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Integration of Just-Noticeable Differences*

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Fechner deduced his logarithmic law from Weber's Law by integrating the equation $du = dx/kx$. Since the work of Luce and Edwards, this method has been regarded as incorrect. Reexamination shows that the method can be reformulated and justified in a rigorous manner.

Let $\Delta_s(x)$ denote the increment in stimulus energy x that is required in order to produce the value s of a certain index of discriminability. (Example: if x is luminance and the discriminability index is the signal-detectability index d', then $\Delta_{1.5}(x)$ is the increment in x required to produce $d' = 1.5$.) The function Δ_s is called the *Weber function* (for discriminability $= s$).

A just-noticeable difference scale (JND scale) is a transformation of the stimulus continuum, $x \to u(x)$, such that

$$u[x + \Delta_s(x)] - u(x) = g(s). \tag{1}$$

That is, the u-scale difference between a stimulus x and the stimulus one s-JND higher is constant, $g(s)$, independent of x; the u-scale difference between two stimuli separated by n s-JND's is $n \cdot g(s)$; $g(s)$ is the u-scale size of one s-JND. This formalization of the concept of JND scale is due to Luce and Edwards (1958), who pointed out that, in general, a JND scale does not exist; Eq. 1 can be solved only for special Weber functions. In particular, if the general linear form of Weber's Law holds,

$$\Delta_s(x) = k_1(s)x + k_2(s),$$

then there are no function-pairs (u, g) satisfying Eq. 1, unless k_2 is propertional to k_1, i.e., unless

$$\Delta_s(x) = k(s)(x + x_0), \tag{2}$$

where x_0 is independent of s. The conditions that must be satisfied by a general Weber function $\Delta_s(x)$, in order for a solution (u, g) to Eq. 1 to exist, have been studied in depth

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by Falmagne (1971) and Levine (1970). They showed, among other things, that under the most commonly encountered conditions, the u-scale is unique (up to linear transformations) when it exists.

If the Weber function is given by Eq. 2, then simple substitution shows that the functions

$$\begin{aligned} u(x) &= a + b \log(x + x_0) \\ g(s) &= b \log[1 + k(s)] \end{aligned} \tag{3}$$

satisfy Eq. 1. In this case (Fechner's Law) the u-scale can be obtained by integrating the reciprocal of the Weber function:

$$\int \frac{dx}{k(s)(x + x_0)} = \frac{1}{k(s)} \log(x + x_0).$$

The purpose of this note is to show that this formula can be generalized to other Weber functions. This is of interest both as a means of obtaining solutions to Eq. 1, when $\Delta_s(x)$ is given in a closed functional form, and because Luce and Edwards explicitly asserted that Fechner's procedure of integrating $dx/\Delta x$ gives wrong results except in the case of Δx given by Eq. 2. In the discussion, I shall try to reconcile Luce's and Edwards' assertion with the present result, and shall point out what *is* special about Weber's Law (Eq. 2).

To begin with, observe that one cannot simply write

$$u(x) = \int \frac{dx}{\Delta_s(x)} \tag{4}$$

for an arbitrary Weber function $\Delta_s(x)$. For one thing, it was just pointed out that solutions to Eq. 1 do not exist for arbitrary Weber functions. Thus, any method of integration can be applied only where we already know that the criteria for existence of a solution to Eq. 1 are satisfied. Secondly, the right side of Eq. 4 is a function of s, while the left side is not. Indeed, the integration performed above led to the u-scale $[1/k(s)] \log(x + x_0)$ which does depend on s. But multiplying by $[1/k(s)]$ is a permissible transformation, amounting to varying b in Eq. 3; thus, for Weber's Law, essentially the same scale was attained, regardless of the choice of s. For more general Weber functions, however, this need not hold.

This last point leads to the idea of normalizing the integral, by fixing the zero and units so as to retain the same u-scale for every s, at least approximately.

Thirdly, the right side of Eq. 4 is differentiable, so the left side must be also; we expect that a JND scale (u-scale) obtained by integration must satisfy some smoothness conditions.

These considerations should lend some intuitive plausibility to the statement of the following theorem:

THEOREM. *Let u be a real-valued function on an open interval (α, β), $-\infty \leqslant \alpha < \beta \leqslant +\infty$, with a continuous, positive first derivative Du; let g be a real-valued function, the range of which has 0 as a limit point; and for all x, s such that $x \in (\alpha, \beta)$ and $u(x) + g(s)$ is in the range of u, let*

$$\Delta_s(x) = u^{-1}[u(x) + g(s)] - x. \tag{5}$$

Then for any x_1, x_2, x_3, x_4 in (α, β), with $x_3 \neq x_4$,

$$\frac{u(x_2) - u(x_1)}{u(x_4) - u(x_3)} = \lim_{g(s)\to 0} \left(\int_{x_1}^{x_2} \frac{dx}{\Delta_s(x)} \Big/ \int_{x_3}^{x_4} \frac{dx}{\Delta_s(x)}\right). \tag{6}$$

Note that Eq. 5 is equivalent to Eq. 1.

The statement of the theorem is intended to include the assertions that both the integrals on the right side of Eq. 6 exist and that the indicated limit exists.

The following line of proof was suggested by Michael Levine.

Proof. Since u has a positive first derivative, it is continuous and strictly increasing. Therefore u^{-1} exists and $\Delta_s(x)$ is well defined. Furthermore, since Du is positive, for every y in the range of u, $Du^{-1}(y)$ exists and is equal to $1/Du[u^{-1}(y)]$.

By the Theorem of the Mean applied to Eq. 5, for every x, s for which $\Delta_s(x)$ is defined and nonzero, we have

$$\Delta_s(x) = Du^{-1}[u(x) + \epsilon(s, x)] \cdot g(s), \tag{7}$$

where $\epsilon(s, x)$ is strictly between 0 and $g(s)$.

For any $x_1 < x_2$ in (α, β), choose $\delta > 0$ so small that $u(x_1) - \delta$, $u(x_2) + \delta$ are in the range of u. Then for $0 < |g(s)| \leqslant \delta$, $x_1 \leqslant x \leqslant x_2$, we have $\Delta_s(x)$ defined and satisfying Eq. 7. Moreover, since Du is continuous, it is bounded on the compact interval $[u^{-1}[u(x_1) - \delta], u^{-1}[u(x_2) + \delta]]$, and, hence, Du^{-1} has a positive lower bound on the interval $[u(x_1) - \delta, u(x_2) + \delta]$. It follows that, for $0 < |g(s)| \leqslant \delta$, the integral

$$\int_{x_1}^{x_2} \frac{dx}{\Delta_s(x)}$$

exists and is equal to

$$\frac{1}{g(s)} \int_{x_1}^{x_2} \frac{dx}{Du^{-1}[u(x) + \epsilon(s, x)]}.$$

Therefore, for x_1, x_2, x_3, x_4 in (α, β) with $x_1 \neq x_2$ and $x_3 \neq x_4$, and for $|g(s)|$ sufficiently small, we have

$$\int_{x_1}^{x_2} \frac{dx}{\Delta_s(x)} \Big/ \int_{x_3}^{x_4} \frac{dx}{\Delta_s(x)} = \int_{x_1}^{x_2} \frac{dx}{Du^{-1}[u(x) + \epsilon(s, x)]} \Big/ \int_{x_3}^{x_4} \frac{dx}{Du^{-1}[u(x) + \epsilon(s, x)]}.$$

It now suffices to show that

$$\lim_{g(s)\to 0} \int_{x_1}^{x_2} \frac{dx}{Du^{-1}[u(x) + \epsilon(s, x)]}$$

exists and is equal to

$$\int_{x_1}^{x_2} \{\lim_{g(s)\to 0} Du^{-1}[u(x) + \epsilon(s, x)]\}^{-1}\, dx$$

for any $x_1 < x_2$ in (α, β). For this latter expression clearly equals

$$\int_{x_1}^{x_2} Du(x)\, dx = u(x_2) - u(x_1);$$

and the theorem follows immediately.

To achieve this, it suffices to show that $1/Du^{-1}[u(x) + \epsilon(s, x)]$ converges uniformly on $[x_1, x_2]$ to $1/Du^{-1}[u(x)]$, as $g(s) \to 0$. We have

$$\left| \frac{1}{Du^{-1}[u(x) + \epsilon(s, x)]} - \frac{1}{Du^{-1}[u(x)]} \right| = \left| \frac{Du^{-1}[u(x)] - Du^{-1}[u(x) + \epsilon(s, x)]}{Du^{-1}[u(x) + \epsilon(s, x)]\, Du^{-1}[u(x)]} \right|.$$

Let $\eta > 0$ be arbitrary. Let δ be chosen as above and let the positive lower bound of Du^{-1} in $[u(x_1) - \delta, u(x_2) + \delta]$ be ρ. Since Du^{-1} is uniformly continuous on this interval, we can choose $\xi > 0$ so small that for $| y_1 - y_2 | < \xi$, $| Du^{-1}(y_1) - Du^{-1}(y_2)| < \eta\rho^2$. Then for $| g(s)| < \inf\{\delta, \xi\}$, we have

$$\left| \frac{1}{Du^{-1}[u(x) + \epsilon(s, x)]} - \frac{1}{Du^{-1}[u(x)]} \right| < \frac{\eta\rho^2}{Du^{-1}[u(x) + \epsilon(x, s)]\, Du^{-1}[u(x)]} \leqslant \eta,$$

uniformly for x in $[x_1, x_2]$. This completes the proof.

Note that the key to this proof is the normalization, which permits the factor $g(s)$ to cancel out of the expression for $\Delta_s(x)$.

If we define

$$u_s(x) = \int_{x_1}^{x} \frac{dy}{\Delta_s(y)},$$

then the theorem asserts that for any x, x',

$$\frac{u(x) - u(x_1)}{u(x') - u(x_1)} = \lim_{g(s)\to 0} \frac{u_s(x) - u_s(x_1)}{u_s(x') - u_s(x_1)}.$$

Thus, in a sense, the function u_s, obtained by integrating the reciprocal of Δ_s, is an approximation to the function u. The sense is the one that matters: all one can hope to know, since u is only an interval scale, is a ratio of differences, $[u(x)-u(x_1)]/[u(x')-u(x_1)]$ and the ratio of differences of u_s-scale values approximates that of u-scale values.

To put matters still another way, suppose that we fix the zero and unit of the u-scale, at x_1 and x_2, respectively. Then our theorem asserts that

$$u(x) = \lim_{g(s)\to 0} \frac{1}{C(s)} \int_{x_1}^{x} \frac{dy}{\Delta_s(y)},$$

where

$$C(s) = \int_{x_1}^{x_2} dy/\Delta_s(y).$$

Thus, we can define a normalized approximation

$$\bar{u}_s(x) = \frac{1}{C(s)} \int_{x_1}^{x} dy/\Delta_s(y)$$

such that $\bar{u}_s(x) \to u(x)$ pointwise as $g(s) \to 0$.

Discussion

There are two cases to discuss. First, in the situation where $\Delta_s(x)$ is given as a function of s and x, then clearly, it makes sense to calculate the normalized integral $u_s(x)$ as above and to evaluate the limit to obtain $u(x)$. To illustrate, consider the Weber function

$$\Delta_s(x) = k(s)(x + x_0)^{\frac{1}{2}} + \tfrac{1}{4}k(s)^2. \tag{8}$$

We calculate the integral of $dx/\Delta_s(x)$ by substituting for x, introducing a new variable w_s:

$$x = [w_s - \tfrac{1}{4}k(s)]^2 - x_0,$$
$$dx = 2[w_s - \tfrac{1}{4}k(s)]\, dw_s.$$

We obtain

$$\int \frac{dx}{\Delta_s(x)} = \int \frac{2[w_s - \frac{1}{4}k(s)]\, dw_s}{k(s)\, w_s} = \frac{2}{k(s)} \int dw_s - \frac{1}{2} \int \frac{dw_s}{w_s} = \frac{2}{k(s)} w_s - \frac{1}{2} \log w_s.$$

As we let $k(s) \to 0$, the first term dominates, and the normalized $\bar{u}_s$ is approximately a linear function of w_s. In turn, w_s converges to $(x + x_0)^{\frac{1}{2}}$. If we substitute

$$u(x) = (x + x_0)^{\frac{1}{2}},$$

we find that Eq. 1 is satisfied with $\Delta_s(x)$ given by Eq. 8 and with $g(s) = \frac{1}{2}k(s)$.

For Weber functions less contrived than Eq. 8, it may be necessary to resort to numerical integration and to numerical approximation of the limit—which may amount merely to choosing a sufficiently fine JND size (index s) and hoping that the resulting integral is close enough to the limit. The estimated u-scale can always be checked by substituting back in Eq. 1.

The special feature of the Weber-Law case (Eq. 2) is that no passage to the limit is needed: the ratio of u_s-differences is independent of s.

For this first case, where $\Delta_s(x)$ is given as a function of s and x, there is even a generalization of Fechnerian integration to a case where Eq. 1 is satisfied only "approximately" as $\Delta_s(x) \to 0$. This was pointed out to me by M. Frank Norman. If we replace Eq. 1 by

$$u[x + \Delta_s(x)] - u(x) = g(s) + o(\Delta_s(x)), \tag{1'}$$

we can calculate u as the integral of the normalized limit of $1/\Delta_s(x)$. More precisely, Norman (personal communication) showed that Eq. 1′ has a solution (u, g) with u continuously differentiable and with $Du(y) \neq 0$ (for some fixed y) if and only if

$$\lim_{\Delta_s(x)\to 0} \frac{\Delta_s(y)}{\Delta_s(x)}$$

exists and is continuous; and u is then the integral of this limit. (Δ_s is assumed strictly positive.)

The second case to discuss is where $\Delta_s(x)$ is given as a function of x, for some fixed value s of the discriminability index. This is the typical case when the Weber function is estimated from data, rather than specified fully either by a theory or an overall generalization from data like Eqs. 2 or 8. Typically, the data yield estimates of a few points on each of several psychometric functions around different stimulus values $x_1 \leqslant \cdots \leqslant x_n$; thus we have estimates of $\Delta_{s_{ij}}(x_i)$, $j = 1,..., m_i$. Frequently, the criterion values s_{ij} are different for different standard stimuli x_i: the comparison stimuli near x_i are chosen in advance by the experimenter, and the performance of the subject on the jth comparison stimulus near x_i determines the value s_{ij} such that $\Delta_{s_{ij}}(x_i) = x_j - x_i$. One must interpolate to estimate $\Delta_s(x_i)$ for some fixed s, independent of i. One can do this for several values of s, but not for values such that Δ_s becomes arbitrarily small (relative errors become too large). The estimation of $\Delta_s(x_i)$ for fixed s is facilitated in the currently popular up-and-down methods, where comparison stimuli are made to hover near a particular value of s. Once $\Delta_s(x_i)$ is obtained, for $i = 1,..., n$, we can interpolate to obtain an estimate of $\Delta_s(x)$, $x_1 \leqslant x \leqslant x_n$. There are two methods of estimating u from Δ_s: integration of $dx/\Delta_s(x)$ or Luce and Edwards' "graphical" method. These must be compared.

The Luce and Edwards method simply cumulates JND's: assuming $\Delta_s(x) > 0$ for all x, let $y_0 = x_1$, $y_1 = y_0 + \Delta_s(y_0)$,...

$$y_i = y_{i-1} + \Delta_s(y_{i-1}). \tag{9}$$

Set $u(y_i) = i$ [this normalizes $u(y_0) = 0$, $u(y_1) = 1$]; interpolate to get u-values for intermediate values of x. This method directly utilizes only the values of Δ_s at y_0, $y_1, \ldots, y_i, \ldots$; but since these values, in general, are not measured empirically the method utilizes the empirical values $\Delta_s(x_i)$ to determine a smooth curve giving the ordinates at the y_i values. Sources of error in the method are: (1) error in determining $\Delta_s(x_i)$; (2) error in interpolating to get $\Delta_s(y_i)$; (3) error in interpolating to get $u(x)$ for $y_{i-1} < x < y_i$. Note that, given the curve Δ_s, there is no further error in determining u at the points $y_0, y_1, \ldots, y_i \ldots$; and error source (3), for points between the y_i, may be rather trivial, if the points y_i are close enough together so that linear interpolation suffices.

If we carry out the first steps, through estimation of $\Delta_s(x)$ as a smooth curve, as in the Luce and Edwards method, then integration can produce an additional source of error. To see this, let $y_0, y_1, \ldots, y_i$ be as above (Eq. 9), so that the correct values are $u(y_i) = i$. Integration [normalized so $u(y_0) = 0$, $u(y_1) = 1$] gives:

$$
\begin{aligned}
u(y_i) &= \int_{y_0}^{y_i} \frac{dx}{\Delta_s(x)} \Big/ \int_{y_0}^{y_1} \frac{dx}{\Delta_s(x)} \\
&= \sum_{\ell=1}^{i} \int_{y_{\ell-1}}^{y_\ell} \frac{dx}{\Delta_s(x)} \Big/ \int_{y_0}^{y_1} \frac{dx}{\Delta_s(x)} \\
&= \sum_{\ell=1}^{i} \frac{(y_\ell - y_{\ell-1})}{\Delta_s(\bar{x}_\ell)} \Big/ \frac{(y_1 - y_0)}{\Delta_s(\bar{x}_1)} \\
&= \sum_{\ell=1}^{i} \frac{\Delta_s(y_{\ell-1})}{\Delta_s(\bar{x}_\ell)} \Big/ \frac{\Delta_s(y_0)}{\Delta_s(\bar{x}_1)} .
\end{aligned} \tag{10}
$$

In the last two expressions, we let $\bar{x}_\ell$ be any value of x between $y_{\ell-1}$ and y_ℓ, where $1/\Delta_s(x)$ assumes its mean value over that interval, i.e., $\bar{x}_\ell$ is defined by

$$
\Delta_s(\bar{x}_\ell) = \frac{1}{y_\ell - y_{\ell-1}} \int_{y_{\ell-1}}^{y_\ell} \frac{dx}{\Delta_s(x)} .
$$

The existence of such an $\bar{x}_\ell$ is assured by the First Mean Value Theorem for integrals.

If the percentage variation in $\Delta_s(x)$ on each interval $(y_{\ell-1}, y_\ell)$ is negligible, then

$$
\Delta_s(y_{\ell-1})/\Delta_s(\bar{x}_\ell) \approx 1, \qquad \ell = 1, \ldots, i,
$$

and the expression in Eq. 10 reduces to $u(y_i) = i$, coinciding with the Luce–Edwards value. In fact, for the case where Δ_s is an increasing function of x for each s, the above fractions tend to unity[1] as $g(s) \to 0$. For this case, the percentage error in $\bar{u}_s(y)$ cannot exceed the maximum percentage deviation of $\Delta_s[y + \Delta_s(y)]$ from $\Delta_s(y)$, over the interval from y_0 to y.

If the terms in Eq. 10 are appreciably different from 1, then there can be a proportionate error in estimating $u(y_i)$ by integration. The case where Weber's Law (Eq. 2) holds is instructive, however. There,

$$\Delta_s(y_{\ell-1})/\Delta_s(\bar{x}_\ell) = \{\log[1 + k(s)]\}/k(s),$$

independent of ℓ. This fraction does approach unity as $g(s)$ and $k(s) \to 0$, but since its value is independent of ℓ, it cancels out of Eq. 10, leaving $u(y_i) = i$ as required. Thus, there is appreciable error introduced by the integration method, in determining $u(y_i)$, only when both Δ_s is far from linear (Weber's Law) and at the same time Δ_s varies widely over some intervals $[y, y + \Delta_s(y)]$.

Another instructive case is the Weber function given by Eq. 8, corresponding to the square-root scale. Here we obtain

$$\frac{\Delta_s(y_{\ell-1})}{\Delta_s(\bar{x}_\ell)} = \int_{y_{\ell-1}}^{y_\ell} \frac{dx}{\Delta_s(x)} = 1 - \frac{1}{2}\log\left(1 + \frac{k(s)/2}{(y_{\ell-1} + x_0)^{\frac{1}{2}} + k(s)^2/4}\right).$$

The logarithmic term goes to zero as $y_{\ell-1}$ becomes large, so asymptotically, the numerator of Eq. 10 is i and the denominator is

$$1 - \frac{1}{2}\log\left(1 + \frac{k(s)/2}{(y_0 + x_0)^{\frac{1}{2}} + k(s)^2/4}\right).$$

Thus, the percentage error decreases as y_i increases, approaching an asymptote determined by y_0. The asymptotic percentage error decreases as y_0 increases or as $k(s) \to 0$. Even for $k(s) = 0.1$ and $y_0 + x_0 = 1.0$ [note that the units of $k(s)$ and x are linked in Eq. 8], the asymptotic error is only about 2.5%.

Note that as $g(s)$ becomes small, the Luce and Edwards method is tantamount to numerical integration of $1/\Delta_s(x)$. For $g(s)$ large, their method is probably still to be preferred. The real advantage of integration emerges when $\Delta_s(x)$ is specified in closed form as a function of both s and x, either by a theory or as a generalization about a broad class of data.

[1] For the proof use Eq. 5, differentiability of Du^{-1}, and L'Hospital's rule:

$$\begin{aligned}\lim_{g(s)\to 0} \frac{\Delta_s[y + \Delta_s(y)]}{\Delta_s(y)} &= \lim_{g(s)\to 0} \frac{u^{-1}[u(y) + 2g(s)] - y}{u^{-1}[u(y) + g(s)] - y} - 1 \\ &= \lim_{g(s)\to 0} 2\,\frac{Du^{-1}[u(y) + 2g(s)]}{Du^{-1}[u(y) + g(s)]} - 1 \\ &= 1.\end{aligned}$$

In concluding, I wish to comment that the return to Fechnerian integration should not be regarded as a repudiation of the functional-equation approach. The integration gives only meaningless results, unless a solution to Eq. 1 *exists*. For example Eisler (1963) pointed out that the integral $\int dx/\Delta_s(x)$ has the Weber–Law property of being independent of s (except for a scale change) whenever

$$\Delta_s(x) = k(s) f(x). \tag{11}$$

This is true, but the resulting u-scale does not solve Eq. 1. Indeed, using Falmagne's criteria, it is easy to show that when Eq. 11 holds, there is no solution to Eq. 1 unless f is linear (back to Weber's Law).

The question of existence of a solution can perhaps be *answered* by integrating and substituting the integral back in the functional equation, but it cannot even be *raised*—as it was not raised by Fechner—without the formulation provided by Luce and Edwards.

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