An Optimal Gas-Fired Heating System*

Un système optimal de chauffage au gaz

Ein optimales Gasheizungssystem

Оптимальная система газового отопления

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A realistic home heating model is derived and optimal control theory is applied to obtain an ideal heating control system against which the performance of conventional and suboptimal systems may be compared.

Summary—Utilizing a prescribed system configuration, this paper discusses the mathematical models of the system components used and formulates a method for controlling a domestic heating system in accordance to a prescribed criterion. The optimal problem treated is one of reducing the room temperature deviation from a prescribed reference value to zero, while at the same time minimizing the value of some predetermined performance or cost functional J. The development proceeds in essentially five steps.

(a) The development of the mathematical models for each of the elements of the heating system;

(b) Combining the mathematical models into a form which is suitable for the application of optimization techniques;

(c) Defining an optimization criterion which incorporates the main objective for minimizing room temperature variations with respect to a prescribed reference temperature;

(d) Choosing the optimization technique best suited for the problem;

(e) Constructing an optimal control system employing the optimization technique developed.

A numerical example compares the performance of the optimal system with a system of the conventional type which can be found in many American homes.

1. INTRODUCTION

THE STUDY of human comfort in a habitable enclosure has consumed the efforts of many individuals over the past two decades. In general, human comfort involves both physiological and psychological factors, many of which are directly related to the aggregate of characteristics that are intrinsic to human beings. Broadly speaking, the requirements are different for males than females; different for the young than for the old; etc. A comprehensive investigation of studies [9] has revealed that air temperature, air temperature gradient, air motion, humidity, radiation are the major environ-

mental factors that effect human comfort. While it may be desirable to control all the above factors, economic considerations have dictated the control of the most important single factor, namely temperature, with humidity ranking as a poor second. It is for this reason that equipment manufacturers control temperatures first and humidity second. In this paper, therefore, temperature was considered to be the one factor that was to be controlled. The immediate problem to be treated is (a) to develop mathematical models for the components of a gas-fired forced-air heating system; (b) to develop a satisfactory state-variable model for the system; (c) to apply optimal control theory techniques to the system in order to minimize temperature variations; (d) apply technique to a specific heating system.

2. MATHEMATICAL MODEL

The fixed portions of the domestic heating system include the following elements:

2.1. *Habitable space.* This is the room space [1, 9–13] in which the temperature is to be controlled. Although the air temperature of a room varies continuously from point to point throughout a room, and therefore is a function of both space and time, it has been found that under forced air operating conditions the temperature in a domestic enclosure can be approximated, for most engineering purposes, by a three region model as will be shown in a subsequent paper. Since under forced air operation, the temperature throughout the habitable region is almost constant, the above three region model in this paper is further simplified in that it is assumed to be a single temperature T_e. The thermostat is assumed to be located in the habitable space having an average space air temperature, T_R.

2.2. *Room boundary characteristics.* The exterior walls of a domestic enclosure are sections of material. The equivalent circuit of which also

possesses distributed heat transfer properties. It has been shown [9], however, that these structures can be satisfactorily modelled as one or more T-sections of an electrical transmission line, in which the wall surfaces are characterized by the outside temperature, T_0; the temperature of inside wall surface, T_w; and the room temperature, T_R. Details of using this approach are summarized in appendix.

The approximate mathematical model of the room with walls as used in this paper is characterized by the following equations:

$$\rho c_p V \dot{T}_R = (\rho c_p Q T_i - \rho c_p Q T_R) - k(T_R - T_W) \quad (1)$$

$$- CR(R + R_0)\dot{T}_R + C(R + R_0)(R_i + R)\dot{T}_W$$
$$- (2R + R_0)T_R + (2R + R_0 + R_i)T_W - R_i T_0 = 0 \quad (2)$$

where ρ = density of the air

V = volume of the room

Q = rate of flow of air

R_i, R, R_0 thermal resistances of walls

C = thermal capacitance of walls

c_p = specific heat at constant pressure.

2.3. Gas-fired forced air furnace.
Gas-fired forced-air furnaces have been studied extensively [5, 9]. The results of the American Gas Association studies in Research Bulletin 63, and Final Report

The equation which describes the dynamic temperature relation in this simplified form of a gas furnace can be expressed by

$$\dot{T}_c = a_1 T_c + a_2 T_R + a_3 T_f \quad (3)$$

$$0 = a_7 T_c + T_a + a_8 T_R \quad (4)$$

where a_1, a_2, a_3, a_7, and a_8 are constants which depend upon the physical dimensions and heat transfer characteristics for a particular furnace.

2.4. Hot-air ducts.
The equation which describes the dynamical characteristics of the hot-air duct is given by

$$T_i = \Phi_1 T_a + \Phi_2 T_R \quad (5)$$

where Φ_1 and Φ_2 depend on the physical characteristics of the duct. This relation, which is empirical in nature, is given in American Society of Heating, Refrigeration and Air Conditioning Engineers Handbook (1963).

2.5. The gas control valve.
Although the dynamical behavior of this component has been determined [9, 14], its response time is negligible compared with the time constants of the rest of the system, and therefore has not been included in this study.

3. FORMULATION OF THE SYSTEM EQUATION

Having obtained the mathematical models for each of the components shown in Fig. 1, they are combined to yield the state equation for the system.
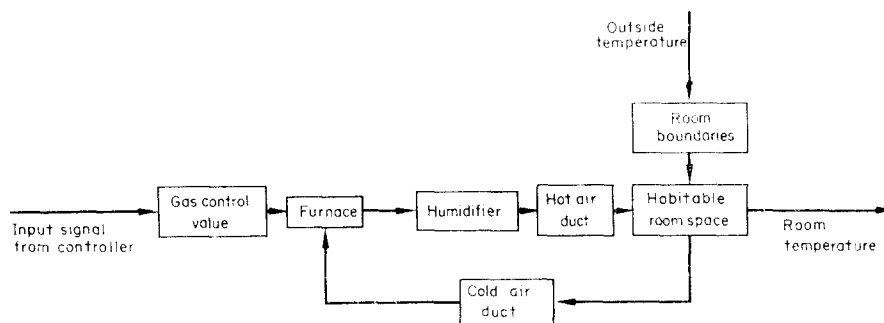

FIG. 1. Components of the heating system.

DO-14-GV of the University of Michigan, Ann Arbor, Michigan, summarized in the appendix, serve as a basis for the model used in the article. It is characterized by the simplified furnace model possessing a combustion chamber which is located inside of another chamber, defined as the heat exchanger, contains the forced-air to be heated which possesses a cold air temperature, T_a; a flame temperature, T_f; and a heat exchanger wall temperature, T_e.

This is accomplished in this case in the following manner. First eliminate T_a and T_i in eqs. (1), (2), and (3) using eqs. (4) and (5). Then, defining

$$u_3 = a_3 T_f \quad (6)$$

and

$$m_2 = \frac{R_i}{C(R_1 + R_0)} T_0 \quad (7)$$

substitute these also into eqs. (1), (2) and (3). This yields

$$\dot{T}_R = a_{11}T_R + a_{12}T_w + a_{13}T_e \qquad (8)$$

$$\dot{T}_W = a_{21}T_R + a_{22}T_W + a_{23}T_e + m_2 \qquad (9)$$

$$\dot{T}_e = a_{31}T_R + a_{33}T_e + u_3 \qquad (10)$$

where a_{ij} are combinations of

$$\Phi_1, \ \Phi_2, \ a_1, \ a_2, \ a_3, \ a_7, \ a_8, \ R_1, \ R_0, \ R, \ C, \ c_p, \ \rho, \ V$$

whose values can be estimated for any given physical gas-fired forced-air heating system that is located in a prescribed habitable enclosure.

Equations (8), (9) and (10) expressed in matrix form yield the following single vector linear differential equation:

$$\dot{x} = Ax + u + m \qquad (11)$$

where

$$x = \begin{bmatrix} T_R \\ T_W \\ T_e \end{bmatrix}, \quad u = \begin{bmatrix} 0 \\ 0 \\ u_3 \end{bmatrix},$$

$$m = \begin{bmatrix} 0 \\ m_2 \\ 0 \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & 0 & a_{33} \end{bmatrix}$$

T_R average temperature of the space to be heated.

T_W average temperature of the inside surface of the outside wall.

T_e average temperature of the heat exchanger wall.

T_f average temperature of the furnace flame.

T_0 outside atmospheric temperature.

$$u_3 = b_3 T_f, \ m_2 = d_2 T_0.$$

The components a_{ij} of the matrix A, b_3, and d_2 are parameters of the system.

4. FORMULATION OF THE PERTURBATION MODEL

In this section a perturbation model is formulated to represent the heating system as referred to some equilibrium position. First, assume that the controlled input u and the uncontrolled input m are such that the system is operating in an equilibrium condition, in other words $\dot{x} = 0$. In this case any

disturbance which occurs in the system, for example an opened door, entering people, additional lighting, etc., causes a deviation in x from its nominal or equilibrium value.

To obtain the equilibrium values, using eq. (11), set $\dot{x} = 0$. This yields

$$Ax_0 + u_0 + m = 0, \qquad (12)$$

where the zero subscript refers to the equilibrium vectors. In order to maintain a desirable room temperature which is a component of the vector x, it is evident that u, the controllable input to the system in the equilibrium state, takes on some value u_0. To determine the value of u_0 required, consider eq. (12):

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & 0 & a_{33} \end{bmatrix} \begin{bmatrix} T_{R0} \\ T_{w0} \\ T_{e0} \end{bmatrix}$$

$$= - \begin{bmatrix} 0 \\ 0 \\ u_{30} \end{bmatrix} - \begin{bmatrix} 0 \\ m_2 \\ 0 \end{bmatrix}.$$

By appropriate manipulations this equation becomes:

$$\begin{bmatrix} a_{12} & a_{13} & 0 \\ a_{22} & a_{23} & 0 \\ 0 & a_{33} & 1 \end{bmatrix} \begin{bmatrix} T_{w0} \\ T_{e0} \\ u_{30} \end{bmatrix}$$

$$= - \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} T_{R0} - \begin{bmatrix} 0 \\ m_2 \\ 0 \end{bmatrix}. \qquad (13)$$

Inspection of eq. (13) reveals that given the desired room temperature T_{R0} and the outside temperature expressed by m, the equilibrium values of the controllable variable T_{f0}, and the various state temperatures are specified. Let $x = x_0 + \delta x$, $\dot{x} = \dot{x}_0 + \delta\dot{x}$ and $u = u_0 + \delta u$ where δx, δu and $\delta\dot{x}$ represent deviations from the nominal values of x_0, u_0, and $\dot{x}_0$ respectively, then

$$\dot{y} = Ay + v \qquad (14)$$

where $\delta x = y$, $\delta u = v$.

This latter equation is the conventional well-known linear first order matrix differential equation. The components of the vectors in this equation represent variations in the state of the system. It is to be noted that in this case this new perturbation model (14) is valid for large swings from equilibrium since the model of the original dynamic system is linear.

5. THE OPTIMIZATION CRITERION

The main objective in the optimization of a gas-fired forced-air heating system is to reduce room temperature variations due to disturbances, and is primarily used here to define the optimization criterion. The square penalizing will discriminate heavily against occasional large room temperature variations. This philosophy is justified as long as the type of control used does not have any significant physical limitations. In a gas-fired heating system, for example, physical limitations are imposed by the size of the heat exchanger, which is a power limitation. Therefore, in order to consider power limitations, a term in the square error criterion is added that is proportional to the square of the control signal. Having these two factors in mind, the optimization criterion for the forced-air heating system can be represented as follows:

$$J(\mathbf{y}, \mathbf{v}) = \sum_{n=1}^{3} \int_{0}^{T} [q_n^2(\sigma) + v_n^2(\sigma)] \mathrm{d}\sigma \qquad (15)$$

where $J(\mathbf{y}, \mathbf{v})$ is the error criterion to be minimized σ is a dummy time variable, T is the period over which the minimization takes place, $q(\sigma)$ and $v(\sigma)$ are defined as:

$$\mathbf{q}(\sigma) = \begin{bmatrix} q_1(\sigma) \\ q_2(\sigma) \\ q_3(\sigma) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1(\sigma) \\ y_2(\sigma) \\ y_3(\sigma) \end{bmatrix} \; ;$$

$$\mathbf{v}(\sigma) = \begin{bmatrix} v_1(\sigma) \\ v_2(\sigma) \\ v_3(\sigma) \end{bmatrix} . \qquad (16)$$

The first term in the integrand of the quadratic criterion (15) represents the penality on the room temperature variations, and the second term is introduced for power limitation. The weighting assigned to $q^2(\sigma)$ and $v^2(\sigma)$ clearly depends on the importance of temperature vs. fuel costs.

Most American home owners would be willing to pay a modest increase in cost to get the desired

comfort. Since no general criterion could be found, for the example given, comfort and power costs were weighted equally the same. For a known set of conditions these weightings could be changed accordingly.

6. THE OPTIMAL CONTROL LAW

The optimization problem at hand is one of starting from some initial temperature disturbance $\mathbf{y}_0$, and driving the system $\dot{\mathbf{y}} = A\mathbf{y} + \mathbf{v}$ to the equilibrium state while constraining the original system to perform in such a way as to minimize the value of the cost functional $J(\mathbf{y}, \mathbf{v})$. Here the period of optimization is allowed to be very large (i.e. $T \to \infty$), since the heating system has to be optimized over a long period of time.

The method of dynamic programming applied to this linear time invariant heating system is guaranteed to provide a closed loop or feedback control law [6, 7] for a given set of heating system parameters, which satisfies the optimization criterion defined in section 5. It does not pose any difficulties such as instability of the resulting equations which could result by applying the calculus of variations to a system to be optimized over a semi-infinite interval (as $T \to \infty$) [6]. For these reasons, the method of dynamic programming is thought to be the most suitable method for the optimization of the heating system under the optimization criterion represented by (15). Bellman's Dynamic programming is basically an optimization process that proceeds backward in time; that is, the solution is computed over the last interval of the process and successive solutions are computed for the remaining intervals of decreasing time until the total solution is obtained for the entire process.

In order to apply the functional equation technique of dynamic programming, this optimization problem is embedded within the wider problem of minimizing:

$$\sum_{n=1}^{3} \int_{t}^{T} [q_n^2(\sigma) + v_n^2(\sigma)] \mathrm{d}\sigma$$

subject to the heating system eq. (14) and the initial condition $\mathbf{y}(0) = \mathbf{y}_0$, with t ranging over the interval $(0, T)$. Let the minimum of this cost functional be:

$$E(y, t) = \min_{v} \sum_{n=1}^{3} \int_{t}^{T} [q_n^2(\sigma) + v_n^2(\sigma)] \mathrm{d}\sigma . \qquad (17)$$

Invoking the principle of optimality to eq. (17) the functional equation becomes:

$$E(y, t) = \min_v \left\{ \sum_{n=1}^{3} \int_{t}^{t+\varepsilon} [q_n^2(\sigma) = v_n^2(\sigma)] d\sigma \right.$$

$$\left. + E(y + \dot{y}\varepsilon, t+\varepsilon) \right\} \qquad (18)$$

where ε is an incremental change in the time t. This equation is reduced, by integration and Taylor series expansion, to the following expression:

$$E(y, t) = \min_v \left\{ \sum_{n=1}^{3} [q_n^2(t) + v_n^2(t)]\varepsilon + E(y, t) \right.$$

$$\left. + \sum_{n=1}^{3} \dot{y}_n \frac{\partial E}{\partial y_n}\varepsilon + \frac{\partial E}{\partial t}\varepsilon \right\} + \Delta(\varepsilon).$$

Simplifying

$$\min_v \left\{ \sum_{n=1}^{3} [q_n^2(t) + v_n^2(t)] \right.$$

$$\left. + \sum_{n=1}^{3} \dot{y}_n \frac{\partial E}{\partial y_n} + \frac{\partial E}{\partial t} \right\} + \Delta'(\varepsilon) = 0.$$

Now since $\Delta(\varepsilon) \to 0$ as $\varepsilon \to 0$, gives

$$\min_v \left\{ \sum_{n=1}^{3} [q_n^2(t) + v_n^2(t)] \right.$$

$$\left. + \sum_{n=1}^{3} \dot{y}_n \frac{\partial E}{\partial y_n} + \frac{\partial E}{\partial t} \right\} = 0. \qquad (19)$$

The minimizing control signal vector $v^*(\sigma)$ is obtained by minimizing the sum of terms within the brackets of eq. (19) with respect to each signal of the control vector. Minimizing now with respect to $v_3(\sigma)$, keeping in mind the relation between the vectors $\mathbf{q}$ and $\mathbf{y}$, the only non zero component of the vector $v(\sigma)$, ... is therefore: $2v_3^* + (\partial E/\partial y_3)$ $= 0$, where $v_3^* =$ optimum control signal. Consequently, the condition for minimum error is:

$$v_3^* = -\tfrac{1}{2} \frac{\partial E}{\partial y_3}. \qquad (20)$$

In order to determine the optimum signal v_3^*,

$$\frac{\partial E}{\partial y_3}$$

for minimum error must be determined first. Substituting eq. (10) and the value of $\mathbf{q}$ in terms of $\mathbf{y}$ into the functional eq. (19), the condition for minimum error becomes:

$$y_1^2 + \tfrac{1}{4}\left(\frac{\partial E}{\partial y_3}\right)^2 + \sum_{n=1}^{3} \dot{y}_n \frac{\partial E}{\partial y_n} + \frac{\partial E}{\partial t} = 0. \qquad (21)$$

As seen from eq. (21), the condition for minimum error is in a partial differential form. To solve such an equation a power series solution is assumed, and the coefficients in the series are found by direct substitution.

Since the integrand of the error criterion function is a quadratic expression and the dynamic system is linear, the minimum error function $E(y, t)$ is also quadratic and can be written as:

$$E(y, t) = k(t) + \sum_{m=1}^{3} k_m(t)y_m(t)$$

$$+ \sum_{m=1}^{3} \sum_{k=1}^{3} k_{mk}(t)y_m(t)y_k(t) \qquad (22)$$

where $k_{mn}(t) = k_{nm}(t)$, and where $k(t)$, $k_m(t)$, $k_{mn}(t)$ are the parameters to be determined from eqs. (21) and (22). By partial differentiation of eq. (22), $[(\partial E(y, t)]/(\partial y_n)$ and $[(\partial E(y, t)]/(\partial t)$ are written as follows:

$$\frac{\partial E(y, t)}{\partial y_n} = k_n(t) + 2\sum_{m=1}^{3} k_{mn}(t)y_m(t) \qquad (23)$$

and

$$\frac{\partial E(y, t)}{\partial t} = k'(t) + \sum_{m=1}^{3} k'_m(t)y_m(t)$$

$$+ \sum_{m=1}^{3} \sum_{k=1}^{3} k'_{mk}(t)y_m(t)y_k(t). \qquad (24)$$

If these partial derivatives are substituted into eq. (21) the condition for minimum error becomes:

$$y_1^2 + \tfrac{1}{4}\left[k_3 + 2\sum_{m=1}^{3} k_{nm}y_m\right]^2 + k' + \sum_{m=1}^{3} k'_m y_m$$

$$+ \sum_{m=1}^{3} \sum_{k=1}^{3} k'_{mk}y_m y_k + \sum_{n=1}^{3} \left[k_n \dot{y}_n\right.$$

$$\left. + 2\dot{y}_n \sum_{m=1}^{3} k_{nm}y_m\right] = 0. \qquad (25)$$

The condition for minimum error expressed by (25) is satisfied for all finite values of $y_n(t)$, assuming the k-parameters are independent of $y_n(t)$, only if each of the coefficients of the constant term, $y_n(t)$, and $y_n(t)y_m(t)$ in eq. (25) vanishes, where $n, m = 1, 2, 3$. Therefore by equating the coefficients of the constant term, y_n and $y_n y_m$ each equal to zero, the following simultaneous first order differential equations in the k-parameters result.

$$f_1(k, k_1, k_2, k_3, k_{11}, k_{22}, k_{33}, k_{12}, k_{13}, k_{23}) = k'$$

$$f_2(k, k_1, k_2, k_3, k_{11}, k_{22}, k_{33}, k_{12}, k_{13}, k_{23}) = k'_1$$

$$f_3(k, k_1, k_2, k_3, k_{11}, k_{22}, k_{33}, k_{12}, k_{13}, k_{23}) = k'_2$$

.

.

.   (26)

.

$$f_{10}(k, k_1, k_2, k_3, k_{11}, k_{22}, k_{33}, k_{12}, k_{13}, k_{23}) = k'_{23}$$

where: $f_1, f_2, \ldots, f_{10}$ are in general non-linear functions of the k-parameters, and the primed k's refer to the derivatives of the k-parameters with respect to time.

This method of assuming a solution leads to the reduction of the problem of solving a partial differential equation to the problem of solving a set of first order ordinary differential equations. The boundary condition for the k-parameters are deduced directly from the required boundary condition on the minimum error function. From the expression for minimum error function for $t = T$, the boundary condition is

$$E[y(T), T] = 0 \text{ which means that } k(T) = k_n(T)$$

$$= k_{nm}(T) = 0. \tag{27}$$

The problem becomes now one of finding the optimum control system of a one-point boundary value problem. The parameters of the optimum control system, $k(t)$, $k_{mn}(t)$ where m, $n = 1$, 2, 3 can be determined from the set of ten differential eqs. (26) with boundary conditions given by (27). It is to be noted that the number of parameters are ten and the number of initial conditions expressed by (27) are ten.

The solution of the set of differential eqs (26) as T tends to ∞, must assume steady state. If the k-parameters assume steady state values, then the differential equations given by (26) reduces to a set of algebraic equations. Therefore, when the dynamic system is time invariant, the error function quadratic, and the optimization process is carried over a semi-infinite time interval, the parameters of the optimal control law become time-invariant.

Since the heating system is to be optimized over a semi-infinite time interval for a quadratic optimization criterion, eqs. (17) through (24) become

$$E(y) = \min_{v} \sum_{n=1}^{3} \int_{t}^{\infty} [q_n^2(\sigma) + v_n^2(\sigma)] d\sigma \tag{17'}$$

$$E(y) = \min_{v} \left\{ \sum_{t}^{3} \int_{t}^{t+\varepsilon} [q_n^2(\sigma) + v_n^2(\sigma)] d\sigma + E(v + v\varepsilon) \right\} \tag{18'}$$

$$\min_{v} \left\{ \sum_{n=1}^{3} [q_n^2(\sigma) + v_n^2(\sigma)] + \sum_{n=1}^{3} \dot{y}_n \frac{\partial E}{\partial y_n} \right\} = 0 \tag{19'}$$

$$v_3^* = -\frac{1}{2} \frac{\partial E}{\partial y_3} \tag{20'}$$

$$y_1^2 + \frac{1}{4} \left(\frac{\partial E}{\partial y_3} \right)^2 + \sum_{n=1}^{3} \dot{y}_n \frac{\partial E}{\partial y_n} = 0 \tag{21'}$$

$$E(y) = k + \sum_{m=1}^{3} k_m y_m(t) + \sum_{m=1}^{3} \sum_{k=1}^{3} k_{mk} y_m(t) y_k(t) \tag{22'}$$

where k, k_m, and k_{mn} where m, $n = 1$, 2, 3 are fixed constants.

$$\frac{\partial E}{\partial y_n} = k_n + 2 \sum_{m=1}^{3} k_{nm} y_m(t) \tag{23'}$$

$$\frac{\partial E}{\partial t} = 0 \tag{24'}$$

also $$\frac{\partial E}{\partial y_3} = k_3 + 2[k_{31} y_1 + k_{32} y_2 + k_{33} y_3]. \tag{28}$$

By substituting $(\partial E)/(\partial y_3)$ from (28) into (20') gives:

$$v_3^* = -\frac{k_3}{2} - k_{31} y_1 - k_{32} y_2 - k_{33} y_3. \tag{29}$$

Therefore it is necessary to determine the parameters k_3, k_{31}, k_{32}, and k_{33} to determine the optimum control signal.

Substituting now from eqs. (23') and (28) into the condition for minimum error (21'), and also using the vector matrix differential equation $\dot{y} = Ay + v$, the following is obtained:

$$y_1^2 - \frac{1}{4} \left[k_3^2 + 4k_3 \sum_{m=1}^{3} k_{m3} y_m + 4 \left(\sum_{m=1}^{3} k_{m3} y_m \right)^2 \right]$$

$$+ \sum_{n=1}^{3} \left(\sum_{m=1}^{3} a_{mn} y_m \right) \left[k_n + 2 \sum_{s=1}^{3} k_{sn} y_n \right] = 0. \tag{30}$$

Since eq. (30) is satisfied for all values of $y_n(t)$, by equating the constant term in this equation to zero, the following is obtained:

$$k_3 = 0. \tag{31}$$

Similarly for the coefficient of y_m:

$$-k_3k_{m1}+\sum_{n=1}^{3}a_{mn}k_n=0 \quad (m=1,2,3)$$

and since this is true for all finite values of a_{mn}, therefore $k_1=k_2=k_3=0$.

For the coefficient of y_1^2:

$$1-k_{13}^2+4\sum_{n=1}^{3}a_{1n}k_n=0. \tag{32}$$

For the coefficient of y_2^2:

$$-k_{23}^2+4\sum_{n=1}^{3}a_{2n}k_{2n}=0. \tag{33}$$

Equation (29) now becomes:

$$v_3^{*}=-[k_{31}y_1+k_{32}y_2+k_{33}y_3]. \tag{38}$$

The control function v_3^{*}, derived here is referred to as the optimum control law.

The optimal control scheme for the variational system may be combined with the equilibrium system developed in section 3 to obtain an optimal feedback system for the heating process. In block diagram form the system may be schematically represented as shown in Fig. 2. In this diagram a controller is provided which compares the values of the environmental state and the desired state and commands the appropriate equilibrium input.
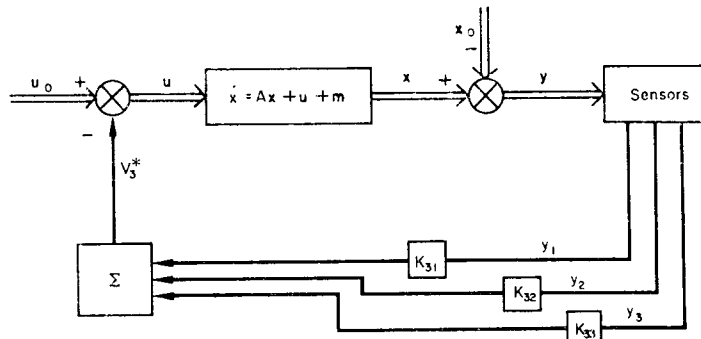

FIG. 2. Block diagram of the optimum heating system.

For the coefficient of y_2^3:

$$-k_{33}^2+4\sum_{n=1}^{3}a_{3n}k_{3n}=0. \tag{34}$$

For the coefficient of y_1y_2:

$$-k_{13}k_{23}+\sum_{n=1}^{3}(a_{1n}k_{2n}+a_{2n}k_{1n})=0. \tag{35}$$

For the coefficient of y_1y_3:

$$-k_{12}k_{33}+\sum_{n=1}^{3}(a_{1n}k_{3n}+a_{3n}k_{1n})=0. \tag{36}$$

For the coefficient of y_2y_3:

$$-k_{23}k_{33}+\sum_{n=1}^{3}(a_{2n}k_{3n}+a_{3n}k_{2n})=0. \tag{37}$$

Equations (32) to (37) are in general non-linear algebraic equations in the parameters k_{mn}, m, $n=1$, 2, 3 and require a digital computer for solution. In the next section a solution of these parameters for a particular heating system on the digital computer will be shown.

It should be noted that the number of feedback loops is equal to the order of the heating system; it is noted also that the feedback signals are measurable state variables.

Since the system was optimized around x_0, the optimum control exists when x_0 remains constant. This, of course, is not in general the case. Thus for values of x_0 not equal to the one chosen only approximate optimization is obtained.

7. EXAMPLE

In this section the optimal control scheme developed above will be applied to a particular heating system. The domestic enclosure used in this study was a $12\times12\times9\cdot7$ ft model room developed for studies of this type. It contained laminated wall construction of $\frac{3}{8}$ in. exterior plywood, 1 in. fiber glass, $0\cdot025$ in. sheet of aluminum, $1\frac{1}{4}$ in. air-space, $0\cdot032$ in. sheet of aluminum, and $\frac{1}{8}$ in. tempered masonite. The construction details are summarized in [1]. The room was instrumented for measuring the physical variables of the system that have an important influence on the thermal behavior, such as inside and outside temperatures, inside and outside air velocities, and quantity of heat supplies, etc. From these measurements the thermal resistances, conductivities, and equivalent capacity of

the walls, and enclosure were determined. These constants along with constants of a model heat exchanger determined in a previous study [9], were used in determining the value of the A matrix, and u as given in eq. (39).

$$A = \begin{bmatrix} -0\cdot191 & 0\cdot0422 & 0\cdot097 \\ 0\cdot2278 & -0\cdot0974 & -0\cdot097 \\ 0\cdot25 & 0 & -0\cdot489 \end{bmatrix},$$

$$\mathbf{u} = \begin{bmatrix} 0 \\ 0 \\ 0\cdot239 \end{bmatrix} T_f, \qquad \mathbf{m} = \begin{bmatrix} 0 \\ 0\cdot0184 \\ 0 \end{bmatrix} T_0 . (39)$$

The University of Michigan Control System Algorithm Program employing a 7090 digital computer was used to solve for the k-parameters. This program was basically obtained from IBM, with some modifications added. The modified program is entitled CSAP and is currently available at the University of Michigan Center Library. This program appears as a subroutine on the system disc and may be entered simply by calling CSAP. Once the program has been called, it will function exactly as described in the user's manual. The solution for this particular system is:

$k_{11} = 1\cdot4449, k_{22} = 1\cdot3273, k_{33} = 0.2065, k_{12} = 0\cdot805$ $k_{31} = 0\cdot4467, k_{32} = 6\cdot36$. The optimal control signal becomes: $v_3^* = -(0\cdot4467y_1 + 0\cdot36y_2 + 0\cdot2065y_3)$, or $0\cdot239\delta T_f = -(0\cdot4467\delta T_R + 0\cdot36\delta T_w + 0\cdot2065\delta T_e)$. Hence, the block diagram of the optimum heating system using this control law follows as shown in Fig. 3. It is to be noted that δT_f, δT_R, δT_w, and
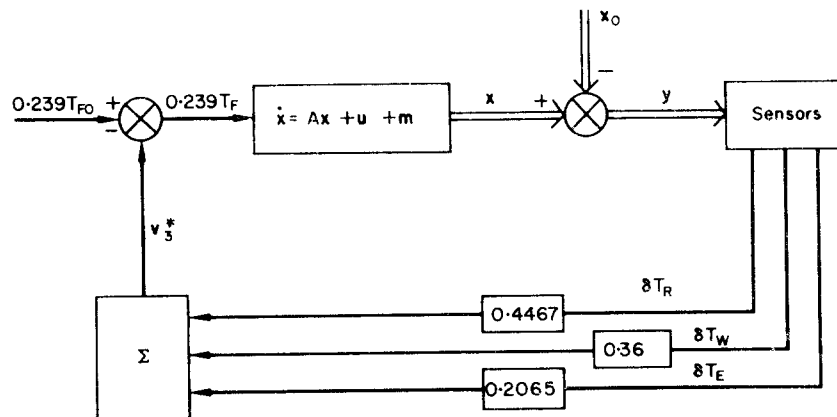

FIG. 3. Block diagram of the optimum heating system.

The variational vector matrix differential equation as derived in section 3 now becomes $\dot{\mathbf{y}} = A\mathbf{y} + \mathbf{v}$ where $v^T = (v_1, v_2, v_3) = (0, 0, 0\cdot239\delta T_f)$ and $\mathbf{y} = (y_1, y_2, y_3) = (\delta T_R, \delta T_W, \delta T_e)$ where $y_1 = \delta T_R$, $y_2 = \delta T_W$, $y_3 = \delta T_e$.

The square matrix A determines the system under consideration, and therefore the k parameters of the system as defined by eqs. (32) to (37) may be written as follows:

$$1 - k_{31}^2 + 0\cdot388k_{31} - 0\cdot764k_{11} + 0\cdot1688k_{12} = 0$$

$$-k_{32}^2 - 0\cdot388k_{32} + 1\cdot112k_{12} - 0\cdot3896k_{22} = 0$$

$$-k_{33}^2 - 1\cdot956k_{33} + k_{31} = 0$$

$$k_{31}k_{32} - 0\cdot2884k_{12} + 0\cdot0422k_{22} + 0\cdot097k_{32} + 0\cdot2278k_{11} - 0\cdot097k_{31} = 0$$

$$-k_{31}k_{33} - 0\cdot68k_{31} + 0\cdot0422k_{32} + 0\cdot097k_{33} + 0\cdot25k_{11} = 0$$

$$-k_{32}k_{33} + 0\cdot2278k_{31} - 0\cdot5864k_{32} - 0\cdot097k_{33} + 0\cdot25k_{12} = 0.$$

δT_e are the variations of temperatures from equilibrium values, and are defined as follows:

$$\delta T_f = T_f - T_{f_0}, \quad \delta T_R = T_R - T_{R_0},$$

$$\delta T_W = T_W - T_{W_0}, \quad \delta T_e = T_e - T_{e_0}.$$

Therefore, to generate v_3^*, the variational signal, it is necessary to first generate the equilibrium values of the temperatures, T_{f_0}, T_{R_0}, T_{W_0}, and T_{e_0}.

For equilibrium conditions: $\dot{T}_R = \dot{T}_W = \dot{T}_e = 0$, then: $T_{f_0} = -1\cdot045T_{R_0} + 2\cdot05T_{e_0}$, $T_{e_0} - 0\cdot43T_{W_0} + 1\cdot97T_{R_0}$, $T_{W_0} = 2\cdot37T_{R_0} - T_{e_0} + 0\cdot19T_0$.

From these latter equations it follows that having T_{R_0}, and T_0 as set inputs, the equilibrium values T_{f_0}, T_{e_0}, and T_{W_0} may be generated.

Having established the equilibrium values, they may be now combined with the fixed portion of the heating system and the optimum controller to provide the optimum control system.

In order to study the performance of this system, a simulation study was carried out on an analog computer. The results for two sets of initial conditions have been included for comparison.

1. *First case.* The room temperature T_R is set at 70°F, and the outside temperature is initially set at 20°F. The system is therefore initially in the equilibrium state of:

$$x_0^T = (70, 52 \cdot 9, 115)°F, \qquad T_{f_0} = 163 \cdot 8°F.$$

The outside temperature T_0 is then suddenly changed from 20°F to 0°F. For these conditions the room temperature T_R, the surface wall temperature T_W, the heat exchanger wall temperature T_e, and the control signal temperature T_f were recorded. These temperature responses are shown in Figs. 4, 5, 6, and 7 respectively.
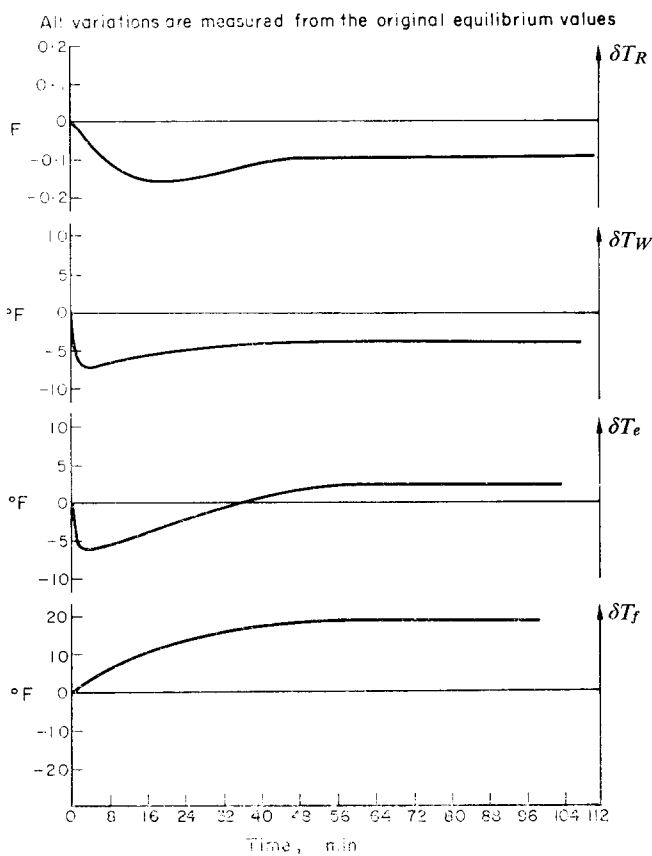

Fig. 4. Room temperature variation response for case 1.
Fig. 5. Surface wall temperature variation response for case 1.
Fig. 6. Heat exchanger temperature variation for case 1.
Fig. 7. Flame temperature variation response for case 1.

From Fig. 4, it is evident that the room temperature T_R decreases gradually from the time the disturbance occurs until the time when the variation δT_R becomes $-0 \cdot 15°F$; a total of 16 min. After this, it begins to increase at a slower rate back toward its original value. In 40 min, the room temperature attains the value of 69·9°F. This is expected, since the optimization criterion was considered over a semi-definite time interval. The optimum control signal T_f as seen in Fig. 7 increases gradually from the time of the drop in the

outside temperature. This effect occurs to compensate for the heat loss caused by sudden disturbance. In Fig. 5, it is noted that the surface temperature of the wall initially falls rapidly to 45·37°F, then it gradually begins increasing until it reaches 49°F. Figure 6 indicates the effect of the disturbance on the heat exchanger temperature T_e. This temperature initially drops to about 109°F because of both the decrease in room temperature and the decrease in surface wall temperature. It then begins to gradually increase until it reaches within 2·4°F of its original value. This is caused by the increase in the flame temperature [5].

2. *Second case.* For this case the room temperature is set at 70°F and the outside temperature is initially at 0°F. The equilibrium values are: $x_0^T = (70, 49, 117)°F$, $T_{f_0} = 182 \cdot 8 \; F$. The outside temperature then rises suddenly to 40°F. Figures 8 to 11 show the state variables and control signal responses to this disturbance. The room temperature response is shown in Fig. 8. This temperature increases gradually to 70·15°F then falls to 70·05°F.
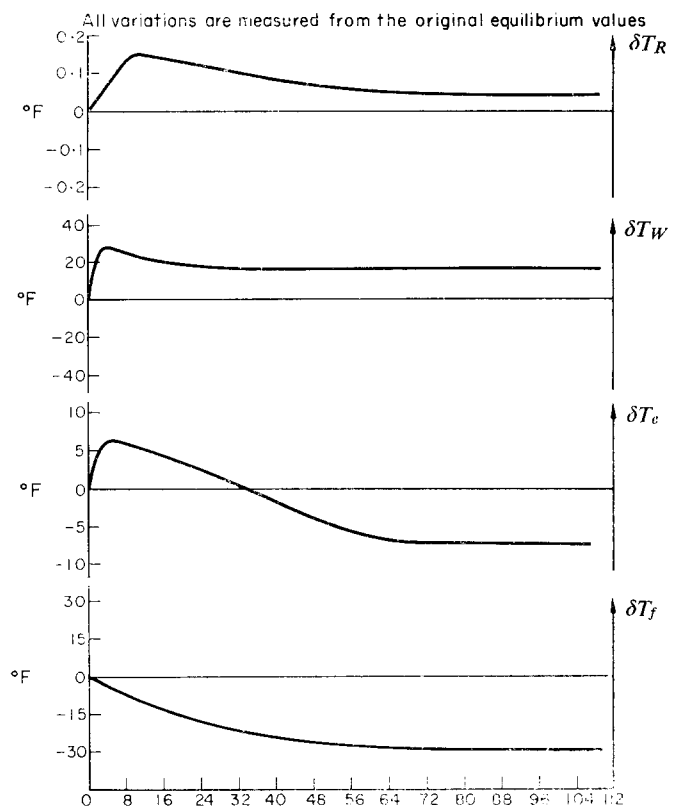

Fig. 8. Room temperature variation response for case 2.
Fig. 9. Surface wall temperature variation response for case 2.
Fig. 10. Heat exchanger temperature variation response for case 2.
Fig. 11. Flame temperature variation response for case 2.

The wall surface temperature shown in Fig. 9 rises as a result of the disturbance and then it decreases

until it reaches 63°F. The heat exchanger temperature also rises by 6°F and then will decrease gradually to 110°F. This result is illustrated in Fig. 10. Figure 11 shows the optimal control signal. It is apparent that the flame temperature changes gradually to 153·5°F, which implies that the disturbance caused by an outside temperature rise from 0°F to 40°F, decreases the flame temperature by 29·3°F to keep the room temperature to within 0·1°F of 70°F.

Comparison. If the conventional heating system is to be compared with the optimal system, the basis of comparison must be the defined performance criterion. It is true by definition that the optimal system developed is the best with respect to this criterion; however, interesting points can be made by analyzing the systems in general.

For a conventional heating system the main properties are: (a) An on-off controller is used. (b) Only the average room temperature T_R is controlled. For analysis purposes [1], the heat output of the furnace is adjusted so that the temperature of the air circulating in the heating system is 120°F when it is leaving the furnace during the on-period. During the off-period the temperature of the air is considered to be 70°F. The outside temperature T_0 is held fixed at 20°F, and then allowed to drop suddenly to zero. Computer runs were made for the conventional heating system for different values of thermostat (controller) time constant τ_T min, furnace time constant τ_F, and hysteresis $q°F$. The peak to peak room temperature variation is measured and is called the cycling amplitude. Also the time for one complete cycle of the room

heating system, when compared to the maximum deviation of the optimal system.

(ii) For an outside temperature disturbance the response and adjustment of the optimum heating system is superior to the corresponding response of the conventional heating system. In the optimum system, the temperature begins to fall gradually, due to an outside temperature drop, until it deviates to $-0·15°F$. Then within about 5 min it tends to remain within $0·1°F$ or less from the original value. The conventional heating system, on the other hand, begins to oscillate. The rates of increase and decrease in the room temperature depend on the thermostat time constant, and thermostat hysteresis. They also depend on the nature of the disturbance. This is shown in Figs. 12, 13, 14, and 15.

(iii) The room temperature is continuously changing in a conventional heating system in a periodic manner. Since this peak-to-peak variation is more than $0·1°F$, it is sensed by the human body as being uncomfortable.
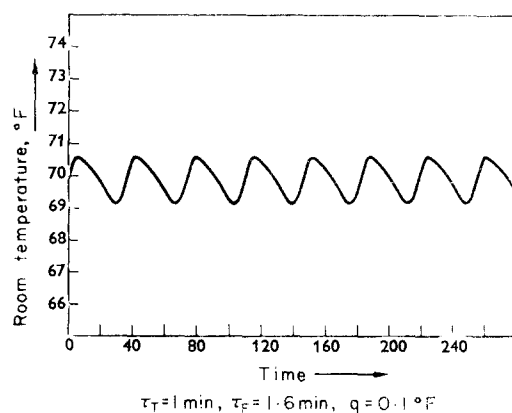

Fig. 12. A sample of recordings.

$\tau_T = 1$ min, $\tau_F = 1·6$ min, $q = 0·1 °F$

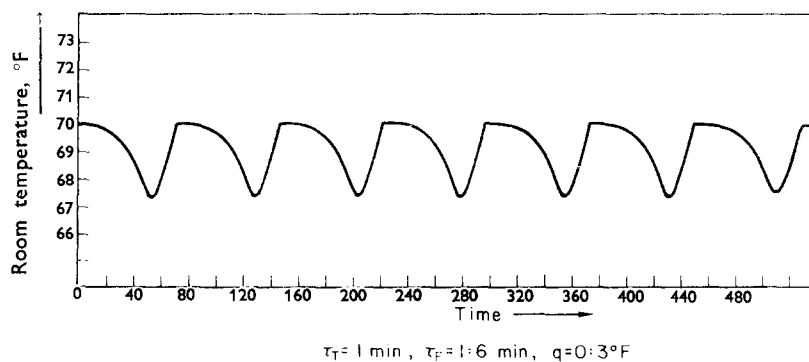$\tau_T = 1$ min, $\tau_F = 1·6$ min, $q = 0·3°F$

Fig. 13. A sample of recordings.

temperature is recorded, and is called the cycling period.

If the conventional heating system is analyzed and compared to the optimum heating system it is found that:

(i) The peak to peak variations of the room temperature are much greater for the conventional

8. CONCLUSION

An optimal heating system for a defined integral quadratic cost function has been developed which incorporates the main objective of minimizing room temperature variations. The optimal control was shown to have the desirable property of providing additional feedback loops to account for disturbances in the system. The feedback portions of the
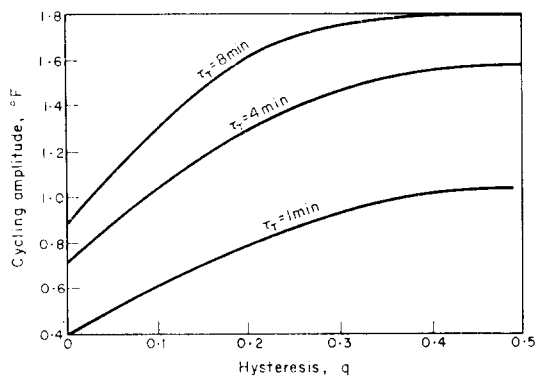
FIG. 14. Conventional system's cycling amplitude–hysteresis curves with τ_T as a parameter.
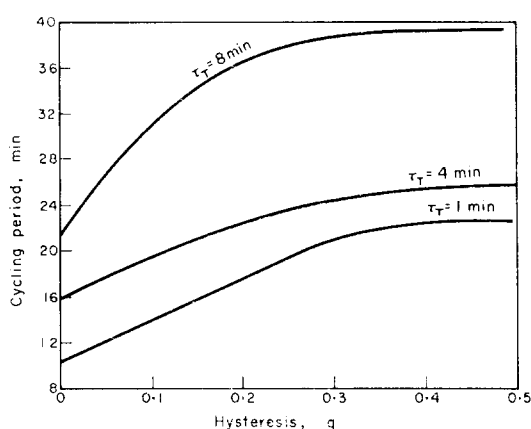

FIG. 15. Conventional system's cycling-period–hysteresis curves with τ_T as a parameter.

optimum control heating system were also shown to be time-invariant, a characteristic which is advantageous in practice. Parameters of the optimum controller were determined through the use of the Control System Algorithm Program (CSAP) on the 7090 digital computer at the University of Michigan. The optimum heating system represents an optimum from the theoretical point of view for the configuration and cost function selected. Therefore it represents an upper bound or standard with which conventional or sub-optimal systems may be compared. However, for some specialized installations possessing rigid performance standards, it may be feasible to utilize a system such as the optimal.

A sub-optimal system embodying the above features has been built. The findings in this study have been reported in another paper [15].

APPENDIX A

Heating system mathematical model

Room model. Let us consider a room with height H, width W, and length L. Let the room have three inside walls and one outside wall. An outside wall is a wall which possesses an exterior exposure to the climatic elements. Let the rest of the house be at a temperature range which is equal to the temperature range of this room. This means that there will be no heat transfer through the three inside walls or the floor or the ceiling. Heat will transfer to the outside only through the outside wall. Let the air inlet to the room be a rectangular opening and let the jet discharge parallel to the inside wall with one edge of the outlet coinciding with the inside wall. To simplify the analysis of the system, no windows or doors are assumed. It can be shown, however, that this does not change the concepts obtained from analysis, since this is essentially equivalent to changing the system parameters: its heating conductance and capacitance.

From the engineering point of view, the room space can be assumed to be at a uniform temperature [1]. In constructing the wall model, the thermal circuit concept is used. Assuming that heat flows only in one direction through the wall, the wall behaves as a distributed parameter RC transmission line.

Consider now Fig. A.1, which represents the thermal circuit of a one T network wall. The outside end of the wall which is exposed to the sun and wind is equivalent to a known temperature source T_0, the effective air temperature, and it is a function of time. R_0 represents the resistance whose conductance represents the heat flow between the outside surface of the wall and the surrounding atmosphere. It is a function of the surface air coefficient. The inside of the wall is facing the room heat capacity. R_i is the resistance whose conductance represents heat transfer between the air in the domestic space and the inside surface of the wall. It is also a function of the air surface coefficient. R and C are the equivalent resistance to heat transfer and heat capacity of the wall respectively. They are functions of the properties of the material of the wall.
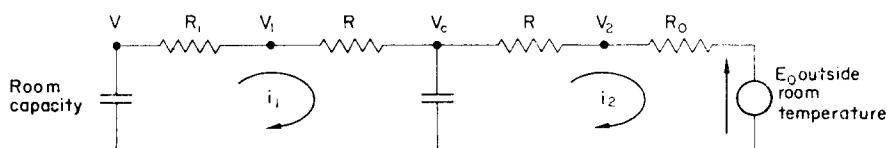

FIG. A.1. The wall thermal circuit.

Wall boundaries. This wall system has two degrees of freedom. Usually i_1, i_2, as defined in the circuit of Fig. A.1, are chosen as the state variables for such circuit. However, since i_1 and i_2 correspond to rate of heat flowing in and out of the wall and are not easily measurable in practice, another set of measurable state variables should be chosen.

Let V be the temperature of the air inside the room: V_1 be the temperature of the inside surface of the wall; V_2 be the temperature of the outside surface of the wall. It can readily be shown that V and V_1 constitute one set of state variable for this system.

Solving the circuit with V and V_1 as state variables yields

$$-CR(R+R_0)\frac{\mathrm{d}V}{\mathrm{d}t}+C(R_i+R)(R+R_0)\frac{\mathrm{d}V_1}{\mathrm{d}t}$$

$$-(2R+R_0)V+(2R+R_0+R_i)V_i-R_iE_0=0.$$

$$\text{(A.1)}$$

In this case T_R corresponds to V and T_w corresponds to V_1.

Applying now the first law of thermodynamics to the domestic air in the room, and rewriting eq. (A.1) we obtain:

$$\rho c_p V \dot{T}_R=(\rho c_p Q T_i-\rho c_p Q T_R)-k(T_R-T_w) \text{ (A.2)}$$

$$-CR(R+R_0)\dot{T}_R+C(R_i+R)(R+R_0)\dot{T}_w$$

$$-(2R+R_0)T_R+(2R+R_0+R_i)T_w-R_iT_0=0$$

$$\text{(A.3)}$$

where

V volume of the room

Q rate of flow of air

k average proportionality constant defining the heat transfer by convection to the inside surface of the outside wall

C equivalent heat capacity of the outside wall

R equivalent resistance of the outside wall

R_0 equivalent outside air to surface resistance of the outside wall

R_i equivalent inside air to surface resistance of the outside wall

ρ density of air flowing

c_p specific heat of air flowing

Heat exchanger model

There are different kinds of gas furnace in practice. A typical gas furnace that is most commonly used would be the vertical tube combustion chamber, and Fig. (A.2) shows a heat transfer schematic for this type of furnace.
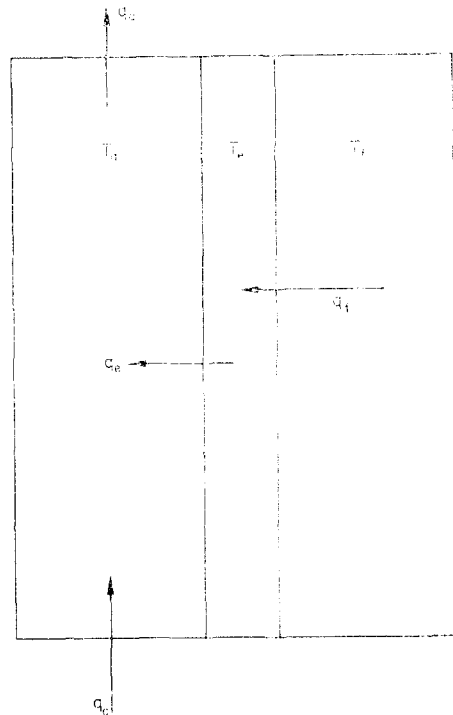

FIG. A.2. Heat transfer in the furnace.

Considering now the thermodynamic control volumes to be the material of the heat exchanger as one and the air circulating around the exchanger as another, the following set of equations can be written for the system with the help of the first law of thermodynamics:

$$q_f(t)=h_f\pi D_e Y[T_f(t)-T_e(t)] \qquad \text{(A.4)}$$

$$q_e(t)=h_a\pi D_e Y[T_e(t)-T_a(t)] \qquad \text{(A.5)}$$

$$q_c(t)=\rho c_p Q_0 T_R(t) \qquad \text{(A.6)}$$

$$q_a(t)=\rho_a c_p Q_0 T_a(t) \qquad \text{(A.7)}$$

$$q_f(t)=q_e(t)+w_e c_e\frac{\mathrm{d}T_e(t)}{\mathrm{d}t} \qquad \text{(A.8)}$$

$$\frac{\mathrm{d}T_a(t)}{\mathrm{d}t}=\frac{1}{w_a c_p}[q_e(t)-q_a(t)+q_c(t)] \qquad \text{(A.9)}$$

where

D_e diameter of the heat exchanger

Y length of the heat exchanger

$T_f(t)$ average temperature of flame

$T_e(t)$ average temperature of the heat exchanger wall

$T_a(t)$ average temperature of the cold air

$q_f(t)$ heat flow rate from flue gas to exchanger wall

$q_e(t)$ heat flow rate from exchanger wall to furnace air

$q_c(t)$ heat flow rate of the cold air

$q_a(t)$ heat flow rate of the hot air duct

ρ_a average air density in the heat exchanger

h_f heat exchanger coefficient between flame and heat exchanger material

h_a heat transfer coefficient between heat exchanger material and circulating air

c_e specific heat of steel

Q_0 average flow rate of air in furnace

w_e mass of heat exchanger

w_a mass of air in heat exchanger

This set of eqs. (A.4) to (A.9) may be summarized by the following two differential equations:

$$h_f \pi D_e Y[T_f(t) - T_e(t)] = h_a \pi D_e Y[T_e(t) - T_a(t)]$$
$$+ w_e c_e \frac{dT_e(t)}{dt} \qquad (A.10)$$

$$w_a c_p \frac{dT_a(t)}{dt} = h_a \pi D_e Y[T_e(t) - T_a(t)] - \rho_a c_p Q_0 T_a(t)$$
$$+ \rho_a c_p Q_0 T_T(t). \qquad (A.11)$$

Transposing and rearranging these equations may be expressed in the following form:

$$\dot{T}_e(t) = a_1 T_e(t) + a_2 T_a(t) + a_3 T_f(t)$$
$$(A.12)$$

$$\dot{T}_a(t) = a_4 T_e(t) + a_5 T_a(t) + a_6 T_R(t). \qquad (A.13)$$

In practice, however, it was found that the heat capacity of the air flowing around the exchanger wall is very small compared with the capacity of the heat exchanger [5]. It is a fact that the time

constant of the heat exchanger, which is defined as the time necessary for the air in the heat exchanger to rise to 63·2% of its final value when the flame temperature changes abruptly, is proportional to its capacity. Therefore the heat exchanger material capacity contributes to most of the exchanger time constant. Therefore the state variable equations for the simplified model take the form:

$$\dot{T}_e(t) = a_1 T_e(t) + a_2 T_a(t) + a_3 T_f(t) \qquad (A.14)$$

$$0 = a_7 T_e(t) + T_a(t) + a_8 T_R(t). \qquad (A.15)$$

From the engineering point of view, the dynamics of the gas valve, humidifier, and air filter can be neglected. For the mathematical model of the thermostat and air ducts the reader is referred to the work done by KAZDA and SPOONER [14].

Rearranging the equations representing the fixed components, the room, air duct and heat exchanger of the heating system, results in:

$$\dot{T}_R = a_{11} T_R + a_{12} T_W + a_{13} T_e \qquad (A.16)$$

$$\dot{T}_W = a_{21} T_R + a_{22} T_W + a_{23} T_e + m_2 \qquad (A.17)$$

$$\dot{T}_e = a_{31} T_R + a_{33} T_e + u_3 \qquad (A.18)$$

where the a_{ij} are constants which may be expressed in terms of Q, ρ, c_p, k, V, R, R_0, R_i, C, and the parameters of the air duct. In addition, the variables u_3 and m_2 appearing in eqs. (A.17) and (A.18) above are defined by:

$$u_3 = a_3 T_f$$

$$m_2 = \frac{R_i}{C(R_i + R)} T_0.$$

REFERENCES

[1] A. H. ELTIMSAHY: The Optimization of Domestic Heating Systems. Ph.D. Dissertation, April, 1967, University of Michigan.

[2] HAROLD E. STRAUB, STANLY F. GILMAN, SEICHI KENZO and MICHAEL MING CHEN: Distribution of Air Within a Room for Year-Round Air Conditioning. Parts I and II. University of Illinois Engineering experiment Station Bulletin Nos. 435 and 442.

[3] R. O. ZERMUEHLEN and H. L. HARRISON: Room Temperature Response to a Sudden Heat Disturbance Input. ASHRAE (1965).

[4] J. R. GARTNER and H. L. HARRISON: Dynamic Characteristics of Water to Air Cross Flow Heat Exchanger. ASHRAE (1965).

[5] Fundamentals of Heat Transfer in Domestic Gas Furnaces. Research Bulletin 63, American Gas Association Laboratories.

[6] C. W. MERRIAM III: Optimization Theory and the Design of Feedback Control Systems. McGraw-Hill, New York (1964).

[7] J. TOU: Modern Control Theory. McGraw-Hill, New York (1964).

[8] *Control Systems Analysis Program II.* (CSAP II), User's Manual.

[9] L. F. KAZDA, G. CASSERLY, A. ELTIMSAHY and R. SPOONER: The optimal control of gas-fired warm air heating systems. AGA Final Report Contract DO-14-GU, University of Michigan.

[10] W. K. ROOTS and J. M. NIGHTINGALE: Two-position discontinuous temperature control in electrical space heating and cooling processes. Part I, The control system. Part II, The control process. *IEEE Trans. Appl. Ind.* **70**, pp. 17–38.

[11] W. K. ROOTS and J. M. NIGHTINGALE: Multiposition discontinuous temperature control in the electric space heating process. *IEEE Trans. Appl. Ind.* **70**, pp. 38–50.

[12] W. K. ROOTS and J. M. NIGHTINGALE: Quasi-continuous temperature control in the electrical space heating process. *IEEE Trans. Appl. Ind.* **70**, pp. 51–69.

[13] W. K. ROOTS and FREDERICK WALKER: Closed-loop controlled electrical space-heating and space cooling processes. *IEEE Trans. Ind. Gen. Appl.*, IGA-2, No. 5, Sept./Oct. (1966).

[14] L. F. KAZDA and R. L. SPOONER: The Study of a Forced-Air Heating System Using Control System Techniques. ISA Transactions, January (1967).

[15] A. H. ELTIMSAHY and L. F. KAZDA: A Sub-Optimal Heating System. Proc. of Allerton Conference, Oct. (1968).

Zusammenfassung—Unter Benutzung einer früher beschriebenen Anordnung wird hier das mathematische Modell eines Regelungssystems für die Heizung eines Wohnhauses diskutiert und zwar auf ein vorgeschriebenes Gütekriterium. Das Optimierungsproblem besteht darin, daß die Abweichung der Raumtemperatur vom Sollwert möglichst gegen Null gehen soll, während gleichzeitung der Wert eines Leistungs- oder Kostenfunktionals J minimiert wird. Die Entwicklung geht im wesentlichen in fünf Schritten vor sich.

(a) Die Entwicklung des mathematischen Modells für jedes der Elemente des Heizungssystems.

(b) Kombination der mathematischen Modelle in einer Form, die für die Anwendung der Optimierungstechnik geeignet ist.

(c) Definition eines Optimierungskriteriums, das dem Hauptziel der Minimierung der Raumtemperaturschwankungen in Bezug auf den Sollwert entspricht.

(d) Wahl der für das Problem am besten geeigneten Optimierungstechnik.

(e) Konstruktion eines optimalen Regelungssystems unter Verwendung der entwickelten Optimierungstechnik.

Ein numerisches Beispiel vergleicht die Leistung des optimalen Systems mit einem System konventionnellen Typs, das man in vielen amerikanischen Wohnungen finden kann.

Résumé—Employant une configuration prescrite du système, cet article discute les modèles mathematiques des composants utilisés dans le système et formule une methode de règlage d'un système de chauffage domestique en accord avec un critère de performance prescrit. Le problème optimal traité est celui de la réduction à zéro de l'écart de la temperature des locaux par rapport à une valeur de référence prescrite, tout en minimalisant simultanement la valeur d'une certaine fonctionnelle de performance ou de coût F. Le developpement a essentiellement lieu en cinq étapes;

(a) Le developpement de modèles mathematiques pour chacun des élements du système de chauffage,

(b) La combinaison des modèles mathématiques sous une forme qui convient à l'application des techniques d'optimalisation.

(c) La définition d'un critère d'optimalisation qui englobe l'objectif principal de minimaliser les variations de la température des locaux rapport à une température de référence prescrite,

(d) Le choix d'une technique d'optimalisation convenant le mieux au problème,

(e) La construction d'un système de commande optimale utilisant la technique d'optimalisation developpée.

Un exemple numérique compare les performances du système optimal à celles d'un système du type conventionnel qui peut être rencontré dans de nombreuses maisons americaines.

Резюме—Используя заданное расположение системы, настоящая статья обсуждает математические модели использованных элементов системы и формулирует метод для управления системой домового отопления в соответствии с заданным критерием работы. Изучаемая оптимальная проблема состоит в сведении к нулю отклонения комнатных температур от заданного значения, минимизируя при этом значение некоторого функционала работы или цены у. Эта выработка происходит на пяти этапах:

(а) Выработка математических моделей для каждого элемента системы отопления,

(б) Комбинация математических моделей в форме подходящей к применению техник оптимизации.

(в) Определение критерия оптимизации включающего в себя главную цель минимизации изменений комнатных температур по отношению к заданной температуре,

(г) Выбор техники оптимизации наиболее подходящей к задаче,

(д) Построение системы оптимального управления использующего выработанную технику оптимизации.

Числовой пример сравнивает работу оптимальной системы с работой системы условного типа встречаемой в многочисленных американских домах.