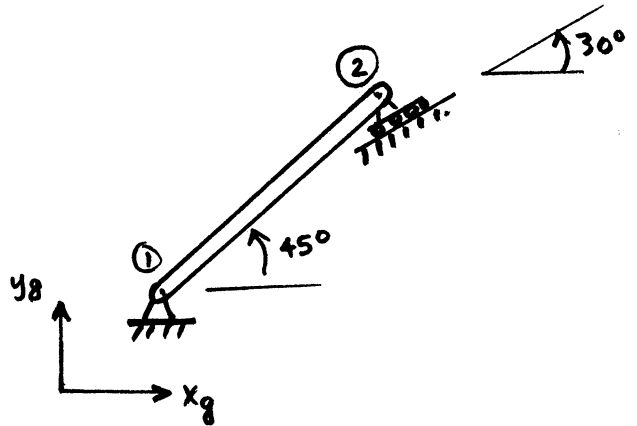
- (a) column 3
- (b) column 4
- (c) column 5
- (d) column 6
- (e) column 7



4. Which column of the $[\phi]$ matrix in problem 3 corresponds to a uniform expansion (dilatation) of the element?
- column 3
 - column 4
 - column 5
 - column 6
 - column 8
5. The set of displacement functions in $[\phi]$ of problem 3 has a problem because
- three of the displacement fields are not linearly independent.
 - there are not enough rigid body modes.
 - the displacements are not geometrically isotropic.
 - all of the above.
 - the displacements are better for a triangle than a rectangle.

6. A line element is aligned 45° from the horizontal with a roller support at 30° from the horizontal as shown. The proper local system for the inclined support at node 2 yields

- $u_\ell = .707 u_g + .707 v_g$
- $v_\ell = .500 u_g + .866 v_g$
- $u_\ell = .866 u_g + .500 v_g$
- $u_\ell = 0.$
- $v_g = 0.$



7. The pinned support at node 1 of problem 6 requires boundary conditions
- $u_g = v_g = 1$
 - $u_g = v_g = 0$
 - $u_\ell = v_\ell = 1$
 - $u_g + v_g = 0$
 - none of the above.
8. One assembles the global stiffness matrix
- in local coordinates.
 - in global coordinates.
 - in global coordinates after eliminating nodes with forces.
 - only after converting to local coordinates.

- 9, A heat-conduction rectangle
10, is to be developed for the
11, rectangle shown. The shape
function includes a term

$$N_3 = \frac{1}{2}xy$$

The element has thickness h .
The heat conduction coefficient
 k is constant. What is
the value of h_{33} ?

- (a) $\frac{1}{2} kh$
(b) $\frac{1}{12} kh_{xy}$
(c) $\frac{9}{12} kh$
(d) $\frac{5}{6} kh$
(e) $2 kh$.
- 12, If a heat flux of $10 \text{ BTU/inch}^2/\text{sec}$
13, is applied into the lower
boundary ($y=0$), how much
equivalent nodal flux
goes into node 3?

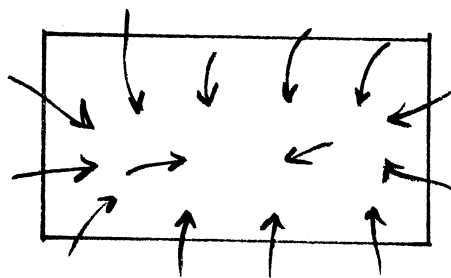
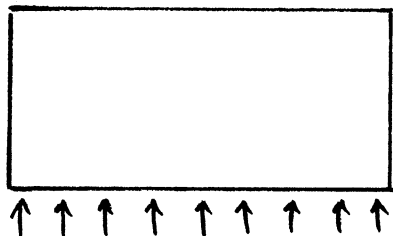
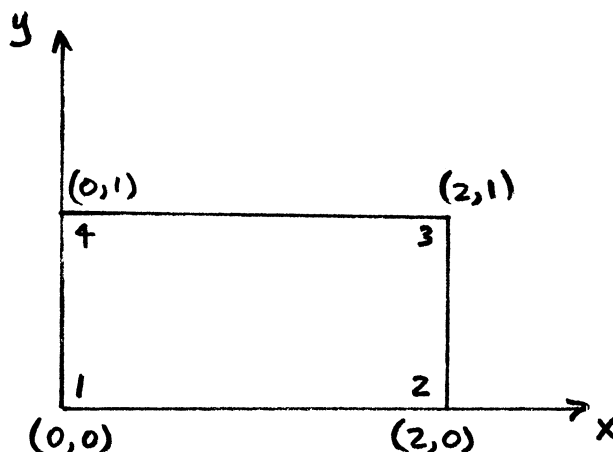
- (a) zero
(b) $5h \text{ BTU/sec}$
(c) $5 kh \text{ BTU/sec}$
(d) $20h \text{ BTU/sec}$
(e) $-5h \text{ BTU/sec}$

- 14, If a uniform heat flux of 100
15, $\text{BTU/inch}^3/\text{sec}$ is applied to the
entire body ($2h \text{ in}^3$), how
much equivalent nodal flux
goes into node 3?

- (a) $-10h \text{ BTU/sec}$
(b) zero
(c) $+10h \text{ BTU/sec}^2$
(d) $+50h \text{ BTU/sec}$
(e) $+100h \text{ BTU/sec}$.

16. The consistent mass matrix

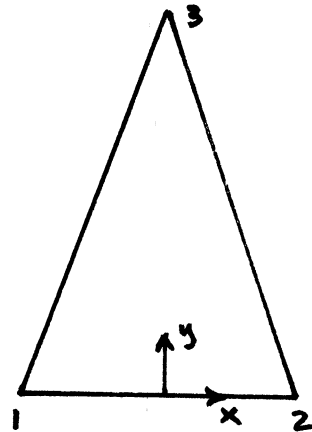
- (a) is always better than a lumped mass matrix.
(b) is always worse than a lumped mass matrix.
(c) has more nonzero terms than a lumped mass matrix.
(d) is not physically meaningful.
(e) has dimensions of FT^2/L^4 .



17. Tangent stiffness can be used
- to calculate stress.
 - to investigate stability of a body.
 - to determine the integration volume.
 - to develop the $[\bar{B}]$ matrix.
 - all of the above.
18. The nonlinear geometric law given in the $[\bar{B}]$ matrix
- is something like a tangent law.
 - is at most linear in the displacements.
 - contributes to the geometric stiffness.
 - all of the above.
 - is a material property.
19. Suppose one wishes to numerically integrate the strain energy in a 4-noded line element, using Gaussian quadrature. How many Gauss points (abscissa) will be needed, at the minimum, for exact integration?
- none - won't work.
 - one
 - two
 - three
 - four.
20. Suppose one uses Radau integration on the line element in problem 19. How many Gauss points are needed for an exact integration?
- none - still won't work
 - one
 - two
 - three
 - four.
21. A Turner triangle is heated 100°F . What are a possible set of equivalent nodal loads due to thermal strain?

(a)
$$\begin{Bmatrix} -1000 \\ +1000 \\ +1000 \\ -1000 \\ 0 \\ 1414 \end{Bmatrix}$$

(b)
$$\begin{Bmatrix} 1000 \\ -1000 \\ -1000 \\ 1000 \\ 0 \\ -1414 \end{Bmatrix}$$



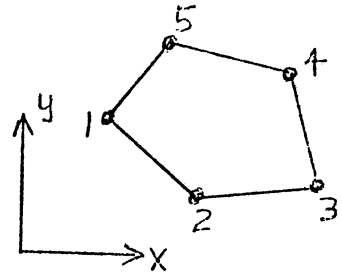
- (c) neither of the above are remotely possible.

22. Bending rigidity of a plate of thickness h , compared to a rectangular cross-section beam of thickness h , will be
- (a) 10% smaller (for a similar amount of material)
 - (b) 10% greater (for a similar amount of material)
 - (c) the same (for a similar amount of material)
23. Successively refined, nested finite element grids, for the displacement method, yield convergence:
- (a) from below on potential energy (monotonically increasing)
 - (b) from above on strain energy (monotonically decreasing)
 - (c) from below on stress (monotonically increasing)
 - (d) all of the above.
24. Conservative force fields
- (a) possess a potential
 - (b) can do no work in a closed cycle of motion
 - (c) can be used in the potential energy method
 - (d) include forces of fixed **direction and magnitude**.
 - (e) all of the above.
25. The serendipity linear mapping functions
- (a) map a line element domain onto a double unit line segment
 - (b) map a triangle onto a double unit square
 - (c) map a tetrahedron onto a double unit cube.

Problem 1 (40 points)

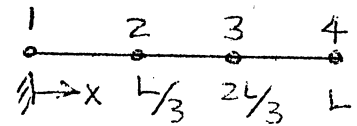
- (a) Propose a displacement function $u(x, y)$ for a 5-sided plane stress element as shown:

$u(x, y) =$



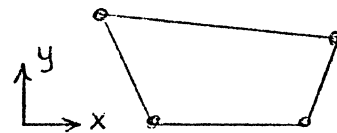
- (b) Propose a shape function $N_1(x)$ for a 4-noded line element as shown:

$N_1(x) =$



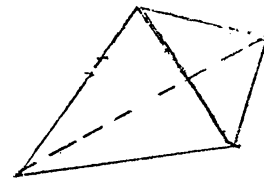
- (c) Give the relation between displacement and shear strain in the quadrilateral shown:

$\gamma_{xy}(x, y) = [\quad] \{ \quad \}$



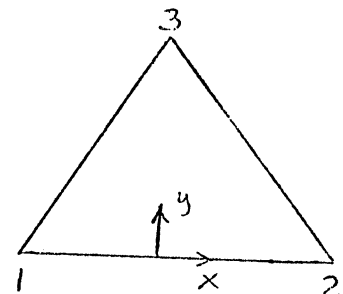
- (d) The tetrahedron shown has 4 sides and is imbedded in 3-D space. If the body is unconstrained, how many rigid body modes will it have ?

Number R. B. Modes =



Problem 2 (20 points)

A constant strain, plane stress triangle is shown. If the 3rd node (at the top) is moved horizontally one unit to the right, and if the other degrees of freedom are constrained, how much force is required to keep node 3 from moving vertically?



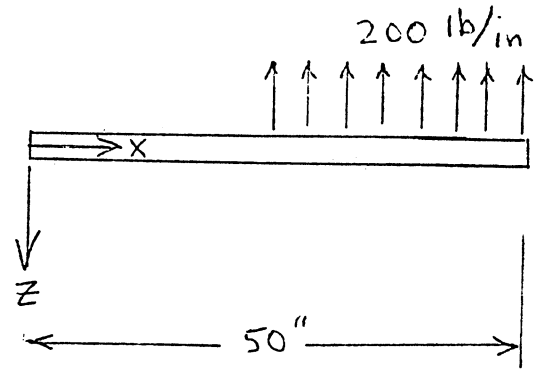
Problem 3 (30 points)

A beam has a load over half its span, as shown. Find the vertical component of the equivalent nodal load on the left end.

$$A = 5 \text{ in}^2$$

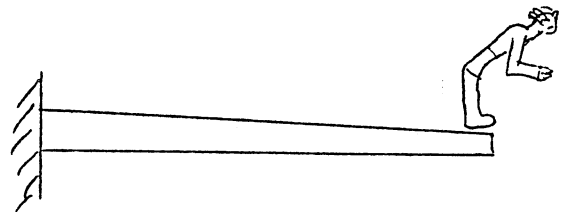
$$E = 10^7 \text{ psi}$$

$$\gamma = 0.3$$



Problem 4 (10 points)

A 200-lb. man is poised (motionless) at the end of a thin, flexible diving board. If the end of the board is given a virtual displacement downward of 1", how much virtual work is done on the board by gravity (the man's weight)?



$$\Delta W_{\text{ext}} =$$

How much work is done by the internal elastic forces during the same 1" displacement?

$$\Delta W_{\text{internal}} =$$

2 Hours
Honor Code
Open Book

AERO 510
ME 557
AMES 510

Dec. 20, 1979
W. J. A.

FINAL EXAMINATION

Please place all answers on the special computer-graded form provided. Your final score will be the total number of right answers. Wrong answers will not count against you, so you should guess answers to any unsolved problems.

Choose the answer supplied which is closest to the correct answer. It is not intended that the list always contains the exact answer.

Each answer will count 2 points.

1. The triangle studied by Turner, Clough, Martin and Topp was originally developed using:
 - a) energy ideas
 - b) equilibrium ideas
 - c) set theory
 - d) integral equations

 2. The Turner triangle is important because of its
 - a) accuracy
 - b) linear strain field
 - c) historical significance
 - d) use in beam theory

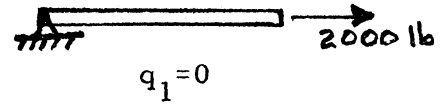
 3. An attempt to separate the shearing and direct stress effects in the Turner triangle stiffness matrix
 - a) is successful
 - b) fails because shear and direct stress effects are always coupled
 - c) is not useful
 - d) can only be done in nonorthotropic cases

 4. Virtual work is the work done:
 - a) during a displacement from deformed to undeformed states
 - b) by external forces
 - c) by internal forces
 - d) during a virtual displacement
-

 A line element has $EA/L = 10^4$ lb/in. The element is loaded with 2000 lb. A virtual displacement of $-0.002''$ is then given at the right node.

5. What is the strain energy in the element at equilibrium?

- a) 20 in·lb
 b) 100 ft·lb
 c) 200 in·lb
 d) 400 in·lb



6. What is the total virtual work done during the virtual displacement?

- a) 0 in·lb
 b) - 20 in·lb
 c) + 20 in·lb
 d) 32 ft·lb

7. What is the value of the work potential of the line element at equilibrium?

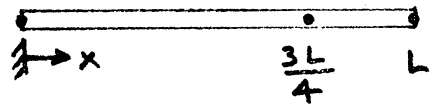
- a) 400 in·lb
 b) 200 in·lb
 c) 0 in·lb
 d) - 400 in·lb

What is the potential energy of the line element at equilibrium?

- a) 400 in·lb
 b) 10 in·lb
 c) 0 in·lb
 d) - 200 in·lb

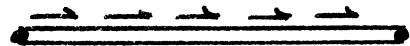
9. A line element with three nodes as shown can have a shape function

- a) $N(x) = x - x^2$
 b) $N(x) = 2 + x$
 c) $N(x) = \frac{4}{3L} \left(x - \frac{3L}{4}\right)(x - L)$
 d) $N(x) = 1 + 2x + 3x^2$



10. An equivalent nodal load vector for a two-noded line element under uniform load could be:

- a) $\begin{Bmatrix} -100 \\ +100 \end{Bmatrix}$
 b) $\begin{Bmatrix} 23 \\ 66 \end{Bmatrix}$
 c) $\begin{Bmatrix} -40 \\ -80 \end{Bmatrix}$
 d) $\begin{Bmatrix} 10 \\ 10 \end{Bmatrix}$



11. An Euler-Bernoulli beam in 2-dimensions, neglecting column effects, has how many degrees of freedom per node?
- a) 1
 - b) 2
 - c) 3
 - d) 4
12. A beam-column in three dimensions has how many degrees of freedom per node?
- a) 2
 - b) 4
 - c) 6
 - d) 8
13. The purpose of the decomposition of a real symmetric matrix before solution is to
- a) reduce storage requirements
 - b) allow iteration
 - c) organize a sequential solution
 - d) improve positive definiteness
14. Solving "in place"
- a) increases the number of equations required to find a total solution
 - b) prevents finding of stresses
 - c) is not practical
 - d) can be done only on real, symmetric matrices
15. For two-dimensional finite elements, there are how many rigid body motions?
- a) One
 - b) Two
 - c) Three
 - d) Six
16. A beam-column in 3 dimensions has how many rigid body modes?
- a) Two
 - b) Four
 - c) Five
 - d) Six
17. Bandwidth
- a) describes the width of the region along the main diagonal of a matrix, containing all nonzero terms
 - b) is defined for rectangular matrices as it is for square matrices
 - c) has no meaning for mass matrices
 - d) has no meaning for damping matrices

18. What is the half bandwidth of the stiffness matrix composed of 6 beams connected end to end in 3-D? *(in detailed notation)*
- a) Two
 - b) Six
 - c) Ten
 - d) Twelve
19. NASTRAN uses a matrix storage method:
- a) based on bandwidth ideas
 - b) based on solving-in-place ideas
 - c) based on sparse matrix concepts, using a pointer system
 - d) based on minimizing input data
20. Natural modes for an element always include:
- a) shape functions
 - b) transcendental functions
 - c) cubics
 - d) rigid body modes
21. The stiffness matrix for a beam
- a) is invariant with respect to translations of a coordinate system
 - b) is invariant with respect to rotations of a coordinate system
 - c) depends on the mass distribution of the beam
 - d) has an odd number of rows and columns
22. Skewed boundary conditions often occur:
- a) because of a mistake in boundary conditions
 - b) in problems solved by SAP6
 - c) because of angled forces
 - d) because of planes of symmetry at angles
23. Nonlinear problems in a network often arise because:
- a) Ohm's law is nonlinear
 - b) the pressure drop across a length of pipe varies as a noninteger power of velocity
 - c) the method of assembly uses a law such as Kirchhoff's current law at a node, which is linear
 - d) the sign convention requires flow into a node to be positive
24. If one wishes to interpolate a field variable in one dimension, and wishes to use four nodes:
- a) a cubic polynomial is needed
 - b) a quartic polynomial is needed
 - c) natural coordinates must be used
 - d) derivatives of the function must be used

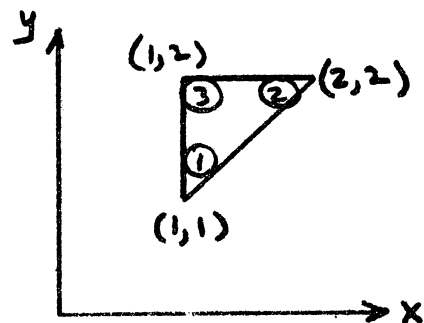
25. Area coordinates for a triangular region
- are difficult to integrate
 - add up to unity
 - are four in number
 - are five in number
26. If one uses Gauss integration with 4 sample points on a one-dimensional region:
- there are 4 corresponding deriving equations in the integration scheme
 - a 7th degree polynomial can be exactly integrated on the region
 - a 5th degree polynomial can be exactly integrated on the region
 - there are 8 corresponding abscissae
27. Gauss quadrature is particularly useful in finite elements because:
- the domain of the problem is complicated
 - the degree of polynomial being integrated is known
 - rigid body modes are automatically included and simplify the problem
 - stress problems don't have to be accurate
28. Equivalent nodal loads:
- can be derived using virtual work
 - are simpler than lumped load ideas
 - allow shorter computer solution times
 - do not work for distributed loads
29. Consistent mass matrices:
- are simpler than lumped mass matrices
 - are often "full" matrices
 - lead to less expensive solutions
 - uncouple the degrees of freedom more than lumped masses

Consider the 3 noded, heat conduction element shown. It has thickness of 2 inches in z direction. The conduction coefficients

$$k_x = k_y = .001 \frac{\text{BTU}}{\text{in} \cdot \text{sec}^\circ\text{F}}$$

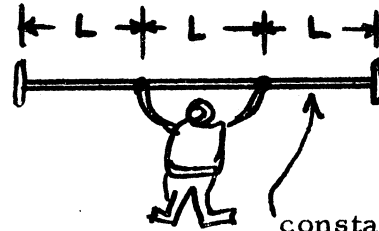
30. The shape functions include:

31.
 - $N_1(x, y) = 2-x$
 - $N_2(x, y) = 2-x$
 - $N_3(x, y) = y-x+1$
 - $N_1(x, y) = 2-y$



32. The value for k_{11} is:
33. a) .001 BTU/(sec·in)
 b) .0005 BTU/(sec⁰F)
 c) .002 BTU/sec
 d) .001 BTU/(sec⁰F)

A bar bell (for weight-lifters) is modelled as a beam using 3 two-dimensional Euler Bernoulli beam elements as shown. The points where the weight-lifter's hands are applied are considered pinned. No axial displacements are considered.



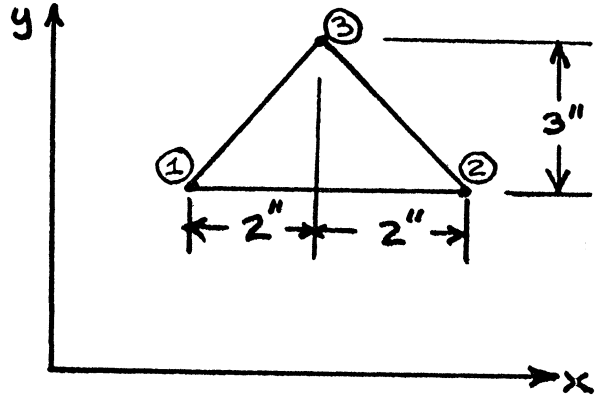
34. What characterizes the term k_{18} in the assembled stiffness matrix? (This term relates the vertical displacement at node 1 to the moment created at node 4.)
- a) it is negative
 b) it is zero
 c) it is positive
 d) one cannot tell from the data given
35. How many of the 8 degrees of freedom are constrained due to boundary conditions?
- a) two
 b) three
 c) four
 d) six
36. From the sketch shown, try to relate k_{14} and k_{58} .
- a) $k_{14} = k_{58}$
 b) $k_{14} = -k_{58}$
 c) $k_{14} = k_{58} = 0$
 d) one cannot tell from the data given
37. What is the relation between k_{14} and k_{76} ?
- a) $k_{14} = k_{76}$
 b) $k_{14} = -k_{76}$
 c) $k_{14} = k_{76} = 0$
 d) one cannot tell from the data given

50. Isoparametric elements are different from "straight-sided" (common) elements in that:
- a) the number of nodes is smaller with isoparametric elements
 - b) isoparametric elements don't use shape functions
 - c) isoparametric elements must be smaller
 - d) straight-sided elements are more accurate
 - e) isoparametric elements almost always use numerical integration

FINAL EXAM

Problem 1. (25 points)

A) A constant strain (Turner) triangle is given as shown. The second node is moved 0.050" to the right while the other nodes are fixed. If the element is made of steel and is 0.100" thick, how much strain energy is stored in the triangle due to this deformation?



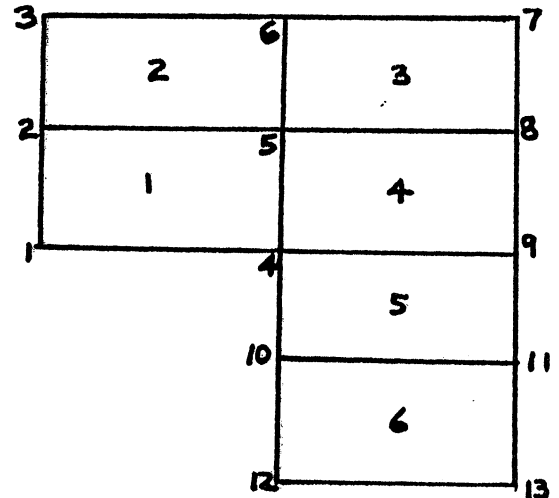
$E = 30 \times 10^6 \text{ psi}$
 $\nu = 0.3$
 $G = 11.5 \times 10^6 \text{ psi}$

B) If such a triangle were actually cut from steel and loaded at one corner to produce the same nodal displacement, would the stored strain energy be less than, equal to, or greater than the value found for the finite element?

Problem 2. (25 points)

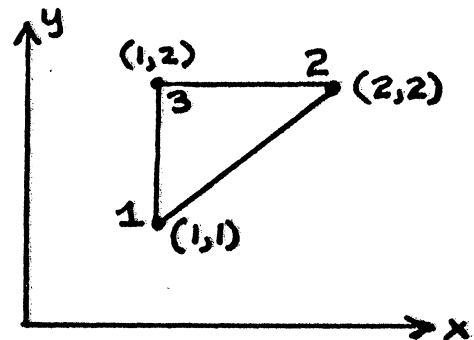
A plane stress problem has the mesh shown.

- A) What is the semi-bandwidth of the assembled set of equations for the nodal numbering given? Use detailed notation.
- B) What is the optimum semi-bandwidth which could be obtained by renumbering the nodes? Again use detailed notation.
- C) For the given element numbering, what is the maximum wavefront for the problem, in terms of number of active equations?



Problem 3. (25 points)

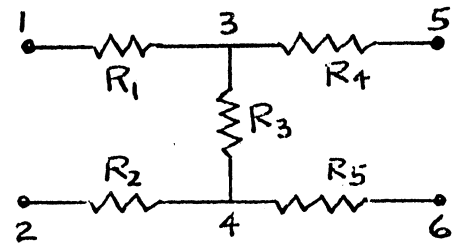
A two-dimensional, constant flux triangle (comparable to a constant strain Turner triangle) is shown. Find the term h_{13} in the internal energy (heat conduction) matrix. Reduce the expression as much as possible for the given triangle.



Problem 4. (25 points)

An electrical network of resistance is shown.

- (a) Write out the matrix of electrical conductivities for this network by inspection.
- (b) What is the semi-bandwidth for the matrix?



Problem 5. (25 points)

A rectangular, thin elastic plate is suspended from its center by a rigid rod as shown. The physical problem involved is to find the deflection of the plate due to gravity. It is desired here, however, to ask only questions about boundary conditions and symmetry.

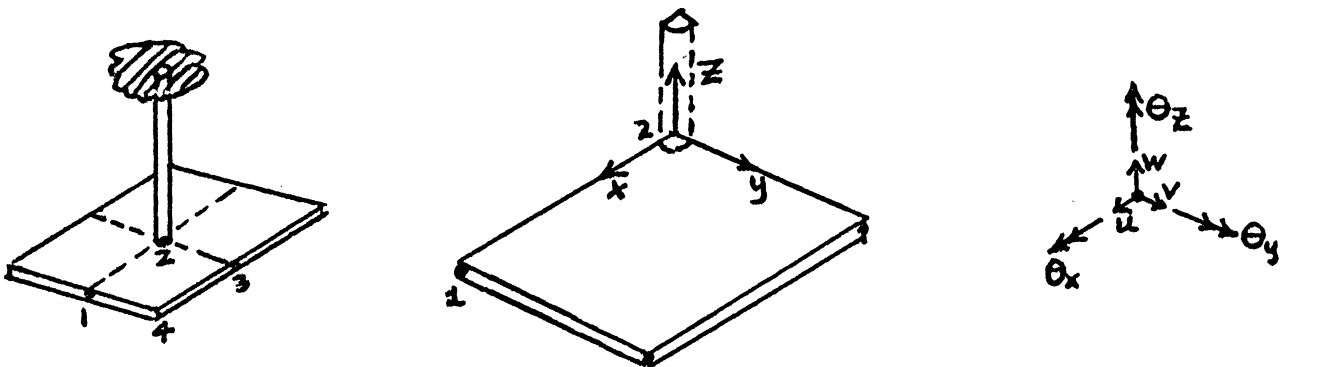


Plate-rod system.

Quarter-plate element.
(enlarged)

Degrees of freedom.

Suppose the plate is modeled by 4 rectangular finite elements as shown, and by symmetry, only 1/4 of the problem (one element) need be solved. Provide the conditions to be placed on the following variables due to symmetry and boundary conditions. Reduce the number of constraints to the minimum number needed to run a typical finite element program such as SAP6.

$$u_1, v_1, w_1, \theta_{x_1}, \theta_{y_1}, \theta_{z_1} \quad u_2, v_2, w_2, \theta_{x_2}, \theta_{y_2}, \theta_{z_2}$$

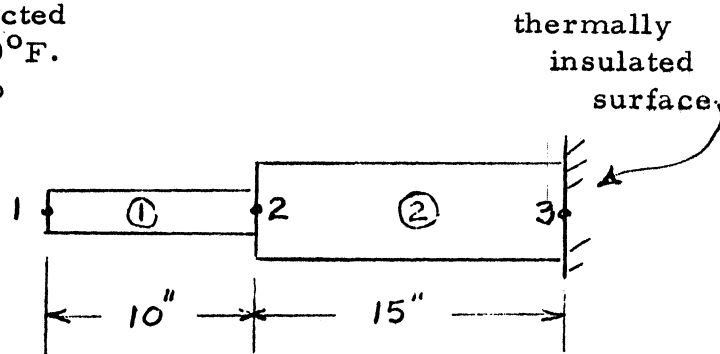
$$F_{x_1}, F_{y_1}, F_{z_1}, M_{x_1}, M_{y_1}, M_{z_1} \quad F_{x_2}, F_{y_2}, F_{z_2}, M_{x_2}, M_{y_2}, M_{z_2}$$

Problem 6. (25 points)

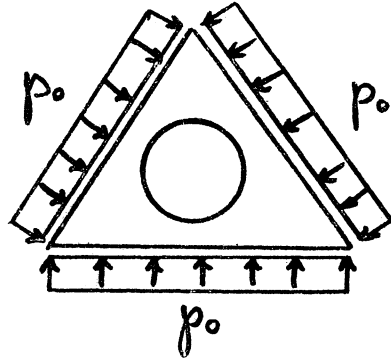
Two thermal line elements are connected as shown. An external heat source at 200°F . is provided at node 2. What heat flux into node 2 at 200°F . is required to maintain the temperature at node 1 at 100°F . ?

$$k_1 = k_2 = 5 \times 10^{-3} \text{ BTU}/(\text{sec. } ^\circ\text{F. in})$$

$$A_1 = 1 \text{ in}^2 \quad A_2 = 4 \text{ in}^2$$



A triangular sheet of metal has a hole punched as shown, centered on the centroid of the triangle. The sheet is put in compression as shown.



38. What is the smallest portion of the problem that can be solved, using symmetry?

- a) 1/12
- b) 1/8
- c) 1/6
- d) 1/3
- e) 1/2

39. If you solved one half of the problem, using symmetry, would you have to constrain any rigid body modes, in addition to applying symmetry conditions?

- a) no
- b) yes, constrain one rigid body mode
- c) yes, constrain two rigid body modes
- d) yes, constrain three rigid body modes
- e) one can't tell - it depends on which way the body is divided

40. If the body is clamped at the lower boundary and subjected to gravity in the vertical direction, what is the smallest portion that can be solved, using symmetry?

- a) 1/8
- b) 1/6
- c) 1/3
- d) 1/2
- e) the entire body

41. Active joints in robots:

- a) require special consideration
- b) can be studied using conventional programs such as SAP6

42. Numerical integration in time:

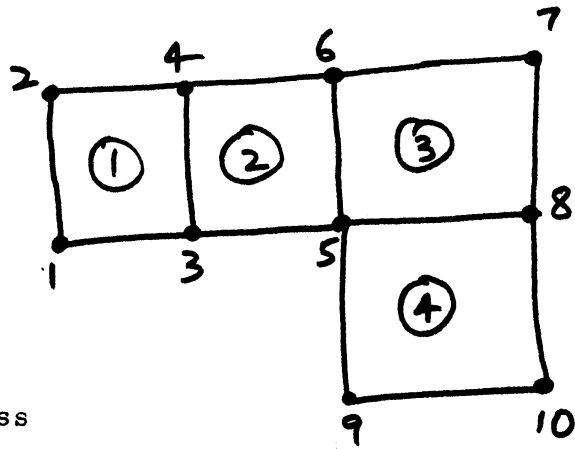
- a) can be explicit, implicit or Gaussian quadrature
- b) is used in static stress analysis
- c) only works for beam elements
- d) is a method peculiar to finite element theory
- e) can have stability problems

43. SAP6 has a problem in that it:
- can't find static stresses
 - doesn't have any dynamics capability
 - doesn't have a plate element
 - doesn't have automatic node generation
 - cannot apply symmetry conditions on a skewed boundary
44. The boundary element in SAP6 is used to:
- impose specific forces at a point
 - impose nonzero displacements at a point
 - apply dynamic loads
 - provide an artificial node for beams

A plane stress problem is shown:

45. What is the maximum wavefront,
46. in detailed notation?

- 4
- 5
- 6
- 8
- 10



47. What is the half bandwidth of the stiffness
48. matrix, using compact notation?
- 4
 - 6
 - 8
 - 10
 - 12

49. The dotted line connecting the origin to a point on the stress-strain law represents an n-dimensional surface with "slope" we call
- elastic stiffness
 - secant stiffness
 - tangent stiffness
 - secant modulus
 - tangent modulus

**IMPORTANT INSTRUCTION: SOLVE ONLY ONE OF PROBLEMS 7 AND 8.
(IF YOU DO TWO, ONLY THE FIRST WILL BE GRADED!)**

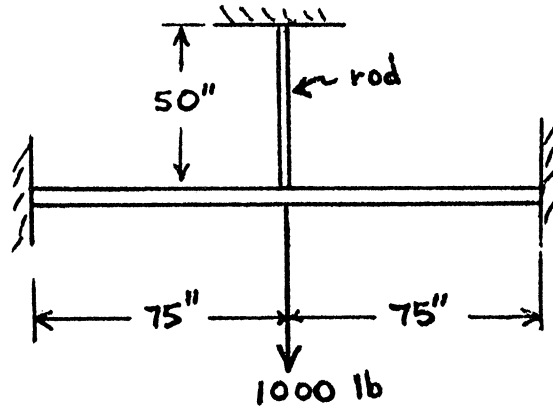
Problem 7. (50 points)

A beam is clamped at both ends. A rod extending from above helps to support it. Find the deflection of the beam at the center when a 1000 lb. load is applied as shown.

$$EI = 2 \times 10^9 \text{ lb.in.}^2 \text{ for the beam}$$

$$EA = 6.5 \times 10^6 \text{ lb for the rod}$$

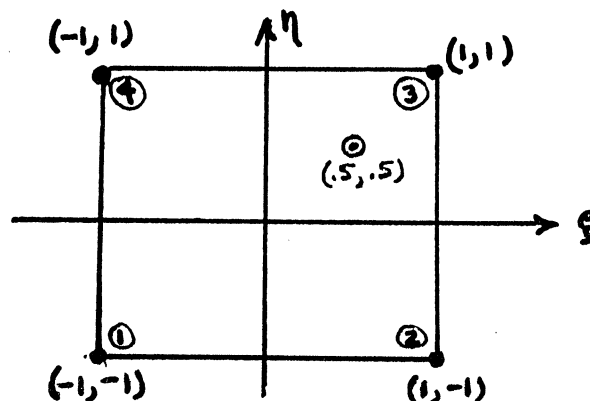
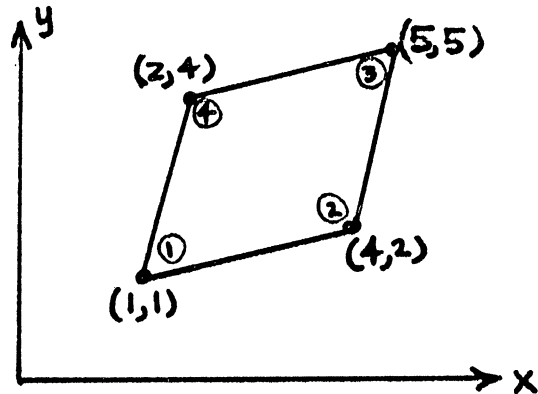
(This problem appeared on the national Professional Engineer's Exam three years ago. It required one hour to solve by conventional analysis.)



Problem 8. (50 points) Alternate problem.

One step in the generation of two-dimensional isoparametric elements is the mapping between a quadrilateral and the square shown.

Suppose a Gauss integration point lies at a point $(0.5, 0.5)$ in the ξ, η plane. To what point in the x, y plane does this correspond?



50 minutes
Honor Code
Open Book & Notes

HOUR EXAM
AFRO 510 ME 557 AMBS 510

December 6, 1978
W.J.A.

Problem 1 (25 points)

- A) Integrate the function $f(x) = 6 + 12x^3 + 4x^6$ over the interval $(-1,1)$ using Gaussian integration and 3 example points.
- B) Is your answer exact? Why?

Problem 2 (25 points)

Calculate the consistent mass matrix term m_{33} for the constant strain triangle shown. The triangle is of constant thickness t and density ρ .

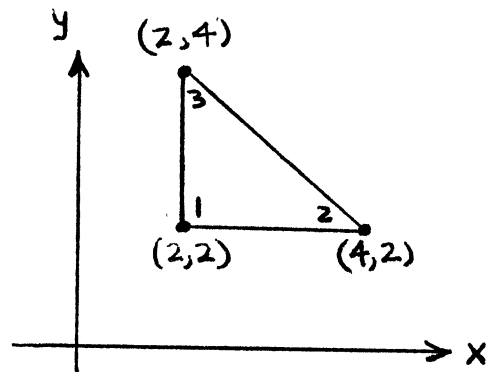


Fig. 1. Constant strain triangle

Problem 3 (25 points)

A truss element is imbedded in 2-D space as shown. If the stiffness matrix in global coordinates has a term $k_{41} = -4.5 \times 10^6 \text{ lb/in}$, what is the basic stiffness $\frac{EA}{L}$ for the truss member?

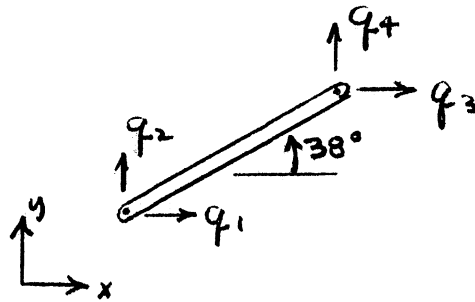


Fig. 2. Truss element

Problem 4 (25 points)

An Euler-Bernoulli beam element in a two dimensional space must have two rigid body motions -- translation and rotation. If α_1 and α_2 are the two generalized coordinates in rigid body translation and rotation respectively, write out a suggested $[A]$ matrix for the beam element.

Open Book.
Honor Code.
50 Minutes.

Aero 510

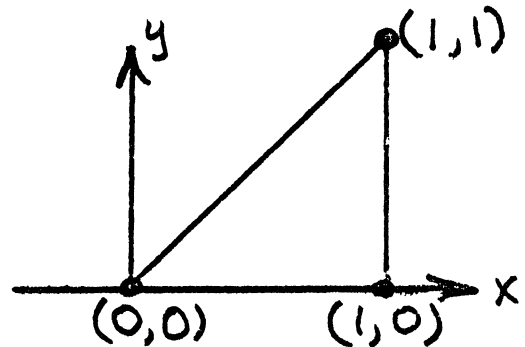
HOURLY EXAM
M. E. 557

AM/ES 510

October 18, 1978
WJA

Problem 1. (20 points)

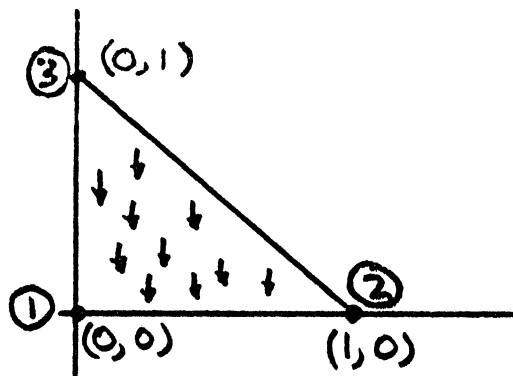
A Turner (constant strain) triangle is shown. Find the shape functions for this triangle.



Problem 2. (40 points)

A Turner (constant strain) triangle is used in a vertical orientation as shown so that gravity loads the triangle at 50 lb/in^2 . Find the equivalent nodal loads on node 2, only. The shape functions are:

$$[N(x,y)] = \begin{bmatrix} (1-x-y) & 0 & x & 0 & y & 0 \\ 0 & (1-x-y) & 0 & x & 0 & y \end{bmatrix}$$



Problem 3. (20 points)

A hypothetical element has

$$[A]^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$[\phi] = [1 \quad y \quad x]$$

$$[S-D] = \left[\frac{\partial}{\partial x} \right]$$

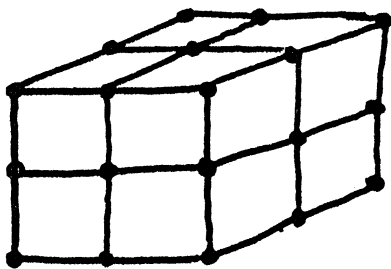
$$[C] = [E]$$

The element volume is V_0 .

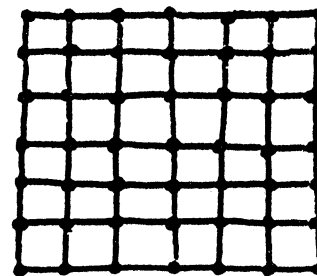
- A) What is the stiffness matrix for the element?
- B) Is there a rigid body mode present in this description of the element? Prove this.

Problem 4. (20 points)

Two problems as shown are to be solved by finite element methods. A Gauss elimination equation solver is to be used. Prove which problem will take more CPU time for the equation solver.



- A) Three-dimensional brick elements. 8 nodes per brick.



- B) Two-dimensional quadrilateral elements. 4 nodes per element.

2 HOURS
Honor Code
Open Book

AERO 510 M E 557 AM/ES 510
FINAL EXAM
FALL 1977

17 Dec 1977
WJA

Problem 1. (20 points)

Given the equations

$$\begin{aligned} 2r_1 + r_2 &= 11 \\ r_1 + 2r_2 &= 13 \end{aligned}$$

carry out a triangular decomposition of the stiffness matrix. Find only the matrices in the form $[L][D][L]^T$

and do not solve for the unknown displacements.

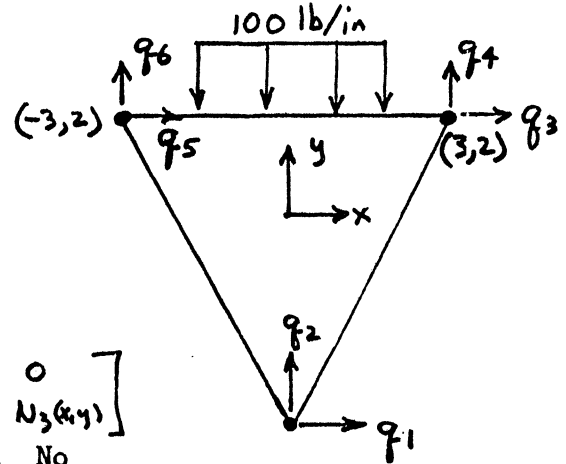
Problem 2. (40 points)

A constant-strain triangle is shown. It uses linear shape functions, of course. Find the equivalent nodal loads for the element due to the distributed load shown, which extends from $x = -2$ to $x = +2$ on the top edge.

Hint: Because both $u(x,y)$ and $v(x,y)$ are interpolated in the same way, the form of the shape function matrix is:

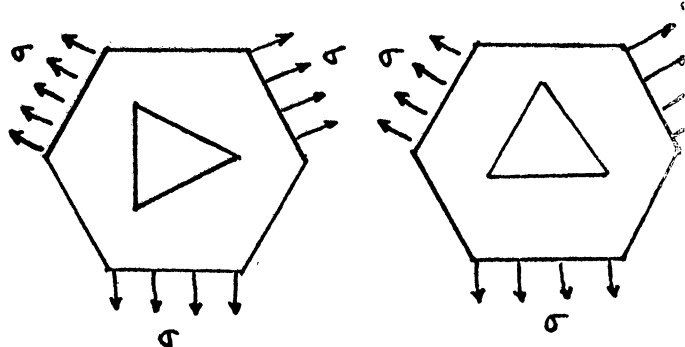
$$[N(x,y)] = \begin{bmatrix} N_1(x,y) & 0 & N_2(x,y) & 0 & N_3(x,y) & 0 \\ 0 & N_1(x,y) & 0 & N_2(x,y) & 0 & N_3(x,y) \end{bmatrix}$$

Comment: You must work this problem out using theory. No intuitive solutions will be accepted.



Problem 3. (20 points)

Two hexagonal bars have been pierced by triangular holes as shown and are loaded on three sides by uniform distributed stresses. The polygons are regular and are concentrically located such that their centroids coincide. Each represents a two-dimensional, plane strain problem, with the only difference being the orientation of the triangle.



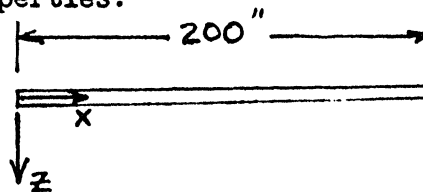
A) Which one(s), if any, of the problems can be solved by modeling a smaller portion of the problem using symmetry? Sketch.

B) If either or both can be solved using a smaller portion, give the type of boundary conditions to be imposed on the cut edges to preserve symmetry.

Problem 4. (30 points)

An Euler-Bernoulli beam has the following properties:

$$\begin{aligned} A &= 10 \text{ in}^2 \\ I_{yy} &= 15 \text{ in}^4 \\ E &= 30 \times 10^6 \text{ lb/in}^2 \\ \nu &= 0.3 \\ L &= 200 \text{ in} \end{aligned}$$



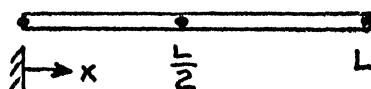
How much strain energy is absorbed by the beam if the right end is rotated -10° and no other end deflections or rotations occur?

Problem 5. (50 points)

A heat conduction line element with three nodes is to be developed. The nodes are to be equally spaced.

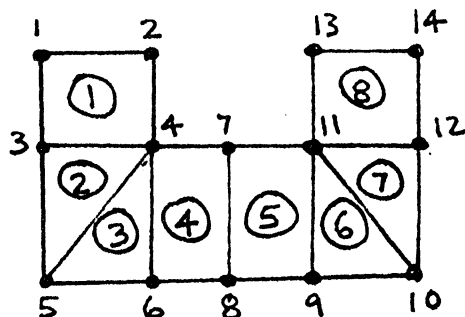
A) Develop the three shape functions from first principles, explaining the geometrical requirements.

B) Using the variational expressions developed in the course, find the term h_{11} , using the proper terminology for the heat conduction problem.

Problem 6. (40 points)

A) A Radau integration is to be used with a total of 5 sample points, including the fixed one. What degree polynomial, at most, can be exactly integrated with this procedure? Explain your reasoning.

B) What is the half-bandwidth, in detailed notation, of the 2-D structure shown? You may use compact notation for intermediate calculations, of course.



C) What is the maximum wavefront, in detailed notation, of the same 2-D structure shown above?

D) If the set of equations shown is to be solved by first modifying them with the Payne and Irons' method for solving "in place", how many equations must be solved simultaneously after the modification?

$$\begin{bmatrix} 1 & 3 & 7 & 9 & 2 \\ 3 & 10 & 5 & 8 & 4 \\ 7 & 5 & 40 & 2 & 4 \\ 9 & 8 & 2 & 100 & 20 \\ 2 & 4 & 4 & 20 & 400 \end{bmatrix} \begin{Bmatrix} r_1 \\ r_2 \\ 10 \\ 12 \\ r_5 \end{Bmatrix} = \begin{Bmatrix} 20 \\ 30 \\ R_3 \\ R_4 \\ 15 \end{Bmatrix}$$

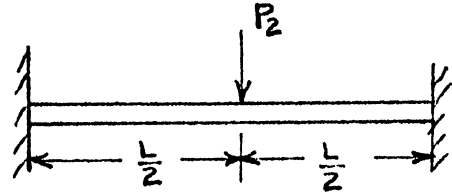
50 minutes
Open Book
Honor Code

ME 557/Aero 510/AMES 510
Second Hour Exam

21 Nov. 1977

Problem 1. (30 points)

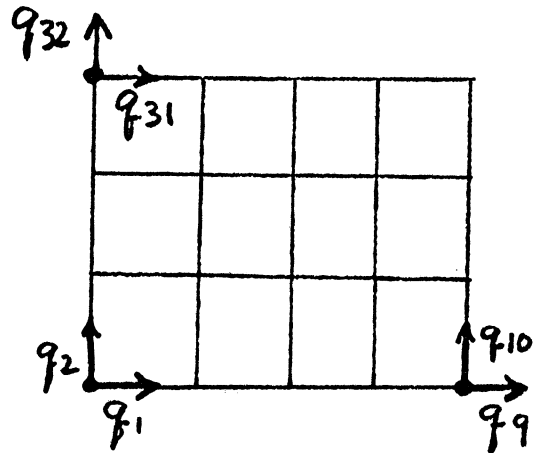
A slender elastic beam is clamped at both ends by rigid walls as shown. This support constrains both the deflections and rotations at the ends. A lateral force of magnitude P_2 acts at the center of the beam. Using known stiffness matrices, calculate the deflection at the center of the beam due to P_2 . Use two Euler-Bernoulli beam elements. The beam has stiffness EI and length L .



Problem 2. (10 points)

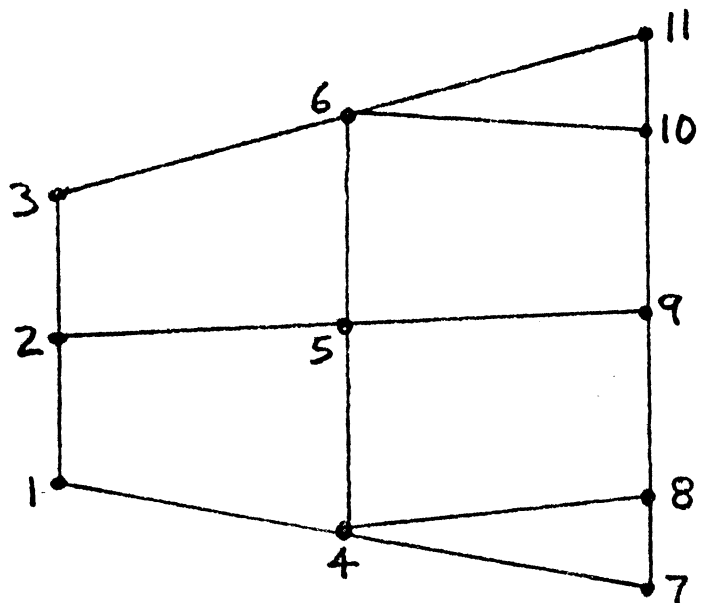
A plane stress problem has been posed where all stress boundary conditions have been given. A 12-element model has been proposed as shown, where only degrees of freedom 1 and 2 have been constrained.

- How many additional degrees of freedom must be constrained to properly remove rigid body modes?
- Name all of the candidate degrees of freedom shown (9, 10, 31 and 32) that are acceptable choices to remove the remaining degree(s) of freedom.



Problem 3. (10 points)

A plane strain problem is modeled as shown. What is the half bandwidth of the given system, in detailed notation? How many zero terms will there be in the stiffness matrix, in detailed notation for the entire matrix?



Problem 4. (40 points)

Two line elements are connected as shown and their outer ends rigidly pinned. The elements are then cooled 20°F . Find the displacement of the center connection.

$$E_1 A_1 / L_1 = 10^6 \text{ lb/in.}$$

$$L_1 = 10 \text{ in.}$$

$$\alpha_1 = 1.2 \times 10^{-5} \text{ 1/}^{\circ}\text{F.}$$

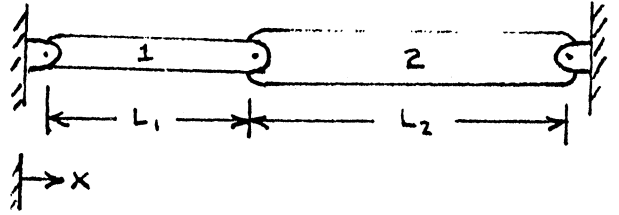
$$\nu_1 = 0.3$$

$$E_2 A_2 / L_2 = 2 \times 10^6 \text{ lb/in.}$$

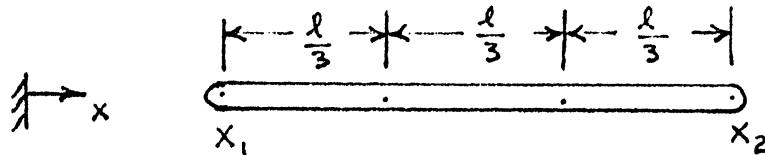
$$L_2 = 20 \text{ in.}$$

$$\alpha_2 = 0.5 \times 10^{-5} \text{ 1/}^{\circ}\text{F.}$$

$$\nu_2 = 0.3$$

Problem 5. (10 points)

A 4-node line element has been proposed as shown. The nodes are equally spaced. The element extends between x_1 and x_2 . Natural length coordinates L_1 and L_2 are defined for the element. Write down the shape function $N_1(L_1, L_2)$ which gives unit displacement at the first node.



Honor Code

50 minutes

Open Book

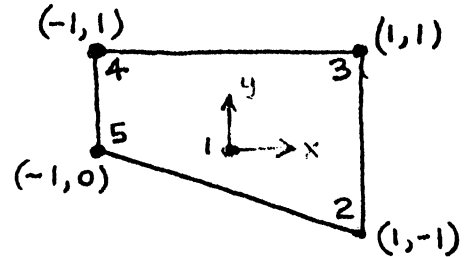
First Hour Exam

Oct 17, 1977

W. J. Anderson

Prob 1 (33 points)

A plane strain quadrilateral has been developed using the following displacement functions:



$$u(x, y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy + \alpha_5 x^2 y^2$$

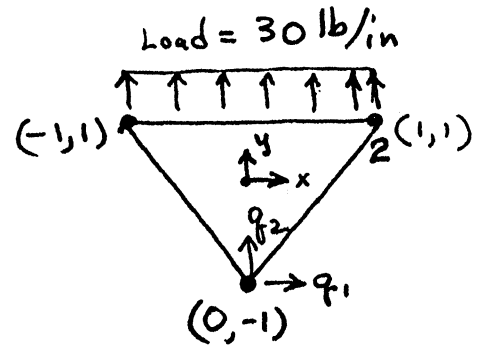
$$v(x, y) = \alpha_6 + \alpha_7 x + \alpha_8 y + \alpha_9 xy + \alpha_{10} x^2 y^2$$

- A) Write down the $[\psi(x, y)]$ matrix, which relates internal displacements to generalized coordinates.
- B) Write down the $[A]$ matrix, which relates generalized coordinates and nodal coordinates, for the specific element shown.

Prob 2 (33 points)

A plane-stress, constant strain triangle is given as shown. If the shape functions for the triangle are

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_{11}(x, y) & 0 & N_{13}(x, y) & \dots \\ 0 & N_{22}(x, y) & \dots & \dots \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}$$



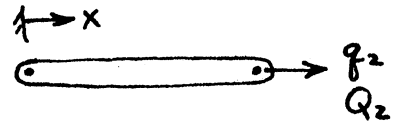
Fill in any missing notation using the obvious generalization of the sketchy version given. Then calculate the equivalent nodal loads on the element due to the given vertical line load on the top edge. Evaluate as completely as possible without finding the explicit polynomials for $N_{ij}(x, y)$, which is a harder problem.

Page 2
10/17/77

Prob 3 (34 points)

Given a line element with two nodes and stiffness

$$[k] = 10^6 \frac{\text{lb}}{\text{in}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$



The left end of the pin is fixed and the right end is loaded so that

$$\{Q\} = \begin{Bmatrix} Q_1 \\ 1000 \text{ lb} \end{Bmatrix}$$

- A) Find the displacement of the right end of the line element using our matrix format (i. e. don't use freshman physics).
- B) If the line element is compressed 10^{-5} in as a virtual displacement (from equilibrium) calculate the virtual work done by the external forces during this virtual displacement.

2 Hours
OPEN BOOK
Honor Code

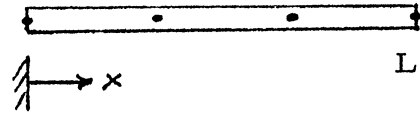
AERO 510
FINAL EXAM

23 Dec. 1976

Problem 1 (25 points)

A constant area line element is to be developed. It will have four equally spaced nodes.

- Write out a suitable displacement function for $u(x)$, using the $[\phi]$ matrix.
- Write out the $[A]$ matrix, relating nodal displacements and generalized coordinates. Use the local coordinate system shown.



Problem 2 (25 points)

Calculate the potential energy at the equilibrium position for a line element with

$$E = 30 \times 10^6 \text{ psi}$$

$$A = 2 \text{ in.}^2$$

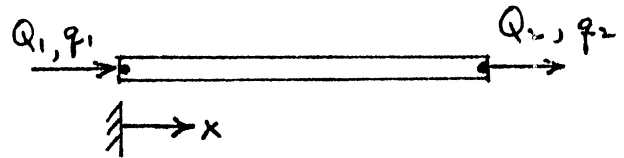
$$L = 6 \text{ in.}$$

$$Q\alpha_1 = 0$$

$$Q\alpha_2 = 1000 \text{ lb.}$$

where $Q\alpha_1$ and $Q\alpha_2$ are the generalized forces corresponding to the natural displacements

$$\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_1 = \begin{Bmatrix} +1 \\ +1 \end{Bmatrix}, \quad \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_2 = \begin{Bmatrix} -1 \\ +1 \end{Bmatrix}$$



Problem 3 (25 points)

A uniform beam is fixed at both ends. Using finite element theory, and using two Euler-Bernoulli beam elements, find the rotation θ_2 at the center of the beam due to a moment

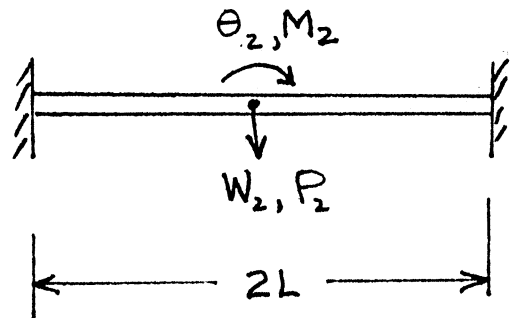
$$M_2 = 5000 \text{ in. lb.}$$

$$E = 10 \times 10^6 \text{ psi}$$

$$I = 400 \text{ in.}^4$$

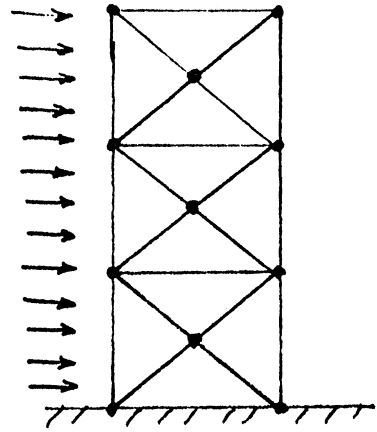
$$\nu = 0.3$$

$$L = 100 \text{ in}$$



Problem 4 (25 points)

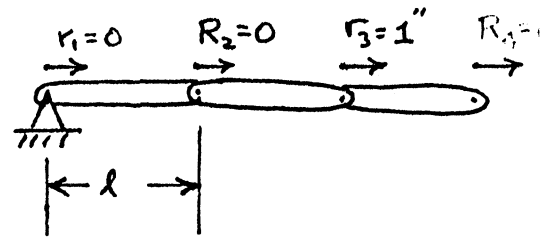
Wind pressure is acting on the left face of a concrete wall as shown. The wall is modelled with constant strain, plane strain triangles (Turner's).



- (a) How many elements are there? _____
- (b) How many nodes, including constrained nodes at the base? _____
- (c) What is the minimum half bandwidth possible, using detailed notation? _____
- (d) How many nodes will be given non-zero equivalent nodal loads due to the wind, including any constrained nodes? _____

Problem 5 (25 points)

Three identical line elements are joined as shown. The left node r_1 is constrained and the third node r_3 is moved 1" to the right. Set up the equations of equilibrium for the 4×4 matrix problem, using the artifice of Payne and Irons to modify the equations for solution "in place." Let $k_1 \equiv E_1 A_1 / L_1$, etc.



$$\begin{bmatrix}
 K_{11} & K_{12} & K_{13} & K_{14} \\
 & \ddots & & \\
 & & \ddots & \\
 K_{41} & & & K_{44}
 \end{bmatrix}
 \begin{Bmatrix}
 \\
 \\
 \\
 \end{Bmatrix}
 =
 \begin{Bmatrix}
 \\
 \\
 \\
 \end{Bmatrix}$$

Express the equations in the simplest modified form, without solving for any of the unknowns.

Problem 6 (50 points)

Three identical segments are assembled in a Y. For an individual pipe element, where $|Q^e|$ is the fluid flow

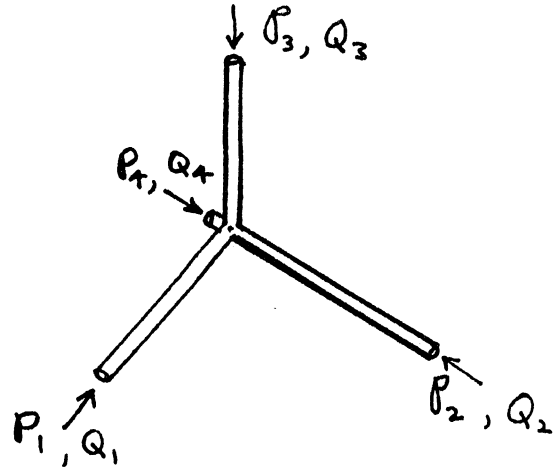
$$\begin{Bmatrix}
 Q_i \\
 Q_j
 \end{Bmatrix}
 =
 \begin{bmatrix}
 \frac{1}{\sqrt{|Q^e|}} & -\frac{1}{\sqrt{|Q^e|}} \\
 -\frac{1}{\sqrt{|Q^e|}} & \frac{1}{\sqrt{|Q^e|}}
 \end{bmatrix}
 \begin{Bmatrix}
 p_i \\
 p_j
 \end{Bmatrix}$$

Problem 6 (continued)

in the element in question.

- (a) Assemble the equations for the "Y", using a node 4 at the junction.
- (b) If $P_1 = 100$ psf
 $P_2 = 200$ psf
 $P_3 = 150$ psf
 $P_4 = 125$ psf

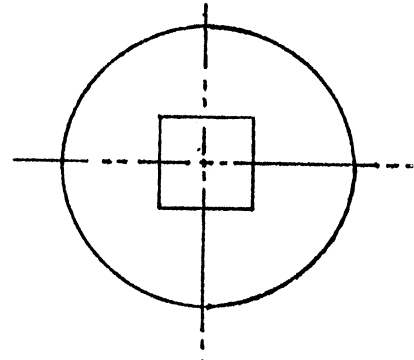
solve for the fluid flows using the "variable stiffness" iteration (secant stiffness approach). Carry the solution out to 3 iterations. It is suggested that you use the notation $1/f^1 \equiv 1/\sqrt{|Q^1|}$, etc., for the assembly process. This will simplify your writing and also grading.



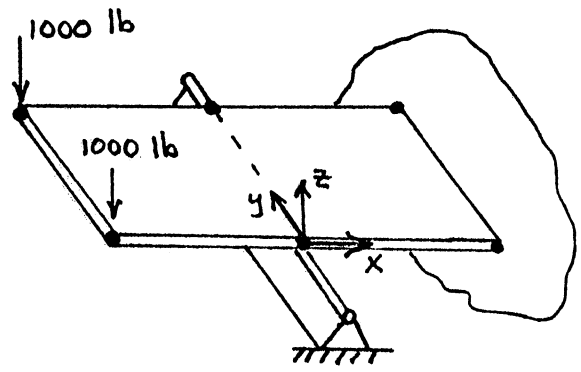
Problem 7 (25 points)

- (a) Does SAPIV require careful numbering of nodes or of elements to minimize high speed storage requirements? _____
- (b) Which takes less CPU time to run on SAPIV: problem 1 with 1000 equations and a bandwidth of 200 or problem 2 with 600 equations and a bandwidth of 300? _____

- (c) A two-dimensional problem involves a square cut out in a round tube. The tube is to be internally pressurized and free on the outer surface. Using symmetry arguments, what is the minimum fraction of the body you would have to model for a solution? _____



- (d) A plate is fixed at a wall and supported by a roller as shown, with two vertical tip loads. If this problem is modelled with two plate elements as shown:
 - How many forces and moments have been prescribed by the problem statement? _____
 - How many displacements and rotations have been prescribed? _____



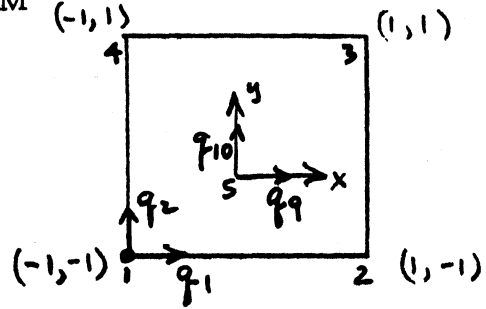
50 Min.
Honor Code
Open Book

AERO 510
SECOND HOUR EXAM

December 3, 1976

PROB 1. (25 pts.)

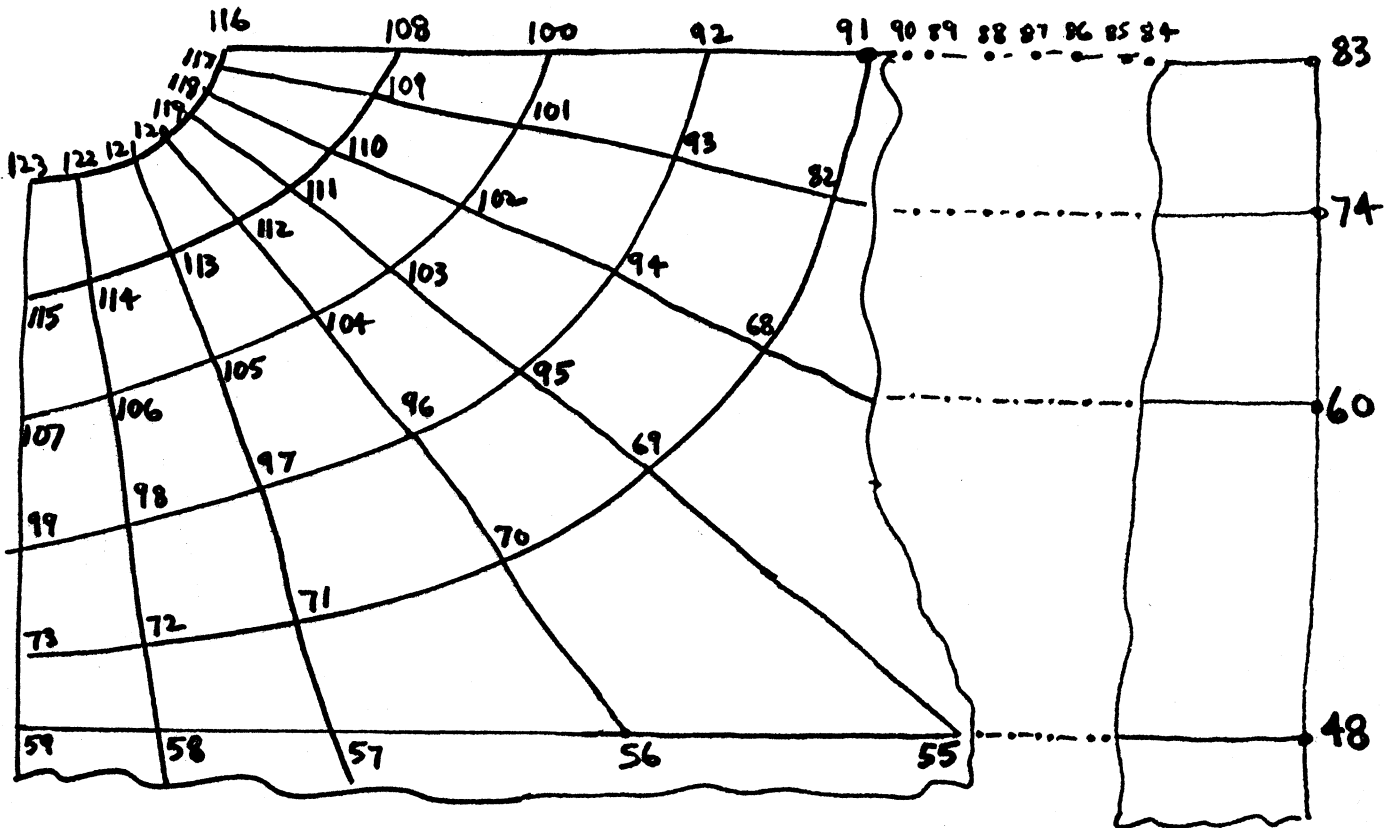
A plane strain square is shown. If this element has 5 nodes, at corners and centroid, write out the relation



$$\{q\} = [A] \{a\}$$

using the natural mode approach. Include numerical values in the A matrix to yield the proper rigid body modes, and include numerical values for one straining mode. Use detailed notation, as suggested in the sketch.

PROB 2.



A certain professor has numbered the nodes for a F.E. study of an octagonal crossarm as shown. Plate elements with 6 D.O.F. per node are used.

- A) What is the current half bandwidth, in detailed notation?
- B) Show a suggested nodal numbering scheme that cuts the bandwidth to below 130. You may use pencil on the existing grid pattern.
- C) What is the half bandwidth of your suggested numbering system, in detailed notation? Show the critical element which causes the maximum difference in node numbers.

PROB 3 (25 points)

A three-point Lobatto integration is to be used to integrate a quartic polynomial (4th degree) on a symmetric domain $[-1, 1]$. The endpoints are taken to be two integration points.

Assume you are asked to develop this 3 point Lobatto integration from "scratch".

- A) How many defining equations are there?
- B) What degree polynomials can be exactly integrated? Is the integration of the quartic going to be exact?
- C) Write out the first defining equation, and reduce it as much as possible to the case at hand.

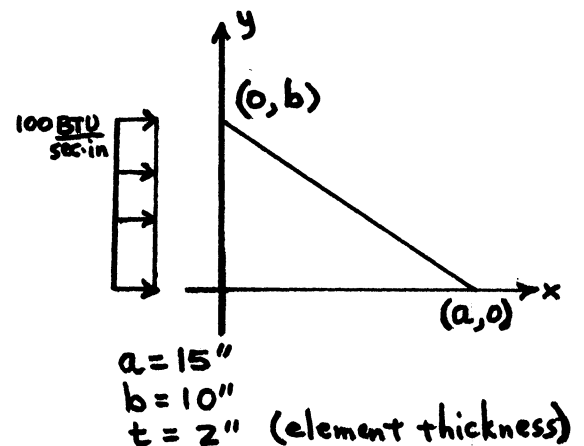
PROB 4 (25 points)

A constant temperature gradient triangle has been developed for steady heat conduction. It is the thermal equivalent of a Turner triangle as seen in stress problems. For simplicity, a right triangle with local coordinate system is considered. The shape functions are

$$N_1 = 1 - x/a - y/b$$

$$N_2 = x/a$$

$$N_3 = y/b$$



Find the equivalent nodal fluxes for the element due to a flux of intensity 100 BTU/SEC/IN at the left boundary as shown.

Honor Code
Open Book
50 minutes

HOUR EXAM
 ME 557 AERO 510 AM/ES 510

15 Oct 76

PROB 1. BEAM ELEMENT (30%)

Given a slender, uniform beam as shown, calculate the rotation θ_2 at the right end if:

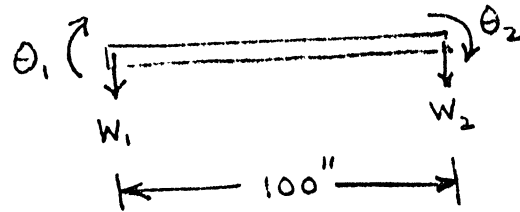
$$W_1 = 0$$

$$\theta_1 = 0$$

$$W_2 = 0$$

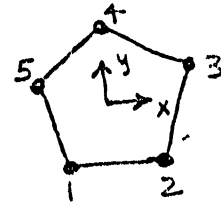
$$M_2 = 10^5 \text{ in}\cdot\text{lb}$$

$$\frac{EI}{L^3} = 10^3 \text{ lb/in}$$



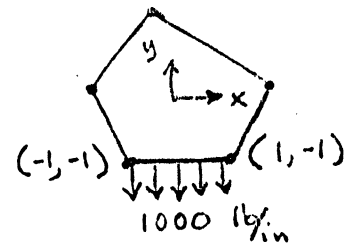
PROB 2. HYPOTHETICAL ELEMENT (30%)

A 5-sided plane stress element is to be developed. A node will be placed at each vertex as shown.



- A.) Write out a proposed set of displacement functions in matrix form, guaranteeing convergence and preserving geometric isotropy.
- B.) Write out the strain-displacement matrix.

C.) Assume a line load is placed on the lower face of the element as shown. The nodes on that boundary are at $(-1, -1)$ and $(1, -1)$. Write out the expression for equivalent nodal loads due to this vertical load as fully as possible, using shape functions $N_1(x, y)$, $N_2(x, y)$... in the form:



$$\begin{Bmatrix} u(x, y) \\ v(x, y) \end{Bmatrix} = \begin{bmatrix} N_1(x, y) & 0 & N_3(x, y) & \dots & \dots \\ 0 & N_2(x, y) & 0 & \dots & \dots \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ \vdots \end{Bmatrix}$$

Do not try to find explicit shape functions; just use their symbols.

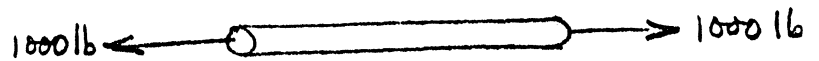
PROB 3. VIRTUAL STRAIN ENERGY (30%)

Let us define virtual strain energy as the change (increment) of strain energy during a virtual displacement. Consider a line element with $EA/l = 10^4 \text{ lb/in}$ under a 1000 lb tensile load as shown. After the line element is in static equilibrium, a virtual displacement

$$\{\delta q\} = \begin{Bmatrix} .001 \\ -.001 \end{Bmatrix} \text{ in.}$$

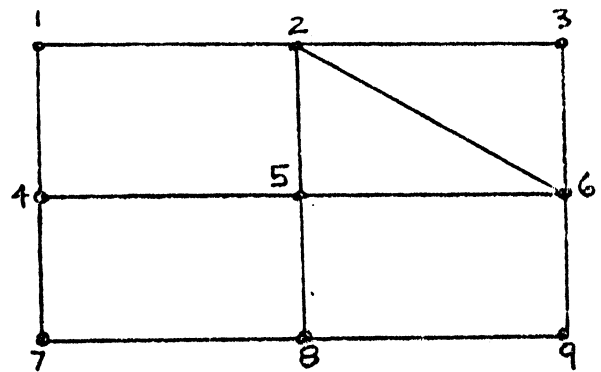
is imposed, What is the resulting virtual strain energy?

Comments: Remember that work is defined as $\left\{ \begin{matrix} \text{INCREMENT} \\ \text{OF} \\ \text{DISPL.} \end{matrix} \right\}^T \left\{ \text{FORCE} \right\}$ and that the elastic force vector is $[k]\{q\}$.



PROB 4. ASSEMBLED STIFFNESS MATRIX (10%)

A plane strain problem is modelled with 3 rectangular and two triangular elements as shown. Using compact notation, how many stiffness terms in the assembled stiffness matrix are zero? Be sure to count both k_{ji} and k_{ij} terms in your answer.



50 Minutes
Open Book
Honor Code

AERO 510 ME 557 AMES 510

October 24, 1975

MIDTERM EXAM

W. J. Anderson

All questions refer to the displacement method, and use detailed (as opposed to compact) notation. Answer on these pages.

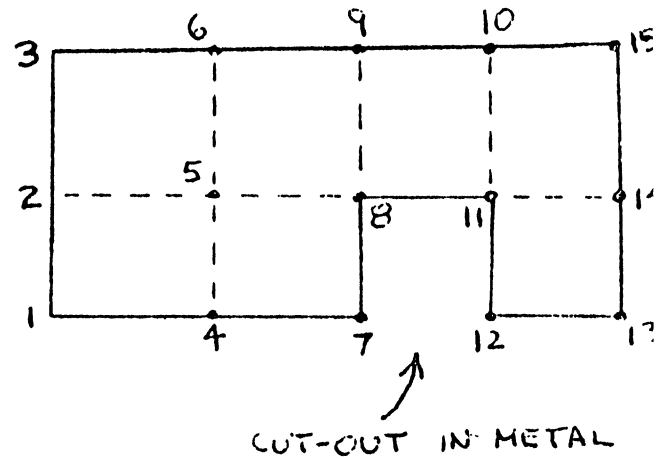
TRUE-FALSE SECTION (30 points - 2 points per problem)

- | | T | F |
|---|-------------------------------------|-------------------------------------|
| 1. The bandwidth of an assembled stiffness matrix depends on the way the elements are numbered. | <input type="checkbox"/> | <input checked="" type="checkbox"/> |
| 2. Higher order plane strain triangles (those that use higher degree polynomials for displacement functions) are less stiff than the constant strain Triangle of Turner, et. al. | <input checked="" type="checkbox"/> | <input type="checkbox"/> |
| 3. The virtual work theorem can be proven from the potential energy theorem. | <input type="checkbox"/> | <input checked="" type="checkbox"/> |
| 4. Usual finite element problems involve more displacements which are specified than forces which are specified. | <input type="checkbox"/> | <input checked="" type="checkbox"/> |
| 5. Prestrain can occur because of temperature effects. | <input checked="" type="checkbox"/> | <input type="checkbox"/> |
| 6. Temperature effects cannot be studied in the same solution as external, prescribed loads because two different stiffness matrices are required. | <input type="checkbox"/> | <input checked="" type="checkbox"/> |
| 7. A useful beam element can be made with as few as 4 d.o.f. | <input checked="" type="checkbox"/> | <input type="checkbox"/> |
| 8. The stress field in an Euler-Bernoulli beam is uniaxial. | <input checked="" type="checkbox"/> | <input type="checkbox"/> |
| 9. A column in the shape function matrix $[N]$ represents internal displacement fields due to a unit nodal displacement at a specific d.o.f. | <input checked="" type="checkbox"/> | <input type="checkbox"/> |
| 10. The strain energy in the body will be lower for an approximate solution than for the exact. | <input checked="" type="checkbox"/> | <input type="checkbox"/> |
| 11. Stiffness matrices for a single element should be singular (have zero determinant) in order to allow convergence. | <input checked="" type="checkbox"/> | <input type="checkbox"/> |
| 12. The global shape function matrix $[\bar{N}]$ is useful for creating assembled equivalent nodal loads in problems where many different local coordinate systems are used. | <input type="checkbox"/> | <input checked="" type="checkbox"/> |
| 13. The work potential as it appears in the potential energy theorem is equal numerically to the work done by the external loads. | <input type="checkbox"/> | <input checked="" type="checkbox"/> |
| 14. The natural mode method of Argyris is useful because the rigid body modes are explicitly seen and because the natural stiffness matrix is simpler than the physically defined stiffness matrix. | <input checked="" type="checkbox"/> | <input type="checkbox"/> |
| 15. The first variation of potential energy yields the equilibrium equations. | <input checked="" type="checkbox"/> | <input type="checkbox"/> |

Section No. 2 (15 points)

A plane stress study uses rectangular elements (see sketch). The nodes are shown.

- A) How many nodal degrees of freedom are there? 30
- B) What is the half bandwidth? 12
- C) If you renumbered, what is the optimum bandwidth? 10
- D) If nodes 1 and 3 are restrained in the horizontal direction and node 2 is constrained in the vertical direction, are rigid body motions possible? NO

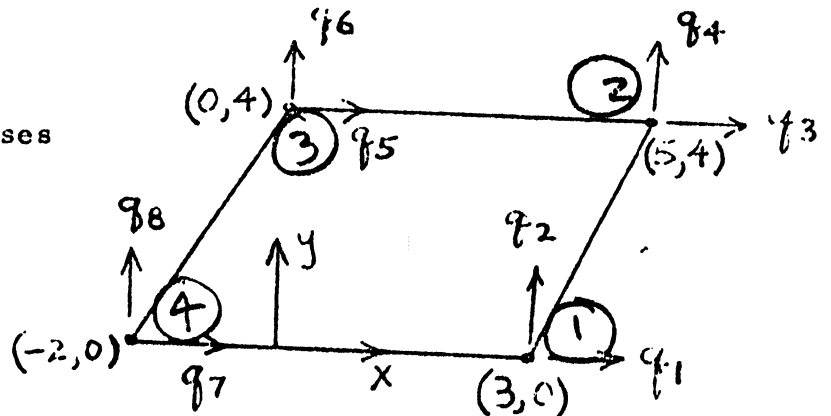


Section No. 3 (15 points)

A plane stress quadrilateral uses displacement functions

$$u(x, y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$$

$$v(x, y) = \alpha_5 + \alpha_6 x + \alpha_7 y + \alpha_8 xy$$



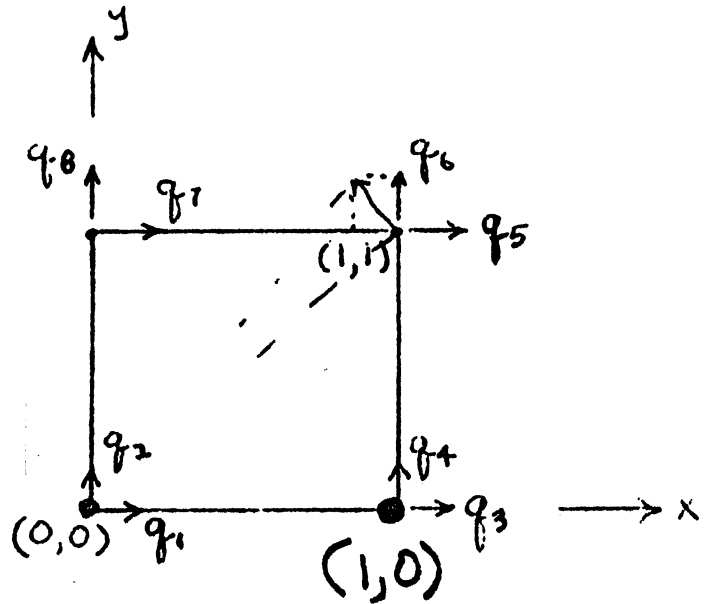
Find the matrix A relating the nodal coordinates to the generalized coordinates.

$$[A] = \begin{bmatrix} 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 & 0 & 0 \\ 1 & 5 & 4 & 20 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 5 & 4 & 20 \\ 1 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 4 & 0 \\ 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 \end{bmatrix}$$

Section 5 (10 points)

A natural mode solution is to be done for a plane problem. Write down the rigid body modes in terms of nodal coordinates for the given square element with unit sides.

$$\begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$



Given in terms of internal displacements. -5

SOLUTION :

$$\begin{aligned} u(x,y) &= [\alpha_1 - \alpha_2 y + \alpha_3 x + \dots] \\ v(x,y) &= [\alpha_2 + \alpha_3 x + \alpha_4 y + \dots] \end{aligned} = \begin{bmatrix} 1 & 0 & x & \dots \\ 0 & 1 & y & \dots \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \end{Bmatrix}$$

Hence α_1 = natural coordinate for rigid body translation: to right.
 α_2 = " " " " " " " " upward.
 α_3 = " " " " " " " " rotation: counterclockwise

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ \vdots \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \end{Bmatrix}$$

Section 6 (20 points)

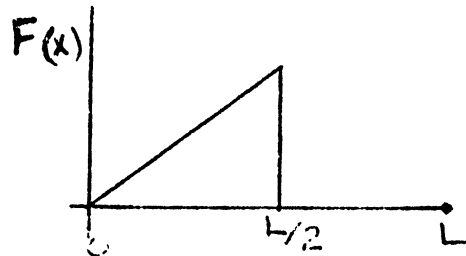
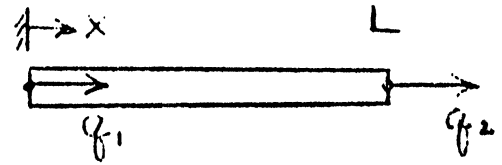
A two-node, constant area, line element has a displacement function

$$u(x) = \alpha_1 + \alpha_2 x$$

A line load is

$$F(x) = \begin{cases} F_0 x/L & 0 \leq x \leq L/2 \\ 0 & L/2 < x \leq L \end{cases}$$

Calculate the equivalent nodal loads for this distributed load.



$$\{Q\}_{e.n.l.} = \int_{\text{space}} [N]^T \{F(x)\} d\text{space}$$

where

$$[N] = \left[1 - x/L, \quad x/L \right]$$

$$\{Q\}_{e.n.l.} = \int_0^{L/2} \begin{bmatrix} 1 - x/L \\ x/L \end{bmatrix} \{F_0 x/L\} dx + \int_{L/2}^L \begin{bmatrix} \\ \end{bmatrix} \{0\} dx$$

$$= \int_0^{L/2} \begin{Bmatrix} F_0 [x/L - (x/L)^2] \\ F_0 (x/L)^2 \end{Bmatrix} dx$$

$$= F_0 \left\{ \begin{array}{c} \frac{x^2}{2L} - \frac{x^3}{3L^2} \\ \frac{x^3}{3L^2} \end{array} \right\} \Big|_{x=0}^{x=L/2}$$

$$= F_0 \left\{ \begin{array}{c} \frac{(L/2)^2}{2L} - \frac{(L/2)^3}{3L^2} \\ \frac{(L/2)^3}{3L^2} \end{array} \right\}$$

$$= F_0 \left\{ \begin{array}{c} \frac{L}{8} - \frac{L}{24} \\ \frac{L}{24} \end{array} \right\} = F_0 L \left\{ \begin{array}{c} \frac{1}{12} \\ \frac{1}{24} \end{array} \right\}$$

Check total loading:

$$\int F dx = \frac{1}{2} \left(\frac{F_0}{2} \right) \frac{L}{2} = \frac{F_0 L}{8}$$

$$\begin{aligned} Q'_{end.} + Q_{e.n.l.} &= \left(\frac{2}{24} + \frac{1}{24} \right) F_0 L \\ &= \frac{3}{24} F_0 L \\ &= \frac{1}{8} F_0 L \quad \checkmark \end{aligned}$$

ME 557

AMES 510

AERO 510

FINAL EXAM

2 HOURS
CLOSED BOOK
HONOR CODE

DECEMBER 20, 1974

Problem 1 25 points

Carry out a formal solution of the equations

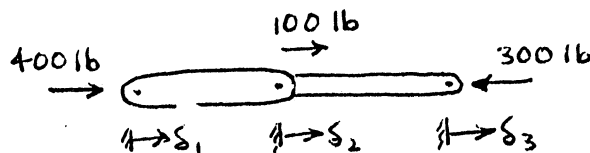
$$\delta_1 + 2\delta_2 = 5$$

$$2\delta_1 + 6\delta_2 = 14$$

by the Choleski decomposition. (Decompose the coefficient matrix into the product of a lower triangular matrix and its transpose.) Show all of your steps. Describe the procedure with proper terminology.

Problem 2 25 points

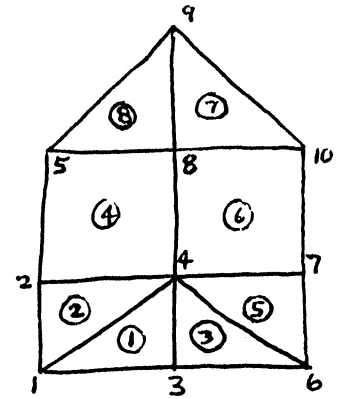
Attack the same set of equations as in Problem 1 with Gauss-Seidel iteration. Assume an initial vector $\begin{Bmatrix} 1 \\ 6 \end{Bmatrix}$ and find the next vector iterate.

Problem 3 25 points

Two prismatic links are connected axially as shown. The three external forces are prescribed. Can you solve for the displacements? Show exactly your calculations and reasoning.

Problem 4 (25 points)

A plane stress problem is solved by finite elements. A total of 8 elements with 10 nodes are used to model the structure.



The half bandwidth of the assembled stiffness matrix
(in compact notation) is _____

The number of zero terms in the assembled
stiffness matrix (in compact notation) is _____

The maximum wavefront (in compact notation) is _____

MULTIPLE CHOICE (5 points per problem)

a b c d

1. Turner, et al developed the constant strain triangle and rectangle using a) Energy arguments b) Galerkin's method c) equilibrium methods d) Lagrange multipliers.
2. Energy approaches in F.E. have advantages in a) development of equivalent nodal loads b) creation of consistent mass matrices c) development of elements with complicated shape functions d) all of the above.
3. Vector force equilibrium at a node is described by a) a single equilibrium equation in "detailed" notation b) a single equilibrium equation in "compact" notation c) a column of the stiffness matrix d) summation of external forces at the node.
4. Fluid and electrical networks a) consist of 1 dimensional elements imbedded in a two or three dimensional space b) must be studied by energy methods since equilibrium doesn't apply to flow c) have a strict sign convention that all flows are positive when to the right d) cannot be a linear problem.
5. Numbering of the nodes affects a) the bandwidth b) the wavefront size c) the number of nonzero terms in the assembled stiffness matrix d) all of the above.
6. Prestrain a) can be handled as an equivalent nodal load b) can be caused by temperature change c) can be caused by shrinkage, such as drying of wood d) all of the above.

7. For an element to provide a solution which converges as more elements are used a) linear strain must be possible when nodal movement allows it b) rigid body motion must result for any combination of nodal movements c) constant strain must result if the nodal displacements are compatible with that condition d) all of the above.
8. In the basic theory of finite elements, an element with n nodes always has a) n generalized coordinates in detailed notation b) $2n$ shape functions in detailed notation c) three displacement fields: $u(x, y)$, $v(x, y)$, $w(x, y)$ d) n nodal displacements in compact notation.
9. Lagrange multipliers are used in F.E. theory to a) form equivalent nodal loads b) calculate residual error c) constrain displacements d) impose material conditions.
10. A method for discretizing a field problem directly from the differential field equations is a) Potential energy theorem b) Galerkin's method c) virtual work theorem d) all of the above.
11. If a material is orthotropic, one must account for this in the a) stress-strain law b) strain-displacement law c) equivalent nodal loads due to volume loads d) all of the above.
12. The size of the element chosen is determined by a) the gradient of the stress field (how rapidly stress changes across the body) b) the accuracy desired c) the amount of

money available for computation d) all of the above.

13. The main reason that 3-D elements (solids) cost so much to run is a) a solid body requires so many equations for its description b) solid elements such as the Tetrahedron must have complicated shape functions c) computers can handle matrices $K(I, J)$ but have more trouble with matrices $K(I, J, L)$ d) the bandwidth (or wavefront) is larger for 3-D bodies.

14. Why should displacement functions be continuous across element faces in plane stress problems? a) they don't have to be! b) to avoid infinite contributions to the potential energy functional at the interface c) so that rigid body modes of the assembled structure are possible d) so that strains (the derivatives of displacements) can be continuous across the interfaces.

15. An acceptable shape function for an Euler-Bernoulli beam in the x, y plane is
 a) $\frac{x(x-L)^2}{L^2}$ b) $\frac{x^2(x-L)^2}{L^2}$ c) $\frac{x^2(x-L)^3}{L^3}$
 d) all of the above.

16. The virtual work theorem requires a) an infinitesimal, virtual displacement b) an energy conserving system c) that one account for work done by external forces on the body in question d) the first and third answers above.

17. The potential energy theorem a) uses a strain energy measured from some initial reference position b) uses a work function defined from some initial reference c) requires a variation of displacement quantities d) all of the above.
18. The natural mode method a) allows solution in terms of generalized coordinates instead of nodal coordinates b) is a version of the Lagrange multiplier method c) creates a natural stiffness matrix which is nonsymmetric d) all of the above.
19. Gaussian elimination is an efficient way to solve:
b) sets of linear, symmetric differential equations c) eigenvalue problems d) sets of linear algebraic equations.
20. Wavefront methods are better than banded decomposition methods because they a) use far less CPU time b) use far less core storage c) don't require efficient numbering of nodes or elements d) all of the above.



REPRINTS

Appendix 1

Stiffness and Deflection Analysis of Complex Structures

M. J. TURNER,* R. W. CLOUGH,† H. C. MARTIN,‡ AND L. J. TOPP**

ABSTRACT

A method is developed for calculating stiffness influence coefficients of complex shell-type structures. The object is to provide a method that will yield structural data of sufficient accuracy to be adequate for subsequent dynamic and aeroelastic analyses.

Stiffness of the complete structure is obtained by summing stiffnesses of individual units. Stiffnesses of typical structural components are derived in the paper. Basic conditions of continuity and equilibrium are established at selected points (nodes) in the structure. Increasing the number of nodes increases the accuracy of results. Any physically possible support conditions can be taken into account. Details in setting up the analysis can be performed by nonengineering trained personnel; calculations are conveniently carried out on automatic digital computing equipment.

Method is illustrated by application to a simple truss, a flat plate, and a box beam. Due to shear lag and spar web deflection, the box beam has a 25 per cent greater deflection than predicted from beam theory. It is shown that the proposed method correctly accounts for these effects.

Considerable extension of the material presented in the paper is possible.

(I) INTRODUCTION

PRESENT CONFIGURATION TRENDS in the design of high-speed aircraft have created a number of difficult, fundamental structural problems for the worker in aeroelasticity and structural dynamics. The chief problem in this category is to predict, for a given elastic structure, a comprehensive set of load-deflection relations which can serve as structural basis for dynamic load calculations, theoretical vibration and flutter analyses, estimation of the effects of structural deflec-

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** Structures Engineer, Structural Dynamics Unit, Boeing Airplane Company, Wichita Division.

tion on static air loads, and theoretical analysis of aeroelastic effects on stability and control. This is a problem of exceptional difficulty when thin wings and tail surfaces of low aspect ratio, either swept or unswept, are involved.

It is recognized that camber bending (or rib bending) is a significant feature of the vibration modes of the newer configurations, even of the low-order modes; in order to encompass these characteristics it seems likely that the load-deflection relations of a practical structure must be expressed in the form of either deflection or stiffness influence coefficients. One approach is to employ structural models and to determine the influence coefficients experimentally; it is anticipated that the experimental method will be employed extensively in the future, either in lieu of or as a final check on the result of analysis. However, elaborate models are expensive, they take a long time to build, and tend to become obsolete because of design changes; for these reasons it is considered essential that a continuing research effort should be applied to the development of analytical methods. It is to be expected that modern developments in high-speed digital computing machines will make possible a more fundamental approach to the problems of structural analysis; we shall expect to base our analysis on a more realistic and detailed conceptual model of the real structure than has been used in the past. As indicated by the title, the present paper is exclusively concerned with methods of theoretical analysis; also it is our object to outline the development of a method that is well adapted to the use of high-speed digital computing machinery.

(II) REVIEW OF EXISTING METHODS OF STRUCTURAL ANALYSIS

(1) *Elementary Theories of Flexure and Torsion*

The limitations of these venerable theories are too well known to justify extensive comment. They are

adequate only for low-order modes of elongated structures. When the loading is complex (as in the case of inertia loading associated with a mode of high order) refinements are required to account for secondary effects such as shear lag and torsion-bending.

(2) *Wide Beam Theory: Schuerch*¹

Schuerch has devised a generalized theory of combined flexure and torsion which is applicable to multi-spar wide beams having essentially rigid ribs. Torsion-bending effects are included but not shear lag. It is expected that wide beam theory will be used extensively in the solution of static aeroelastic problems (effect of air-frame flexibility on steady air loads, stability, etc.). However, the rigid rib assumption appears to limit its utility rather severely for vibration and flutter analysis of thin low aspect ratio wings.

(3) *Method of Redundant Forces: Levy, Bisplinghoff and Lang, Langefors, Rand, Wehle and Lansing*²⁻⁴

These writers have contributed the basic papers leading to the present widespread use of energy principles, matrix algebra, and influence coefficients in the solution of structural deflection problems. Redundant internal loads are determined by the principle of least work, and deflections are obtained by application of Castigliano's theorem. The method is, of course, perfectly general. However, the computational difficulties become severe if the structure is highly redundant, and the method is not particularly well adapted to the use of high-speed computing machines. Rand has suggested a method of solution for stresses in highly redundant structures which might also be used for calculating deflections. Instead of using member loads as redundants, he proposes to employ systems of self-equilibrating internal stresses. These redundant stresses may be regarded as perturbations of a primary stress distribution that is in equilibrium with the external loads (but does not generally satisfy compatibility conditions). The number of properly chosen redundants required to obtain a satisfactory solution may be considerably less than the "degree of redundancy." Successful application of this method requires a high degree of engineering judgment, and the accuracy of the results is very difficult to evaluate.

(4) *Plate Methods: Fung, Reissner, Bencotter, and MacNeal*⁵⁻⁷

As the trend toward thinner sections approaches the ultimate limit, we enter first a regime of very thick walled hollow structures, such that the flexural and torsional rigidities of the individual walls make a significant contribution to the overall stiffness of the entire wing. Finally we come to the solid plate of variable thickness. During the past few years a substantial research effort has been devoted to the development of methods of deflection analysis for these structural types, and important contributions have been made by all of the aforementioned authors.

(5) *Direct Stiffness Calculation: Levy, Schuerch*^{10, 11}

In a recent paper Levy has presented a method of analysis for highly redundant structures which is particularly suited to the use of high-speed digital computing machines. The structure is regarded as an assemblage of beams (ribs and spars) and interspar torque cells. The stiffness matrix for the entire structure is computed by simple summation of the stiffness matrices of the elements of the structure. Finally, the matrix of deflection influence coefficients is obtained by inversion of the stiffness matrix. Schuerch has also presented a discussion of the problem from the point of view of determining the stiffness coefficients.

(III) SOME UNSOLVED PROBLEMS

At the present time, it is believed that the greatest need is to derive a numerical method of analysis for a class of structures intermediate between the thin stiffened shell and the solid plate. These are hollow structures having a rather large share of the bending material located in the skin, which is relatively thick but still thin enough so that we may safely neglect its plate bending stiffness. In order to cope with this class of structures successfully, we must base our analysis upon a structural idealization that is sufficiently realistic to encompass a fairly general two-dimensional stress distribution in the cover plates; and our method of analysis must yield the load-deflection relations associated with such stresses. It is characteristic of these problems that the directions of principal stresses in certain critical parts of the structure cannot be determined by inspection. Hence, the familiar methods of structural analysis based upon the concepts of axial load carrying members, joined by membranes carrying pure shear, are not satisfactory, even if we employ effective width concepts to account for the bending resistance of the skin. We should like to include shear lag, torsion-bending, and Poisson's ratio effects to a sufficient approximation for reliable prediction of vibration modes and natural frequencies of moderate order. Also, we should like to avoid any assumptions of closely spaced rigid diaphragms or of orthotropic cover plates, which have been introduced in many papers on advanced structural analysis. The actual rib spacing and finite rib stiffnesses should be accounted for in a realistic fashion. In summary, what is required is an approximate numerical method of analysis which avoids drastic modification of the geometry of the structure or artificial constraints of its elastic elements. This is indeed a very large order. However, modern developments in high-speed digital computing machines offer considerable hope that these objectives can be attained.

(IV) METHOD OF DIRECT STIFFNESS CALCULATION

For a given idealized structure, the analysis of stresses and deflections due to a given system of loads is a purely mathematical problem. Two conditions

must be satisfied in the analysis: (1) the forces developed in the members must be in equilibrium and (2) the deformations of the members must be compatible—i.e., consistent with each other and with the boundary conditions. In addition, the forces and deflections in each member must be related in accordance with the stress-strain relationship assumed for the material.

The analysis may be approached from two different points of view. In one case, the forces acting in the members of the structure are considered as unknown quantities. In a statically indeterminate structure, an infinite number of such force systems exist which will satisfy the equations of equilibrium. The correct force system is then selected by satisfying the conditions of compatible deformations in the members. This approach has been widely used for the analysis of all types of indeterminate structures but is, as already noted, particularly advantageous for structures that are not highly redundant.

In the other approach, the displacements of the joints in the structure are considered as unknown quantities. An infinite number of systems of mutually compatible deformations in the members are possible; the correct pattern of displacements is the one for which the equations of equilibrium are satisfied. The concept of static determinateness or indeterminateness is irrelevant when the analysis is considered from this viewpoint. This approach is the basis for many relaxation type analyses (such as moment distribution) and has been applied to the analysis of complex aircraft structures by Levy in the aforementioned paper. This will be called the method of direct stiffness calculation hereafter.

After reviewing the various methods available to the dynamics engineer for computing load-deflection relations of elastic structures, it is concluded that the most promising approach to our present difficulties is to extend further the method of direct stiffness calculation. The remainder of this paper is concerned with methods by which that extension may be accomplished.

(V) SIMPLE EXAMPLES OF STIFFNESS INFLUENCE COEFFICIENTS

(1) Elastic Spring

If an elastic spring deflects an amount δ under axial load F , Hooke's Law applies and

$$F = k\delta \tag{1}$$

Here k can be regarded as the force required to produce a unit deflection; hence it can be considered to be a stiffness influence coefficient.

Eq. (1) can also be written as

$$\delta = (1/k)F = cF \tag{2}$$

where c is the deflection due to a unit force (deflection influence coefficient).

(2) Two-Dimensional Elastic Body

Extending the above relations to the two-dimensional body is most conveniently accomplished by introducing

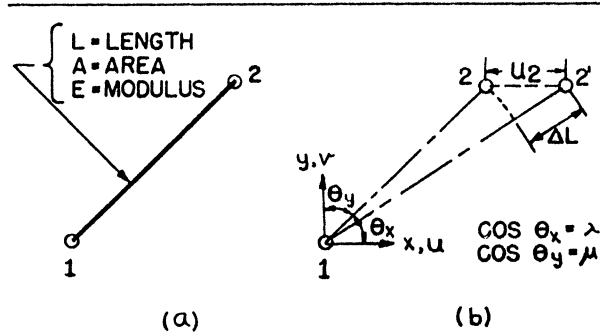


FIG. 1. Typical pin-ended truss member.

matrix notation. Eqs. (1) and (2) become, respectively,

$$\{F\} = [K] \{\delta\} \tag{3}$$

$$\{\delta\} = [K]^{-1} \{F\} = [C] \{F\} \tag{4}$$

Here $[K]$ is the matrix of stiffness influence coefficients. A typical element of $[K]$ is $k_{ij}^{\xi\eta}$ = force required at i in the ξ -direction, to support a unit displacement at j in the η -direction. If ξ and η always refer to the same direction, we can use the simpler form k_{ij} . In either case an element of $[K]$, and also of $[C]$, must obey the well-known reciprocal relations. In other words, the $[K]$ and $[C]$ matrices are symmetric, provided they are referred to orthogonal coordinate systems. As will be seen later, the symmetry condition does not apply if oblique coordinates are used.

(3) Truss Member.

Fig. 1(a) shows a typical pin ended truss member. We wish to determine its matrix of stiffness influence coefficients. Loads may be applied at points (nodes) 1 and 2. Each node can experience two components of displacement. Therefore, prior to introducing boundary conditions (supports), $[K]$ for this member will be of order 4×4 .

To develop one column of $[K]$, subject the member to $u_2 \neq 0, u_1 = v_1 = v_2 = 0$. Then

$$\Delta L = u_2 \cos \theta_x = u_2 \lambda$$

The axial force needed to produce ΔL is

$$P = (AE/L)\Delta L = (AE/L)\lambda u_2$$

The components of P at node 2 are

$$F_{x_2} = P \cos \theta_x = (AE/L) \lambda^2 u_2$$

$$F_{y_2} = P \cos \theta_y = (AE/L) \lambda \mu u_2$$

Equilibrium gives the forces at node 1 as

$$F_{x_1} = -F_{x_2}$$

$$F_{y_1} = -F_{y_2}$$

Eq. (3) for this member then takes the form

$$\begin{Bmatrix} F_{x_1} \\ F_{y_1} \\ F_{x_2} \\ F_{y_2} \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} . & -\lambda^2 & . & . \\ . & \lambda^2 & . & . \\ . & -\lambda\mu & . & . \\ . & \lambda\mu & . & . \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{Bmatrix} \tag{5}$$

The other elements in $[K]$ are found in a similar manner. We get

$$[K]_{\text{truss member}} = \frac{AE}{L} \begin{bmatrix} u_1 & u_2 & v_1 & v_2 \\ \lambda^2 & & & \\ -\lambda^2 & \lambda^2 & & \\ \lambda\mu & -\lambda\mu & \mu^2 & \\ -\lambda\mu & \lambda\mu & -\mu^2 & \mu^2 \end{bmatrix} \quad (6)$$

As given in Eq. (6), $[K]$ is singular—that is, its determinant vanishes and its inverse does not exist. This is overcome by supplying boundary conditions or supports for the bar sufficient to prevent it from moving as a rigid body. For example, we may choose $u_1 = v_1 = u_2 = 0, v_2 \neq 0$. Node 1 is then fixed, while node 2 is provided with a roller in the y -direction. The only force component now capable of straining the bar is F_{v_1} . The force in the bar and the reactions are given by Eqs. (5) and (6).

Any other physically correct boundary conditions can be imposed. In other words, once $[K]$ has been determined, a solution can be found for any set of support conditions. The only requirement is that the structure be fixed against rigid body displacement.

(VI) STIFFNESS ANALYSIS OF SIMPLE TRUSS

Once stiffness matrices for the various component units of a structure have been determined, the next step of finding the stiffness of the composite structure may be taken. The procedure for doing this is essentially independent of the complexity of the structure. As a result, it will be illustrated for a simple truss as shown in Fig. 2.

The stiffness of any one member of the truss is given by Eq. (6). Since length varies for the truss members, this term should be brought inside the matrix. It is then convenient to call the elements of the stiffness matrix $\bar{\lambda}^2 = \lambda^2/\text{length}$, etc. Then $\bar{\lambda}^2, \bar{\mu}^2$, and $\bar{\lambda}\bar{\mu}$ represent the essential terms defining the stiffness of the separate truss members. These are conveniently calculated by setting up Table 1.

From the last three columns of Table 1 the truss stiffness matrix can be written directly. This is best seen by forming the truss equation $[Eq. (7a)]$ analogous to Eq. (5) for the single member.

The formation of all columns in Eq. (7a) can be explained by considering any one of them as an example. The second column will be chosen. It represents the case for which $v_1 \neq 0$, all other node displacements = 0.

$$\begin{Bmatrix} F_{x_1} \\ F_{y_1} \\ F_{x_2} \\ F_{y_2} \\ F_{x_3} \\ F_{y_3} \end{Bmatrix} = AE \begin{bmatrix} \frac{1}{2\sqrt{2}L} & -\frac{1}{2\sqrt{2}L} & 0 & 0 & -\frac{1}{2\sqrt{2}L} & \frac{1}{2\sqrt{2}L} \\ -\frac{1}{2\sqrt{2}L} & \frac{1}{L} + \frac{1}{2\sqrt{2}L} & 0 & -\frac{1}{L} & \frac{1}{2\sqrt{2}L} & -\frac{1}{2\sqrt{2}L} \\ 0 & 0 & \frac{1}{L} & 0 & -\frac{1}{L} & 0 \\ 0 & -\frac{1}{L} & 0 & 1 & 0 & 0 \\ -\frac{1}{2\sqrt{2}L} & \frac{1}{2\sqrt{2}L} & -\frac{1}{L} & 0 & \frac{1}{L} + \frac{1}{2\sqrt{2}L} & -\frac{1}{2\sqrt{2}L} \\ \frac{1}{2\sqrt{2}L} & \frac{1}{2\sqrt{2}L} & 0 & 0 & \frac{1}{2\sqrt{2}L} & \frac{1}{2\sqrt{2}L} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \quad (7a)$$

or $\{F\} = [K] \{\delta\}$ (7b)

In this second column the y -components of force are given by the $\bar{\mu}^2$ terms in Table 1; the x -components of force are given by the $\bar{\lambda}\bar{\mu}$ terms. Thus F_{y_1} is the sum of $\bar{\mu}^2$ for members 1-2 and 1-3 since these are strained due to displacement v_1 . Also F_{y_2} is $-\bar{\mu}^2$ for member 1-2, and F_{y_3} is $-\bar{\mu}^2$ for member 1-3. The

signs follow from the basic stiffness matrix given in Eq. (6). Since equilibrium must hold, the sum of these y -components of force must vanish.

Similarly, F_{x_1} is the sum of the $\bar{\lambda}\bar{\mu}$ terms for members 1-2 and 1-3. Likewise, F_{x_2} is the negative value of $\bar{\lambda}\bar{\mu}$ for member 1-2. Finally F_{x_3} is $-\bar{\lambda}\bar{\mu}$ for member 1-3. These forces must also sum to zero if equilibrium is to hold.

TABLE 1

Member	x	y	Length	λ	μ	λ^2	μ^2	$\lambda\mu$	$\bar{\lambda}^2$	$\bar{\mu}^2$	$\bar{\lambda}\bar{\mu}$
1-2	0	-L	L	0	-1	0	1	0	0	$\frac{1}{L}$	0
1-3	L	-L	$\sqrt{2}L$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{2}L}$	$\frac{1}{2\sqrt{2}L}$	$-\frac{1}{2\sqrt{2}L}$
2-3	L	0	L	1	0	1	0	0	$\frac{1}{L}$	0	0

This process is repeated for all columns. In this way all possible node displacement components are taken into account. In each case the displacements are compatible ones for all members of the truss.

A structure having various kinds of structural components—beams as well as axially loaded members, for example—would be treated in the same manner. However, the basic stiffness matrix for each type of member would have to be known. Deriving these for units of interest in aircraft design represents a major part of this paper.

The matrix of Eq. (7a) is singular. This is altered by providing supports for the truss sufficient to prevent it from displacing as a rigid body when loads are applied. Any sufficient set of supports may be imposed; here we choose to put

$$u_1 = v_1 = u_2 = v_2 = 0$$

In other words, nodes 1 and 2 are fixed, while 3 is left free.

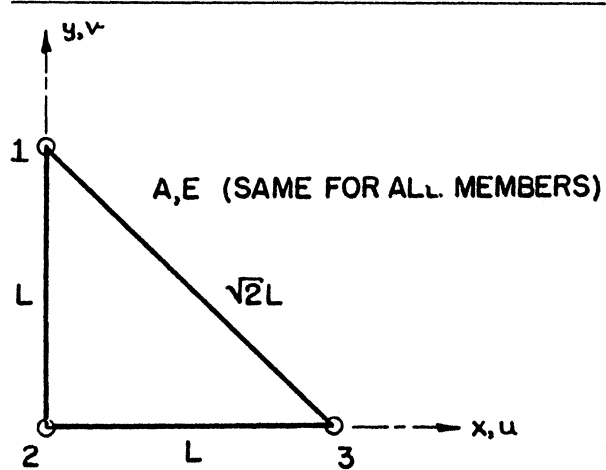


FIG. 2. Simple truss.

It is now convenient to rewrite Eq. (7a) and simultaneously partition it as shown by the broken lines in Eq. (7c).

$$\begin{Bmatrix} F_{x_1} \\ F_{y_1} \\ F_{x_2} \\ F_{y_2} \\ F_{x_3} \\ F_{y_3} \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 + \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -1 & 0 \\ -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & 0 & 0 \\ -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & 0 & 0 \\ \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & 1 + \frac{1}{2\sqrt{2}} & 0 & -1 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_3 \\ v_3 \\ u_1 = 0 \\ v_1 = 0 \\ u_2 = 0 \\ v_2 = 0 \end{Bmatrix} \quad (7c)$$

If the partitioned square (stiffness) matrix is designated by

$$\begin{bmatrix} A_{2 \times 2} & B_{2 \times 4} \\ B'_{4 \times 2} & D_{4 \times 4} \end{bmatrix}$$

expanding Eq. (7c) leads to the following two sets of equations:

$$\begin{Bmatrix} F_{x_1} \\ F_{y_1} \end{Bmatrix} = [A] \begin{Bmatrix} u_3 \\ v_3 \end{Bmatrix} \quad (8a)$$

and

$$\begin{Bmatrix} F_{x_2} \\ F_{y_2} \\ F_{x_3} \\ F_{y_3} \end{Bmatrix} = [B]' \begin{Bmatrix} u_3 \\ v_3 \end{Bmatrix} \quad (8b)$$

Eq. (8a) gives unknown node displacements in terms of applied forces,

$$\begin{Bmatrix} u_3 \\ v_3 \end{Bmatrix} = [A]^{-1} \begin{Bmatrix} F_{x_1} \\ F_{y_1} \end{Bmatrix} \quad (9a)$$

while Eq. (8b), together with Eq. (9a), gives unknown reactions in terms of applied forces,

$$\begin{Bmatrix} F_{x_2} \\ F_{y_2} \\ F_{x_3} \\ F_{y_3} \end{Bmatrix} = [B]' [A]^{-1} \begin{Bmatrix} F_{x_1} \\ F_{y_1} \end{Bmatrix} \quad (9b)$$

In dynamic analyses of aircraft structures it is ordinarily sufficient to determine $[A]^{-1}$. This is the flexibility matrix. It is interesting to note that $[A]$ can be found from the complete $[K]$ matrix by merely striking out columns and rows corresponding to zero displacements as prescribed by the support conditions.

A complete stress analysis leading to the truss member forces can also be carried out. It is merely necessary to know the force-deflection relations for the individual members, or components, of the structure. This is a straightforward problem for the truss and, therefore, will not be discussed further in this paper.

It is worth while to notice that once the stiffness matrix has been written, the solution follows by a series of routine matrix calculations. These are rapidly carried out on automatic digital computing equipment. Changes in design are taken care of by properly modifying the stiffness matrix. This cuts

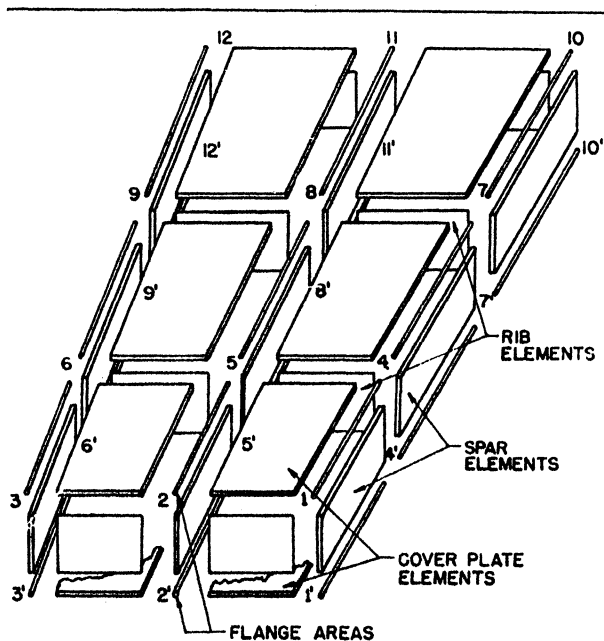


FIG. 3. Wing structure breakdown.

analysis time to a minimum, since development of the stiffness matrix is also a routine procedure. In fact, it may also be programmed for the digital computing machine.

(VII) SUMMARY—METHOD OF DIRECT STIFFNESS CALCULATION

(1) A complex structure must first be replaced by an equivalent idealized structure consisting of basic structural parts that are connected to each other at selected node points.

(2) Stiffness matrices must be either known or determined for each basic structural unit appearing in the idealized structure.

(3) While all other nodes are held fixed, a given node is displaced in one of the chosen coordinate directions. The forces required to do this and the reactions set up at neighboring nodes are then known from the various individual member stiffness matrices. These forces and reactions determine one column in the overall stiffness matrix. When all components of displacement at all nodes have been considered in this manner, the complete stiffness matrix will have been developed. In the general case, this matrix will be of order $3n \times 3n$, where n equals the number of nodes. The stiffness matrix so developed will be singular.

(4) Desired support conditions can be imposed by striking out columns and corresponding rows, in the stiffness matrix, for which zero displacements have been specified. This reduces the order of the stiffness matrix and renders it nonsingular.

(5) For any given set of external forces at the nodes, matrix calculations applied to the stiffness matrix then yield all components of node displacement plus the external reactions.

(6) Forces in the internal members can be found by applying the appropriate force-deflection relations.

The primary functions of the engineer will be to provide the information required in steps (1) and (2) above and to provide the individual member force-deflection relations if a stress analysis is to be carried out. Steps (3) through (6) can be performed by non-engineering trained personnel. Changes in design can be taken into account by correcting local stiffness contributions to K . Node densities can be increased in regions of maximum complexity and importance. If vertical deflections only are required, as in the case of the aircraft wing problem, the $3n \times 3n$ matrix for K can be reduced to order $n \times n$ by a sequence of matrix calculations. Physically, continuity of displacements in three directions at each node will still be maintained.

(VIII) STIFFENED SHELL STRUCTURES

In carrying the above procedure over to stiffened shell structures, it is first necessary to perform steps (1) and (2) of the previous outline.

For a wing structure the idealization will be made by replacing the actual structure by an assemblage of spar segments, rib segments, stiffeners, and cover plate elements, joined together at selected nodes. Fig. 3 shows the proposed idealized structure. The decomposition of the structure can be carried further with some increase in accuracy (for example, by decomposing spar segments into spar caps and shear webs), or it can be simplified by treating the structure as an assemblage of spars and torque boxes. The degree of breakdown should be consistent with the complexity of structural deformations required by the problem at hand. (In a vibration analysis the order of the highest mode is a determining factor.) In light of the proposed idealization, it is necessary that stiffness matrices be developed for the following components: beam segments consisting of flanges joined by thin webs, and plate elements of arbitrary shape. In addition, provision must be made for taking stiffeners into account and possibly for including the effect of sandwich type skin panels.

In the general case, spars will be swept, nonparallel, and not necessarily orthogonal to ribs. It will generally be convenient to transfer stiffness values for any given member to a fixed set of reference axes. These reference axes will be chosen as rectangular Cartesian (x, y, z) in order to preserve symmetry in the total K -matrix.

An outline of the determination of member stiffness for simple structural elements is given in the paper. Further details are presented in Appendixes. Derivation of stiffness matrices for more complex elements can be accomplished in a straightforward manner. However, in the analysis of an actual structure, it will be necessary to weigh the relative advantages of employing a small number of large complex elements against the advantages of using a larger number of small elements for which simple stiffness coefficients

may be employed. The main criterion to be observed in resolving this issue is that the problem must be programmed so that as much as possible of the data processing is performed automatically by the computer and not by human operators substituting in complex formulas.

(IX) SPARS AND RIBS

First we consider the untapered beam segment of uniform cross section shown in Fig. 4. Its stiffness matrix will be determined by application of beam theory, which is extended, however, to include shear web flexibility.

Nodes, 1, 1', 2, and 2' are established as shown in Fig. 4. The following notation is used:

- I = moment of inertia of beam section about neutral (y) axis
- t_w = t = thickness of shear web
- E = modulus of elasticity of flange material
- G = modulus of rigidity of shear web material
- ν = Poisson ratio

Displacements are assumed such as to be compatible with elementary beam theory. In other words,

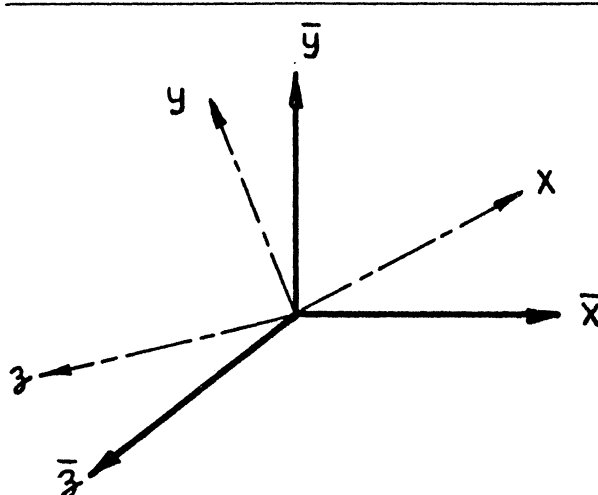


FIG. 5. Rectangular Cartesian axes systems.

$$\left. \begin{aligned} w_1 &= w_{1'}, & w_2 &= w_{2'} \\ u_1 &= -u_{1'}, & u_2 &= -u_{2'} \end{aligned} \right\} \quad (10)$$

Stiffness in the y -direction is assumed negligible.

An outline of the derivation of the stiffness matrix for the above beam segment is given in Appendix (A). It is shown to be of the form

$$[K] = \frac{6EI}{Lh^2(1+4n)} \begin{bmatrix} (4/3)(1+n) & 0 & (2/3)(1-2n) & 0 \\ 0 & -(h/L) & 0 & 0 \\ -(h/L) & h^2/L^2 & 0 & 0 \\ (2/3)(1-2n) & 0 & (4/3)(1+n) & 0 \\ 0 & 0 & 0 & 0 \\ h/L & 0 & -(h^2/L^2) & h/L \\ 0 & 0 & 0 & 0 \\ 0 & -(h^2/L^2) & h/L & 0 \\ 0 & h^2/L^2 & 0 & h^2/L^2 \end{bmatrix} \quad (11a)$$

where $n = 3(E/G) [I/(htL^2)]$ (11b)

Contribution of shear web deformation to the above stiffness matrix is indicated by values of $n > 0$; for a rigid shear web $n = 0$.

As a simple example of the use of the beam stiffness matrix, we consider a cantilever of length L and loaded by force P at the free end (nodes 2 and 2'). Putting $n = 0$ and applying Eq. (11a) gives:

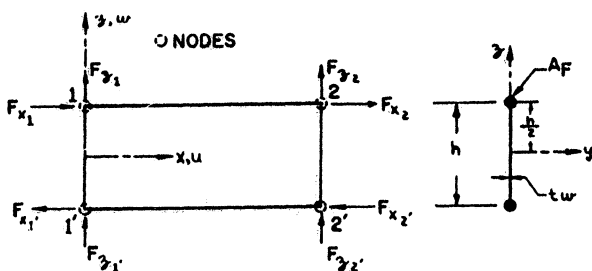


FIG. 4. Beam (spar or rib) segment.

$$\begin{Bmatrix} F_{x1} \\ F_{y1} \end{Bmatrix} = \frac{6EI}{L^3} \begin{bmatrix} 4L^2 & L \\ 3h^2 & h \\ L & 1 \\ h & \end{bmatrix} \begin{Bmatrix} u_2 \\ w_2 \end{Bmatrix} \quad (12)$$

Eq. (12) may be inverted to yield tip displacements u_2 and w_2 in terms of applied load P ($F_{x1} = 0, F_{y1} = P/2$). The results are

$$u_2 = -(PL^2/2EI)(h/2), \quad w_2 = PL^3/3EI$$

which agree with known results.

In an actual wing structure, spar and rib segments will be more or less randomly oriented with respect to a set of standard reference axes. As a result, transformation of stiffness matrices for these members to the standard set of axes will generally be necessary. The basis for such transformations is given below.

Let the direction cosines of x, y, z -axes with respect to standard $\bar{x}, \bar{y}, \bar{z}$ -axes, Fig. 5, be

(X) STIFFENED PLATES

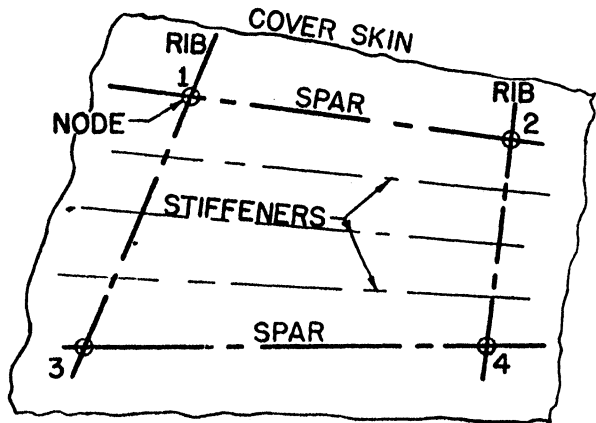


FIG. 6. Stiffened cover skin element.

$$\begin{array}{c|ccc} & x & y & z \\ \hline \bar{x} & \lambda_x & \lambda_y & \lambda_z \\ \bar{y} & \mu_x & \mu_y & \mu_z \\ \bar{z} & \nu_x & \nu_y & \nu_z \end{array}$$

Simple geometrical considerations then give the following equation for relating forces in the x, y, z system to forces in the $\bar{x}, \bar{y}, \bar{z}$ system:

$$\begin{pmatrix} F_{x1} \\ F_{y1} \\ F_{z1} \\ F_{x2} \\ F_{y2} \\ F_{z2} \end{pmatrix} = \begin{bmatrix} \lambda_x & \lambda_y & \lambda_z & 0 & 0 & 0 \\ \mu_x & \mu_y & \mu_z & 0 & 0 & 0 \\ \nu_x & \nu_y & \nu_z & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_x & \lambda_y & \lambda_z \\ 0 & 0 & 0 & \mu_x & \mu_y & \mu_z \\ 0 & 0 & 0 & \nu_x & \nu_y & \nu_z \end{bmatrix} \begin{pmatrix} F_{x1} \\ F_{y1} \\ F_{z1} \\ F_{x2} \\ F_{y2} \\ F_{z2} \end{pmatrix} \quad (13a)$$

or, $\{F\} = [\Phi] \{F\}$ (13b)

Displacements are vectors similarly related to the coordinate systems as forces and hence transform under a rotation of axes in the same manner. Consequently,

$$\{\delta\} = [\Phi] \{\delta\} \quad (14)$$

where $\{\delta\} = \begin{pmatrix} u_1 \\ v_1 \\ \cdot \\ \cdot \\ w_2 \end{pmatrix}$ etc.

From the above and Eq. (3) it follows that,

$$[\bar{K}] = [\Phi] [K] [\Phi]^{-1} = [\Phi] [K] [\Phi]' \quad (15)$$

where $[\bar{K}]$ is the stiffness matrix referred to the standard $\bar{x}, \bar{y}, \bar{z}$ set of axes. Beam segments encountered in the analysis of real structures will be tapered in depth, and flange areas will be variable; generally the segments will be taken short enough so that the variation in depth may be assumed linear. Derivation of stiffness matrices for elements of this kind is straightforward, and details will not be included in the present paper.

(1) Stiffeners

A plan view of a typical portion of stiffened cover skin structure is shown in Fig. 6. Nodes are initially established at points 1, 2, 3, and 4. The included structure then consists of spar segments (1-2 and 3-4), rib segments (1-3 and 2-4), and stiffened plate element 1-2-3-4. Stiffeners may be conveniently lumped with spar caps and, if desired, into one or more equivalent stiffeners located between spars. In this latter event additional nodes must be established, as at the intersections of these equivalent stiffeners with the ribs. The stiffness matrix for a lumped stiffener of constant area A , length L , and modulus E is

$$[K]_{\text{stiffener}} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (16)$$

Derivation of a similar matrix for a tapered member is straightforward; the area A is replaced by a suitable mean value. The influence of shear lag effects on load-deflection relations for the panel and stiffeners can only be included if nodes are established at intermediate points on the ribs, between spars.

(2) Plate Stiffness

The quadrilateral plate element 1-2-3-4 of Fig. 6 is assumed to possess in-plane stiffness only. Since two independent displacement components can occur at each node, the order of the K -matrix for this plate element will be 8×8 . The problem of calculating K is not an easy one, and the solution offered here is felt to have potential usefulness for finding approximate solutions to many two-dimensional problems in elasticity.

Before proceeding with the method developed for calculating K of the plate element, it is pointed out that a so-called framework analogy¹² exists, which permits one to replace the elastic plate with a lattice of elastic bars. Under certain conditions the framework then deforms as does the plate and hence can be used to calculate the plate stiffness. The determination of a lattice representation for a rectangular plate is relatively straightforward; however, plate elements of nonrectangular form present basic difficulties. For example, if one attempts to apply the rectangular gridwork to a nonrectangular plate, difficulties arise in attempting to satisfy boundary conditions. On the other hand, if one goes to nonrectangular lattice forms, difficulties arise when attempting to satisfy the stress-strain relations in the interior of the plate. Considerations such as these led to eventual abandonment of this approach.

The concept finally employed for determining plate stiffness is based on approximating actual plate strains by a restricted strain representation. In other words, no matter what the actual strains in the plate may be, these will be approximated by a superposition of several simple strain states. The method for doing

this and the accuracy of results based on such a representation form an important portion of this paper.

To give an initial illustration, the actual strain distribution in a rectangular plate element can be approximated by superimposing the strains that correspond to each of the simple external load states shown in Fig. 7. These load states are seen to represent uniform and linearly varying stresses plus constant shear, along the plate edges. Later it will be seen that the number of load states must be $2n - 3$, where n = number of nodes.

Before commenting further on the scheme suggested here for analyzing plate elements, the method will be applied to the triangular plate of Fig. 8. The triangle is not only simpler to handle than the rectangle but later it will be used as the basic "building block" for calculating stiffness matrices for plates of arbitrary shape.

We start by assuming constant strains, or

$$\left. \begin{aligned} \epsilon_x &= a = (1/E) (\sigma_x - \nu\sigma_y) = \partial u/\partial x \\ \epsilon_y &= b = (1/E) (\sigma_y - \nu\sigma_x) = \partial v/\partial y \\ \gamma_{xy} &= c = (1/G)\tau_{xy} = (\partial u/\partial y) + (\partial v/\partial x) \end{aligned} \right\} \quad (17a)$$

Later it will be pointed out why we are restricted in the choice of strain expressions. Integrating we find the displacements to be

$$\left. \begin{aligned} u &= ax + Ay + B \\ v &= by + (c - A)x + C \end{aligned} \right\} \quad (17b)$$

where, A , B , and C are constants of integration which define rigid body translation and rotation of the tri-

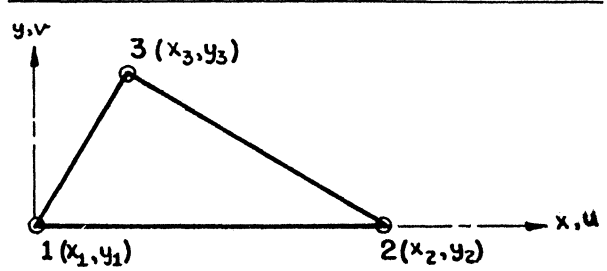


FIG. 8. Node designation for triangular plate element.

angle. Hence the triangle can displace as a rigid body in its own plane and undergo uniform straining according to Eq. (17a).

Displacements at the nodes can be determined by inserting applicable node coordinates into Eq. (17b). In this way six equations occur which are just sufficient for uniquely determining the six constants of Eq. (17b). As a result the constants become known in terms of node displacements and coordinates. It is this part of the solution which determines the number of terms which must be chosen in the strain expressions or alternatively the number of applied edge stress states which must be used. The number is always twice the number of nodes minus three. Hence, for the triangle we require three terms and five for the rectangle (or quadrilateral).

To proceed with the solution, we solve directly for stresses in terms of node displacements $u_1, v_1, u_2, etc.$ If $x_{ij} = x_i - x_j$ and $\lambda_i = (1 - \nu)/2$, this gives

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1 - \nu^2} \begin{bmatrix} -\frac{1}{x_2} & \frac{\nu x_{32}}{x_2 y_3} \\ \frac{\nu}{x_2} & \frac{x_{32}}{x_2 y_3} \\ \lambda_1 x_{32} & -\lambda_1 \\ x_2 y_3 & x_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \quad (18a)$$

$$\text{or} \quad \{\sigma\} = [S] \{\delta\} \quad (18b)$$

The next step is to obtain the concentrated forces at the nodes which are statically equivalent to the applied constant edge stresses. The procedure for doing this will be briefly illustrated for the case of the shear stress.

Fig. 9(a) shows the shear stresses on the circumscribed rectangular element, and Fig. 9(b) shows the corresponding edge shear forces on the triangle. As before x_i, y_i refer to coordinates of node points.

Forces on any edge are equally distributed between nodes lying on that edge. For the forces as given in Fig. 9(b), this leads to

$$\left. \begin{aligned} F_{x_1}^{(3)} &= -(x_2 - x_3) (t/2) \tau_{xy} \\ F_{y_1}^{(3)} &= -y_3 (t/2) \tau_{xy} \\ F_{x_2}^{(3)} &= -x_3 (t/2) \tau_{xy} \\ F_{y_2}^{(3)} &= +y_3 (t/2) \tau_{xy} \\ F_{x_3}^{(3)} &= +x_2 (t/2) \tau_{xy} \\ F_{y_3}^{(3)} &= 0 \end{aligned} \right\} \quad (19)$$

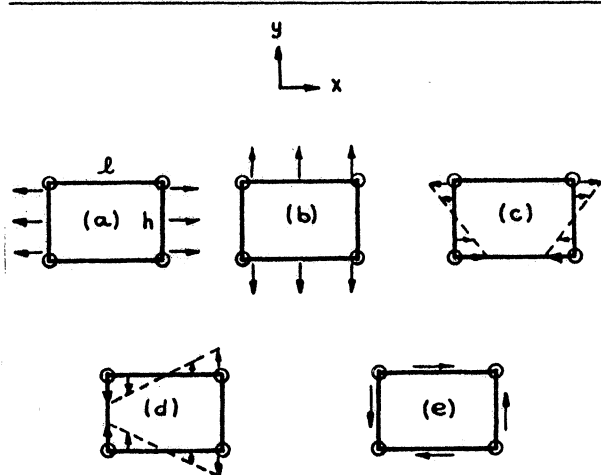


FIG. 7. Applied loads on edges of rectangular plate element.

Two simple subdivisions of the quadrilateral into triangles are shown in Figs. 10(a) and 10(b). These lead to different stiffness matrices for the quadrilateral. A unique result is obtained by using the subelements shown in Fig. 10(c). The interior node will be located at the centroid, although any other choice could be used.

For the general quadrilateral plate it has proved to be preferable to program the calculation of the stiffness matrix for high-speed computing equipment. In the case of the rectangle, however, an explicit derivation can be readily carried out. The necessary calculations, included below, are given here, since the end result is useful and since these calculations serve to illustrate a step of some importance in carrying out the analysis of a more complete structure—for example, a wing or tail surface.

The rectangle and its four triangular subelements, with interior node number 5 at the centroid, is shown in Fig. 11. Stiffness matrices for the triangles can be calculated from Eq. (22), or more conveniently from Eq. (B-3) of Appendix (B). In determining K of the rectangle, superposition in the following form is used:

$$K_{\text{rectangle}} = K_I + K_{II} + K_{III} + K_{IV}$$

Since five nodes have been established, K for the rectangle will initially be of order 10×10 . This will later be reduced to order 8×8 to give a result consistent with the choice of four external nodes; only at these external nodes is contact implied with adjoining structure. The immediate point is, however, that K for each triangle must be increased to order 10×10 before superposition is carried out. This is accomplished in the usual way—that is, by introducing appropriate rows and columns of zero elements.

In order to simplify the expressions for elements appearing in the stiffness matrices the derivation of K for the rectangle will be restricted to $\nu = 1/3$.

On superimposing stiffnesses for the component triangles of Fig. 11 it becomes possible to express Eq. (3) in the form

$$\begin{Bmatrix} F_{x_1} \\ F_{x_2} \\ F_{x_3} \\ F_{x_4} \\ F_{y_1} \\ F_{y_2} \\ F_{y_3} \\ F_{y_4} \\ F_{x_5} \\ F_{y_5} \end{Bmatrix} = \begin{bmatrix} A_{8 \times 8} & B_{8 \times 2} \\ B'_{2 \times 8} & C_{2 \times 2} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ u_5 \\ v_5 \end{Bmatrix} \quad (23)$$

Since forces are to be applied to the rectangle by stresses equivalent to forces acting at nodes 1, 2, 3, and 4, the condition

$$F_{x_i} = F_{y_i} = 0$$

can be applied to Eq. (23). Doing this results in the two sets of equations written below:

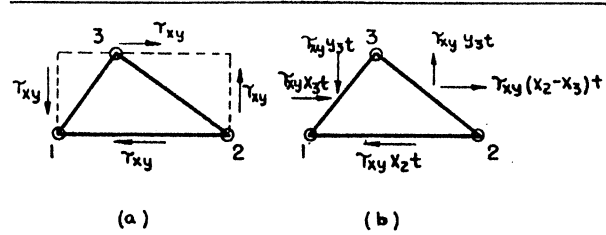


FIG. 9. Shear loading on triangular plate element.

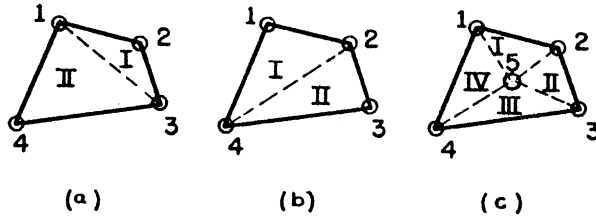


FIG. 10. Decomposition of quadrilateral plate into triangular subelements.

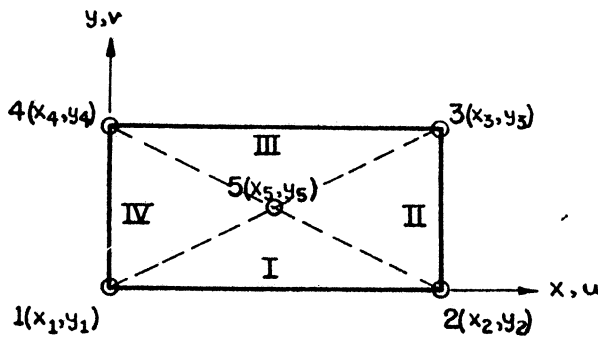
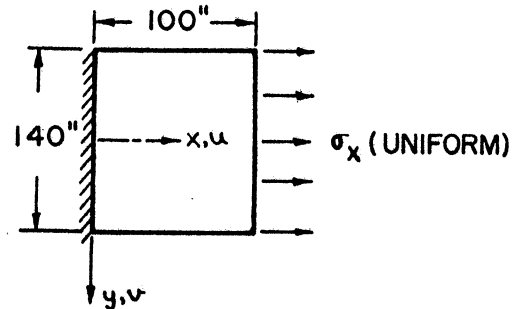


FIG. 11. Triangular subelements for rectangular plate.



$$\begin{aligned} t &= 0.050 \text{ IN.} \\ E &= 10.5 \times 10^6 \text{ PSI} \\ \nu &= 1/3 \end{aligned}$$

TOTAL LOAD = 2 LBS.

FIG. 12. Clamped rectangular plate subjected to uniform tensile loading.

STIFFNESS AND DEFLECTION ANALYSIS

TABLE 2

Solution No.	Method	Fig.	u_1	u_2	u_3	u_4	u_5	v_1	v_2	v_4
1	Relaxation	13	2.703	2.607	2.703	1.391	1.248	0.686	-0.685	0.562
2	Simple theory	13	2.721	2.721	2.721	1.360	1.360	0.635	-0.635	
3	Plate K -matrix	13a	2.595		2.595			0.740	-0.740	
4	Plate K -matrix	13b	2.692	2.578	2.692	1.355	1.199	0.680	-0.680	0.568
5	Plate K -matrix	13c	2.718		2.697			0.686	-0.717	
6	Plate K -matrix	13d	2.714		2.712			0.688	-0.691	

will be taken as those calculated by applying the relaxation method to the fundamental equations governing this problem. Although details of these calculations are not presented, results are listed in Table 2.

The problem is interesting for at least two reasons. First, the accuracy obtainable using various numbers of subelements can be observed, and second, the effect of using random orientation of subelements—with respect to the plate edges—can be observed.

Results of all calculations are summarized in Table 2. Node locations and subelements are illustrated in Fig. 13.

In Table 2 the solution based on simple theory was obtained from $u = PL/AE$ and $\epsilon_v = -\nu \epsilon_r$. It is observed that on this basis both u_1 and v_1 agree quite well with the relaxation solution.

The crudest plate matrix solution is listed in Table 2 as Solution No. 3. It was obtained by considering the plate as a single element whose stiffness is given by Eq. (27). The results for u_1 and v_1 are seen to be reasonably good. Solution No. 4 considers the plate as consisting of four rectangular subelements as shown in Fig. 13(b). Again the stiffness matrix was obtained by using Eq. (27), this time for each subelement. Agreement with relaxation results is seen to be satisfactory, particularly in regard to u_1 . Also the differences between u_1 and u_2 are approximated accurately by this solution. It is to be remembered that the actual strain distribution in the plate is complex in nature.

Solutions 5 and 6 in Table 2 were carried out in a matter of minutes on a high-speed digital computer.

Each subquadrilateral was considered as consisting of four triangles in a manner analogous to the treatment described previously for the rectangle of Fig. 11. In Solution No. 5 we note that u_1 and u_3 are not equal, a consequence of the random nature of orientation of the subelements. By increasing the number of random subelements as in Solution No. 6, this lack of symmetry in results is virtually removed. Comparison with relaxation values is seen to be very good for both Solutions 5 and 6.

A more comprehensive example is given in the next section of the paper.

(XI) ANALYSIS OF BOX BEAM

As a final example, the box beam of Fig. 14 will be analyzed for deflections, using the stiffness matrices previously derived.

The box is uniform in section, unswept, and contains a rib at the unsupported end. The following dimensions apply: $a/b = \pi$, $2b/h = 10$, $t_c = t_w = t = 0.05$ in., $A_r = bt/2$, $a = 400$ in.

As the simplest possible breakdown, we consider the box to consist of two spars, one rib, and two cover skins. The nodes are then as shown in Fig. 15. Forces may be applied at the nodes at the free end. Two cases will be investigated: (1) up loads at each spar (bending) and (2) up load on one spar and a down load at the other spar (twisting).

The spar matrix is given by Eq. (11a). Calculation shows it to be

$$[K]_{\text{spar}} = \frac{Et}{2} \begin{bmatrix} & u_1 \text{ OR } u_2 & & w_1 \text{ OR } w_2 & & u_3 \text{ OR } u_4 & & w_3 \text{ OR } w_4 \\ & 1.13903 & & & & & & \\ & 0.05227 & & 0.00333 & & & & \\ & 0.50303 & & 0.05227 & & 1.13903 & & \\ -0.05227 & & -0.00333 & & -0.05227 & & 0.00333 & \end{bmatrix} \quad (28)$$

Cover plate stiffness is given by Eq. (27a) and for this case becomes

$$[K]_{\text{cover plate}} = \frac{Et}{2} \begin{bmatrix} & u_1 & & v_1 & & u_2 & & v_2 & & u_3 & & v_3 & & u_4 & & v_4 \\ & 0.90878 & & & & & & & & & & & & & & \\ -0.37500 & & 1.39778 & & & & & & & & & & & & & \\ -0.19329 & & 0 & & 0.90879 & & & & & & & & & & & \\ 0 & & -1.15928 & & 0.37500 & & 1.39778 & & & & & & & & & \\ -0.31916 & & 0 & & -0.39634 & & -0.37500 & & 0.90879 & & & & & & & \\ 0 & & 0.37109 & & -0.37500 & & -0.60959 & & 0.37500 & & 1.39778 & & & & & \\ -0.39634 & & 0.37500 & & -0.31916 & & 0 & & -0.19329 & & 0 & & 0.90879 & & & \\ 0.37500 & & -0.60959 & & 0 & & 0.37109 & & 0 & & -1.15928 & & -0.37500 & & 1.39778 & \end{bmatrix} \quad (29)$$

The rib has not been defined as yet. Two possible rib configurations will be analyzed in this paper. In the first case, the rib is considered as a beam identical in section to the spar. This leads to the following stiffness matrix for the rib:

less flanges.

$$[K]_{\text{rib}} = \frac{Et}{2} \begin{bmatrix} & v_1 & w_1 & v_2 & w_2 \\ 0.13086 & & & & \\ -0.00976 & 0.00098 & & & \\ 0.06413 & -0.00976 & 0.13086 & & \\ 0.00976 & -0.00098 & 0.00976 & 0.00098 & \end{bmatrix} \quad (30a)$$

In the second case, the rib is treated as a flat plate. The general stiffness matrix which has been derived for a rectangular flat plate is of order 8×8 . However, in the present instance, the following conditions must be introduced to insure compatibility with the other portions of the structure (see Fig. 15 for subscript locations):

$$\begin{aligned} w_1 &= w_1' & \text{and} & & v_1 &= -v_1' \\ w_2 &= w_2' & & & v_2 &= -v_2' \end{aligned}$$

and, likewise, for the forces

$$\begin{aligned} F_{z_1} &= F_{z_1}' & \text{and} & & F_{v_1} &= -F_{v_1}' \\ & & & & & \\ F_{z_2} &= F_{z_2}' & & & F_{v_2} &= -F_{v_2}' \end{aligned}$$

Treating the rib as a flat plate ($t = 0.050$ in.) and applying the above conditions leads to the following rib stiffness matrix:

$$[K]_{\text{rib}} = \frac{Et}{2} \begin{bmatrix} & v_1 & w_1 & v_2 & w_2 \\ 5.65088 & & & & \\ -0.37500 & 0.03754 & & & \\ 1.84181 & -0.37500 & 5.65088 & & \\ 0.37500 & -0.03754 & 0.37500 & 0.03754 & \end{bmatrix} \quad (30b)$$

It is anticipated that the choice of rib will have little effect on deflections due to the bending-type loading and a more pronounced effect on the twisting-type loading.

Using the same technique as described for the simple truss, it is now a straightforward matter to form the stiffness matrix for the complete box. Advantage can be taken of the following: (1) structural symmetry that exists for the box with respect to the xy -midplane and (2) restriction in this problem to loads that act normal to this plane. Under these conditions each pair of upper and lower surface nodes will experience, in addition to equal vertical deflections, equal but opposite displacements with respect to the xy -midplane. In other words, the box will deflect in the sense of a conventional beam. The spar and rib stiffness matrices already provide for such elastic behavior. The plate stiffness matrices make no distinction, other than in the sign of the node forces, for a reversal in direction of node displacement. Consequently, if the normal loading is carried equally by upper and lower nodes, only the upper set will need be considered when forming the box stiffness matrix. Due to the division of loading, correct deflections will result. In this manner the stiffness matrix for the box is found to be [Eq. (30a) used for rib stiffness]

$$[K]_{\text{box}} = \frac{Et}{2} \begin{bmatrix} & u_1 & v_1 & w_1 & u_2 & v_2 & w_2 \\ 2.04782 & & & & & & \\ -0.37500 & 1.52864 & & & & & \\ +0.05227 & -0.00976 & 0.00430 & & & & \\ -0.19329 & 0 & 0 & 2.04782 & & & \\ 0 & -1.09515 & -0.00976 & 0.37500 & 1.52864 & & \\ 0 & 0.00976 & -0.00098 & +0.05227 & 0.00976 & 0.00430 & \end{bmatrix} \quad (31)$$

The inverse of this matrix is the flexibility matrix.

$$[K]_{\text{box}}^{-1} = [C] = \frac{2}{Et} \begin{bmatrix} & F_{z_1} & & & & & \\ & & F_{z_2} & & & & \\ & & & F_{v_1} & & & \\ & & & & F_{z_1} & & \\ & & & & & F_{v_1} & \\ & & & & & & F_{z_2} \end{bmatrix} \quad (32)$$

From the flexibility matrix, deflections due to applied loads can be found at once. For the two cases of applied loadings we find the following (rib treated as beam).

Case 1 (bending):

Forces of 1 lb. acting upward at each spar (nodes 1 and 2).

$$\begin{array}{lll} w_1 = 11,041.55/E & u_1 = -320.47/E & v_1 = -45.80/E \\ w_2 = 11,041.55/E & u_2 = -320.47/E & v_2 = 45.80/E \end{array}$$

Case 2 (twisting):

Force of 1 lb. upward at node 1 and 1 lb. downward at node 2.

$$\begin{array}{lll} w_1 = 5,334.45/E & u_1 = -98.46/E & v_1 = 154.99/E \\ w_2 = -5,334.45/E & u_2 = 98.46/E & v_2 = 154.99/E \end{array}$$

Similar results may be calculated for the case when the rib is assumed as a plate. Complete details are not given. In bending we get $w_1 = 10,888.12/E$, $u_1 = -310.56/E$, and $v_1 = -18.25/E$. Twisting results are $w_1 = 3615.72/E$, $u_1 = -25.84/E$, and $v_1 = 349.52/E$.

It is now advisable to select additional nodes and recalculate the previous deflection data. When added nodes have little effect on results, the process can be considered to have converged. Whether convergence be to the correct values requires additional information. These questions are now examined.

First, solutions are found for the node patterns shown in Fig. 16. Vertical deflections at node 1 for bending-type loading are as follows:

Fig. 16(a) $w_1 = 8558.0/E$

Fig. 16(b) $w_1 = 8591.2/E$

Fig. 16(c) $w_1 = 8548.4/E$

It is seen that the change in w_1 in going from the node pattern of Fig. 16(b) to 16(c) is about 1/2 per cent. Consequently convergence can be assumed to have been attained with the solution found from Fig. 16(b).

Obviously the first solution, based on Fig. 15, is in considerable error. This is due to the poor tie between spars and cover plate. Fig. 16(a) introduces an additional tie between these two components. The decreased value of w_1 for this case therefore reflects the added stiffness due to including the two nodes at the mid-span location.

An unexpected result is the close agreement between the solutions based on Figs. 16(a) and 16(b). In fact it would seem reasonable to expect Fig. 16(b) to lead to a smaller value for w_1 than that given by Fig. 16(a). Careful scrutiny, however, indicates that these results are quite reasonable. Whereas the node pattern of Fig. 16(b) accounts for shear lag in the cover plate, this is not the case with Fig. 16(a). As a result, the added stiffness in Fig. 16(b), due to the additional nodes connecting spars and cover skins, is offset by the added flexibility introduced by shear lag in cover skins. The results indicate these factors to be nearly equal; hence the reason for the nearly correct values given by Fig. 16(a).

Fig. 16(c) allows for shear lag and, at the same time, provides for adequate tie between spars and cover

plates. It can therefore be felt that this node pattern will give final results which represent convergence of the method. As mentioned previously, this is substantiated by comparison with values obtained from Fig. 16(b).

There remains the question as to what is the correct value for w_1 for this problem. Elementary beam theory gives $w_1 = 6,900/E$, and, if extended to include shear distortion of spar webs, gives $w_1 = 7,740/E$. Using Reissner's shear lag theory,¹³ the tip deflection is obtained as $w_1 = 7,900/E$. Finally if Reissner's shear lag theory is modified to include spar shear web deformation, the result is $w_1 = 8,740/E$. This is the most accurate theory available. It agrees to approximately 2 per cent with the numerical solution based on stiffness matrices.

The pronounced shear lag effect in this problem and its marked influence on the vertical tip deflection are significant. It is precisely this effect that produces a very complex stress distribution in the cover skins. Nevertheless the plate stiffness matrix developed in Eq. (27a) and based on triangular subelements represents this stress pattern with gratifying effectiveness.

The solution for the node pattern of Fig. 16(c) was obtained in a few minutes by utilizing a program for a high-speed digital computer that computed individual plate and spar stiffnesses and then combined these into the stiffness matrix for the complete box.

(XII) REDUCTION IN ORDER OF STIFFNESS MATRIX

(1) Eliminating Components of Node Displacement

In an actual problem—as a wing analysis—the number of nodes to be used can become quite large. If, for purposes of discussion, 50 nodes are assumed, the stiffness matrix becomes of order 150×150 . By eliminating u and v components of displacement at each node, the stiffness matrix can be reduced to order 50×50 . However, this reduction process [see treatment of Eq. (23), for example] can require the calculation of the inverse of a 100×100 matrix. Such calculations are best avoided at present.

The problem that arises in eliminating the u and v components can be handled satisfactorily in any one of several ways. First, the calculation of the inverse of a large-order matrix can be avoided by eliminating a single component at a time. This is a practical expedient when automatic digital computing equipment

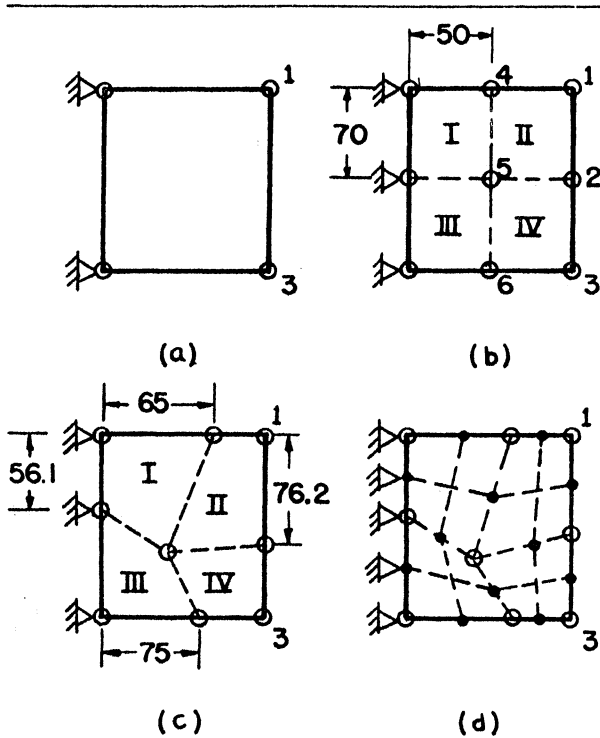


FIG. 13. Nodes and supports for clamped rectangular plate.

is available. Second, in some cases it may be feasible to eliminate "blocks" of u and v components at a time, thereby reducing the order of matrices to be inverted at any one time to a reasonable size (say 20×20). Third, the analysis can be carried out for sections of the structure, taken one by one. For each section, as a spanwise portion of the wing, the complete stiffness matrix can be determined. Elimination of u and v components can then be carried out at any selected nodes, except those common to two distinct sections of the structure. Each section can be treated in this manner. By properly adding the individual section stiffness matrices, the total stiffness matrix can be obtained. Finally u and v displacements at nodes where the sections join together can be eliminated. The stiffness matrix that remains will apply to w deflections only.

From a practical standpoint, the method just described has several worth-while features. For example all components of displacement at a given node may be eliminated. This can be useful when additional nodes are felt to be necessary in order to account properly for regions of maximum structural complexity. Even though eventually eliminated, these nodes will have contributed to the elements retained in the stiffness matrix.

(2) *Inversion of Stiffness Matrix*

Ordinarily, only the first few low-order vibration nodes and frequencies are required for the purpose of carrying out subsequent dynamic analyses. Using the stiffness matrix directly in the matrix iteration

method leads to the highest frequency and corresponding mode. If the order of the stiffness matrix is high (say, 50×50), it becomes impractical to eliminate successively the higher modes and so eventually obtain the lowest modes.

Inversion of the stiffness matrix leads to the flexibility matrix. This matrix used in the matrix iteration procedure yields results for the lowest mode. Therefore, it is ordinarily preferable to know the flexibility matrix.

If the stiffness matrix is of high order (say, 50×50), inverting it becomes a major problem in itself. This can be overcome to some extent by employing the capabilities of present-day digital computing equipment. However, in many instances an alternative procedure may either be useful or necessary. Consequently, a possible approach to overcoming this difficulty will be outlined here.

The proposed method consists of converting the original stiffness matrix K into a lower order stiffness matrix K^* . This is accomplished by introducing a set of generalized coordinates which are related to the original displacements (on which K is based) through a set of appropriately chosen functions. The accuracy inherent in K will have a direct influence on K^* .

Suppose K is known for the cantilever beam of Fig. 17. The order of K is 10×10 . Now assume a set of polynomials of the form

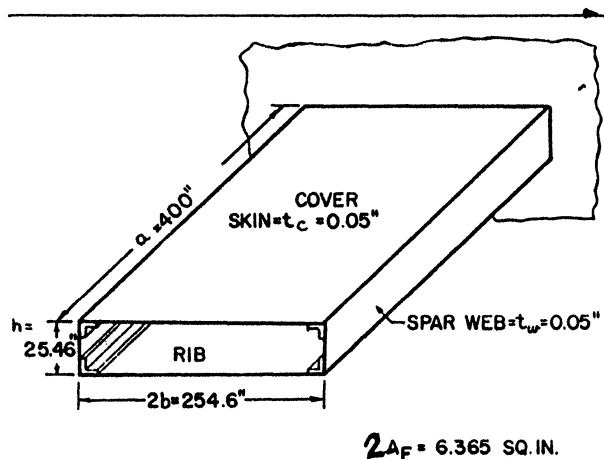


FIG. 14. Cantilevered box beam.

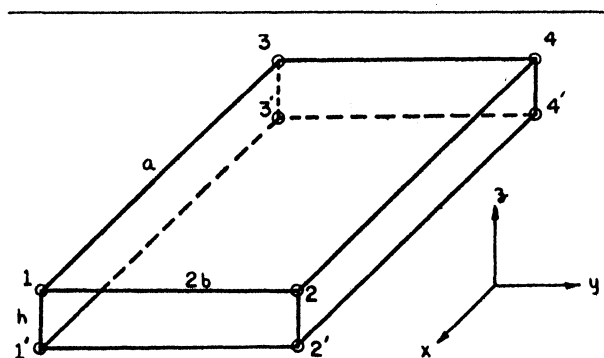


FIG. 15. Simplest node pattern for box beam.

$$\left. \begin{aligned} P_1(x) &= a_1x^2 + b_1x^3 + c_1x^4 \\ P_2(x) &= a_2x^2 + b_2x^3 + c_2x^5 \\ &\vdots \\ P_5(x) &= a_5x^2 + b_5x^3 + c_5x^8 \end{aligned} \right\} \quad (33)$$

Each of these will be made to satisfy the boundary conditions of the cantilever which are,

$$P_i(0) = P_i'(0) = P_i''(L) = P_i'''(L) = 0$$

Applying these conditions results in

$$\left. \begin{aligned} P_1(x) &= 6(x/L)^2 - 4(x/L)^3 + (x/L)^4 \\ P_2(x) &= 20(x/L)^2 - 10(x/L)^3 + (x/L)^5 \\ &\vdots \\ P_5(x) &= 140(x/L)^2 - 56(x/L)^3 + (x/L)^8 \end{aligned} \right\} \quad (34)$$

We now introduce generalized coordinates q_i which are related to the displacements y_i through the above polynomials. This relationship is established through the equations

$$\begin{Bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{10} \end{Bmatrix} = \begin{bmatrix} P_1(x_1) & P_2(x_1) & \dots & P_5(x_1) \\ P_1(x_2) & P_2(x_2) & \dots & P_5(x_2) \\ \vdots & \vdots & \dots & \vdots \\ P_1(x_{10}) & P_2(x_{10}) & \dots & P_5(x_{10}) \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{Bmatrix} \quad (35)$$

It is seen that the ten displacements y_1, y_2, \dots, y_{10} are to be replaced by the five coordinates q_1, q_2, \dots, q_5 .

The free vibration problem for the cantilever can be set up in terms of kinetic and potential energies. In terms of original displacements y_1, y_2, \dots, y_{10} , these energies are, respectively,

$$T = (1/2) \{\dot{y}\}' [M] \{\dot{y}\} \quad \text{and} \quad V = (1/2) \{y\}' [K] \{y\} \quad (36)$$

where $[M]$ is the inertia (mass) matrix and $[K]$ the original 10×10 stiffness matrix.

Writing Eq. (35) as

$$\{y\} = [P] \{q\}$$

and substituting into Eqs. (36),

$$T = (1/2) \{q\}' [P]' [M] [P] \{q\}$$

$$V = (1/2) \{q\}' [P]' [K] [P] \{q\}$$

from which we define

$$\left. \begin{aligned} [K^*] &= [P]' [K] [P] \\ [M^*] &= [P]' [M] [P] \end{aligned} \right\} \quad (37)$$

If K is of order 10×10 and P of order 10×5 , K^* will be of order 5×5 . The vibration analysis is now performed using K^* and M^* . By inverting K^* the lower modes can be calculated directly. Or alternatively, K^* can be used and all modes and frequencies

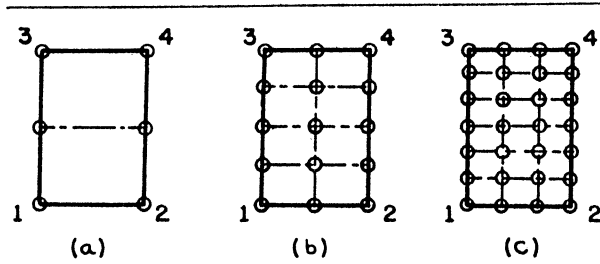


FIG. 16. Additional node patterns for box beam.

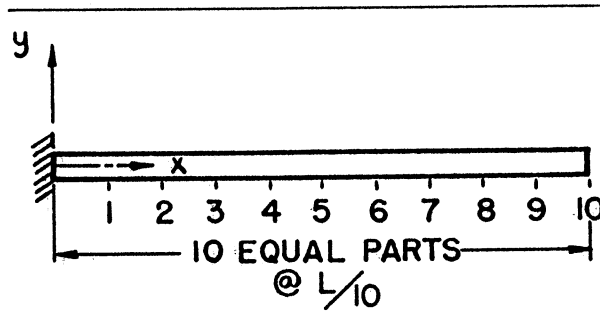


FIG. 17. Station selections on cantilever beam.

determined, starting with the highest. This is feasible if K^* is of sufficiently low order (say, 10×10).

This process can be modified in several respects, and the purpose here is not to give an exhaustive treatment but rather to simply point out a possible approach to the problem. Preliminary calculations indicate that the idea may possess practical value. Extension to a two-dimensional grid can be made by generalizing the procedure suggested above.

APPENDIX (A)

DERIVATION OF SPAR STIFFNESS MATRIX

The structure and notation are described in Section (IX) and Fig. 4.

Flanges are assumed to carry axial stresses, while the web carries shear stresses. Cover plate material is not included as part of spar flanges. Derivation below is based on conventional beam theory.

Case 1

$u_1 = -u_1' \neq 0$; all other components of node displacement for the beam = 0.

The deflected beam and necessary forces and reactions are shown in Fig. A-1. Due to forces F_x at the left end, the beam deflects upward. The F_x forces cause a downward deflection. Beam theory, including effects of uniformly distributed shear in web, gives

$$w = \frac{F_x h L^2}{2EI} + \frac{2F_x L^3}{3EI} (1 + n) \quad (A-1)$$

$$\theta = \frac{F_x h L}{EI} + \frac{F_x L^2}{EI} \quad (A-2)$$

where w and θ are deflection and slope at the left end of

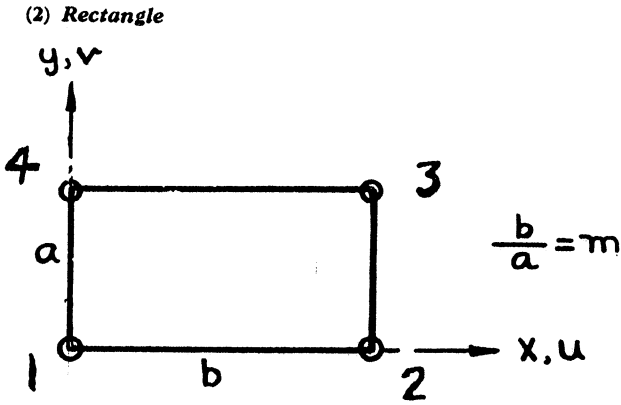


FIG. B-2. Node locations for rectangular plate element.

The stiffness matrix given below for the rectangle is based on the load states shown in Fig. 7. As a result this matrix is more general than that given in Eq. (27) due to the inclusion of linear terms in the strain expressions.

Again the stiffness matrix is arranged to agree with the equation

$$\begin{Bmatrix} F_{x_1} \\ F_{y_1} \\ F_{x_2} \\ F_{y_2} \\ F_{x_3} \\ F_{y_3} \\ F_{x_4} \\ F_{y_4} \end{Bmatrix} = [K] \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} \quad (B-4)$$

in which $[K]$ is given by

$$[K] = \frac{Et}{8(1-\nu^2)} \begin{bmatrix} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 & u_4 & v_4 \\ a_1 + b_1 & & & & & & & \\ 1 + \nu & a_2 + b_2 & & & & & & \\ a_1 - b_1 & 1 - 3\nu & a_1 + b_1 & & & & & \\ 3\nu - 1 & c_2 - a_2 & -1 - \nu & a_2 + b_2 & & & & \\ -a_1 - c_1 & -1 - \nu & c_1 - a_1 & 1 - 3\nu & a_1 + b_1 & & & \\ -1 - \nu & -a_2 - c_2 & 3\nu - 1 & a_2 - b_2 & 1 + \nu & a_2 + b_2 & & \\ c_1 - a_1 & 3\nu - 1 & -a_1 - c_1 & 1 + \nu & a_1 - b_1 & 1 - 3\nu & a_1 + b_1 & \\ 1 - 3\nu & a_2 - b_2 & 1 + \nu & -a_2 - c_2 & 3\nu - 1 & c_2 - a_2 & -1 - \nu & a_2 + b_2 \end{bmatrix} \quad (B-5)$$

where, in the above matrix,

$$\left. \begin{aligned} a_1 &= m(1 - \nu), & b_1 &= (2/3m)(4 - \nu^2), & c_1 &= (2/3m)(2 + \nu^2) \\ a_2 &= (1 - \nu)/m, & b_2 &= (2m/3)(4 - \nu^2), & c_2 &= (2m/3)(2 + \nu^2) \end{aligned} \right\} \quad (B-6)$$

$$m = l/h \quad (\text{see Fig. 7}) \quad (B-7)$$

Eq. (B-5) simplifies to the following if $\nu = 1/3$:

$$[K] = \frac{Et}{96} \begin{bmatrix} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 & u_4 & v_4 \\ \varphi_1(m) & & & & & & & \\ 18 & \varphi_1(1/m) & & & & & & \\ \varphi_2(m) & 0 & \varphi_1(m) & & & & & \\ 0 & \varphi_3(1/m) & -18 & \varphi_1(1/m) & & & & \\ \varphi_4(m) & -18 & \varphi_3(m) & 0 & \varphi_1(m) & & & \\ -18 & \varphi_4(1/m) & 0 & \varphi_2(1/m) & 18 & \varphi_1(1/m) & & \\ \varphi_3(m) & 0 & \varphi_4(m) & 18 & \varphi_2(m) & 0 & \varphi_1(m) & \\ 0 & \varphi_2(1/m) & 18 & \varphi_4(1/m) & 0 & \varphi_3(1/m) & -18 & \varphi_1(1/m) \end{bmatrix} \quad (B-8)$$

where

$$\left. \begin{aligned} \varphi_1(m) &= 9m + (35/m), & \varphi_1(1/m) &= (9/m) + 35m \\ \varphi_2(m) &= 9m - (35/m), & \varphi_2(1/m) &= (9/m) - 35m \\ \varphi_3(m) &= -9m + (19/m), & \varphi_3(1/m) &= (-9/m) + 19m \\ \varphi_4(m) &= -9m - (19/m), & \varphi_4(1/m) &= (-9/m) - 19m \end{aligned} \right\}$$

(3) Other Shapes

Although the parallelogram and arbitrary quadrilateral can be treated in a manner similar to that used for the rectangle, the individual elements in $[K]$ tend to become unwieldy. For that reason use of automatic digital computing equipment is considered to offer the practical means for obtaining stiffnesses of such plates. Programs for carrying out such calculations can be de-

termined by following the basic ideas developed in this paper.

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DIFFERENCE SCHEMES OR ELEMENT SCHEMES?†

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SUMMARY

Several examples are presented to illustrate how standard finite difference schemes for the wave equation (e.g. Lax-Wendroff, Leapfrog, etc.) can be developed from finite element analysis. The development of the difference schemes from the element schemes is made possible by using Galerkin's method on both the spacial and temporal dimensions.

INTRODUCTION

There is general disagreement about the advantages or disadvantages of using a finite difference technique in preference to a finite element method and vice versa. It is generally felt, however, that for irregular domains finite element analysis is often easier to use, and that for regular domains finite difference methods are more easily programmed.

The purpose of this paper is to illustrate that when regular meshes are used for the wave equation (we consider only the one-dimensional problems, although multi-dimensional problems can be handled similarly), it is possible to generate the standard finite difference schemes as special cases of finite element schemes.

The tool that makes the analysis possible was developed in Reference 7. The idea is simply to apply the finite element discretization process to time as well as to space. This technique has rarely been used because of the lack of theoretical results on stability and convergence. There is also a general belief that there is little to be gained by the method. Another reason people rarely use the technique is that it appears that the storage and computational requirements are increased. These reasons for not using finite elements in time will be dispelled in this paper.

In the literature one can find very little concerning the use of finite elements on the temporal dimension, although there are exceptions.^{1-6,9,10} However, the manner in which we subdivide time here will be quite novel.

ANALYSIS

Since the mathematical details of this method are straightforward we will, in general, overlook them.

The finite element method is justified in any finite dimensional topological space and hence is valid when used on the temporal dimension.⁷

In this paper, time and space will be partitioned in some very unusual ways. Then our residue vector (coming from a trial solution) will be made orthogonal in the L_2 inner product to a set of trial functions by the use of Galerkin's method.⁸ This minimization technique, when used properly on the temporal dimension, will produce the desired finite difference schemes.

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The equation which we will examine is simply the one-dimensional wave equation

$$u_t + cu_x = 0, \quad 0 \leq x \leq L \tag{1}$$

Let $H_1(\Omega)$ be the Sobolev space of order one and let $\Omega = X \times T$ be such that

$$X = \{x: 0 < x < L\}$$

and

$$T = \{t: 0 < t < t' \leq \infty\}$$

Then if $u \in H_1(\Omega)$ and N is our vector of linear interpolation functions (shape functions),⁸ Galerkin's method may be stated as

$$\int_{\Omega'} N^T(u_t + cu_x) dA = 0 \tag{2}$$

where $\Omega' \subset \Omega$.

The stronger requirement of $u \in H_2(\Omega)$ is sufficient if u is to satisfy the following form of the wave equation:

$$u_{tt} = c^2 u_{xx} \tag{3}$$

After partitioning Ω' into elements and letting $u = N\bar{u}$ (\bar{u} being the vector of known (unknown) values of u at the nodes), equation (2) may be written as

$$\sum_e \int_{\Omega_e} N_e^T \left(\frac{\partial N_e}{\partial t} + c \frac{\partial N_e}{\partial x} \right) dA \bar{u} = 0 \tag{4}$$

where the sum is over the number of elements and the elements may cover only a subset Ω' of Ω .

In the past, when Galerkin's method was used on time \times space, the temporal dimension had been partitioned into layers of uniform height;^{1-6,9,10} the solution then proceeded layer by layer (Figure 1).

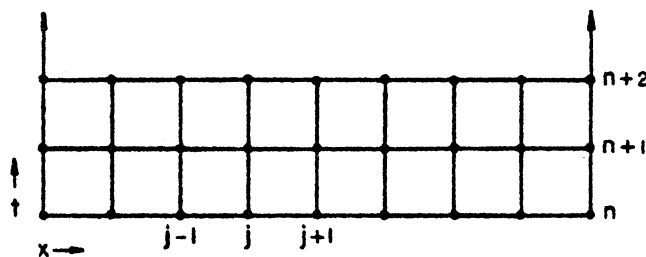


Figure 1. Marching of elements in time

In two instances^{6,9} the layers were allowed to overlap, producing what have been called lag elements. We will generalize and extend this technique much further.

The most concise and easiest method to describe how we can construct duplicates of finite difference schemes via finite element analysis is by examples. Hence the bulk of the remainder of this paper will be so oriented.

EXAMPLES: EXPLICIT CASES

Example 1: unstable Euler scheme

The finite difference form of the wave equation using Euler's method is

$$u_j^{n+1} = u_j^n - \nu(u_{j-1}^n - u_j^n) \tag{5}$$

where j denotes the x -location, n denotes the time step, and the Courant number $\nu = (c \Delta t) / \Delta x$.

We will next show how we can derive equation (5) via finite elements in time \times space.

Figure 2 represents the time-stepping scheme we will use. Note that only the shaded regions are used and that the solution proceeds level by level. The novelty to this approach is that we are using (for any one time level) only half of $\Omega|_n^{n+1}$ in our analysis.

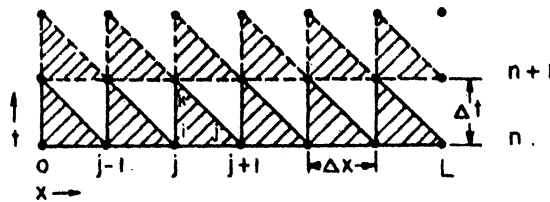


Figure 2. The Euler method

As can be seen, the elements we are using are linear and triangular. To derive the equations for the unknown nodal values (\bar{u} at the $(n + 1)$ st level) it is only necessary to consider one element since, for example, u_j^{n+1} (Figure 2) is only dependent on u_j^n and u_{j+1}^n . Had we been using a procedure outlined in Reference 7 this would not be the case.

We will thus confine our attention to one element. Integration of the element equation (see any finite element text) and neglecting known values of u (i.e. at nodes (j, n) and $(j + 1, n)$) the element stiffness matrix can be shown to have the form

$$\begin{matrix} & i & j & k \\ \begin{matrix} i \\ j \\ k \end{matrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ (cb_i + c_i) & (cb_j + c_j) & (cb_k + c_k) \end{pmatrix} \end{matrix}$$

where i, j, k denote the associated nodes (Figure 2), $b_i = T_j - T_k$, $b_j = T_k - T_i$, $b_k = T_i - T_j$, $c_i = X_k - X_j$, $c_j = X_i - X_k$, $c_k = X_j - X_i$, and (X_i, T_i) denotes the (x, t) -co-ordinate of node i .

Note that there is only one non-zero row since there is only one unknown per element, and this is at the $(n + 1)$ st time level. With this in mind and substitution for the $\{b_i\}$ and $\{c_i\}$ one gets the following equation:

$$(-c \Delta t - \Delta x)u_j^n + c \Delta t u_{j+1}^n + \Delta x u_j^{n+1} = 0 \tag{6}$$

or, after rearranging,

$$u_j^{n+1} = u_j^n - \nu(u_{j+1}^n - u_j^n) \tag{7}$$

which is the desired result. Note that although it appears that equation (6) was derived from the element stiffness matrix, this is in fact equivalent to the global stiffness matrix equation (assuming that an interior element was used).

The reason we were able to obtain equation (7) is that a node at the $(n + 1)$ st level of the e th element was only directly dependent on the two nodes in the e th element at the n th level.

Example 2: Upstream differencing

The finite difference form is

$$u_j^{n+1} = u_j^n - \nu(u_j^n - u_{j-1}^n) \tag{8}$$

If one proceeds as in example 2 but with the elements in Figure 3, the required equation will be obtained.

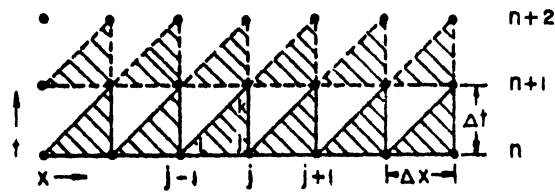


Figure 3. Upstream differencing

Example 3: Two-step Lax-Wendroff

Step 1.

$$u_{j+1/2}^{n+1/2} = \frac{u_{j+1}^n + u_j^n}{2} - \frac{\nu}{2}(u_{j+1}^n - u_j^n) \tag{9}$$

Step 2.

$$u_j^{n+1} = u_j^n - \nu(u_{j+1/2}^{n+1/2} - u_{j-1/2}^{n+1/2}) \tag{10}$$

Figures 4, 5 and 6 represent the required finite element scheme. Figure 4 corresponds to the first required step. The elements have width Δx and height $\Delta t/2$.

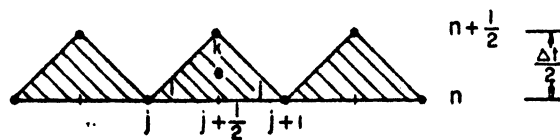


Figure 4. Lax-Wendroff, step 1

Again, as in the previous examples, it is only necessary to examine one element (e) and in particular the node located at $(j + \frac{1}{2}, n + \frac{1}{2})$. The integrations will not be carried out here; they are, however, straightforward and can be verified by the reader.

Step 2 is more interesting and is depicted in Figure 5. After completion of step 1 all nodes on the n th and $(n + \frac{1}{2})$ th level have known u values. The only unknowns are at $(n + 1)$ st level. Let us examine node $(j, n + 1)$. This nodal value is dependent on both elements 1 and 2 (Figure 5) and thus both elements enter into the local stiffness matrix necessary to calculate u_j^{n+1} . Integration of

Example 4: Leap frog

The finite difference method for this equation is

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} + c \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = 0 \tag{11}$$

where initially two time levels must be specified.

Figure 7 represents the required discretization to obtain equation (11).

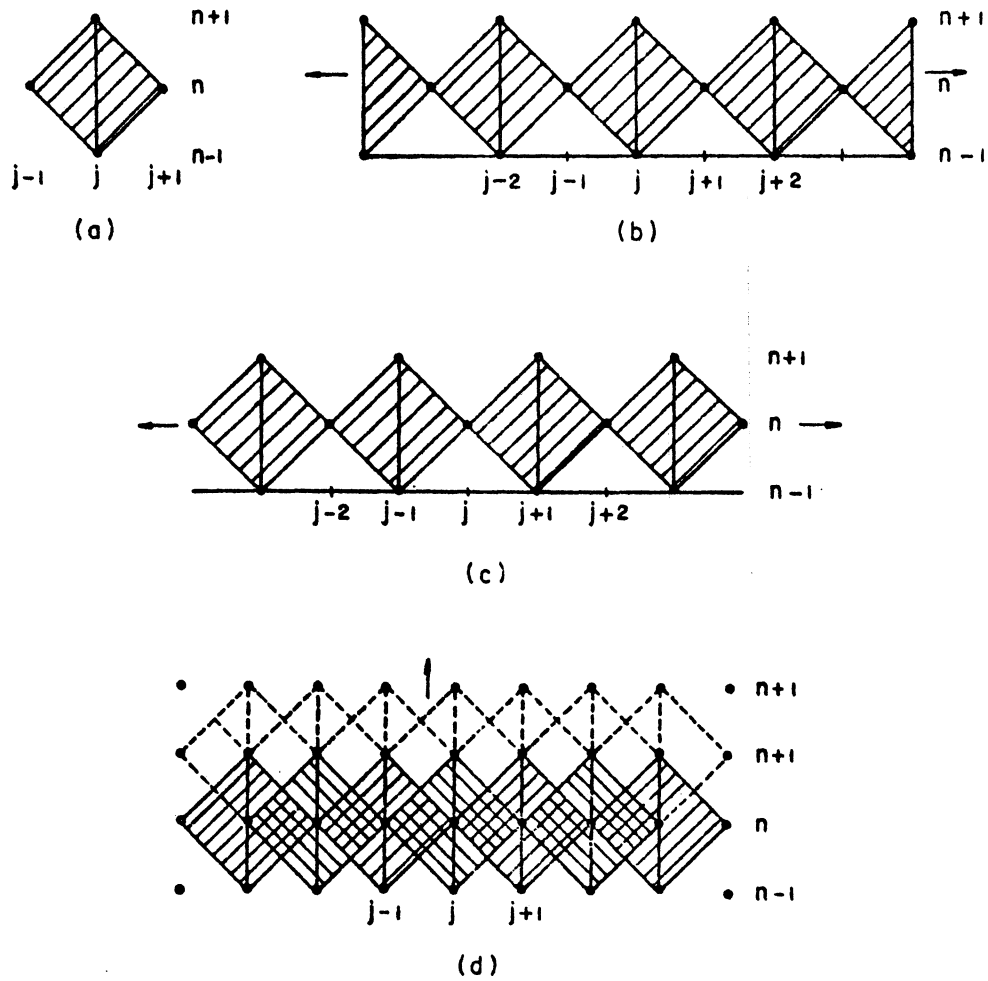


Figure 7. Leap frog: (a) primary elements; (b) first sweep; (c) second sweep; (d) marching in time

The primary elements are the same as in step 2 of example 3, but they are used in a different fashion and are larger—Figure 7(a). Since initially two time levels must have known nodal values there is only one unknown associated with a pair of elements. Noticing that the elements have a width of $2\Delta x$, we see that the method must be broken into two steps, of which both are at the same time level. Step 1(a) is depicted in Figure 7(b) and step 1(b) is depicted in Figure 7(c)

Note that the only difference in steps 1(a) and 1(b) is the shift required to cover all nodes. The reader can verify that the required results are obtained.

Figure 7(d) illustrates how the method proceeds in time. In using the elements in this fashion we obtain a more sophisticated form of lag elements than depicted in Reference 6, and thus are able to maintain an explicit scheme.

Example 5: MacCormack method

The MacCormack difference scheme can be written as follows:

$$\text{Predictor: } \bar{u}_i^{n+1} = u_i^n - \nu(u_{j+1}^n - u_j^n) \quad \text{Euler scheme} \quad (12)$$

$$\text{Corrector: } u_i^{n+1} = \frac{1}{2} \left[u_i^n + \bar{u}_i^{n+1} - \nu(u_j^{n+1} - \bar{u}_{j-1}^{n+1}) \right] \quad (13)$$

where the bar denotes a temporary solution to u .

Example 1 illustrates how the predictor step is obtained from an element point of view.

The corrector step is somewhat complicated and involves a special interpretation of u .

Step 2 must be broken into two steps, steps 2(a) and 2(b), to obtain the required result.

Step 2(a). This step should actually be carried out before step 1. Figure 8 illustrates step 2(a). Here we are using two layers of one-dimensional linear vertically oriented elements of length

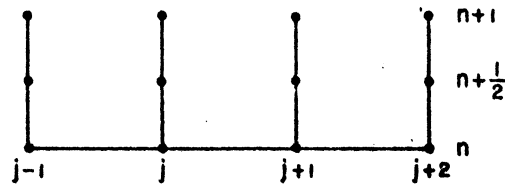


Figure 8. MacCormack, step 2(a)

$\Delta t/2$. Again, since at any particular node at the $(n + \frac{1}{2})$ th level, the node is a function of two elements only (thus the node immediately below and above it), we must consider both associated elements. Integration of equation (4) over these two elements produces (upon combining the element matrices) the equation

$$\begin{matrix} 1 & 2 & 3 \\ 1 & \left(\begin{matrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{matrix} \right) & \begin{pmatrix} u_i^{n+1} \\ u_j^{n+1/2} \\ u_j^n \end{pmatrix} & = & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{matrix} \quad (14)$$

and thus

$$u_j^{n+1/2} = \frac{u_j^n + u_j^{n+1}}{2} \quad (15)$$

Step 2(b). This step, illustrated in Figure 9, requires special interpretation, as will now be outlined. If u is defined as

$$u = N_i u_j^{n+1/2} + N_j \bar{u}_j^{n+1} + N_k \bar{u}_{j-1}^{n+1}$$



Figure 9. MacCormack, step 2(b), $(n+1)$ and $(\bar{n}+1)$ overlay

(where $u_j^{n+1/2}$ and $u_{j-1}^{\bar{n}+1}$ are given by steps 2(b) and 1, respectively) for the integral

$$\int_A N^T \frac{\partial N}{\partial x} dA \bar{u}$$

and u is defined by $u = N_i u_j^{n+1/2} + N_j u_j^{n+1} + N_k u_{j-1}^{\bar{n}+1}$ for

$$\int_A N^T \frac{\partial N}{\partial t} dA \bar{u}$$

then the elements in step 2(b) are independent. Performing the element integrations and after simplifying one gets the corrector step in the MacCormack schemes. This was possible only because we have interpreted u in a different and new fashion.

IMPLICIT EXAMPLE

Example 6: Time-centred implicit

The finite difference scheme is given by

$$u_j^{n+1} = u_j^n - \frac{\nu}{2} \left(\frac{u_{j+1}^{n+1} + u_{j+1}^n}{2} - \frac{u_{j-1}^{n+1} + u_{j-1}^n}{2} \right)$$

Although this is a one-step finite difference scheme it takes essentially two steps in finite elements.

The first step is the same as step 2(a) in example 3, and is illustrated in Figure 10(b). It is necessary to use all elements (1 and 2 of Figure 10(b)) to determine u_j^{n+1} , since u_j^n is known initially and $u_{j+1}^{n+1/2}$ are known as functions of u_j^n and u_j^{n+1} . Integration over all elements will produce the required implicit scheme. Note, however, that step 2 needs to be carried out twice before the system is solved. This is necessary since the pairs of elements in step 2 have a combined width of $2\Delta x$.

CONCLUSIONS

In the previous examples, for brevity, we have only sketched the procedures for developing the various difference schemes; the details are available upon request.

We have not encountered a difference equation representing the wave equation for which we were unable to develop a corresponding finite element scheme. The examples presented are only representative of those we tried, and the only difficulties we encountered were on interpreting the interpolated function u for predictor-corrector schemes.

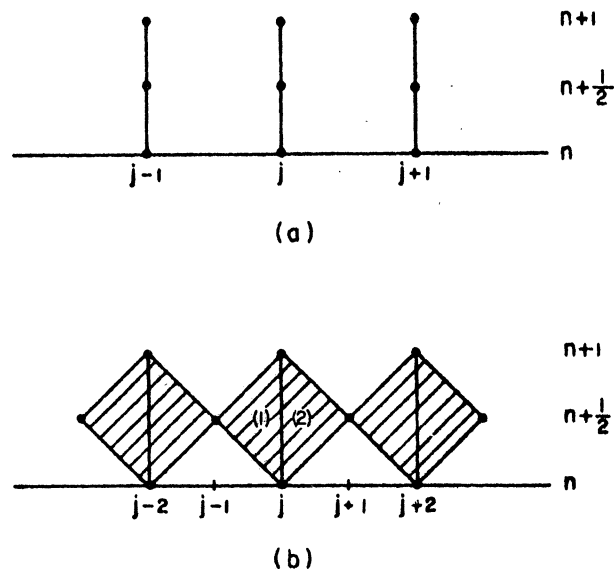


Figure 10. Time-centred implicit: (a) two layers of one-dimensional elements; (b) this layer of elements must be used twice—once shifted to get all nodes

The examples presented show the power of using Galerkin's method on the temporal as well as the spatial dimensions in finite element analysis, and should dispel most of the negative attitudes toward using it.

The examples presented serve to tie together the stability theory of finite differences to that of the finite elements. This implies a better understanding of finite element schemes.

With the presented results one can go in many directions, and ask many questions. For example, can all difference schemes be interpreted as finite element schemes? If not, which cannot and why? What effect would using a curved temporal element have?, etc.

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MATRICES
AND
MATRIX EQUATIONS

J.G. EISLEY

MATRICES AND MATRIX EQUATIONS

A.1 Introduction

Elementary matrix theory has become an essential tool for all engineers. To read this text, only an elementary knowledge of matrix definitions and operations is necessary. These definitions are summarized here to serve as a ready reference. A reader totally unfamiliar with matrices should consult one of the many excellent texts available, a few of which are listed at the end of this appendix. In addition, many books on solid mechanics and other engineering subjects have included chapters on matrix theory.

A.2 Matrix Definitions and Notation

A matrix of order $m \times n$ is an array of quantities in m rows and n columns as follows:

$$[a] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad (A-1)$$

An element in the i -th row and j -th column is represented by the notation a_{ij} . A square bracket will denote a matrix of any order. For convenience, we shall define some special cases of the general

matrix and use some special notation.

A column matrix has m rows and 1 column, or order $m \times 1$, and is denoted by $\{a\}$. For example, if $a_{11} = 1$, $a_{21} = 3$ and $a_{31} = 2$ and the matrix is of order 3×1 and

$$\{a\} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

A row matrix has 1 row and n columns or order $1 \times n$, and is denoted by $[a]$. For example,

$$[a] = [1 \ 3 \ 2]$$

A square matrix has the same number of rows and columns, or $m = n$. For example,

$$[a] = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 7 & 6 & 4 \end{bmatrix}$$

where a matrix of order 3×3 is shown.

A diagonal matrix is a square matrix in which all the elements are zero except those on the principal diagonal. It is denoted by $\uparrow a \downarrow$. Another way to say this is $a_{ij} = 0$ if $i \neq j$. For example,

$$\uparrow a \downarrow = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -5 \end{bmatrix}$$

The identity matrix is a special case of the diagonal matrix

and is denoted by $[I]$. For example,

$$[I] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It is also frequently called the unit matrix.

The null or zero matrix is one for which all elements are zero or $a_{ij} = 0$. It may be of any order.

A single element matrix is a scalar and may be written without brackets.

The determinant $|a|$ formed from the elements of a square matrix is known as the determinant of $[a]$.

A symmetric matrix is a square matrix for which

$$a_{ij} = a_{ji}$$

A diagonal matrix is always symmetric but so may others be.

The transpose of a matrix is found by interchanging rows and columns and is denoted by $[a]^T$. The elements of the transpose of a matrix are found by setting

$$a_{ij}^T = a_{ji}$$

For example,

$$[a] = \begin{bmatrix} 1 & 2 \\ 7 & 9 \\ 6 & 3 \end{bmatrix} \qquad [a]^T = \begin{bmatrix} 1 & 7 & 6 \\ 2 & 9 & 3 \end{bmatrix}$$

The transpose of a column matrix is a row matrix and vice-versa.

A symmetric matrix is identical to its transpose.

A.3 Matrix Algebra

We now define certain rules which make matrices useful.

1. Equality. Two matrices are equal if they are of the same order and all corresponding elements are equal. That is,

$$[a] = [b] \text{ if } a_{ij} = b_{ij} \quad (\text{A-2})$$

2. Addition and subtraction. Addition and subtraction is defined only for matrices of the same order. Addition is performed by adding corresponding elements, subtraction by subtracting corresponding elements. Thus,

$$[a] + [b] = [c] \text{ if } a_{ij} + b_{ij} = c_{ij}$$

For example

$$\begin{bmatrix} 2 & 1 & 7 \\ 3 & 6 & 9 \\ 1 & -4 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 6 & 7 \\ 1 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 7 & 14 \\ 4 & 4 & 9 \\ 1 & -4 & 1 \end{bmatrix}$$

3. Multiplication by a scalar. Any matrix may be multiplied by a scalar. The result is that each element of the matrix is multiplied by the scalar,

$$a[b] = [c] \text{ if } ab_{ij} = c_{ij}$$

4. Multiplication. We denote multiplication of two matrices, say $[a]$ and $[b]$, by

$$[a]^{-1}[a] = [I] \quad (A-8)$$

Thus the matrix equation

$$[a]\{x\} = \{c\}$$

becomes

$$[a]^{-1}[a]\{x\} = [I]\{x\} = \{x\} = [a]^{-1}\{c\} \quad (A-9)$$

A.4 Partitioned Matrices

A useful property of matrices is their ability to be partitioned into submatrices. These submatrices may then be treated as elements of the parent matrix and manipulated by the rules just reviewed. For example

$$[a] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ \text{-----} & \text{-----} & \text{-----} & \text{-----} & \text{-----} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \end{bmatrix} = \begin{bmatrix} [A]_{11} & [A]_{12} \\ [A]_{21} & [A]_{22} \end{bmatrix} \quad (A-10)$$

where

$$[A]_{11} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \quad [A]_{12} = \begin{bmatrix} a_{14} & a_{15} \\ a_{24} & a_{25} \end{bmatrix}$$

$$[A]_{21} = [a_{31} \quad a_{32} \quad a_{33}] \quad [A]_{22} = [a_{34} \quad a_{35}]$$

$$[a][b] = [c] \quad (A-5)$$

provided certain conditions exist. The number of columns in [a] must equal the number of rows in [b]. Each element in [c] is obtained by multiplying the elements of the corresponding row in [a] by the elements of the corresponding column in [b] and adding the results according to the rule

$$c_{ik} = \sum_{j=1}^n a_{ij} b_{jk} \quad (A-6)$$

For example,

$$\begin{bmatrix} 3 & 2 \\ 1 & 1 \\ 7 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 6 & -3 & 1 \end{bmatrix} = \begin{bmatrix} (3 \times 1 + 2 \times 6)(3 \times 2 - 2 \times 3)(3 \times 1 + 2 \times 1) \\ (1 \times 1 + 1 \times 6)(1 \times 2 - 1 \times 3)(1 \times 1 + 1 \times 1) \\ (7 \times 1 - 1 \times 6)(7 \times 2 + 2 \times 3)(7 \times 1 - 1 \times 1) \end{bmatrix} = \begin{bmatrix} 15 & 0 & 5 \\ 7 & -1 & 2 \\ 1 & 20 & 6 \end{bmatrix}$$

Note that if the order of [a] is $m \times n$ and the order of [b] is $n \times r$, the order of [c] is $m \times r$.

Matrix multiplication is associative, distributive, but, in general, not commutative. That is,

$$[a]([b][c]) = ([a][b])[c]$$

$$[a]([b] + [c]) = [a][b] + [a][c] \quad (A-7)$$

$$[a][b] \neq [b][a]$$

5. Inversion. Division, as such, is not defined for matrices but is replaced by something called inversion. We define an inverse matrix $[a]^{-1}$ such that

A. 5 Differentiating a Matrix

To differentiate a matrix we differentiate each element in a conventional manner. For example, if

$$[a] = \begin{bmatrix} x & x^2 & 3x \\ x^2 & x^4 & 2x \\ 3x & 2x & x^3 \end{bmatrix}$$

then

$$\frac{d}{dx}[a] = \begin{bmatrix} 1 & 2x & 3 \\ 2x & 4x^3 & 2 \\ 3 & 2 & 3x^2 \end{bmatrix}$$

Partial derivatives are as simple. For example,

$$\frac{\partial}{\partial x} \begin{bmatrix} xy & y & 2 \\ x & xy^2 & y \\ 2 & y & x^2y^2 \end{bmatrix} = \begin{bmatrix} y & 0 & 0 \\ 1 & y^2 & 0 \\ 0 & 0 & 2xy^2 \end{bmatrix}$$

In structural theory we often encounter an expression of the following form

$$U = \frac{1}{2} [a] [c] \{a\} \quad (\text{A-11})$$

where $[c]$ is symmetric and we wish to find its derivative $\partial U / \partial a_i$.

By expanding the above form, differentiating and reassembling to form a new matrix, the results we get are

$$\frac{\partial U}{\partial a_i} = [c] \{a\} \quad (\text{A-12})$$

For example, if

$$U = [a_1 \quad a_2] \begin{bmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$\frac{1}{2} (c_{11} a_1^2 + 2c_{12} a_1 a_2 + c_{22} a_2^2)$$

Differentiation yields

$$\frac{\partial U}{\partial a_1} = c_{11} a_1 + c_{12} a_2 \qquad \frac{\partial U}{\partial a_2} = c_{12} a_1 + c_{22} a_2$$

which, in matrix form, is

$$\begin{bmatrix} \frac{\partial U}{\partial a_1} \\ \frac{\partial U}{\partial a_2} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

We note further that the second derivatives

$$\frac{\partial^2 U}{\partial a_1^2} = c_{11} \qquad \frac{\partial^2 U}{\partial a_1 \partial a_2} = c_{12} \qquad \frac{\partial^2 U}{\partial a_2^2} = c_{22}$$

or, in general

$$\frac{\partial^2 U}{\partial a_i \partial a_j} = c_{ij} \qquad (A-13)$$

A.6 Integrating a Matrix

Matrix integration is defined to be consistent with matrix differentiation. To integrate a matrix we integrate term by term.

$$[a] = \begin{bmatrix} 1 & 2x & 3 \\ 2x & 4x^3 & 2 \\ 3 & 2 & 3x^2 \end{bmatrix}$$

then

$$\int_0^x [a] dx = \begin{bmatrix} x & x^2 & 3x \\ x^2 & x^4 & 2x \\ 3x & 2x & x^3 \end{bmatrix}$$

A.7 Summary of Useful Matrix Relations

$$[a][I] = [I][a] = [a]$$

$$a([b] + [c]) = a[b] + a[c]$$

$$[a]([b] + [c]) = [a][b] + [a][c] \quad (\text{A-14})$$

$$[a] + [b] + [c] = [a] + ([b] + [c]) = ([a] + [b]) + [c]$$

$$[a][b][c] = [a]([b][c]) = ([a][b])[c]$$

$$[a] + [b] = [b] + [a]$$

$$[a][b] \neq [b][a]$$

$$([a][b])^T = [b]^T [a]^T$$

$$([a][b])^{-1} = [b]^{-1} [a]^{-1}$$

$$([a]^T)^{-1} = ([a]^{-1})^T$$

MISCELLANEOUS
NOTES

Appendix 3

Details of the derivation of the plate equations are given in Chapter 3 (to be handed out later) and Chapter 7. For the present, let's adopt a system point of view and see how to handle the plate response and stability problems.

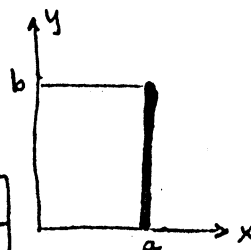
First of all, note that Eqn's 1, 2 and 3 (the in-plane equations) are coupled to Eqn 4' (the out-of-plane equation) in a special way. The in-plane solution affects the out-of-plane, but not vice-versa. This is because of the range of loads and deflections we are interested in for this study. At really large deflections (not studied here) the w deflection would cause substantial changes in N_x, N_{xy} and N_y and this would have to be accounted for. As it stands, we have

$$w(x, y) = f_{out}(N_x, N_{xy}, N_y)$$

$$N_x, N_{xy}, N_y \neq f_{in} w(x, y)$$

BOUNDARY CONDITIONS (HOMOGENEOUS)

On an edge $x=a$ (say), one can prescribe only certain boundary conditions:



PHYSICAL QUANTITY	EXPRESSION	CLAMPED	S.S.	FREE
Deflection	$w(a, y)$	0	0	
Slope	$\frac{\partial w}{\partial x}(a, y)$	0		
M_x Moment	$-D \left[\frac{\partial^2 w}{\partial x^2}(a, y) + \nu \frac{\partial^2 w}{\partial y^2}(a, y) \right]$		0	0
Q_x Shear	$-D \frac{\partial}{\partial x} \left[\frac{\partial^2 w}{\partial x^2}(a, y) + \frac{\partial^2 w}{\partial y^2}(a, y) \right]$			
M_{xy} Twist	$-D(1-\nu) \frac{\partial^2 w}{\partial x \partial y}(a, y)$			
Shear + $\frac{\partial}{\partial y}$ (Twist)	$-D \left[\frac{\partial^3 w}{\partial x^3}(a, y) + (2-\nu) \frac{\partial^3 w}{\partial x \partial y^2}(a, y) \right]$			0

Note that no B.C. are applied directly to shear or moment, but only to a combination of shear and twist. This can be proven only by a general variational approach beyond the scope of the course.

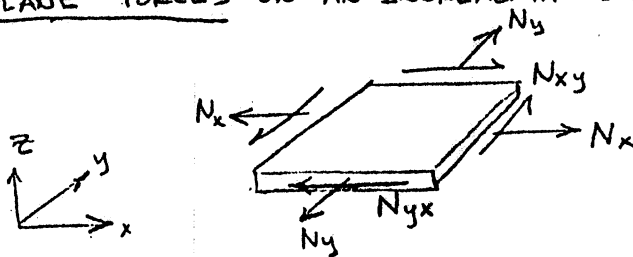
The student should be able to construct such a table for any edge of the plate.

Many people call this theory "small deflection plate theory". The theory is valid for plate deflections less than a plate thickness, $w < h$.

KIRCHHOFF-LOVE PLATE THEORY

- First plate equations were derived in France in response to a prize offered by French government.
- Sophie St. Germain won the prize but was later proven wrong.
- Disputes over boundary conditions lasted for decades.
- Kirchhoff's energy approach solved the problem.

IN PLANE FORCES ON AN INCREMENTAL ELEMENT:



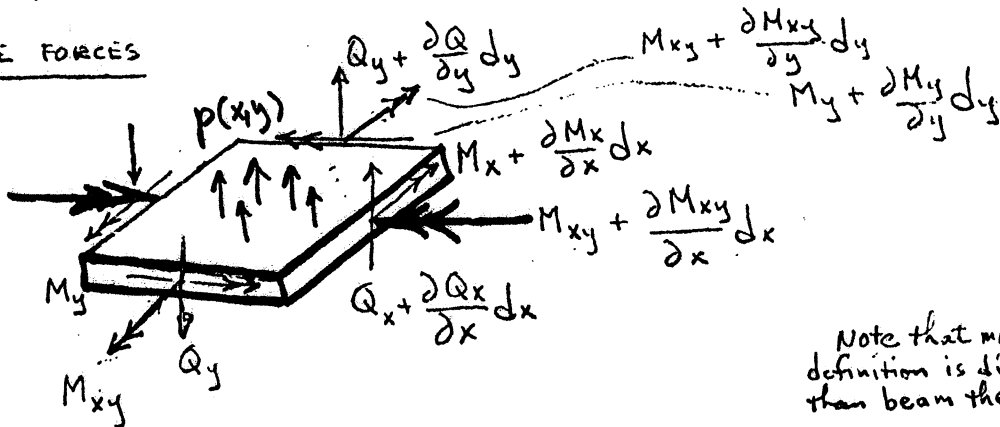
$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \quad (1)$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0 \quad (2)$$

$$\frac{\partial^2 (N_y - \nu N_x)}{\partial x^2} - 2(1+\nu) \frac{\partial^2 N_{xy}}{\partial x \partial y} + \frac{\partial^2 (N_x - \nu N_y)}{\partial y^2} = 0 \quad (3)$$

For standard problems with no body moments, $N_{yx} = N_{xy}$

OUT-OF-PLANE FORCES



Note that moment definition is different than beam theory.

RESULTING EQUATION OF EQUILIBRIUM IS

$$\frac{Eh^3}{12(1-\nu^2)} \left(\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) - N_x \frac{\partial^2 w}{\partial x^2} - 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} - N_y \frac{\partial^2 w}{\partial y^2} = p(x,y) \quad (4)$$

OR

$$D \nabla^4 w - N_x \frac{\partial^2 w}{\partial x^2} - 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} - N_y \frac{\partial^2 w}{\partial y^2} = p(x,y) \quad (4')$$

April 12, 1976
W. J. Anderson

PLATE FLEXURE THEORY

Ref: Gallagher, Chapter 12
 Zienkiewicz, Chapter 10
 Aero 414 Notes, Lectures 15, 16, 17

General Comments

Gallagher's book is excellent for a short review of "pure bending", i.e. flexure theory. Zienkiewicz's book has a good discussion of interelement compatibility. The Aero 414 notes include in-plane forces and allow a classical study of buckling. The current state of the art is that conforming elements in the displacement theory of FE are now available, and I believe they should be demanded. There was some trouble with triangular elements for a while, but the Olson-Lindberg-Couper triangle should eliminate that.

Definitions for small deflection, plate flexure

Referring to the sketches in Lec 15, Aero 414 notes, the line loads are defined:

$$M_x = \int_{-h/2}^{h/2} \sigma_x z dz \quad (1a)$$

$$M_y = \int_{-h/2}^{h/2} \sigma_y z dz \quad (1b)$$

$$M_{xy} = M_{yx} = \int_{-h/2}^{h/2} \tau_{xy} z dz \quad (1c)$$

$$Q_x = \int_{-h/2}^{h/2} \tau_{xz} dz \quad (1d)$$

$$Q_y = \int_{-h/2}^{h/2} \tau_{yz} dz \quad (1e)$$

These have dimension of force/unit length, and play a role comparable to stress in the usual F.E. theory. The quantities that play the role of strain are the curvatures in the two orthogonal directions,

plus the twist. These are defined, for small displacements from the initially flat shape as:

$$\begin{Bmatrix} K_x \\ K_y \\ K_{xy} \end{Bmatrix} = \begin{Bmatrix} -\frac{\partial^2 w}{\partial x^2} \\ -\frac{\partial^2 w}{\partial y^2} \\ +2\frac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} \quad (2)$$

This kinematic definition of curvature also gives the strain-displacement matrix (for which we have assigned no special symbol)

$$\begin{Bmatrix} K_x \\ K_y \\ K_{xy} \end{Bmatrix} = \begin{bmatrix} -\frac{\partial^2}{\partial x^2} \\ -\frac{\partial^2}{\partial y^2} \\ -2\frac{\partial^2}{\partial x \partial y} \end{bmatrix} \begin{Bmatrix} w \end{Bmatrix} \quad (3)$$

\uparrow displacement
 \uparrow strain-displacement
 \uparrow generalized strain

The constitutive relation between generalized stress and generalized strain is the moment-curvature relation:

$$\{M\} = [C] \{K\} \quad (4)$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} D_x & D_1 & 0 \\ D_1 & D_y & 0 \\ 0 & 0 & D_{xy} \end{bmatrix} \begin{Bmatrix} K_x \\ K_y \\ K_{xy} \end{Bmatrix} \quad (5)$$

For isotropic thin plate flexure,

$$[C] = \begin{bmatrix} D & \nu D & 0 \\ \nu D & D & 0 \\ 0 & 0 & D\left(\frac{1-\nu}{2}\right) \end{bmatrix} \quad (6)$$

where $[D] = \frac{E h^3}{12(1-\nu^2)}$ (7)

Equilibrium of the out-of-plane forces (see Fig in Aero414 notes) in the z direction yields

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + p(x,y) = 0 \quad (8)$$

Equilibration of moments about the x and y axes yields

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0 \quad (9)$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = 0 \quad (10)$$

The problem has been posed for the interior of the plate. (No boundary conditions have yet been discussed.) There are two routes now available - we can proceed with derivation of the differential equations and boundary conditions from equilibrium arguments or we can go to an energy derivation of equations. The former leads one to a Galerkin approach to F.E. but is good for intuition. The latter is a better method for creating F.E.

Derivation of Differential Equations

The field variable $w(x,y)$ is the important quantity in the problem. The small deflection, plate flexure problem uncouples in a remarkable way! The steps are:

- 1) Use Eqn 3 to express curvature as a function of $w(x,y)$
- 2) Insert $\{K(w)\}$ into Eqn 5 to get moments as a function of w .
- 3) Insert $\{M(K(w))\}$ into Eqns 9 and 10 to get shears as a function of w .
- 4) Insert $\{Q(M(K(w)))\}$ into Eqn 8 to get:

$$D_x \frac{\partial^4 w}{\partial x^4} + 2(D_1 + D_{xy}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} = p(x,y) \quad (11)$$

It is a remarkable result to be able to uncouple all these equations and end up with a single 4th order equation. (The fact that it is fourth order can be rationalized by seeing that Eqn 3 is 2nd order, Eqns 9 & 10 are 1st order and Eqn. 8 is 1st order. The uncoupling adds each order in turn.)

For isotropic plate material, we have $D_x = D_y \equiv D$, $D_1 = \nu D$, $D_{xy} = (\frac{1-\nu}{2})D$ and

$$D \left(\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) = p(x,y) \quad (12)$$

This equation is accepted widely and used widely in many applications. The operator in parentheses is called the "biharmonic" operator and written ∇^4 :

$$D \nabla^4 w = p(x,y) \quad \leftarrow \begin{array}{l} \text{Small deflection;} \\ \text{Plate flexure,} \\ \text{Isotropic material} \\ \text{and construction.} \end{array} \quad (13)$$

Boundary conditions are difficult to argue from equilibrium considerations (a general variational approach is better). The homogeneous boundary conditions are pointed out in Lecture 15, Aero 414 notes. Particularly difficult is the two term condition on shear and twist at a free edge. Each edge of a plate requires specification of two boundary conditions involving displacements and derivatives of displacement taken parallel and normal to the edge. This means a plate cut in a polygonal shape with n edges will have $2n$ boundary conditions, where $n \geq 3$.

This is certainly different than beam theory where one always has precisely 4 boundary conditions.

Potential Energy of a Plate in Flexure.

Ref. Gallagher, pp 331
Aero 414 notes, lecture 17 (also includes N_x)

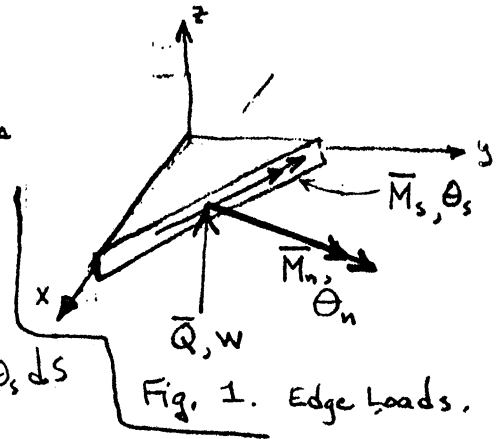
The strain energy in pure bending of a plate is

$$U = \frac{1}{2} \iint \{ \kappa \}^T [C] \{ \kappa \} dx dy \quad (14)$$

\uparrow generalized strain (curvature) \uparrow stress-strain law (moment-curvature) \uparrow generalized strain

The potential of the various external loads is

$$(15) \quad W = - \int_0^b \int_0^a p(x,y) w(x,y) dx dy - \int_{S_r} \bar{Q} w ds - \int_{S_r} \bar{M}_n \theta_n ds - \int_{S_r} \bar{M}_s \theta_s ds$$



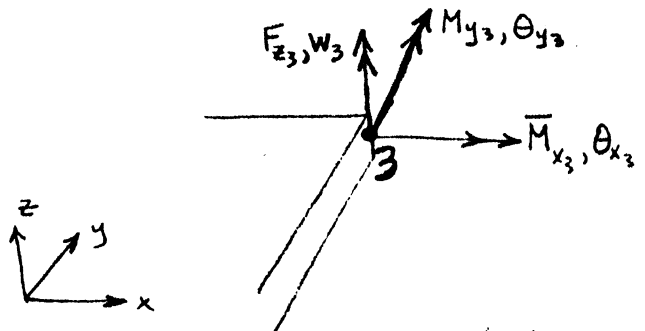
where \bar{Q} , \bar{M}_n and \bar{M}_s are specified running loads on the edge of the plate. Because concentrated nodal loads may be present (either physically or as a finite element idealization) we add

$$W = - \sum_{i=1}^r F_{x_i} w_i - \sum_{i=1}^r \bar{M}_{x_i} \theta_{x_i} - \sum \bar{M}_{y_i} \theta_{y_i} \quad (16)$$

The nodal rotations are referenced to a Cartesian coordinate system where

$$\theta_x = \frac{\partial w}{\partial y} \quad (17a)$$

$$\theta_y = - \frac{\partial w}{\partial x} \quad (17b)$$



In each term of the potential of the loads V one demands that the force and displacement terms have the same sign convention, also moments and rotations.

The strain energy in a plate can be evaluated for the isotropic case by inserting the curvature and stress strain law in Eqn. 14. One obtains

$$U = \frac{D}{2} \int_0^b \int_0^a \left\{ \underbrace{\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right]^2}_{\text{Bending}} - 2(1-\nu) \underbrace{\left[\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial w}{\partial x \partial y} \right)^2 \right]}_{\text{Twisting}} \right\} dx dy \quad (1)$$

This form shows the breakdown between bending and twisting effects. If the Rayleigh-Ritz method is to be used for an entire plate (global solution, not finite elements) and if the plate has zero deflection at its boundaries, the twisting term can be shown to drop out. This proof can be done by integration by parts, and shows an interesting case where finite element theory, using pieces, cannot take advantage of a global consequence.

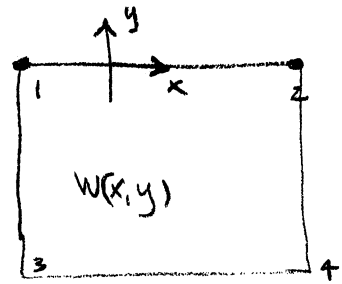
The above discussion is from classical plate theory. For finite element work, it is better to work directly with Eqn 14.

Continuity Requirements for Compatible Elements

Ref: Zienkiewicz, pp 175, 177

The variational form $\delta \Pi = 0$ for the plate element requires continuity of first derivatives across interelement boundaries, or else something will be "missed" in Eqn 18. To have continuity of any function along an element interface requires that the function be uniquely determined by nodes on that interface. This must be true for both elements contacting that interface, but since similar elements are assumed on either side, the discussion centers on one element at a time.

Consider a rectangular plate with nodes 1 and 2 as shown. The displacement field $w(x, y)$ is given in polynomial form. What type of polynomial is needed to ensure compatibility?



We wish $w(x, 0)$ to be determined by 'specification' of the following nodal data:

$$\begin{aligned} w(x_1, y_1) &= \text{given} \\ \frac{\partial w}{\partial y}(x_1, y_1) &= \text{given} & \frac{\partial w}{\partial x}(x_1, y_1) &= \text{given} \\ w(x_2, y_2) &= \text{given} \\ \frac{\partial w}{\partial y}(x_2, y_2) &= 0 & \frac{\partial w}{\partial x}(x_2, y_2) &= \text{given} \end{aligned}$$

The polynomial would hence say

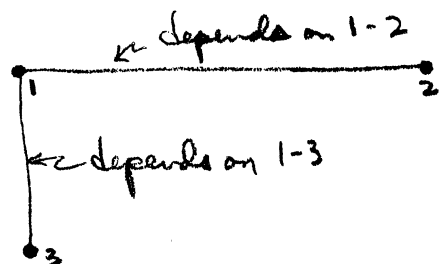
$$w(x, 0) = A_1 + A_2 x + A_3 x^2 + \dots$$

$$\frac{\partial w}{\partial y}(x, 0) = B_1 + B_2 x + B_3 x^2 + \dots$$

Because of the way the y polynomial terms enter, the B_i are independent of the A_i at this stage. For this two node example, $w(x, 0)$ could have a cubic variation because there are 4 equations to determine four constants A_i . On the other hand, there are only two equations to evaluate $\frac{\partial w}{\partial y}$ and hence only two nonzero B_i are allowed.

A similar approach can be taken on the left boundary between nodes 1 and 3. This will give that $w(x, y)$ and $\frac{\partial w}{\partial x}(x, y)$ depends only on nodes 1 and 3.

A problem arises in the corner if one attempts to calculate the cross derivative $\frac{\partial^2 w}{\partial x \partial y}$ at a corner



and within this element. (It is not required that this cross derivative have any properties, such as continuity across interelement lines, but just that it exist.) If one looks at the field variable at node one and calculates the cross derivative in two ways:

$$\frac{\partial^2 w(x, y)}{\partial x \partial y} \equiv \frac{\partial}{\partial x} \left(\frac{\partial w}{\partial y}(x, y) \right)$$

↑
depends only on 1-2 edge

$$\frac{\partial^2 w(x, y)}{\partial x \partial y} \equiv \frac{\partial}{\partial y} \left(\frac{\partial w}{\partial x}(x, y) \right)$$

↑
depends only on 1-3 edge

These cross derivatives cannot be equal in general because each depends on data from a separate, remote point. Therefore, in a 4 node, rectangular plate element, ^{conforming} ~~applying~~ polynomial cannot be constructed for the case where each node has one displacement and 2 rotations.

Zienkiewicz notes that there are additional problems with creating higher order nodal degrees of freedom because specifying higher derivatives can over constrain the element.

See Zienkiewicz, Hübner, Przemienicki, Deraï & Abel, Gallagher & Martin for an amazing assortment of plate elements. I personally prefer conforming elements and am suspicious of nonconforming elements.

16 April 1976

PLATE FLEXURE
CONTINUED

ALRO 610

REF. Gallagher, pp 338-369

Pascal's Triangle

Discussion about compatibility centers on the displacement function

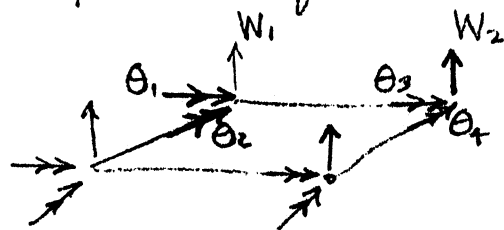
$$W(x,y) = a_1 + a_2x + a_3y + a_4xy + a_5x^2 + \dots$$

usually taken in such a polynomial form. It is best to take all the terms up to a given degree, say cubic, in order to have a complete polynomial. This will guarantee geometric isotropy, that is, the stiffness of the element will not depend on the coordinate system. (Of course, the question of material non-isotropy is separate and can still cause the element to be non-isotropic.)

Pascal's Triangle is a means of grouping polynomial terms

$$\begin{array}{cccc}
 & & a_1 & & \\
 & & a_2x & a_3y & \\
 & a_4x^2 & a_5xy & a_6y^2 & \\
 \hline
 & a_7x^3 & a_8x^2y & a_9xy^2 & a_{10}y^3 & \\
 a_{11}x^4 & a_{12}x^3y & a_{13}x^2y^2 & a_{14}xy^3 & a_{15}y^4 & \\
 a_{16}x^5 & & & & &
 \end{array}$$

The solid line separates those terms required for a rectangular element with 4 nodes having 3 D.O.F. per node (see sketch). This is the oldest and most obvious plate bending element, but unfortunately (from preceding arguments) is not compatible. The element can be shown to violate interelement continuity of the slope normal to the edge.



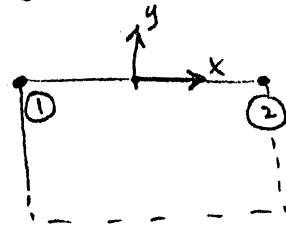
M-Z-C Rectangle

This can be shown easily with such a concrete example at hand. Taking various derivatives along the edge $y=0$, say,

$$W|_{(x,0)} = a_1 + a_2 x + a_4 x^2 + a_7 x^3$$

$$\frac{\partial W}{\partial x} \Big|_{(x,0)} = a_2 + 2a_4 x + 3a_7 x^2$$

$$\frac{\partial W}{\partial y} \Big|_{(x,0)} = a_3 + a_5 x + a_8 x^2 + a_{12} x^3$$



Note that there are 6 pieces of information from nodes ① and ②.

Applying $W(x_1,0) \equiv W_1 = a_1 + a_2 x_1 + a_4 x_1^2 + a_7 x_1^3$

$W(x_2,0) \equiv W_2 = a_1 + a_2 x_2 + a_4 x_2^2 + a_7 x_2^3$
↑ nodal coordinate

$\frac{\partial W}{\partial x}(x_1,0) \equiv \theta_1 = a_2 + 2a_4 x_1 + 3a_7 x_1^2$

$\frac{\partial W}{\partial x}(x_2,0) \equiv \theta_3 = a_2 + 2a_4 x_2 + 3a_7 x_2^2$

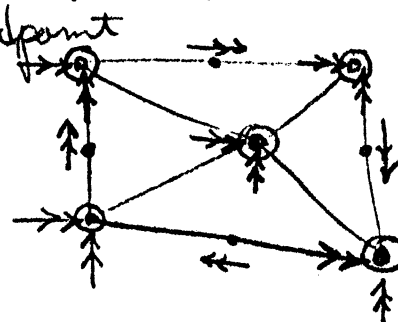
Note that these constitute 4 equations in the 4 unknowns a_1, a_2 , and hence the function $W(x,y)$ along the cut edge ①-③ is determined completely by nodal data on that edge. The problem is now with the "normal" slope

$\frac{\partial W}{\partial y}(x_1,0) \equiv \theta_2 = a_3 + a_5 x_1 + a_8 x_1^2 + a_{12} x_1^3$

$\frac{\partial W}{\partial y}(x_2,0) \equiv \theta_4 = a_3 + a_5 x_2 + a_8 x_2^2 + a_{12} x_2^3$

In this case there are two equations in four unknowns and the slope normal to the cut edge ①-③ is not determined by information at nodes lying on that edge. Therefore, the slope is incompatible. The element is widely used, however, and a convergence proof for this type of element has been established (although not the classical potential energy proof for nested elements by Melosh).

The element discussed above is called a "single field" element, because one domain was used for the field variable. Another popular approach is to use a subdomain approach in which one quadrilateral is made from assembly of 4 triangles. Each triangle has a complete cubic with 10 terms and an extra midpoint node at the external boundary



When the middle node is condensed out, a total of 16 D.O.F. remain.

This element is conforming and hence always overestimates the structural stiffness, guaranteeing convergence from below (on displacements and stresses) when the successive meshes are "nested".

The figure below compares several quadrilateral plate bending elements convergence properties. The two cases discussed are accentuated. The figure is from Gallagher.

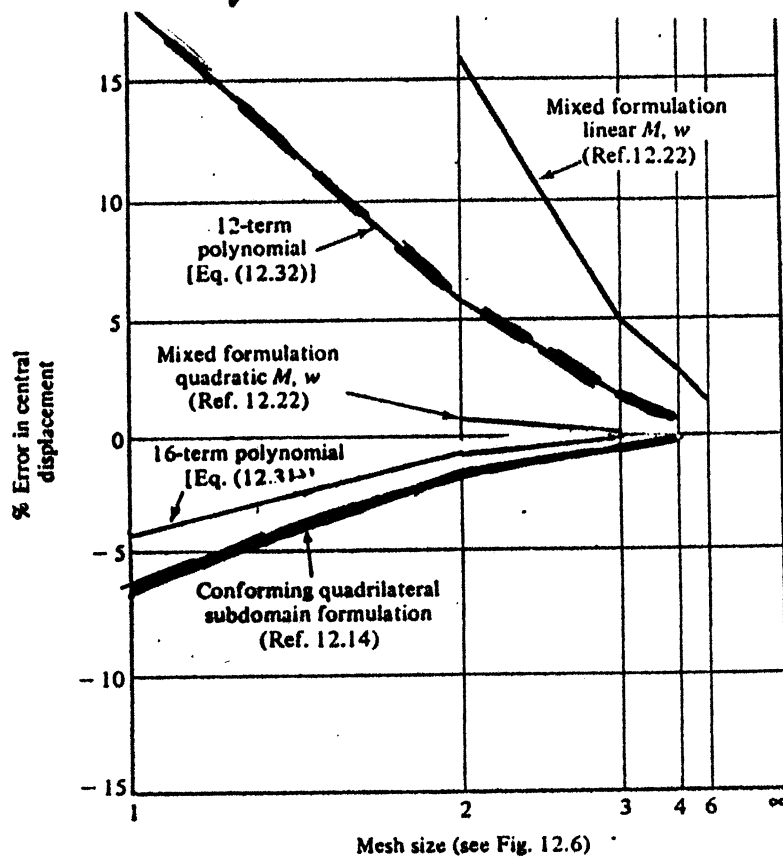


Fig. 12.7 Numerical comparisons: quadrilateral plate element formulations.

6 DEC 1976

LECTURE 27
AERO 510

VIRTUAL WORK & POTENTIAL ENERGY

REF: ZIEKIEWICZ SEC 2.4, 2.7
DFA SEC 4.2

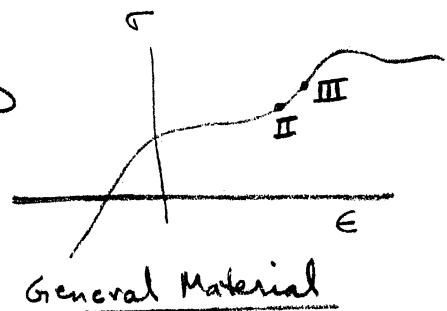
I. Introduction

The virtual work theorem was used in Lectures 7 & 8 of Aero 510 to derive the finite element relations for elastic bodies. Although I prefer virtual work as the most direct way to derive equilibrium equations, and although virtual work applies to all situations, there are times when the potential energy theorem is preferable. These include convergence proofs, where a minimization principle is helpful.

Let us start with the virtual work statement

$$1) \quad \Delta W_{\text{internal}} + \Delta W_{\text{external}} = 0$$

$\text{II} \rightarrow \text{III} \qquad \qquad \qquad \text{II} \rightarrow \text{III}$



which is valid for any material (linear or nonlinear, dissipative or not) and any load situation (conservative or not, prestress & prestrain or not). One can proceed to the case of an elastic material (not necessarily linear) to get

$$2) \quad -\Delta U + \Delta W_{\text{external}} = 0$$

$\text{II} \rightarrow \text{III} \qquad \qquad \qquad \text{II} \rightarrow \text{III}$

where the quantity ΔU is defined as $\int_{\text{vol}} \{\sigma\}^T \{\Delta \epsilon\} dV$. A rationale for equation 2) can be a simple conservation of energy and is possibly a better place for many students to start!

If we now emphasize the operator character of the increment above and rewrite it with a δ ,

$$3) \quad \delta(U_{\text{II}}) = (\Delta U)_{\text{I} \rightarrow \text{II}}$$

This creates a new creature, U_{II} , the strain energy at state II.

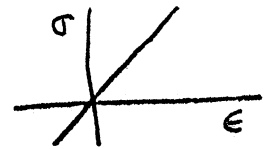
242

Now, my feeling is that strain energy is best defined in an incremental way (see above), as in the case of work. If the operator δ is to operate on all displacement-like quantities, then we could define

$$U_{II} = \int_V \{\sigma\}^T \{\epsilon\} dV$$

which would be of limited usefulness, or we could define

$$4) \quad U_{II} = \frac{1}{2} \int_V \{\epsilon\}^T [C] \{\epsilon\} dV$$



which is valid for linear, elastic bodies with no prestress or prestrain.

We have a similar problem in developing a work potential. We need to express the increment in work (a physical quantity) as a variation of a work potential (a fictitious quantity). This is only possible for conservative force fields, and not circulatory or dissipative fields.

$$\delta(W)_{II} = (\Delta W_{\text{external}})_{II \rightarrow III}$$

work potential \curvearrowright

This work potential can usually be derived. It is usually of the form (force) \times (distance). Two sign conventions are possible; the physicists', where forces can be derived

$$\text{Force} = -\vec{\nabla} W$$

\uparrow gradient vector
 \nwarrow work potential

and the engineers'

$$\text{Force} = +\vec{\nabla} W$$

The work potential differs only in sign and is usually evident from the context.

Once U and W have been so defined, the potential energy

theorem can be written:

$$\boxed{\delta U - \delta W = 0}$$

engineers version,
where $\bar{z} = +\nabla W$

$$\boxed{\delta U + \delta W = 0}$$

physicists version,
where $\bar{z} = -\nabla W$

Let us use the physicists' version, since both Desai & Abel and Zienkiewicz use it.

Desai: define $\Pi \equiv U + W$

Zienkiewicz: define $\chi \equiv U + W$

$$\boxed{\delta(\Pi) = 0}$$

Potential energy
theorem. (Desai & Abel, physicist's
notation.)

II. Minimization of Potential Energy

Consider a system which has been discretized by the finite element method, perhaps by virtual work as we have done earlier. Then

$$U \equiv \frac{1}{2} \{r\}^T \int_{Vol} [\bar{B}]^T [C] [B] dV \{r\} = \frac{1}{2} \{r\}^T [K] \{r\}$$

and

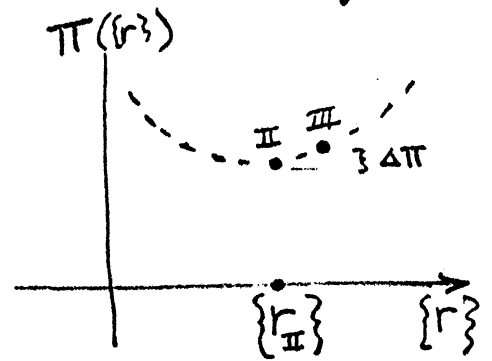
$$W = - \{R\}^T \{r\}$$

Note that the concept of a global \bar{B} matrix is used here, where $[\bar{B}] = [S-D][N]$ and $[N]$ is a global shape function (Z, sec 2.3). The matrix $[K]$, the assembled stiffness matrix, is positive definite if rigid body modes have been removed from the system (e.g., by using one of the methods mentioned in lecture 18 for solving in place)

One can prove the potential energy theorem

Potential energy takes a minimum value at the equilibrium configuration of a linear elastic structure under conservative loading (derivable from a work potential)

Proof: The proof will be done for a discretized system, using an incremental approach. First of all, the Kirchhoff's uniqueness theorem says that a well posed, linear, elastic problem has one unique equilibrium solution. We call the displacement there $\{r_{II}\}$. Calculate the potential energy at a nearby point in configuration space



$$\begin{aligned}
 \Delta\pi &= \pi(\{r_{II}\} + \{\Delta r\}) - \pi(\{r_{II}\}) \\
 &= \frac{1}{2}(\{r_{II}\} + \{\Delta r\})^T [K] (\{r_{II}\} + \{\Delta r\}) - \{R\}^T (\{r_{II}\} + \{\Delta r\}) \\
 &\quad - \frac{1}{2} \{r_{II}\}^T [K] \{r_{II}\} + \{R\}^T \{r_{II}\} \\
 &= \frac{1}{2} \{r_{II}\}^T [K] \{r_{II}\} + \{r_{II}\}^T [K] \{\Delta r\} + \frac{1}{2} \{\Delta r\}^T [K] \{\Delta r\} - \{R\}^T \{r_{II}\} - \{R\}^T \{\Delta r\} \\
 &\quad - \frac{1}{2} \{r_{II}\}^T [K] \{r_{II}\} + \{R\}^T \{r_{II}\} \\
 &= \underbrace{(\{r_{II}\}^T [K] - \{R\}^T)}_{\text{ZERO AT EQUILIBRIUM!}} \{\Delta r\} + \frac{1}{2} \{\Delta r\}^T [K] \{\Delta r\}
 \end{aligned}$$

$$= \frac{1}{2} \{\Delta r\}^T [K] \{\Delta r\} > 0 \quad \text{because } [K] \text{ is positive definite Q.E.D.}$$

POTENTIAL ENERGY

I. DISCUSSION OF POTENTIALS

The potential energy theorem says that at equilibrium:

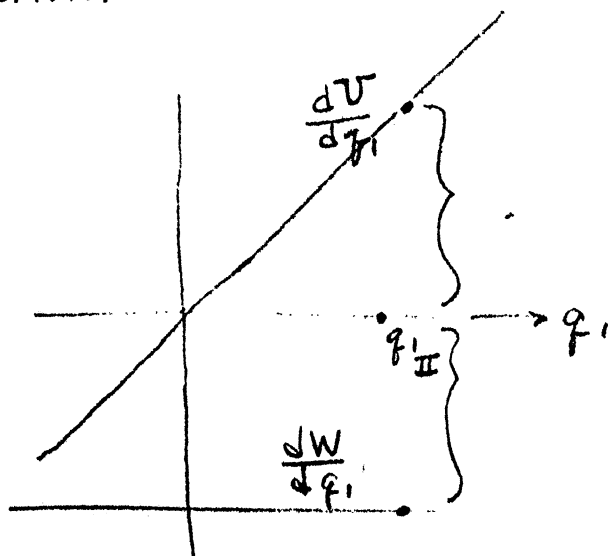
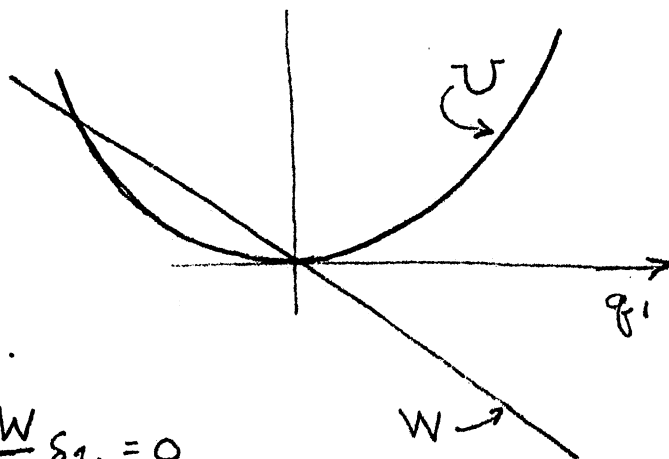
$$\delta(U+W) = 0$$

If U and W were functions of one spatial variable q_1 , say, then at equilibrium

$$\delta U + \delta W \equiv \frac{dU}{dq_1} \delta q_1 + \frac{dW}{dq_1} \delta q_1 = 0$$

or

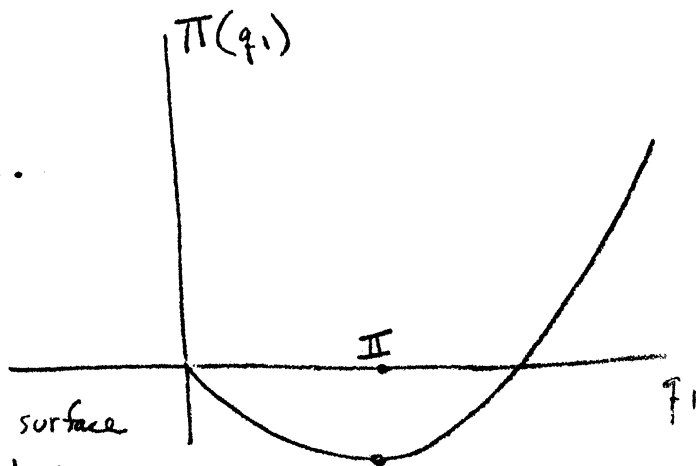
$$\frac{dU}{dq_1} + \frac{dW}{dq_1} = 0 \text{ at equilibrium.}$$



Surprisingly, the equilibrium condition, for a linear, elastic solid with no prestress or prestrain occurs at a negative potential energy Π , always.

Note that the physicists' sign convention gives the proper sense to the slope of the work potential.

A ball placed on the work potential surface tends to move away from the origin, but strain energy eventually restrains it.



II. Alternate form of Potential Energy

The P.E. in a single element is easily rewritten in terms of generalized coordinates

$$\Pi = \frac{1}{2} \{q\}^T [k] \{q\} - \{Q\}^T \{q\}$$

where $\{q\} = [A] \{\alpha\}$

Hence

$$\begin{aligned} \Pi &= \frac{1}{2} ([A] \{\alpha\})^T [k] [A] \{\alpha\} - \{Q\}^T [A] \{\alpha\} \\ &= \frac{1}{2} \{\alpha\}^T [A]^T [k] [A] \{\alpha\} - ([A]^T \{Q\})^T \{\alpha\} \\ &\equiv \frac{1}{2} \{\alpha\}^T [k_\alpha] \{\alpha\} - \{Q_\alpha\}^T \{\alpha\} \end{aligned}$$

\uparrow generalized stiffness \uparrow generalized force.

III. Different levels of approximation

Suppose two finite elements are created, with no rigid body modes, one with

$$\{u\} = [\phi_1 \ \phi_2 \ \dots \ \phi_{20}] \begin{Bmatrix} \alpha_1 \\ \vdots \\ \alpha_{20} \end{Bmatrix}$$

and the second with

$$\{u\} = [\phi_1 \ \phi_2 \ \dots \ \phi_{10}] \begin{Bmatrix} \alpha_1 \\ \vdots \\ \alpha_{10} \end{Bmatrix}$$

(where each ϕ_i is a column vector itself)

Then one can form the potential energy for the first case

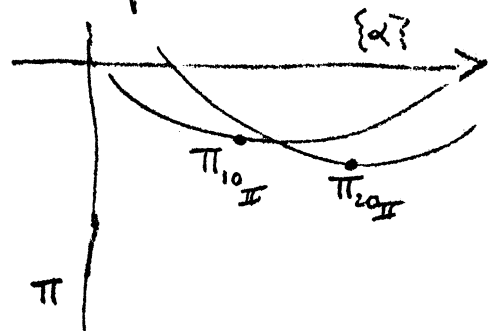
$$\{\Pi_{20}\} = \frac{1}{2} \begin{Bmatrix} \alpha_1 \\ \vdots \\ \alpha_{20} \end{Bmatrix}^T \begin{bmatrix} k_{11} & \dots \\ & \dots \\ & & k_{20,20} \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \vdots \\ \alpha_{20} \end{Bmatrix} - \begin{Bmatrix} Q_{\alpha_1} \\ \vdots \\ Q_{\alpha_{20}} \end{Bmatrix}^T \begin{Bmatrix} \alpha_1 \\ \vdots \\ \alpha_{20} \end{Bmatrix}$$

The second case is

$$\begin{Bmatrix} \pi_{10} \end{Bmatrix} = \frac{1}{2} \begin{Bmatrix} \alpha_1 \\ \vdots \\ \alpha_{10} \end{Bmatrix}^T \begin{bmatrix} k_{11} & \dots \\ \vdots & \ddots \\ & & k_{10,10} \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \vdots \\ \alpha_{10} \end{Bmatrix} - \begin{Bmatrix} Q_1 \\ \vdots \\ Q_{10} \end{Bmatrix}^T \begin{Bmatrix} \alpha_1 \\ \vdots \\ \alpha_{10} \end{Bmatrix}$$

The potential energy method proven in Lecture 27 applies to each case separately, that is, the equilibrium position Π provides the minimum potential energy for any choice of the

$\{\alpha\}$ vector in each case. In the sketch at the right, each curve is guaranteed to be cup shaped as shown, but with no hint as to whether $\pi_{10,II} \stackrel{?}{\leq} \pi_{20,II}$.



The simpler approximation however can be imbedded in the more general case

$$\pi_{10} = \frac{1}{2} \begin{Bmatrix} q_1 \\ q_2 \\ q_{10} \\ 0 \\ 0 \\ 0 \end{Bmatrix}^T \begin{bmatrix} k_{11} & k_{12} & \dots & k_{1,20} \\ k_{2,1} & k_{22} & \dots & k_{2,20} \\ k_{31} & & & \\ \vdots & & & \\ & & & k_{20,1} \\ & & & & k_{20,20} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ \vdots \\ q_{10} \\ 0 \\ \vdots \\ 0 \end{Bmatrix} - \begin{Bmatrix} Q_{q_1} \\ \vdots \\ Q_{q_{10}} \\ Q_{q_{20}} \end{Bmatrix}^T \begin{Bmatrix} q_1 \\ \vdots \\ q_{10} \\ 0 \\ 0 \\ \vdots \end{Bmatrix}$$

Note that

$$\{u_{10}\} = [\phi_1 \ \phi_2 \ \dots \ \phi_{10} \ 0 \ 0 \ \dots \ 0] \begin{Bmatrix} \alpha_1 \\ \vdots \\ \alpha_{20} \end{Bmatrix}$$

$$\begin{aligned} \left\{ Q_\alpha \right\} &= \begin{Bmatrix} Q_{\alpha 1} \\ Q_{\alpha 2} \\ \vdots \\ Q_{\alpha 10} \\ 0 \\ \vdots \\ 0 \end{Bmatrix} = \int_{\text{space}} [\phi]^T \{ \ddot{x}(\text{space}) \} d\text{space} \\ &= \int_{\text{space}} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_{10} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \{ \ddot{x}(\text{space}) \} d\text{space} \end{aligned}$$

and

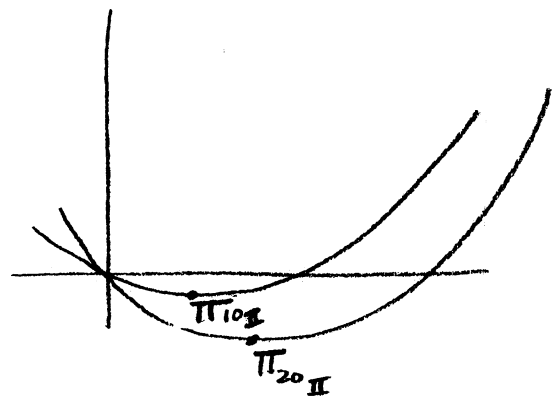
$$[k_\alpha] = \int_{\text{Vol}} \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_{10} \\ 0 \\ \vdots \\ 0 \end{bmatrix} [S-D]^T [C] [S-D] [\phi_1 \dots \phi_{10} 0 0 \dots 0] d\text{Vol}$$

This means that we have a complete "nesting" of the u_{10} solution within the u_{20} solution. Therefore, the $\{u_{10}\}$ solution at equilibrium $\{u_{10}\}_{II}$ can be viewed as a constrained version of $\{u_{20}\}_{II}$ and causes

$$\pi_{10_{II}} \geq \pi_{20_{II}}$$

From our minimization proof in lecture 27, which said that $\pi_{20_{II}}$ is less than or equal to any other π in the N dimensional (20 dim.) space.

This proves that the 20 d.o.f. model is a better model than the 10 d.o.f. model.



IV. Maximization of Strain Energy

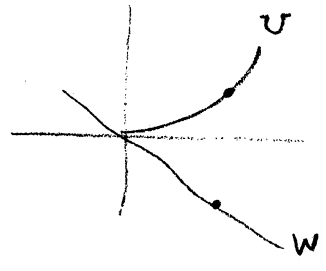
(Reference: Zienkiewicz, Section 2.7)

For a linear elastic system with no prestress or prestrain, subjected to a constant force field, one can call on conservation of energy for the load path from state I to state II.

$$(\text{work done by external force}) = (\text{strain energy stored in body})$$

Notice that this is not an infinitesimal increment.

$$-\frac{1}{2} W_{II} = U_{II}$$



(This is also not a virtual work idea.)

We can now compare strain energy at the equilibrium position by writing separately for two levels of approximation

$$\pi_{10_{II}} \equiv U_{10_{II}} + W_{10_{II}} = U_{10_{II}} - 2U_{10_{II}} = -U_{10_{II}}$$

$$\pi_{20_{II}} \equiv U_{20_{II}} + W_{20_{II}} = U_{20_{II}} - 2U_{20_{II}} = -U_{20_{II}}$$

We just proved in the last section that

$$\pi_{20_{II}} \leq \pi_{10_{II}}$$

Hence

$$-U_{20_{II}} \leq -U_{10_{II}}$$

which is rewritten

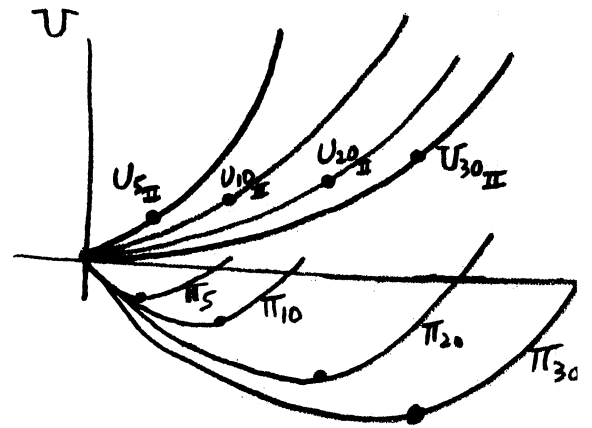
$$U_{20_{II}} \geq U_{10_{II}}$$

We conclude that, as a finer, nested grid is used, the strain energy at equilibrium becomes greater.

In the potential energy theorem, potential energy is minimized at each level of refinement and as the grid is refined. Strain energy

is not maximized at each level of approximation, when compared to nearby configurations, but the strain energy at equilibrium is maximized as the grid is refined in a nested way.

The pattern of convergence is sketched at the right.



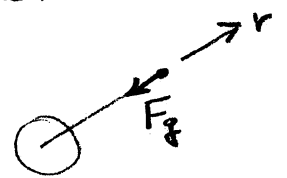
V. Existence of Work Potential

Students in engineering are often hazy about the work potential, whether in a continuum or a discrete system.

Example 1 Gravitational potential; a continuum example.

Viewed as a scalar problem, the gravitational force

$$F_g = -\frac{G m_1 m_2}{r^2}$$



where G is the universal gravitational constant and m_1 and m_2 are the masses of the two bodies involved. A potential exists, which we would call $W(r)$ such that $F = -\frac{d}{dr}(W(r))$. By integrating and discarding the constant of integration

$$W(r) = -\frac{G m_1 m_2}{r}$$

check:

$$F_g = -\frac{d}{dr}\left(-\frac{G m_1 m_2}{r}\right) = ++ - \frac{G m_1 m_2}{r^2} \quad \checkmark$$

Reference: Greenwood, "Principles of Dynamics", p 186.

Example 2 Gravity in 3 dimensions, spherical coordinates

$$F_g = -\frac{G m_1 m_2}{r^2} \vec{e}_r + (0) \vec{e}_\theta + (0) \vec{e}_\psi$$

If
$$F_g = -\vec{\nabla} W(r, \theta, \psi)$$

Then choose $W = -\frac{G m_1 m_2}{r}$ again, because

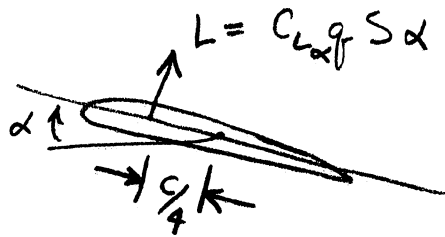
$$\vec{F}_g = - \left\{ \begin{array}{l} \vec{e}_r \frac{d}{dr} \\ \vec{e}_\theta \frac{1}{r} \frac{d}{d\theta} \\ \vec{e}_\psi \frac{1}{r \sin\psi} \frac{d}{d\psi} \end{array} \right\} \left(- \frac{G m_1 m_2}{r} \right)$$

$$= \left\{ \begin{array}{l} - \frac{G m_1 m_2}{r^2} \vec{e}_r \\ 0 \\ 0 \end{array} \right\}$$

Note that since the work potential doesn't depend on θ and ψ , no forces are generated in those coordinate directions.

Example 3 Wing in rotation (pitch)

Consider an airfoil pinned at the 50% chord and allowed to rotate only. The generalized force of interest is the moment about the pin



$$M = (C_{L\alpha} q S \alpha) \frac{c}{4} \quad (\uparrow \downarrow)$$

$C_{L\alpha}$ = slope of lift curve
 q = dynamic pressure
 S = area of wing
 α = angle of attack

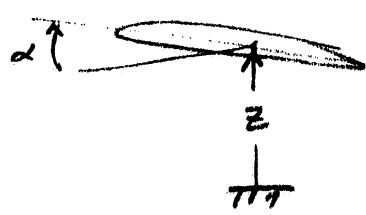
A work potential is defined

$$W(\alpha) = - \frac{1}{2} C_{L\alpha} q S \frac{c}{4} \alpha^2$$

such that $M = - \frac{dW(\alpha)}{d\alpha}$ Q.E.D.

Example 4 Pitching and Plunging Wing

For the same wing as in Ex 3, except with freedom to translate vertically:



$$M = C_{L\alpha} q S \alpha \frac{c}{4}$$

$$L = C_{L\alpha} q S \alpha$$

$$\begin{Bmatrix} L \\ M \end{Bmatrix} \stackrel{?}{=} - \begin{Bmatrix} \frac{d}{dz} \\ \frac{d}{d\alpha} \end{Bmatrix} W(z, \alpha)$$

This proves to be impossible! The force system is circulatory.



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