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MUON ENERGY LOSSES AND STRAGGLING FOR HIGH ENERGY MUONS

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The non-Gaussian nature of energy loss straggling is examined. At NAL large muon shields must be designed to reduce flux by about a factor of 10^{12}. Fortunately it is shown here that on the long range end, the straggling falls off much faster than Gaussian and the amount of shielding needed is significantly reduced.

1. Introduction

The purpose of this article is to look at range and straggling effects as they influence design of NAL muon shields. The major problem here is that one is trying to shield down to levels of around 1 in 10^{12}. I will generally limit myself to collision losses and direct pair production only. Fluctuations on bremsstrahlung losses are known to be so large that they are useless to this kind of level. (The shield is ≈ 1 muon radiation length.) Nuclear interactions should probably also be examined as the nuclear cross sections are dominated by low energy transfers, due to giant dipole resonance, and may be useable.

We are used to taking errors as Gaussian. However, if true this would be disastrous for shielding to this level. For the 10^{-12} region we want to go out some 7 standard deviations. Since at NAL energies $(\Delta R/R)^2$ is typically 10% one would have to add 70% to the calculated range.

Fortunately as we shall see the errors are not Gaussian and a much smaller number suffices.

2. Source of straggling

To begin with we will consider collision loss only, ignoring the effects of direct pair production. Let us first ask what fraction of the energy loss is due to high transferred energies. We use the results quoted by Rossi[1]) modified slightly for spin $\frac{1}{2}$ particles:

$$\frac{\mathrm{d}E}{\mathrm{d}x}(\mathrm{gcm}^2) = \frac{2Cm_e}{\beta^2} \times \left[\ln A - \beta^2\left(1+\frac{E'_{\mathrm{MAX}}}{E'_{\mathrm{m}}}\right) + \frac{1}{4}\frac{E'^2_{\mathrm{MAX}}}{(E_\mu+m_\mu)^2}\right],$$

where

$$A = \frac{2m_e p_\mu^2 E'_{\mathrm{MAX}}}{m_\mu^2 I^2(Z)};$$

E'_{MAX} = the maximum energy transfer we wish to consider;

E'_{m} = the upper limit of E'_{MAX}:

$$E'_{\mathrm{m}} = \frac{2m_e p_\mu^2}{m_e^2+m_\mu^2+2m_e E_\mu}$$

(for $E_\mu \gg 20$ GeV, $E'_{\mathrm{m}} \sim E_\mu$);

I = ionization potential of the atom;

$C = \pi N(Z/A) r_e^2 = 0.15(Z/A)$.

This formula ignores the density effect. The density effect only involves low energy transfers where the statistical fluctuation is small. The influence of this effect will be considered later.

The results are given in table 1. We first note that over most of the range the probabilities of different percentages of energy loss remain approximately constant. This occurs since A is such a huge number (around 10^{20}) that $\ln A$ doesn't change by a large percentage even if A changes by a factor of a hundred or so. The major cause of straggling is now clear. For instance look at the top 2 lines of table 1. 2.3% of the energy loss occurs because of collisions with more than 20% of the muon energy transferred. This is far less than 1 collision per stopping muon. The approximate numbers of collisions in each interval are given in the last column for 500 GeV muons. Until transfer gets well below 1% of the incident energy one is dealing with statistics of very small numbers. However, this accounts for only around 10% of the energy loss and the rest is subject to much smaller fluctuations.

3. Straggling due to collision loss

We will now develop a straggling formula valid for that part of the collision loss caused by low energy transfers.

The energy loss in $\mathrm{d}x$ from collisions transferring between E' and $E'+\mathrm{d}E'$ is $E'N(E')\mathrm{d}E'\mathrm{d}x$ where N = number of these collisions (GeV/gcm^2). We approximate the error in this by taking the square root of the number of collisions i.e. $E'^2N(E')\mathrm{d}E'\mathrm{d}x = (\delta E)^2$. The total error is obtained by taking

TABLE 1
Energy loss by collision. Fraction of energy loss due to large energy collisions.

E_μ (GeV)	E'_{MAX}	E'_{MAX}/E_μ	$\ln A - \beta^2\left(\frac{1+E'_{MAX}}{E'_m}\right) + \frac{1}{4}\frac{E'_{MAX}}{(E_\mu+m_\mu)^2}$	% dE/dx from energy transfers > E'_{MAX}	No. of collisions in interval
500	500	1	44.95	0	0.045
	100	0.2	43.9	2.3	0.08
	50	0.1	43.4	3.5	0.7
	10	0.02	41.8	7.0	1
	5	0.01	41.1	8.6	7
	1	0.002	39.5	12.1	10
	0.5	0.001	38.8	13.7	
100	100	1	41.65	0	
	20	0.2	40.6	2.6	
	10	0.1	40.1	3.7	
	2	0.02	38.5	7.6	
	1	0.01	37.8	9.3	
	0.2	0.002	36.2	13.1	
	0.1	0.001	35.5	14.8	

$$\sum(\delta E)^2 = (\Delta E)^2 = \iint_0^{E_{MAX}} (E')^2 N(E')\,dE'\,dx,$$

where E_{MAX} is the maximum energy transfer we wish to consider. We then approximate $\Delta R/R \approx \Delta E/E_\mu$ where E_μ is the initial energy.

For most of the range the energy loss is very crudely constant. (Again this occurs as A is approximately constant, see table 1. The inclusion of the density effect only helps this approximation.)

Hence

$$E = \overline{\left(\frac{dE}{dx_{total}}\right)}(R-x), \quad \text{where } R = \text{range.}$$

We use[1])

$$N_{coll}(E,E')\,dE' = \frac{2Cm_e}{\beta^2}\frac{dE'}{(E')^2} \times \left[1 - \beta^2\frac{E'}{E'_m} + \frac{1}{2}\left(\frac{E'}{E+m_\mu}\right)^2\right].$$

(Note: the last term is specific to spin $\frac{1}{2}$ particles.) We ignore very low energy collisions as a source of straggling. (For these collisions ($E' < \eta$) atomic effects are important.) We also set $\beta = 1$ and ignore m_μ compared to E. Then

$$\Delta E^2 = \iint_\eta^{E'_{MAX}} E'^2 N\,dE'\,dx$$
$$= \frac{2Cm_e}{\beta^2}\frac{E'_{MAX}}{2}\left(1 - \frac{E'_{MAX}}{2E'm} + \frac{1}{6}\frac{E'^2_{MAX}}{E^2_\mu}\right)\frac{E_\mu}{(dE/dx)_{tot}}.$$

This derivation has assumed that at any point in the path we consider losses due to collisions below a constant fraction of the energy at that point. The fraction $F = (E'_{MAX}/E_\mu)_{original}$. Using our previous expression for dE/dx (including only collision losses and ignoring the density effect for now) we obtain:

$$\left(\frac{\Delta R}{R}\right)^2 = \left(\frac{\Delta E}{E_\mu}\right)^2 = \frac{E'_{MAX}}{E_\mu}\left[2\left(\left\{\ln\frac{2m_e\bar{E}^3_\mu}{m^2_\mu I^2(Z)}\right\} - 1.75\right)\right]^{-1} \times \left(1 - \frac{E'_{MAX}}{2E'_m} + \frac{1}{6}\frac{E'^2_{MAX}}{E_{\mu^2}}\right),$$

$E_\mu > 20$ GeV.

The density effect reduces the denominator somewhat increasing the error. At 500 (100) GeV $(dE/dx)_{no\ density\ effect} = 3.17$ (2.82) and the Sternheimer[2]) density effect is for iron 0.88 (0.65). $(\Delta R/R)^2$ above should then be multiplied by a factor $3.17/(3.17-0.88) = 1.38$ et 500 GeV. The important thing to note here is that the error decreases as $E'_{MAX}/E_\mu = F$. For $F = 1$, $\Delta R/R = 0.087$ at 500 GeV (0.10 if density effect is included.)

Suppose we want to shield 500 GeV muons to the 10^{-12} level. If we set E'_{MAX}/E_μ at 0.002 then (including density effect) $\Delta R/R \approx 0.0055$. Using 7 standard deviations we see $7 \times 0.55\% = 3.9\%$ and we must also include the 16.4%[3]) of the loss due to high energy transfers. Hence we should use

$$\frac{1-0.16}{1+0.039}\left(\frac{dE}{dx}\right)_{full\ col.\ loss} = 0.80\left(\frac{dE}{dx}\right)_{full\ col.\ loss}$$

for designing shielding to be safe if only collision losses are included.

TABLE 2
Direct pair production in earth as a function of muon energy (GeV) and α.

E_μ(GeV)/α	1	1.5	3
100	0.17	0.19	0.22
200	0.37	0.41	0.48
300	0.58	0.64	0.74
400	0.79	0.87	1.01
500	1.00	1.10	1.28

In passing we note that the above procedure can easily be modified for $E_\mu < 20$–40 GeV and a closed form obtained for straggling at all relativistic energies. (This just involves replacing E_{MAX} by E'_m and replacing the log by the appropriate terms with $E'_m < E_\mu$.)

There is a broad maximum in the straggling introduced by the fact that E'_m approaches E_μ as E_μ increases. This has the effect of increasing the straggling due to more high energy collisions. However, the logarithm in the denominator also increases and eventually takes over. The maximum occurs at around 150 GeV (8.7%) if the density effect is ignored or 500 GeV (10%) if one includes the density effect.

4. Straggling due to direct pair production

Next we turn to direct pair production. The theory here is not in as good shape as that for collision losses and the existing formulas can only be integrated numerically. Most existing calculations[4-7]) are based on the work of Bhabha[8]) and Racah[9]) which are subject to several approximations. I will use the more modern treatment given by Murota, Ueda and Tanaka[10]). There is still much room for further improvement. A treatment using one of the computerized reductions of the matrix elements is clearly called for if the calculations are to get to 10% accuracy. In addition the atomic shielding has to be included in a better manner than at present. The formula I use is given in the appendix[11]). I use the completely shielded form whenever it gives a smaller result than the unshielded form. This should be a safe approximation since partial shielding should increase the energy loss over complete shielding.

There is a parameter, α, in the treatment which is arbitrary but of the order of 1. Results for earth $\bar{Z} = 12$, $\bar{Z}/A = 0.5$ for different values of α give the results shown in table 2. I settled on $\alpha \approx 1.5$ since this seemed to give in the logarithm $137\,\alpha \approx 206$ which is close to the 183 used in many of the calculations.

Table 3 gives results for iron and table 4 gives results for earth (units are MeV/gcm^2). The last column in table 3 gives the results of a crude integration to get the average $(\mathrm{d}E/\mathrm{d}x)_{\text{direct pair}}$ over the interval 0–500 GeV.

The values I obtain for $\mathrm{d}E/\mathrm{d}x$ are somewhat higher than those of Theriot[7]). For instance, at 500 GeV on iron I find 2.03 while Theriot obtains 1.53 and at 100 GeV I obtain 0.35 while Theriot gets 0.275.

Using the same procedure for $\Delta R/R$ as outlined for the collision loss case, but using numerical integration, I obtained the results shown in table 5 for 500 GeV muons. Although not as good as collision loss again I find most of the energy is lost in low energy transfer collisions. The R used for normalising here is the range from collision loss only. Fig. 1 shows some old results[11]) obtained at 200 GeV for typemetal. The shape of the direct pair cross section can be seen here. It is clear that very low and very high energy transfers are both decreased relative to collision loss.

If we use R from the total $\mathrm{d}E/\mathrm{d}x$ our errors in table 5 should be reduced by $\sqrt{(2.3+0.98)/2.3} = 1.20$ ($\Delta R_{new}/R = \Delta R_{old}/R/1.20$).

TABLE 3
Direct pair production in iron as a function of muon energy (GeV) and fractional energy transfer.

$\mathrm{d}E/\mathrm{d}x$ for energy transfers less than E_{pair}

$\frac{E_{pair}}{E_\mu}$ / E_μ	100.00	200.00	300.00	400.00	500.00	$\overline{\left(\frac{\mathrm{d}E}{\mathrm{d}x}\right)}$
0.0005	0.017	0.048	0.086	0.126	0.168	0.072
0.0010	0.038	0.100	0.169	0.241	0.315	0.141
0.0020	0.075	0.182	0.297	0.414	0.533	0.247
0.0050	0.147	0.334	0.528	0.725	0.925	0.44
0.0100	0.208	0.461	0.722	0.986	1.252	0.60
0.0200	0.263	0.574	0.894	1.217	1.543	0.74
0.0500	0.313	0.678	1.052	1.429	1.809	0.88
0.1000	0.334	0.723	1.120	1.521	1.924	0.93
0.2000	0.344	0.746	1.156	1.571	1.987	0.96
1.0000	0.350	0.760	1.180	1.603	2.029	0.98

Fraction of $\mathrm{d}E/\mathrm{d}x$ due to energy losses greater than E_{pair}

$\frac{E_{pair}}{E_\mu}$ / E_μ	100.00	200.00	300.00	400.00	500.00
0.0005	0.951	0.936	0.927	0.921	0.917
0.0010	0.891	0.868	0.857	0.850	0.845
0.0020	0.785	0.760	0.749	0.742	0.737
0.0050	0.581	0.561	0.553	0.548	0.544
0.0100	0.406	0.394	0.388	0.385	0.383
0.0200	0.249	0.245	0.242	0.241	0.240
0.0500	0.106	0.108	0.109	0.109	0.109
0.1000	0.047	0.050	0.051	0.051	0.052
0.2000	0.017	0.019	0.020	0.020	0.021
1.0000	0.000	0.000	0.000	0.000	0.000

TABLE 4
Direct pair production in earth as a function of muon energy (GeV) and fractional energy transfer.

$\mathrm{d}E/\mathrm{d}x$ for energy transfers less than E_{pair}

$\frac{E_{\mathrm{pair}}}{E_\mu} \Big/ E_\mu$	100.00	200.00	300.00	400.00	500.00
0.0005	0.009	0.025	0.045	0.066	0.089
0.0010	0.020	0.052	0.089	0.128	0.168
0.0020	0.039	0.097	0.159	0.223	0.288
0.0050	0.078	0.179	0.285	0.393	0.503
0.0100	0.111	0.249	0.391	0.536	0.682
0.2000	0.141	0.311	0.485	0.662	0.841
0.0500	0.168	0.367	0.571	0.778	0.986
0.1000	0.179	0.391	0.608	0.827	1.048
0.2000	0.184	0.403	0.627	0.854	1.082
1.0000	0.187	0.410	0.639	0.871	1.104

Fraction of $\mathrm{d}E/\mathrm{d}x$ due to energy losses greater than E_{pair}

$\frac{E_{\mathrm{pair}}}{E_\mu} \Big/ E_\mu$	100.00	200.00	300.00	400.00	400.00
0.0005	0.953	0.939	0.930	0.924	0.920
0.0010	0.894	0.872	0.860	0.853	0.848
0.0020	0.790	0.764	0.752	0.745	0.740
0.0050	0.584	0.564	0.554	0.549	0.545
0.0100	0.406	0.394	0.388	0.385	0.382
0.0200	0.247	0.243	0.241	0.240	0.239
0.0500	0.104	0.106	0.107	0.107	0.107
0.1000	0.045	0.048	0.050	0.050	0.051
0.2000	0.017	0.018	0.019	0.019	0.020
1.0000	0.000	0.000	0.000	0.000	0.000

TABLE 5
Straggling for 500 GeV muons in iron from direct pair production.

$\frac{E'}{E_\mu}$	$\frac{\Delta R}{R}$	% of total $(\mathrm{d}E/\mathrm{d}x)_{\mathrm{dp}}$ with $E' >$ this value	$\overline{\left(\frac{\mathrm{d}E}{\mathrm{d}x}\right)}$
0.0005	0.00186	0.917	0.072
0.001	0.00353	0.845	0.141
0.002	0.00716	0.737	0.247
0.005	0.0122	0.544	0.44
0.01	0.0187	0.383	0.60
0.02	0.0264	0.240	0.74
0.05	0.0375	0.109	0.88
0.1	0.0455	0.052	0.93
0.2	0.053	0.021	0.96
1.0	0.0645	0	0.98

5. Conclusions

What is a safe fraction of the $(\mathrm{d}E/\mathrm{d}x)_{\mathrm{dp}}$ to use in shielding calculations? We look at table 5, at the changes in $\Delta R/R$, i.e. $\Delta(\Delta R/R)$. We ask where $\Delta(\Delta R/R)\cdot 7\cdot\overline{(\mathrm{d}E/\mathrm{d}x)}$ is equal to the $\Delta(\overline{\mathrm{d}E/\mathrm{d}x})$ from row to row. $E'_{\mathrm{MAX}}/E_\mu = 0.05$ seems about best. $7\cdot 0.0375\cdot(1/1.20)\cdot(3.28/0.98) = 0.73$. We then include 11% loss with energy transfers > 0.05. Hence for shielding it would appear safe to use

$$\left(\frac{\mathrm{d}E}{\mathrm{d}x}\right)_{\text{full collision loss}} 0.79 + \left(\frac{\mathrm{d}E}{\mathrm{d}x}\right)_{\text{full direct pair loss}} 0.52.$$

The above fractions should be about right independently of whether Theriot or I have the correct direct pair $\mathrm{d}E/\mathrm{d}x$. Furthermore this should hold unchanged for earth since as seen in tables 3 and 4 the percentage of loss due to high energy transfers remains the same as for iron.

Using the present numbers for losses averaged 0–500 GeV: $2.26\cdot 0.79 + 0.98\cdot 0.52 = 2.30$ MeV/gcm^2

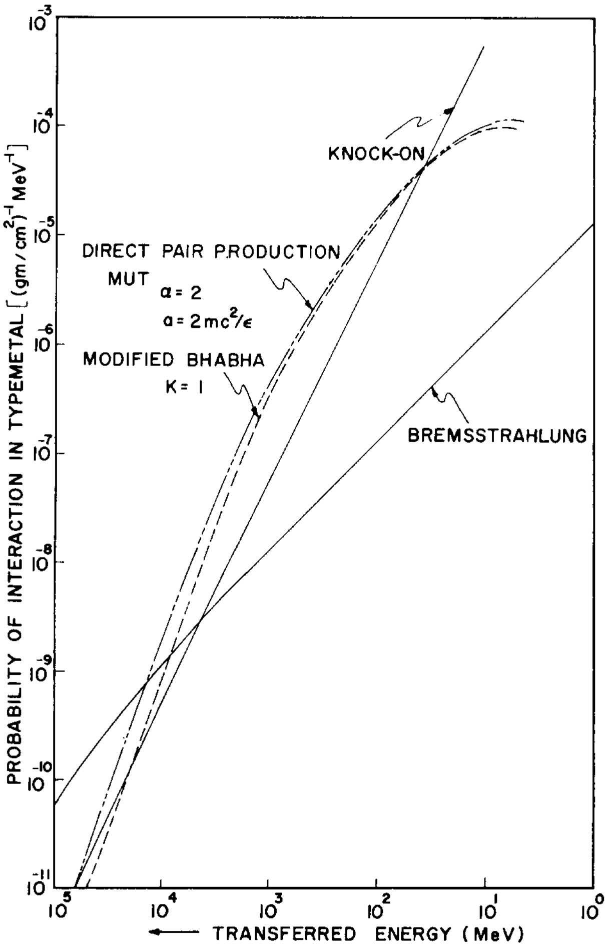

Fig. 1. Direct electron pair production by 200 GeV muons in typemetal (90% lead) as a function of energy transferred to the pair.

should be a reasonable $\overline{(\mathrm{d}E/\mathrm{d}x)}$ to use for shielding calculations in iron for shielding down to the 10^{-12} level. (Using Theriot values we obtain $2.20 \cdot 0.79 + 0.74 \cdot 0.52 = 2.12$ MeV/gcm^2.) Hence we find with an iron density of 7.87 that 285 m (300 m with Theriot values) of iron is adequate for 500 GeV.

For earth we find at 500 GeV using my numbers (Theriot numbers) $\overline{(\mathrm{d}E/\mathrm{d}x)} = 2.55 \cdot 0.79 + 0.574 \cdot 0.52 = 2.31$ $(2.51 \cdot 0.79 + 0.421 \cdot 0.52 = 2.20)$ MeV/gcm^2. Hence for density 2 we find 1085 (1140) m of earth is adequate for 500 GeV.

Appendix

NUMERICAL EVALUATION AND INTEGRATION OF THE MUROTA-UEDA-TANAKA DIRECT PAIR PRODUCTION CROSS SECTION

The formula is given by[10])

$$\sigma(\varepsilon, s)\,\mathrm{d}\varepsilon\,\mathrm{d}s = \frac{2}{\pi}\frac{N}{A}\left(\frac{Zr_0}{137}\right)^2 L\frac{1}{\varepsilon}$$

$$\times\Bigg[\Bigg\{\left[(1+\tfrac{4}{3}\chi)\log\left(1+\frac{1}{\chi}\right)-\tfrac{4}{3}\right]$$

$$-\tfrac{4}{3}(s-s^2)\left[(1+\chi)\log\left(1+\frac{1}{\chi}\right)-1\right]\Bigg\}$$

$$\times\left[\frac{E^2+(E-\varepsilon)^2}{E^2}\right]+\tfrac{8}{3}(s-s^2)\frac{1}{1+\chi}\frac{E-\varepsilon}{E}$$

$$+\Bigg\{\left[\frac{1}{3}\frac{1}{(1+\chi)}+\frac{1}{\chi}-\tfrac{4}{3}\log\left(1+\frac{1}{\chi}\right)\right]$$

$$+(s-s^2)\left[-\frac{4}{3\chi}+\tfrac{4}{3}\log\left(1+\frac{1}{\chi}\right)\right]\Bigg\}$$

$$\times\chi\frac{\varepsilon^2}{E^2}\Bigg]\,\mathrm{d}s\,\mathrm{d}\varepsilon \text{ per gcm}^{-2},$$

where

$s = \varepsilon^-/\varepsilon$,

$\chi = A(s-s^2)$,

$A = \dfrac{\mu^2}{m^2}\dfrac{\varepsilon^2}{E(E-\varepsilon)}$,

ε = combined energy of pair,

ε_- = energy of electron,

m = mass of electron,

μ = mass of incident particle,

E = energy of incident particle,

r_0 = classical radius of electron,

α = constant of order magnitude 1,

$$L = \log\left[\frac{2\alpha E(s-s^2)}{Mc^2}\right] - 1 \quad \text{for non-screening,}$$

$$= \log\left[\alpha 137 Z^{-\frac{1}{3}}\frac{Mc^2}{mc^2}\right] \quad \text{for complete screening,}$$

$$M^2 = m^2\left[1+\frac{(1-s)s\varepsilon^2}{E(E-\varepsilon)}\frac{\mu^2}{m^2}\right].$$

This formula is valid for the case in which the incident particle has $\mu \gg m$, that $E-\varepsilon \gg \mu c^2$, $\varepsilon \ll E$ and $\varepsilon \gg mc^2$. For evaluation, $Z(Z+1)$ was used instead of Z^2 in order to get a rough correction for electron contribution. The incident particle was considered to be a muon. The process was considered as either completely screened or completely unscreened, the criterion being that the smallest of the two possible values for L were chosen unless the smallest value was < 0 in which case 0 was chosen.

The integrals tend to peak near the lower limit and we integrated $\int_{\varepsilon_{\min}}^{E'}\int_{s_{\min}}^{\frac{1}{2}} 2\sigma\,\mathrm{d}s\,\mathrm{d}\varepsilon$ by using a trapedzoidal rule with varying bin widths. Parameter $(N) = [\text{parameter}(N-1)] \cdot [1+\Delta]$. $\Delta = 0.1$ was used. Changing Δ to 0.05 made about a $\frac{1}{2}\%$ change.

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