STRUCTURE FUNCTIONS IN AN INTERACTING BOSON SYSTEM*

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A microscopic analysis of the dynamic structure factor in a simple interacting Boson system at $T = 0$ shows that the coherent two-phonon backflow leads to the breakdown of the one-phonon Feynman relation for the statis structure factor at order k^3.

We report the results of a microscopic analysis of the dynamic structure factor [1] $S(k, \omega)$ and its moments $S_m(k) = \int_0^\infty d\omega\, \omega^m S(k, \omega)$ for a system of Bosons of mass m at zero temperature and density n to the first approximation beyond the Bogoliubov (zeroth) approximation. The two-body interaction v is characterized by the s-wave scattering length and is taken to be a constant in k-space. The small dimensionless parameter $g = 4\pi a m s_0$, where $s_0 = (4\pi a n)^{1/2}/m$ is the phonon speed in the zeroth approximation. We choose units such that $\hbar = m = s_0 = 1$.

We write $S(k, \omega)$ as a coherent one-phonon part, $gS_\mathrm{I}(k, \omega) = Z(k)\,\delta(\omega - \omega_k)$, and a multi-phonon part [2, 3]. In order to isolate the effects of backflow, S_I is constructed to exhaust the f-sum rule, $gS_\mathrm{I}(k) = \frac{1}{2}k^2$ thus, $Z(k) = k^2/2\omega_k$ and ω_k is the energy of the one-phonon excitations. The form of the multiphonon term and the k-dependence of $S_m(k)$ is of interest.

The method of calculation of $S(k, \omega)$ is based on the generalized dielectric formulation [4] of Bose systems and can be outlined as follows. We relate $S(k, \omega)$ to the density-density response function [4] $F(k, \omega)$; $S(k, \omega) = -\mathrm{Im}\, F(k, \omega)/\pi$. The effects of backflow are conveniently taken into account by the identity [4] $gF(k, \omega) = (k^2/\omega^2)[1 + gF^{33}(k, \omega)]$ based on the continuity equation, where F^{33} is the longitudinal current response function [4]. A perturbation expansion for $S(k, \omega)$ is developed by expanding F^{33}, F, S, ω_k^2 and v in powers of g, e.g., $gS(k, \omega) = S^{(0)}(k, \omega) + gS^{(1)}(k, \omega) + \ldots$; $\omega_k^2 = \omega_k^{2(0)} + g\omega_k^{2(1)} + \ldots$; $v/g = 1 + gv^{(1)} + \ldots$. We find that to $O(g^0)$ $S(k, \omega)$ has only a coherent one-phonon part $S_\mathrm{I}^{(0)}(k, \omega) = Z^{(0)}(k)\,\delta(\omega - \omega_k^{(0)})$, where $Z^{(0)}(k) = k^2/2\omega_k^{(0)}$ and $\omega_k^{(0)} = k(1 + k^2/4)^{1/2}$. To $O(g)$ we obtain

$$F^{(1)}(k, \omega) = k^2\omega^{-2}(\omega^2 - \omega_k^{2(0)})^{-2} N^{33(1)}(k, \omega) \tag{1}$$

$$N^{33(1)}(k, \omega) = \omega^2 k^2 v + \tfrac{1}{2}k^6(S - \mu) + \tfrac{1}{2}\omega k^4 A + (\omega^2 - k^4/4)[\tfrac{1}{2}k^2(S + M_2 - \mu) - \tfrac{1}{4}k^4 n' + k\omega(\Lambda_+^3 + \Lambda_-^3) + \tfrac{1}{2}k^3(\Lambda_+^3 - \Lambda_-^3)] + + (\omega^2 - k^4/4)^2 F^{33\mathrm{r}}, \tag{2}$$

where the integrals on the right hand side of eq. (2) are given explicitly in ref. [5] with $Q^{\pm} = (\omega - \omega_{p+k}^{(0)} - \omega_p^{(0)})^{-1} + \mp (\omega + \omega_{p+k}^{(0)} + \omega_p^{(0)})^{-1}$. To $O(g)$ the dynamic structure factor is given by $S^{(1)}(k, \omega) = -\mathrm{Im}\, F^{(1)}(k, \omega)/\pi$. The one-phonon contribution $S_\mathrm{I}^{(1)}$ is simply the expansion ot $O(g)$ of $gS_\mathrm{I}(k, \omega)$; we find $Z^{(1)}(k) = -k^2\omega_k^{2(1)}/4\omega_k^{3(0)}$, where $\omega_k^{2(1)}$ [5] and hence $Z(k)$ are not analytic functions of k. The multiphonon contribution to $S^{(1)}(k, \omega)$ can be written as a sum of two terms. The coherent two-phonon part S_II is defined as the remaining coherent part

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$$S_{\mathrm{II}}^{(1)}(k,\omega) = Y^{(1)}(k)\delta(\omega-\omega_k^{(\mathrm{o})}) \tag{3}$$

where

$$Y^{(1)}(k) = \frac{1}{4}(k^2/\omega_k^{2(\mathrm{o})})\,\frac{\mathrm{d}}{\mathrm{d}\omega}(\omega^{-2}N^{33(1)})\,\omega_k^{(\mathrm{o})} \tag{4}$$

and $Y^{(1)}(k) = y_3k^3 + y_{\mathrm{L}5}k^5 \ln 1/k + \ldots$ with $y_3 = -(7/1440)\pi^{-2}$ and $y_{\mathrm{L}5} = -(3/640)\pi^{-2}$. The incoherent two-phonon part $X(k,\omega)$ is found to be

$$X^{(1)}(k,\omega) = -(k^2/\pi\omega^2)(\omega^2-\omega_k^{2(\mathrm{o})})^{-2}\,\mathrm{Im}\,N^{33(1)}(k,\omega)\ . \tag{5}$$

For large ω and $k \to 0$, $X^{(1)}(k,\omega)$ scales as $(7/120\pi^2)\omega^{-3/2}(k^2/\omega)^2\,[1+O(k^2/\omega)]$.

The first-order processes corresponding to $S_{\mathrm{I}}^{(1)}$, $S_{\mathrm{II}}^{(1)}$, and $X^{(1)}$ can be given a simple physical interpretation. $S_{\mathrm{I}}^{(1)}$ represents the production of a real phonon to $O(g)$ that exhaust the f-sum rule. $S_{\mathrm{II}}^{(1)}$ corresponds to the production of a real phonon in $O(g^{\mathrm{o}})$ that is surrounded in $O(g)$ by two virtual phonons, i.e., backflow. $X^{(1)}$ can be interpreted as the production of a virtual phonon that subsequently decays into two real phonons, giving rise to a background.

It can be seen from the above that

$$S_m^{(1)}(k) = (\omega_k^{(\mathrm{o})})^m(1-m)Z^{(1)}(k) + (\omega_k^{(\mathrm{o})})^m Y^{(1)}(k) + X_m^{(1)}(k) \tag{6}$$

where $X_m^{(1)}(k) = \int_0^\infty \mathrm{d}\omega\,\omega^m X^{(1)}(k,\omega) = x_m k^4 + \ldots$ represents the contribution from incoherent two-phonon background. From the large ω behavior of $X^{(1)}(k,\omega)$ we see that $X_m^{(1)}(k)$ and hence $S_m^{(1)}(k)$ is divergent for $m \geqslant 3$. Thus our perturbation expansion for the moments $S_m(k)$ breaks down for $m \geqslant 3$ and our form for $S^{(1)}(k,\omega)$ holds only for $\omega < 1/g$. For $-1 \leqslant m \leqslant 2$ the perturbation expansion of $S_m(k)$ is well-defined through $O(g)$, and $x_{-1} = (7/7680)\pi^{-2}\ln 1/k$, $x_{\mathrm{o}} = (7/10240)\pi^{-1}$, $x_1 = (7/1440)\pi^{-2}$, $x_2 = (7/480)\pi^{-1}$. It is easy to check that $S_{-1}(k)$ and $S_1(k)$ are consistent with the compressibility and f-sum rules respectively. Since the leading contribution of the coherent two-phonon backflow to $S_m^{(1)}(k) \propto k^{3+m}$, the one-phonon Feynman relation for the static structure factor $S_{\mathrm{o}}(k)$, $gS_{\mathrm{o}}(k) = k^2/2\omega_k$, breaks down at $O(k^3)$. Note that even with our simple form for v, $S_{\mathrm{o}}(k)$ includes both even and odd terms in k and has an inflection point. We also see that for $m = -1, 0, 2$, $S_m^{(1)}(k)$ is not an analytic function of k, but includes terms $\propto k^{5+m}\ln 1/k$.

As discussed in ref. [5] the logarithmic term in $S_m^{(1)}(k)$ might be expected in superfluid helium. In ref. [2] general arguments for the leading k^4 dependence of $X_m^{(1)}(k)$ were given; we now present similar arguments for the k^{3+m} contribution of the coherent multiphonon backflow which was negelcted in ref. [2]. We write [2] $S_m(k) = \sum_n |\langle n|\rho_k^+|0\rangle|^2(\omega_{no})^m$. The contribution of the multiphonon backflow can be written

$$\sim \langle 0|\rho_k|1\rangle\langle 1|B_k\rho_k^+|0\rangle(\omega_{10})^m$$

where B_k is the matrix element of the square of the backflow operator $\rho_k^B = kJ_k^B/\omega$. In the limit $k \to 0$, the backflow current J_k^B is finite from hydrodynamical considerations and the excitation energy ω is finite because of the multiphonons; thus $\rho_k^B \sim k$ and $B_k \sim k^2$. It is easy to show [2] that $\langle 1|\rho_k^+|0\rangle \sim k^{1/2}$, and since $\omega_{10} \sim k$ the multiphonon backflow contribution to $S_m(k) \propto k^{3+m}$.

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