Minimum Energy Problems in Hilbert Function Space

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ABSTRACT: *Minimum energy problems in Hilbert function space are formulated. The methods presented are applicable to both continuous and discrete linear systems. Transformation of a particular operator equation into a set of 2n linear differential equations is described. Applications to physical systems are illustrated.*

Introduction

The term "minimum energy" is derived from the fact that many physical problems involving energy can be formulated such that the signals $u(t)$ involved are considered as elements in a Hilbert function space of functions of t with the energy of the signal being

$$\| u \|^2 = \langle u, u \rangle = \int | u(t) |^2 \, dt. \tag{1}$$

In a fairly obvious manner this concept may be generalized to the case such that the energy of signals may be described by

$$\| u \|_{H_1}^2 + \| Fu \|_{H_2}^2, \tag{2}$$

where the input signal $u \in H_1$ and the output signal $x \in H_2$ are related by $x = Fu \in H_2$.

The methods presented here are strongly motivated by and closely related to the previous work of Balakrishnan (1) and Porter (2). It is the purpose of this paper to generalize the concept of minimum energy problems from the functional analysis view of point with emphasis on both mathematical theories and physical applications.

Statement of Problem

Let H_1 and H_2 be two Hilbert function spaces and F be a transfer operator carrying H_1 into H_2. Let F^t be a bounded linear operator of H_1 into R^n and is

defined by

$$F^t = \sum_{i=1}^{n} \phi_i(t) \rangle \langle f_i{}^t, \tag{3}$$

where the set of range vectors, $\{\phi_i\}_1^n$, and the set of functionals, $\{\langle f_i{}^t\}_1^n$, are assumed to be linearly independent. If $x(t)$ is an arbitrary element of R^n and $\mathring{u}$ and $\hat{x}$ are arbitrary elements of H_1 and H_2, respectively, from the set $U_x \subset H_1$ defined by

$$U_x = \{u : u \in H_1, F^t u = x(t)\} \tag{4}$$

find the element u which minimizes the functional $J(u)$ where

$$J(u) = \| u \|_{H_1}^2 \tag{5}$$

$$J(u) = \| Fu \|_{H_2}^2 + \| u \|_{H_1}^2 \tag{6}$$

$$J(u) = \| Fu - \hat{x} \|_{H_2}^2 + \| u - \mathring{u} \|_{H_1}^2. \tag{7}$$

Remarks:

1) The phrase "Hilbert function space" is defined over a finite interval $\tau = [t_0, T]$ of reals, or $\sigma = [t_0, t_1 \cdots t_f]$ to be $L_2(\tau)$, or $l_2(\sigma)$, respectively.

2) $F^t u$ is the value of Fu at a point when the time is equal to t and Fu is the unique continuous image of u under F.

Definitions and Theorems

In order to deal more concisely with the concept of generalized minimum energy problems the results of the solutions are summarized in the following theorems:

Theorem I.

Let M denote the linear manifold spanned by the set $\{f_i{}^t\}_1^n$, then $M^\perp$, the orthogonal complement of M in H_1, is the null space of F^t and the restriction of the operator F^t to M is nonsingular.

Proof:

Let N_{F^t} denote the null space of F^t. If $u \in M^\perp$ then $\langle f_i{}^t, u \rangle = 0$, $i = 1 \cdots n$ and hence $F^t u = 0$ which implies that $u \in N_{F^t}$ hence $M^\perp \subset N_{F^t}$. On the other hand if $u \in N_{F^t}$ then

$$F^t u = \sum_{i=1}^{n} \phi_i \rangle \langle f_i{}^t, u \rangle = 0$$

and since the set $\{\phi_i\}_1^n$ is linearly independent $\langle f_i{}^t, u \rangle = 0$, $i = 1, \cdots, n$, which implies that $u \in M^\perp$ hence $M^\perp \supset N_{F^t}$ and the first part of Theorem I is proved. Let F_M denote the restriction of F^t to the subspace M that F_M is nonsingular

(3) is immediate from the fact that F_M is a linear mapping between finite-dimensional spaces with a domain and range of equal dimensions.

Theorem II.

The set U_x has a unique element in M which is exactly u^0, the solution of the problem for the minimization of the functional given by Eq. 5.

Proof:

Since F_M is nonsingular F_M^{-1} exists and the element defined by $u = F_M^{-1}x$ is the unique preimage of x in M. Since M is finite dimensional it is closed and we may consider the orthogonal decomposition $H_1 = M + M^\perp$ of H_1. Let u^0 denote an arbitrary element of H_1 which may then be uniquely expressed as

$$u^0 = u_1 + u_2, \tag{8}$$

where $u_1 \in M$ and $u_2 \in M^\perp$. Since $M^\perp$ is the null space of F^t the additional relation

$$x(t) = F^t u^0 = F^t(u_1 + u_2) = F^t u_1 + F^t u_2 = F^t u_1 + 0$$

holds. Since u_1 is a preimage of $x(t)$ in M it is unique and given by

$$u_1 = F_M^{-1}x. \tag{9}$$

From the relation

$$\| u^0 \|^2 = \| F_M^{-1}x \|^2 + \| u_2 \|^2 \tag{10}$$

it is apparent that

$$\| u^0 \|^2 \geq \| F_M^{-1}x \|^2 \tag{11}$$

holds for every other $u^0 \in H_1$ with equality if and only if $u^0 = F_M^{-1}x$ and hence the theorem is proved.

Let $x(t) \in R^n$ be a given arbitrary vector and take the set $\{ f_i{}^t \}_1^n$ as a basis for M. Then u^0 may be expressed by

$$u^0 = \sum_{i=1}^n c_i f_i{}^t \tag{12}$$

where c_i is a scalar quantity to be determined. It can be shown (2) that

$$c = \mathfrak{F}^{-1}\beta, \tag{13}$$

where $\mathfrak{F}^{-1}$ is the inverse matrix of $\mathfrak{F}$ which is defined by

$$\mathfrak{F} = [\langle f_i{}^t, f_j{}^t \rangle], \tag{14}$$

$$\beta = \text{col.} (\beta_1 \cdots \beta_n), \qquad \beta_i = \langle \phi_i{}^+, x \rangle, \tag{15}$$

$$c = \text{col.} (c_1 \cdots c_n).$$

Thus

$$J^0 = \| u^0 \|_{H_1}{}^2 = \langle \beta, \mathfrak{F}^{-1}\beta \rangle. \tag{16}$$

Theorem III.

The element u^0 which minimizes the functional Eq. 6 is given by

$$u^0 = (I + F^*F)^{-1}v, \qquad v \in M.$$

From the operation of adjoint operator Eq. 6 may be rewritten as follows:

$$J(u) = \langle (I + F^*F)u, u \rangle_{H_1}. \tag{17}$$

Since $I + F^*F$ is a positive operator it can be expressed as the product of an operator K and its adjoint operator K^*, i.e.,

$$I + F^*F = K^*K, \qquad \text{where } K^{-1} \text{ exists.} \tag{18}$$

From Eqs. 17 and 18, we obtain

$$J(u) = \langle Ku, Ku \rangle_{H_2}. \tag{19}$$

Let $w = Ku \in H_2$ and let G^t denote the linear transformation of H_2 onto R^n with the property that

$$G^t w = F^t u = x(t) \tag{20}$$

and

$$G^t = \sum_{i=1}^{n} \phi_i \rangle \langle g_i{}^t. \tag{21}$$

It is clear that G^t is linear and well defined on H_2. Thus, if $x(t) \in R^n$ and if $w \in H_2$ is the element with minimum norm mapping into $x(t)$ under G^t then $u = K^{-1}w$ is the $u \in H_1$ which minimizes

$$J(u) = \| Fu \|_{H_2}^2 + \| u \|_{H_1}^2.$$

While doing so the problem of minimization of the functional Eq. 6 reduces to the problem of minimization of Eq. 5 with

$$G^t : H_2 \to R^n$$

being the operator in question.

Proof of Theorem III:

Let Q denote the linear manifold spanned by the set $\{g_i\}_1^n$. By applying

Theorems I and II the element w^0 with minimum norm is given by

$$w^0 = \sum_{i=1}^{n} c_i g_i{}^t \tag{22}$$

where c_i is a constant to be determined.

From Eqs. 3, 20 and 21, we obtain

$$G^t w = \sum_{i=1}^{n} \phi_i \langle g_i{}^t, w \rangle = \sum_{i=1}^{n} \phi_i \langle K^* g_i{}^t, u \rangle = \sum_{i=1}^{n} \phi_i \langle f_i{}^t, u \rangle. \tag{23}$$

Since the set $\{\phi_i\}_1^n$ is linearly independent and $u = 0$ is a trivial solution it follows from Eq. 23 that

$$f_i{}^t = K^* g_i{}^t. \tag{24}$$

From Eqs. 22 and 24, there results

$$K^* w^0 = \sum_{i=1}^{n} c_i f_i{}^t. \tag{25}$$

Thus

$$K^* K u^0 = v, \qquad\qquad v \in M \tag{26}$$

or

$$u^0 = (I + F^* F)^{-1} v, \qquad v \in M. \tag{27}$$

Lemma 1.

If F is a linear transformation of H_1 into H_2 then $I + F^*F$ is nonsingular.

Proof:

We must show that $I + F^*F$ is one to one onto as a mapping of H_1 into itself.

(i) It is one to one, because $(I + F^*F)u = 0$, $u \in H_1$, implies that **(4)**

$$\begin{aligned}
0 &= \langle (I + F^*F)u, u \rangle_{H_1} \\
&= \langle u, u \rangle_{H_1} + \langle F^*Fu, u \rangle_{H_1} \\
&= \langle u, u \rangle_{H_1} + \langle Fu, Fu \rangle_{H_2} \\
&= \| u \|_{H_1}^2 + \| Fu \|_{H_2}^2 \text{ \textbf{(5)}}
\end{aligned} \tag{28}$$

from which we have $u = 0$.

(ii) We next show that the range R of $I + F^*F$ is close. It follows from

$$\| (I + F^*F)u \|_{H_1}^2 = \| u \|_{H_1}^2 + \| F^*Fu \|_{H_1}^2 + 2\langle F^*Fu, u \rangle_{H_1}$$

that

$$\| (I + F^*F)u \|_{H_1} \geq \| u \|_{H_1}, \tag{29}$$

since F^*F is positive. By this inequality and the completeness of H_1, R is com-

plete and therefore closed. We conclude the proof that $R = H_1$; for otherwise there would exist a nonzero vector u_0 orthogonal to R and this contradicts the fact that

$$\langle (I + F^*F) u_0, u_0 \rangle_{H_1} = 0 \Rightarrow \| u_0 \|_{H_1}^2 = -\langle F^*F u_0, u_0 \rangle_{H_1} < 0 \Rightarrow u_0 = 0. \quad (30)$$

Definition:

Let F be a linear mapping of H_1 into H_2. The set $H(F)$ defined by $H(F) = \{ (u, v) : v = Fu, u \in H_1 \}$ is called the graph of F. The graph of a transformation is thus the subset of

$$H_1 \times H_2 = \{ (u, v) : u \in H_1, v \in H_2 \} \quad (31)$$

for which the ordinate v is related the abscissa u by $v = Fu$. The basic properties pertinent to the present undertaking are summarized by the following lemmas:

Lemma 2.

If addition and scalar multiplication are defined on the set $H(F)$ by

$$k(f, g) = (kf, kg) \quad (32)$$

$$(f_1, g_1) + (f_2, g_2) = (f_1 + f_2, g_1 + g_2) \quad (33)$$

then $H(F)$ is a linear space.

Lemma 3.

If H_1 and H_2 are Hilbert spaces and an inner product on $H(F)$ is defined by

$$\langle (f_1, g_1), (f_2, g_2) \rangle_{H(F)} = \langle f_1, f_2 \rangle_{H_1} + \langle g_1, g_2 \rangle_{H_2} \quad (34)$$

then $H(F)$ is a Hilbert space.

Lemma 4.

If F is bounded then $H(F)$ is a closed linear subspace of $H_1 \times H_2$.
The above lemmas will be proved in Appendix A.

Theorem IV.

If $(\hat{u}, \hat{x}) \in H(F)$ then the element u^0 which minimizes the functional Eq. 7 is satisfied by

$$(I + F^*F)(u^0 - \hat{u}) = v, \qquad v \in M. \quad (35)$$

Proof:

Since $(\hat{u}, \hat{x}) \in H(F)$

$$\hat{x} = F\hat{u}. \quad (36)$$

Let $u - \hat{u} = \bar{u}$. Then the functional Eq. 7 may be written as

$$J(\bar{u}) = \| F\bar{u} \|_{H_2}^2 + \| \bar{u} \|_{H_1}^2. \tag{37}$$

It follows directly from Theorem III that

$$(I + F^*F)\bar{u} = v, \qquad v \in M.$$

Theorem V.

If $(\hat{u}, \hat{x}) \in H_1 \times H_2$ then the element u^0 which minimizes the functional Eq. 7 is given by

$$u^0 = (I + F^*F)^{-1}(v + \hat{u} + F^*\hat{x}), \qquad v \in M. \tag{38}$$

Since $H(F)$ is a closed linear subspace of $H_1 \times H_2$ it follows that the decomposition

$$H_1 \times H_2 = H(F) + H(F)^\perp \tag{39}$$

may be defined along with the orthogonal projection P_H of $H_1 \times H_2$ on $H(F)$. Let $(\hat{u}, \hat{x})$ be defined by

$$(\bar{u}, \bar{x}) = P_H(\hat{u}, \hat{x}), \tag{40}$$

then $(\bar{u}, \bar{x})$ is in $H(F)$ and $(\hat{u}, \hat{x}) - (\bar{u}, \bar{x}) \in H(F)^\perp$ and the

$$
\begin{aligned}
\| Fu - \hat{x} \|_{H_2}^2 &+ \| u - \hat{u} \|_{H_1}^2 \\
&= \| (Fu - \bar{x}) + (\bar{x} - \hat{x}) \|_{H_2}^2 + \| (u - \bar{u}) + (\bar{u} - \hat{u}) \|_{H_1}^2 \\
&= \| Fu - \bar{x} \|_{H_2}^2 + \| \bar{x} - \hat{x} \|_{H_2}^2 + \| u - \bar{u} \|_{H_1}^2 + \| \bar{u} - \hat{u} \|_{H_1}^2 \\
&= \| Fu - \bar{x} \|_{H_2}^2 + \| u - \bar{u} \|_{H_1}^2 + \| \bar{x} - \hat{x} \|_{H_2}^2 + \| \bar{u} - \hat{u} \|_{H_1}^2
\end{aligned} \tag{41}
$$

holds for all $u \in H_1$. The last two terms are independent of u and hence it is apparent that Theorem V reduces to the case of Theorem IV by taking the projection of $(\hat{u}, \hat{x})$ on the graph of F.

Let the operator V of the Hilbert space $H_1 \times H_2$ be defined by the relation

$$V(u, v) = (-v, u), \qquad (u, v) \in H_1 \times H_2. \tag{42}$$

It can be proved (see Appendix B) that the orthogonal complement of $H(F)$ in $H_1 \times H_2$ is given by

$$H(F)^\perp = VH(F^*). \tag{43}$$

A consequence of this result is that the orthogonal decomposition of $H_1 \times H_2$ on $H(F)$ is given explicitly by the formula (6)

$$H_1 \times H_2 = H(F) + VH(F^*). \tag{44}$$

This formula will prove useful in defining the orthogonal projection of $H_1 \times H_2$ on $H(F)$.

Let $(\hat{u}, \hat{x}) \in H_1 \times H_2$ denote an arbitrary vector. Then in view of Eq. 44 a unique decomposition exists of the form

$$
\begin{aligned}
(\hat{u}, \hat{x}) &= \bar{u}, F\bar{u}) + V(u_1, F^*u_1) \\
&= (\bar{u}, F\bar{u}) + (-F^*u_1, u_1) \\
&= (\bar{u} - F^*u_1, F\bar{u} + u_1).
\end{aligned}
\tag{45}
$$

This equality implies the two equalities

$$
\hat{u} = \bar{u} - F^*u_1
$$

and

$$
\hat{x} = F\bar{u} + u_1.
\tag{46}
$$

Operating on the second equation with F^* and using the result to eliminate F^*u_1 in the first equation produces the result

$$
\bar{u} = (I + F^*F)^{-1}(\hat{u} + F^*\hat{x}).
\tag{47}
$$

Replacing $\hat{u}$ in Eq. 35 by $\bar{u}$ expressed in Eq. 47 gives

$$
(I + F^*F)[u^0 - (I + F^*F)^{-1}(\hat{u} + F^*\hat{x})] = v, \qquad v \in M.
\tag{48}
$$

or

$$
u^0 = (I + F^*F)^{-1}(v + \hat{u} + F^*\hat{x}), \qquad v \in M.
\tag{49}
$$

A Method of Evaluating the Optimal Element u^0 without Inversion of Operator

It was shown that the optimal element u^0 which minimizes the functional Eq. 7 is given by Eq. 49. Therefore, the evaluation of u^0 involves the inversion of $(I + F^*F)$. However, there is a simple method of finding the element u^0 which avoids the inversion of the operator $(I + F^*F)$ if the operator F^t is defined by (7)

$$
F^t u = \int_{t_0}^{t} \Phi(t, t_0) B(s) u(s) \, ds,
\tag{50}
$$

which is a bounded linear operator and is well known to be the solution of the system equation as given by

$$
\dot{x}(t) = x(t)A(t) + B(t)u(t), \qquad x(t_0) = 0.
\tag{51}
$$

This method is carried out by transforming the operator Eq. 49 into $2n$ linear differential equations.

Since $v \in M$ has a unique expansion in terms of any basis for that space. In the present instance the rows of matrix $\Phi(t_0, t)B(t)$ provide a convenient basis. By introducing the scalar $c = \text{col.}\ (c_1, c_2 \cdots c_n)$ we may write

$$v(t) = B^*(t)\Phi^*(t_0, t)c \tag{52}$$

as a defining relation for an arbitrary $v \in M$. Along the optimal trajectory $x = Fu^0$ Eq. 49 may be written as

$$u^0 = \hat{u} + v + F^*(\hat{x} - x). \tag{53}$$

It is not difficult to show (see Appendix C) that

$$(F^*x)(t) = B^*(t)\Phi^*(t_0, t) \int_t^T \Phi^*(s, t_0)x(s)\ ds \tag{54}$$

for $t \in \tau = [t_0, T]$ is the defining equation for F^* as t sweeps through τ. Letting $\alpha(t)$ denote the tuplet

$$\alpha(t) = c - \int_t^T \Phi^*(s, t_0)[x(s) - \hat{x}(s)]\ ds$$

$$= c - \int_{t_0}^T \Phi^*(s, t_0)[x(s) - \hat{x}(s)]\ ds + \int_{t_0}^t \Phi^*(s, t_0)[x(s) - \hat{x}(s)]\ ds$$

$$= \alpha(t_0) + \int_{t_0}^t \Phi^*(s, t_0)[x(s) - \hat{x}(s)]\ ds. \tag{55}$$

It is apparent that Eq. 53 may be written in the form

$$u^0 - \hat{u} = B^*(t)\Phi^*(t_0, t)\alpha(t). \tag{56}$$

Consider now the vector $\lambda(t) = \Phi^*(t_0, t)\alpha(t)$. Using the property that $\Phi^*(t_0, t) = \Psi(t, t_0)$, where $\Psi(t, t_0)$ is the transition matrix for the adjoint system

$$\dot{\Psi}(t, t_0) = -A^*(t)\Psi(t, t_0), \qquad \Psi(t_0, t_0) = I \tag{57}$$

(I is the identity matrix), it is apparent from Eq. 55 that

$$\lambda(t) = \Phi^*(t_0, t)\alpha(t_0) + \Phi^*(t_0, t) \int_{t_0}^t \Phi^*(s, t_0)[x(s) - \hat{x}(s)]\ ds$$

$$= \Psi(t, t_0)\alpha(t_0) + \Psi(t, t_0) \int_{t_0}^t \Psi(t_0, s)[x(s) - \hat{x}(s)]\ ds \tag{58}$$

which implies immediately that $\lambda(t)$ is the solution to the differential system

$$\dot{\lambda}(t) = -A^*(t)\lambda(t) + [x(t) - \hat{x}(t)], \qquad \lambda(t_0) = \lambda_0. \qquad (59)$$

The original differential system for this example along the optimal trajectory is given by

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)u^0(t) \\ &= A(t)x(t) + B(t)[B^*(t)\lambda(t) + \hat{u}].\end{aligned} \qquad (60)$$

Thus, we have the $2n$ linear differential equations

$$\dot{x}(t) = A(t)x(t) + B(t)[B^*(t)\lambda(t) + \hat{u}], \qquad x(t_0) = x_0$$

and

$$\dot{\lambda}(t) = x(t) - \hat{x}(t) - A^*(t)\lambda(t), \qquad \lambda(t_0) = \lambda_0, \qquad (61)$$

the solution that determines $\lambda(t)$, which in turn determines $u^0(t)$ by

$$u^0(t) = B^*(t)\lambda(t) + \hat{u}. \qquad (62)$$

One important point to be noted is that λ_0 is unknown. Indeed, λ_0 is a function of c which we originally set out to determine. The solution of these equations may be expressed in terms of the transition matrix $\Theta(t, t_0)$ which is defined by

$$\dot{\Theta}(t, t_0) = \begin{bmatrix} A(t) & B(t)B^*(t) \\ \hline I & -A^*(t) \end{bmatrix} \Theta(t, t_0)$$

with

$$\Theta(t_0, t_0) = I. \qquad (63)$$

The transition matrix has $2n$ rows and $2n$ columns. It can be partitioned into four $n \times n$ submatrices

$$\Theta(t, t_0) = \begin{bmatrix} \Theta_{11}(t, t_0) & \Theta_{12}(t, t_0) \\ \hline \Theta_{21}(t, t_0) & \Theta_{22}(t, t_0) \end{bmatrix}. \qquad (64)$$

Solution to the system Eq. 61 thus can be expressed by

$$\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = \begin{bmatrix} \Theta_{11}(t, t_0) & \Theta_{12}(t, t_0) \\ \Theta_{21}(t, t_0) & \Theta_{22}(t, t_0) \end{bmatrix} \begin{bmatrix} x(t_0) \\ \lambda(t_0) \end{bmatrix} + \begin{bmatrix} q_1(t, t_0) \\ q_2(t, t_0) \end{bmatrix}$$

where

$$\begin{bmatrix} q_1(t, t_0) \\ q_2(t, t_0) \end{bmatrix} = \int_{t_0}^{t} \begin{bmatrix} \Theta_{11}(t, s) & \Theta_{12}(t, s) \\ \Theta_{21}(t, s) & \Theta_{22}(t, s) \end{bmatrix} \begin{bmatrix} B(s)\hat{u}(s) \\ -\hat{x}(s) \end{bmatrix} ds. \qquad (65)$$

In particular we have

$$x(T) = \Theta_{11}(T, t_0)x(t_0) + \Theta_{12}(T, t_0)\lambda(t_0) + q_1(T, t_0). \qquad (66)$$

Thus, if $x(T)$ and $x(t_0)$ are known this expression determines $\lambda(t_0)$. If $x(t_0) = 0$ and $\Theta_{12}(t, t_0)$ is nonsingular then

$$\lambda(t_0) = \Theta_{12}^{-1}(T, t_0)[x(T) - q_1(T, t_0)]. \qquad (67)$$

This gives

$$\lambda(t) = \Theta_{22}(t, t_0)\Theta_{12}(T, t_0)^{-1}[x(T) - q_1(T, t_0)] + q_2(t, t_0). \qquad (68)$$

Substituting Eq. 68 into Eq. 56 gives

$$u^0(t) = B^*(t)\{\Theta_{22}(t, t_0)\Theta_{12}(T, t_0)^{-1}[x(T) - q_1(T, t_0)] + q_2(t, t_0)\} + \hat{u}. \qquad (69)$$

Physical Applications

Example I.

Consider the continuous second-order position control servomechanism shown in Fig. 1 where the gain, mass, and viscous damping which, in general, are
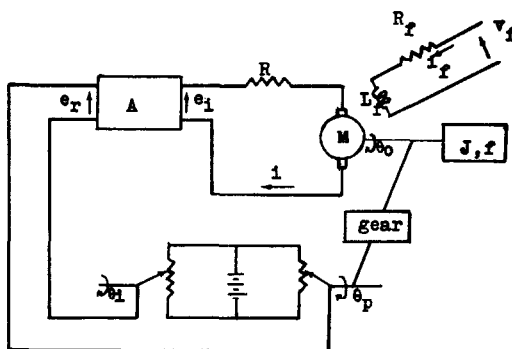

FIG. 1. Second-order position control servomechanism.

deterministic differentiable functions of time are assumed to be constant. The motor armature inductance and backlash of the gear are neglected. Now, by fixing i_f the following basic equations describe this system:

$$e_i = Ae_r = Ak_4(\theta_i - \theta_p) = Ri + k_1\dot{\theta}_0 \qquad (70)$$

$$T = k_2 i = J\ddot{\theta}_0 + f\dot{\theta}_0 \qquad (71)$$

$$\theta_p = k_3\theta_0 \qquad (72)$$

where A = gain of amplifier

k_1 = motor constant

T, k_2 = developed torque, torque constant

k_3 = gear ratio

k_4 = proportionality constant of potentiometers

ϕ = air gap flux of motor

R = resistance of armature circuit of servomotor M

R_f, L_f = resistance, inductance of motor field circuit

J, f = inertia, viscous damping referred to motor shaft

It is desired to find $\theta_i(t)$ such that

$$\theta_0(t_0) = \theta_0, \qquad \dot{\theta}_0(t_0) = \dot{\theta}_0, \qquad \theta_0(T) = \theta_T, \qquad \dot{\theta}_0(T) = \dot{\theta}_T$$

and

$$J = \int_{t_0}^{T} \theta_i^2 \, dt \quad \text{has a minimum value.} \tag{73}$$

From Eqs. 70 and 71 we have

$$\frac{JR}{k_2} \ddot{\theta}_0 + \left(\frac{fR}{k_2} + k_1 \right) \dot{\theta}_0 + A k_4 k_3 \theta_0 = A k_4 \theta_i \tag{74}$$

or

$$\ddot{\theta}_0 + a_1 \dot{\theta}_0 + a_0 \theta_0 = d \theta_i \tag{75}$$

where

$$a_1 = \frac{k_2}{JR} \left(\frac{fR}{k_2} + k_1 \right)$$

$$a_0 = \frac{A k_2 k_3 k_4}{JR}$$

$$d = A k_2 k_4 \left(\frac{1}{JR} \right).$$

Let $x_1(t) = \theta_0(t)$, $x_2(t) = \dot{\theta}_0(t)$ and $u(t) = \theta_i(t)$. Then in matrix notation Eq. 75 may be written as

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{76}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ d \end{bmatrix}. \tag{77}$$

The transition matrix $\Phi(t, t_0)$ of the system, Eq. 76, is given by

$$\Phi(t, t_0) = \begin{bmatrix} \phi_{11}(t, t_0) & \phi_{12}(t, t_0) \\ \phi_{21}(t, t_0) & \phi_{22}(t, t_0) \end{bmatrix}$$

where

$$\phi_{11}(t, t_0) = c_1 \exp\left[-b(t - t_0)\right] \sin\left[w(t - t_0) + \psi_1\right]$$
$$\phi_{12}(t, t_0) = c_2 \exp\left[-b(t - t_0)\right] \sin\left[w(t - t_0)\right]$$
$$\phi_{21}(t, t) = c_3 \exp\left[-b(t - t_0)\right] \sin\left[w(t - t_0)\right]$$
$$\phi_{22}(t, t_0) = c_4 \exp\left[-b(t - t_0)\right] \sin\left[w(t - t_0) + \psi_2\right]$$

$$c_1 = \frac{(b^2 + w^2)^{1/2}}{w}, \qquad c_2 = \frac{-1}{w}, \qquad c_3 = \frac{a_0}{w}, \qquad c_4 = \frac{[(a_1 - b)^2 + w^2]^{1/2}}{w}$$

$$w = (a_0 - b^2)^{1/2}, \qquad b = \frac{a_1}{2}, \qquad \psi_1 = \tan^{-1}\left(\frac{w}{-b}\right),$$

and

$$\psi_2 = \tan^{-1}\left(\frac{w}{a_1 - b}\right).$$

In this example we take the columns of the state transition matrix $\Phi(t, t_0)$ as $\phi_1(t)$ and $\phi_2(t)$ and the vectors $f_1(t)$ and $f_2(t)$ as the rows of the matrix $\Phi(t_0, t)B$. We consider the control function $u(t)$ as the element in the Hilbert function space $H_1 = L_2(\tau)$. It is not difficult to show that $(2, 8)$

$$\beta = \Phi(t_0, T)\left[x(T) - \Phi(T, t_0)x(t_0)\right]$$

$$\mathfrak{F} = \int_{t_0}^{T} \Phi(t_0, s)BB^*\Phi^*(t_0, s) \, ds$$

$$= \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$$

where

$$f_{11} = \int_{t_0}^{T} d^2\phi_{12}{}^2(t_0, s) \, ds$$

$$f_{12} = \int_{t_0}^{T} d^2\phi_{12}(t_0, s)\phi_{22}(t_0, s) \, ds$$

$$f_{21} = \int_{t_0}^{T} d^2\phi_{22}(t_0, s)\phi_{12}(t_0, s) \, ds$$

$$f_{22} = \int_{t_0}^{T} d^2\phi_{22}{}^2(t_0, s) \, ds.$$

The optimal control function $u^0(t)$ or $\theta_i{}^0(t)$ is given by

$$u^0(t) = B^*\Phi^*(t_0, t)\mathfrak{F}^{-1}\beta$$

$$= \frac{d}{(f_{11}f_{22} - f_{12}f_{21})} (d_1 f_{22}\phi_{12} - d_2 f_{12}\phi_{12} - d_1 f_{21}\phi_{22} + d_2 f_{11}\phi_{22})$$

and the optimal performance index is

$$J^0 = \langle \beta, \mathfrak{F}^{-1}\beta \rangle$$

$$= \frac{1}{(f_{11}f_{22} - f_{12}f_{21})} (d_1{}^2 f_{22} - d_1 d_2 f_{12} - d_1 d_2 f_{21} + d_2{}^2 f_{11}).$$

Given specific values of these constants J, f, R, A, k_1, k_2, k_3 and k_4 we can compute $\theta^0(t)$ and J^0. This method is applicable to high order systems, only the size of the related transition matrix increases.

Example II.

Given a plant whose dynamic characteristic is defined by

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \qquad t \in \tau = [t_0, T] \tag{78}$$

and the reference trajectory $\hat{x}(t)$. Find the control function u such that $x(t)$ transfers from $x(t_0) = 0$ to $x(T) = x_T$, and

$$J(u) = \int_{t_0}^{T} \{[(x - \hat{x}), (x - \hat{x})] + [u, u]\} \, ds \tag{79}$$

is minimized.

We may reformulate the above problem as follows: Given $\hat{x} \in H_1$ and $x(t) \in R^n$ and let F^t, defined by Eq. 50, be the transformation from $H_1 = [L_2(\tau)]^m$ onto R^n to find the element $u \in H_1$ such that

$$J(u) = \| x - \hat{x} \|_{H_2}^2 + \| u \|_{H_1}^2 \tag{80}$$

is minimized.

For the present case $\hat{u} = 0$ and $(\hat{x}, 0)$ is a tuplet of $H_1 \times H_2$. The relation $Fx = u$ does not hold. Theorem V, therefore, is applied. The element u^0 is given by Eq. 38 or, after transformation, Eq. 69.

Conclusion

Minimum energy problems in Hilbert function space for three types of index of performance have been formulated. A method of evaluating the optimal element u^0 without the inverse of the operator $(I + F^*F)$ has been studied.

These methods are applicable to both continuous and discrete linear systems. In the continuous case we consider $H = [L_2(\tau)]^m$, $t \in \tau = [t_0, T]$ and in the discrete case, $H = [l_2(\sigma)]^m$, $\sigma = [t_0, t_1, \cdots t_f]$. Two examples of continuous

linear systems have been chosen for the purpose of illustration. Discrete linear systems can be similarly described.

Appendix A

A Proof of Lemmas 2, 3, and 4. Lemma 2 follows directly from the definition. To prove Lemma 3 we note that the function

$$\langle (f_1, g_1), (f_2, g_2) \rangle_{H(F)} = \langle f_1, f_2 \rangle_{H_1} + \langle g_1, g_2 \rangle_{H_2}$$

is an inner product on $H(F)$ as is easily shown. For instance, if $w_1 = (f_1, g_1)$ and $w_2 = (f_2, g_2)$ then

$$\begin{aligned} (w_1, w_2)_{H(F)} &= \langle f_1, f_2 \rangle_{H_1} + \langle g_1, g_2 \rangle_{H_2} = \langle \overline{f_2, f_1} \rangle_{H_1} + \langle \overline{g_2, g_1} \rangle_{H_2} \\ &= \overline{\langle w_2, w_1 \rangle_{H(F)}} \end{aligned}$$

where $\bar{w}$ denotes the complex conjugate of w. Thus, it remains only to show that $H(F)$ is complete in order to prove that $H(F)$ is a Hilbert space. Let $\{w_n = (f_n, g_n)\}$ denote a Cauchy sequence in $H(F)$ then for arbitrary e there exists N such that

$$\| w_m - w_n \|^2_{H(F)} = \| f_m - f_n \|^2_{H_1} + \| g_m - g_n \|^2_{H_2} < e \qquad \text{for} \quad m, n > N.$$

Clearly, this implies both

$$\| f_m - f_n \|^2_{H_1} < e_1 \quad \text{and} \quad \| g_m - g_n \|^2_{H_2} < e_2,$$
$$e_1 + e_2 = e, \qquad \text{for} \quad m, n > N.$$

Since H_1 and H_2 are both complete the sequence f_n and g_n both converge to points f and g of those spaces. Since $g_n = Ff_n$ for all n it follows that $g = Ff$ and hence the element w which is the limit of the sequence $\{w_n\}$ is given by $w = (f, g) \in H(F)$ and hence $H(F)$ is complete.

To prove Lemma 4, let $(u_1, Fu_1), (u_2, Fu_2) \cdots$ be the sequence of points of $H(F)$. Since F is a bounded linear operator this implies that F is continuous, therefore

$$Fu_n \to Fu \quad \text{as} \quad u_n \to u.$$

This leads to

$$\lim_{n \to \infty} (u_n, Fu_n) = (u, Fu) \in H(F) \subset H_1 \times H_2.$$

We must show that $H(F)$ is closed or equivalently that its complement, denoted by $H(F)^c$, is open. If $H(F)^c$ is not open then there must be a point $(c, Fc) \in H(F)^c$ such that for each neighborhood N of (c, Fc)

$$N \not\subset H(F)^c.$$

In particular, the sphere $S((c, Fc), 1/n) \not\subset H(F)^c$ for $n = 1, 2, 3, \cdots$. Thus, there is a sequence of points $(u_1, Fu_1), (u_2, Fu_2) \cdots$ such that

$$(u_n, Fu_n) \in H(F) \quad \text{and} \quad \lim_{n \to \infty} (u_n, Fu_n) = (c, Fc) \in H(F)$$

which contradicts to the assumption that $(c, Fc) \in H(F)^c$. Therefore, $H(F)^c$ is open and $H(F)$ is closed.

Appendix B

A Proof of $H(F)^{\perp} = VH(F^*)$. Let (x, y) be an arbitrary element of $H(F)^{\perp}$, then for every $u \in H_1$

$$\begin{aligned} 0 = \langle (x, y), (u, Fu) \rangle_{H(F)} &= \langle x, u \rangle_{H_1} + \langle y, Fu \rangle_{H_2} \\ &= \langle x, u \rangle_{H_1} + \langle F^*y, u \rangle_{H_1} = \langle x + F^*y, u \rangle_{H_1} \end{aligned} \tag{B-1}$$

which implies $x = -F^*y$ or that (x, y) is of the form

$$(x, y) = (-F^*y, y) = V(y, F^*y). \tag{B-2}$$

Thus, $(x, y) \in VH(F^*)$ and hence $H(F)^{\perp} \subset VH(F^*)$. Conversely, if $(x, y) \in VH(F^*)$ then

$$(x, y) = V(y, F^*y) = (-F^*y, y).$$

For any (u, Fu), however, we have $(u, Fu) \perp (-F^*y, y)$ since

$$\begin{aligned} \langle (u, Fu), (-F^*y, y) \rangle_{H(F)} &= \langle u, -F^*y \rangle_{H_1} + \langle Fu, y \rangle_{H_2} \\ &= -\langle Fu, y \rangle_{H_2} + \langle Fu, y \rangle_{H_2} = 0. \end{aligned}$$

Hence, $(-F^*y, y) \in H(F)^{\perp}$ which implies $VH(F^*) \subset H(F)^{\perp}$, and the lemma is proved.

Appendix C

A proof of Eq. 54. From the definition of adjoint operator we have

$$\langle x, Fy \rangle_{H_1} = \langle F^*x, y \rangle_{H_2}.$$

In particular, if

$$H_1 = H_2 = L_2(\tau) \quad \text{then} \quad \langle x, Fy \rangle_{L_2} = \langle F^*x, y \rangle_{L_2}. \tag{C-1}$$

In real space $L_2(t_0, T)$ the inner product of two elements u and v is defined by

$$\langle u, v \rangle_{L_2} = \int_{t_0}^{T} [u, v] \, ds.$$

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Thus

$$\langle x, Fy \rangle_{L_2} = \int_{t_0}^{T} \left[x(s_1), \ \Phi(s_1, t_0) \int_{t_0}^{s_1} \Phi(t_0, s_2) B(s_2) y(s_2) \ ds_2 \right] ds_1$$

$$= \int_{t_0}^{T} \int_{t_0}^{s_1} y^*(s_2) B^*(s_2) \Phi^*(t_0, s_2) \Phi^*(s_1, t_0) x(s_1) \ ds_2 ds_1$$

$$= \int_{t_0}^{T} \int_{s_2}^{T} y^*(s_2) B^*(s_2) \Phi^*(t_0, s_2) \Phi^*(s_1, t_0) x(s_1) \ ds_1 ds_2$$

$$\left\langle \int_{s_2}^{T} B^*(s_2) \Phi^*(t_0, s_2) \Phi^*(s_1, t_0) x(s_1) \ ds_1, \ y(s_2) \right\rangle_{L_2}. \qquad \text{(C-2)}$$

From (C-1) and (C-2) we have

$$(F^*x)(s_2) = \int_{s_2}^{T} B^*(s_2) \Phi^*(t_0, s_2) \Phi^*(s_1, t_0) x(s_1) \ ds_1$$

or

$$(F^*x)(t) = B^*(t) \Phi^*(t_0, t) \int_{t}^{T} \Phi^*(s, t_0) x(s) \ ds.$$

Acknowledgment

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