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CONDITIONS FOR THE EXISTENCE OF DYNAMIC SNAP-THROUGH OF A SHALLOW CYLINDRICAL SHELL UNDER IMPULSIVE LOADING

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Abstract—Shallow shell theory is used to investigate the non-linear plane deformation of a circular cylindrical panel elastically restrained against rotation at the supports. The critical or equilibrium configurations which may exist at zero load are determined. By examining the local stability of the various configurations, the critical rotational stiffness is obtained, above which the shell cannot exhibit dynamic snap-through under impulsive load. Finally for the range of geometries and rotational stiffness for which snap-through may exist, a sufficient condition for stability is given.

INTRODUCTION

IN A PREVIOUS paper [1] the authors used shallow shell theory to investigate the dynamic stability of simply supported cylindrical panels subject to impulsive loading. The actual impulse to produce snap-through was obtained by direct integration of the equations of motion. The results implied that the critical impulse is strongly dependent upon the spatial distribution of the loading. The values obtained were compared to the values associated with a sufficient condition for stability. Although the sufficiency condition may be conservative, it does provide a bound valid for all distributions.

The sufficiency condition was based on the theory introduced by Hsu [2]. For the problem of impulsive loading of an initially undeformed shell, the condition is equivalent to the statement that the total energy imparted to the shell is less than the potential energy associated with the first non-trivial equilibrium configuration encountered in the phase space by successive energy surfaces expanding about the undeformed state. It is necessarily an unstable configuration.

Implied in the above statement is the assumption that at least one non-trivial equilibrium configuration exists which is locally stable. Dynamic snap-through instability cannot occur if the only locally stable equilibrium state is the undeformed configuration. Hsu [3] showed that simply supported sinusoidal and parabolic arches have for a range of geometries a stable non-trivial equilibrium configuration, whereas the corresponding clamped arches do not. Vahidi [4] also concluded that clamped shallow arches cannot exhibit snap-through.

Based on the formulation in [1], the present paper investigates a shallow cylindrical

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panel which is elastically restrained against rotation at the supports. From the above discussion it is clear that a critical restraint stiffness must exist, above which dynamic snap-through cannot occur. After obtaining the equilibrium configurations, this critical stiffness is determined by examining the local stability of the various equilibrium states. The sufficient condition for stability is then obtained for the range of geometries and restraint stiffness for which dynamic snap-through can exist.

FORMULATION

The shell geometry is shown in Fig. 1. We consider plane motion of the shell. Before deformation a mid-surface point P has polar coordinates (a, θ). The point in the deformed shell is located by the displacement vector

$$\delta = \beta^2 \alpha(\beta\psi \boldsymbol{t} + \phi \boldsymbol{n}), \tag{1}$$

where a is the radius of the mid-surface, β is the semi-opening angle, and $\boldsymbol{t}$ and $\boldsymbol{n}$ are the unit tangent and normal vectors to the undeformed mid-surface. Thus ϕ and ψ are dimensionless radial and tangential displacement components.

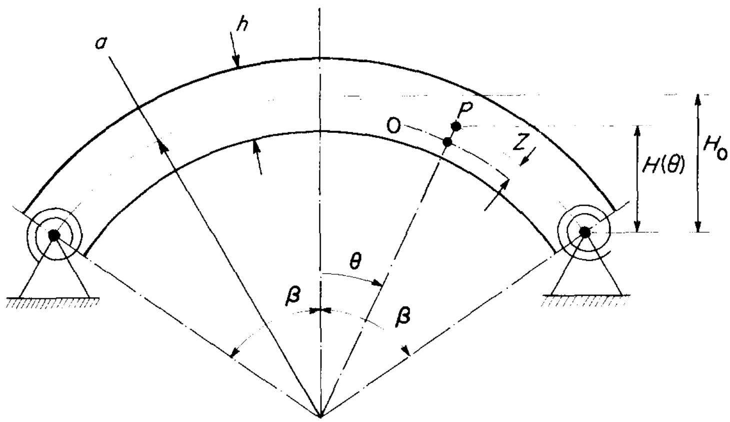

FIG. 1. Shell geometry.

The membrane force N and bending moment M per unit length of cylinder are

$$N = \frac{Eh}{(1 - \nu^2)} \beta^2 \eta \tag{2}$$

$$M = \frac{Eh^3}{12(1 - \nu^2)a} \overline{M}, \tag{3}$$

where h is the shell thickness, E is Young's modulus, ν is Poisson's ratio and

$$\eta = \psi' - \phi + \tfrac{1}{2}\phi'^2 \tag{4}$$

$$\overline{M} = -\phi'', \tag{5}$$

in which the prime denotes $\partial/\partial\Gamma$ where

$$\Gamma = \theta/\beta. \tag{6}$$

In (2) the quantity $\beta^2\eta$ is the mid-surface strain, and in (3) the quantity $\overline{M}/a$ is the change in curvature.†

The equations governing the free plane motion are given in [1]. Consistent with the use of (5), they are in the present notation

$$\ddot{\phi} + \lambda^{-4}\,\phi'''' - \eta - (\eta\,\phi')' = 0 \tag{7a}$$

$$\beta^2\ddot{\psi} - \eta' = 0 \tag{7b}$$

in which

$$\lambda^2 = \beta^2/\alpha, \qquad \alpha^2 = \tfrac{1}{12}(h/a)^2. \tag{8}$$

The dot denotes $\partial/\partial\tau$ where

$$\tau = ct/a, \qquad c^2 = E/\rho\,(1 - \nu^2) \tag{9}$$

in which t is real time.

The boundary conditions considered in the present paper are

$$\phi(\pm\,1, \tau) = 0 \tag{10a}$$

$$\psi(\pm\,1, \tau) = 0 \tag{10b}$$

$$\overline{M}(\pm\,1, \tau) = \pm\,K\,\phi'(\pm\,1, \tau), \tag{10c}$$

in which

$$K = k\,\beta\,a/[E\,h^3/12\,(1 - \nu^2)], \tag{11}$$

where k is the torsional spring constant of the rotational restraint per unit length.

THE CRITICAL CONFIGURATIONS

To obtain a sufficient condition for snap-through stability we first obtain the critical or equilibrium configurations. Setting the time derivatives to zero and using (7b), equation (7a) reduces to

$$\phi'''' - \eta\,\lambda^4\,\phi'' = \eta\,\lambda^4. \tag{12}$$

It follows from (7b) that η is a constant. Integrating (4) and using (10b) yields

$$\eta = -\tfrac{1}{2}\int_{-1}^{1}(\phi - \tfrac{1}{2}\phi'^2)\,\mathrm{d}\Gamma. \tag{13}$$

Equations (12) and (13) together with the boundary conditions (10a) and (10c) determine the critical configurations. The trivial solution ($\phi = \psi = 0$) is the undeformed state. Depending upon the geometric and stiffness parameters, there may be zero, two, or more additional configurations. The solution is

$$\phi = \frac{A}{q^2}\left[b\,(\sin q\Gamma - \Gamma\sin q) + 1 - \frac{\cos q\Gamma}{\cos q}\right] + \tfrac{1}{2}(1 - \Gamma^2), \tag{14}$$

† In [1] the expression used was equivalent to $\overline{M} = -(\phi'' + \beta^2\phi)$. Here the second term is dropped since for numerical computation it is negligible in comparison to ϕ''.

where q is the eigenvalue. b is a constant associated with a specific configuration and

$$A = \frac{1 + K}{1 + K \tan q/q}. \tag{15}$$

The eigenvalue is related to the membrane strain through the relation

$$\eta = - q^2/\lambda^4. \tag{16}$$

There are two distinct cases. For symmetric configurations

$$b = 0, \tag{17}$$

and q is a root of the characteristic equation

$$g = B, \tag{18a}$$

where

$$g = 2\,q^2/3 - 4\,q^4/\lambda^4 \tag{18b}$$

$$B = \left(\frac{1 + K}{1 + K \tan q/q}\right)^2 [\sec^2 q - \tan q/q]. \tag{18c}$$

There may also exist non-symmetric configurations for which

$$b^2 = \frac{g - A^2(\sec^2 q - \tan q/q)}{A^2\left(1 + \frac{\sin 2q}{2q} - 2\frac{\sin^2 q}{q^2}\right)} \tag{19}$$

and q is a root of the characteristic equation

$$(q^2 + K)\sin q - K\,q\cos q = 0. \tag{20}$$

Clearly, non-symmetric configurations exist only when the right hand side of (19) is positive. Setting b^2 equal to zero and solving for λ for a specific root to (20), $q = \tilde{q}_N$, we obtain

$$\tilde{\lambda}_N = q\left[\frac{6}{q^2 - \frac{3}{2}A^2(\sec^2 q - \tan q/q)}\right]^{\frac{1}{4}}. \tag{21}$$

The roots of (20) are admissable only if (21) yields a real number. It can be shown that when $q = \tilde{q}_N$

$$b^2 > 0 \text{ if } \lambda > \tilde{\lambda}_N$$

$$b^2 < 0 \text{ if } \lambda < \tilde{\lambda}_N.$$

Thus an additional pair† of non-symmetric configurations exist as λ is increased beyond $\tilde{\lambda}_N$.

To investigate the existence of symmetric configurations we examine the characteristic equation (18) with K held fixed. The function g defined by (18b) increases monotonically with the parameter λ, and completely fills the space under the curve

$$y = 2\,q^2/3.$$

† The pair arises, of course, from taking the square root to obtain b. One member of the pair is the mirror image of the other reflected in the ray $\theta = 0$.

as λ takes on all positive values. The function B defined by (18c) has, for $q > 0$, a series of positive branches separated by poles. For all finite K, each branch beyond a certain one drops below $2q^2/3$. We denote $\bar{\lambda}_N$ as a constant such that when $\lambda = \bar{\lambda}_N$ the function g is tangent to the N-th branch of B. Thus two additional roots to (18a) exist as λ is increased beyond each $\bar{\lambda}_N$. Associated with each $\bar{\lambda}_N$ are constants $\bar{q}_N$ that locate the tangency points.

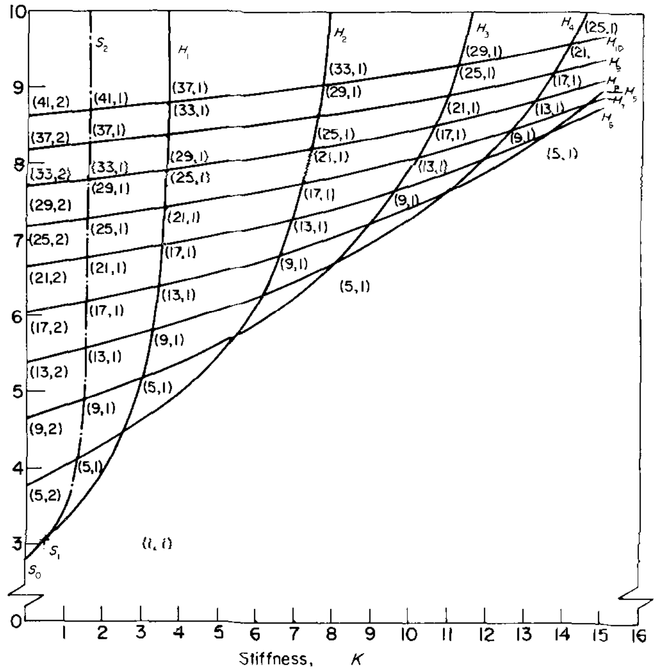

FIG. 2. Number of equilibrium configurations as a function of geometrical parameter λ and rotational restraint stiffness K.

In the present investigation the constants $\tilde{\lambda}_N$ were determined numerically through iteration. If the sequences for $\bar{\lambda}_N$ and $\tilde{\lambda}_N$ are both arranged in ascending order, then

$$\bar{\lambda}_N < \tilde{\lambda}_N < \bar{\lambda}_{N+1},$$

with $\tilde{\lambda}_N$ only slightly larger than $\bar{\lambda}_N$. The conditions for the existence of various kinds of equilibrium configurations is summarized in Fig. 2. The $\lambda - K$ plane is divided into regions by the curves labeled $H_1, H_2, \ldots$. The curve H_i is associated with the root $\tilde{q}_i$, the roots being ordered by increasing magnitude. Because the numerical difference between $\tilde{\lambda}_N$ and $\bar{\lambda}_N$ is very small, the curve associated with $\bar{q}_i$ is indistinguishable from H_i in the figure. Thus, at least two symmetric and two non-symmetric configurations exist above and to the left of each curve. The undeformed configuration, of course, exists everywhere. As λ increases or K decreases, an increasing number of configurations may exist. The total number existing in each region is shown as the first number in the parentheses.

We next examine the energy state associated with each critical configuration. We introduce the dimensionless potential energy

$$\tilde{U} = \frac{U(1 - \nu^2)}{Eah\,\beta\alpha^2} = \tilde{V} + \tilde{V}_c. \tag{22}$$

where U is the total potential energy, $\tilde{V}$ is the dimensionless strain energy, and $\tilde{V}_c$ is due to the external elastic restraints against rotation at $\Gamma = \pm 1$. We have

$$\tilde{V} = \tfrac{1}{2}\int_{-1}^{1} [\lambda^4 \eta^2 + \overline{M}^2]\, d\Gamma \tag{23a}$$

$$\tilde{V}_c = \tfrac{1}{2} K[\phi'^2(-1, \tau) + \phi'^2(+1, \tau)]. \tag{23b}$$

Substituting (14) into (23) gives, after simplification:

$$\tilde{V}_c = \frac{KA^2}{q^2}[b^2(\cos q - \sin q/q)^2 + (\tan q - q/A)^2] \tag{24a}$$

$$\tilde{V} = \frac{q^4}{\lambda^4} + 1 + A\left\{\frac{A}{2}\left[b^2\left(1 - \frac{\sin 2q}{2q}\right) + \sec^2 q + \frac{\tan q}{q}\right] - \frac{\tan q}{q}\right\}. \tag{24b}$$

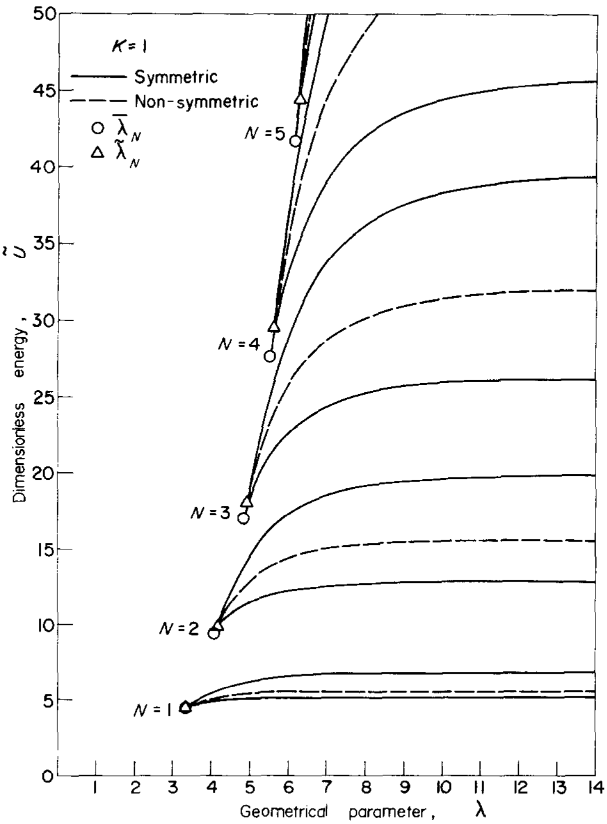

FIG. 3. Energy of critical configurations for $K = 1$ vs. geometrical parameter λ.

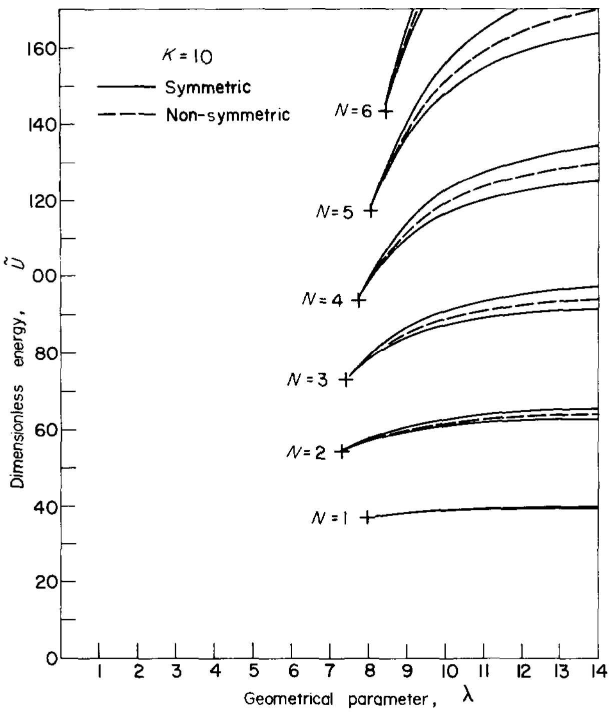

FIG. 4. Energy of critical configurations for $K = 10$ vs. geometrical parameter λ.

Plots of $\tilde{U}$ vs. λ are shown in Fig. 3 and Fig. 4 for $K = 1$ and $K = 10$ respectively.† The distribution of critical configurations as a function of λ is evident in the figures. In Fig. 4 the differences between $\bar{\lambda}_N$ and $\tilde{\lambda}_N$ are too small to distinguish.

LOCAL STABILITY OF CRITICAL CONFIGURATIONS

For snap-through to occur, a stable non-trivial equilibrium configuration must exist. Thus we must investigate the stability character of each equilibrium configuration. We denote the perturbation from a given configuration ϕ_c by ϕ_d. Thus the perturbed motion is

$$\phi = \phi_c + \phi_d. \tag{25}$$

Substituting this into the equation of motion, and noting that ϕ_c is a solution, we obtain

$$E(\phi_d) - \phi_d'' F(\phi_c) - \phi_c'' F(\phi_d) - \tfrac{1}{2}(1 + \phi_c'' + \phi_d'')\int_{-1}^{1} \phi_c' \phi_d' \, \mathrm{d}\Gamma = 0, \tag{26a}$$

where

$$E(\phi) = \ddot{\phi} - \frac{1}{\lambda^4}\phi'''' - (1 + \phi')\,F(\phi) \tag{26b}$$

$$F(\phi) = -\tfrac{1}{2}\int_{-1}^{1} (\phi - \tfrac{1}{2}\phi'^2)\,\mathrm{d}\Gamma. \tag{26c}$$

† The plot corresponding to $K = 0$ is given in [1].

We introduce the representation

$$\phi_d = \sum_{j=1}^{c} T_j(\tau)\, Z_j(\Gamma). \tag{27}$$

where the Z_j are the natural modes of infinitesimal vibration about the undeformed configuration. For convenience they are listed in the Appendix. Substituting (27) into (26a), using Galerkin's method, and retaining only the terms linear in the $T_i(\tau)$ gives the governing equations for small perturbations. The result is

$$\ddot{T}_i + \sum_{j=1}^{c} C_{ij}\, T_j = 0, \tag{28a}$$

where

$$C_{ij} = \mu_{ij}^2 - \frac{q^2}{\lambda^4} P_{ij} + n_i Q_j + n_j Q_i + \tfrac{1}{2} Q_i Q_j \tag{28b}$$

$$P_{ij} = \int_{-1}^{1} Z_i' Z_j' \,\mathrm{d}\Gamma \tag{28c}$$

$$Q_i = \int_{-1}^{1} z_i' \phi_c' \,\mathrm{d}\Gamma \tag{28d}$$

$$\mu_{ij}^2 = \begin{matrix} \mu_j^2 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{matrix} \tag{28e}$$

$$n_i = -\tfrac{1}{2} \int_{-1}^{1} Z_i \,\mathrm{d}\Gamma. \tag{28f}$$

The constant μ_j in (28e) is the natural frequency of the j-th mode of infinitesimal vibration about the undeformed configuration. Expressions for P_{ij}, Q_i and n_i may be obtained by direct integration. For brevity they are omitted here. The eigenvalues of the matrix C_{ij} are the squares of the natural frequencies of the perturbed motion. Hence the critical configuration is unstable if any of the eigenvalues are negative. For the present analysis the first twenty eigenvalues were determined numerically with the Jacobi Method, after truncating C_{ij} to a 20×20 array. Using additional modes did not significantly change their values. In Fig. 2 the second number in the parentheses indicate the number of stable configurations which exist in each region. At most, only one configuration (in addition to the undeformed one) is stable. It exists in the region above and to the left of the curve $S_0S_1S_2$; everywhere else in the $\lambda - K$ plane only the undeformed configuration is stable. When the second stable equilibrium configuration exists, it is symmetric and corresponds to $q < q_s$ where q_s is the lowest root to (18) associated with points along the curve S_1S_2. Our numerical investigation indicates that q_s is approximately equal to π.

CONCLUSIONS

Dynamic snap-through instability cannot occur if the undeformed configuration is the only locally stable equilibrium configuration. Thus in the present problem, dynamic snap-through can only occur in the region above the curve $S_0S_1S_2$ in the $\lambda - K$ plane (Fig. 2). The critical spring stiffness is the value of K on the curve $S_0S_1S_2$. It is a function of λ, but it is clear from the figure that snap-through will never occur if K exceeds 1·6.

For the region where snap-through may occur we can state a sufficient condition for stability. The unstable configuration with the lowest energy level is non-symmetric,

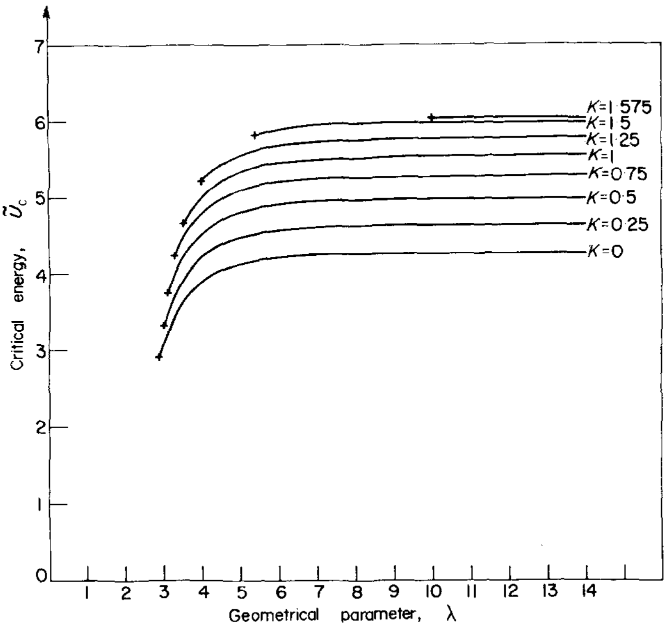

FIG. 5. Sufficient condition for stability against snap-through.

except for a very narrow range of λ between $\lambda_\perp$ and $\lambda_\perp$. We denote this energy level by $\tilde{U}_c$. The shell cannot snap-through under impulsive load if the initial energy imparted to the shell is less than $\tilde{U}_c$. This sufficiency condition is shown in Fig. 5 for several values of K for which snap-through is possible.

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APPENDIX

Infinitesimal vibrations about the undeformed position

Neglecting tangential inertia and retaining only linear terms reduces the equations of motion to

$$\ddot{\phi}' = \frac{1}{\lambda^4}\phi'''''. \tag{A.1}$$

For the boundary conditions (10), the solution is

$$\phi = \sum_{n=1}^{\infty} [\bar{C}_n \sin(\bar{\mu}_n\tau + \bar{\Delta}_n)\,\bar{Z}_n(\Gamma) + \tilde{C}_n \sin(\tilde{\mu}_n\tau + \tilde{\Delta}_n)\,\tilde{Z}_n(\Gamma)]. \tag{A.2}$$

where the bar denotes an asymmetric mode and the tilde denotes a symmetric mode. The quantities C_n and Δ_n are arbitrary constants determined from the initial conditions.

The dimensionless circular natural frequencies are

$$\bar{\mu}_n = \bar{v}_n^2/\lambda^2 \tag{A.3a}$$

$$\tilde{\mu}_n = \tilde{v}_n^2/\lambda^2. \tag{A.3b}$$

where $\bar{v}_n$ and $\tilde{v}_n$ are roots to the characteristic equations

$$(2v \tanh v + K)\sin v - (K \tanh v)\cos v = 0 \tag{A.4a}$$

and

$$\begin{aligned}&[v + 2K \tanh v - Kv(1 - v^4/\lambda^4] \sin v \\ &- v[-\tanh v + (2v + K \tanh v)(1 - v^4/\lambda^4)] \cos v = 0,\end{aligned} \tag{A.4b}$$

respectively.

The asymmetric mode shapes are

$$\bar{Z}_n = \frac{\sin v_n \Gamma}{\left[1 - \left(\dfrac{\sin 2v_n}{2v_n}\right)\right]^{\frac{1}{2}}}. \tag{A.5}$$

The symmetric mode shapes are

$$\tilde{Z}_n = A_n(\cos v_n\Gamma + R_n \cosh v_n\Gamma + S_n), \tag{A.6a}$$

where

$$R_n = \frac{(v_n \cos v_n + K \sin v_n)}{(v_n \cosh v_n + K \sinh v_n)} \tag{A.6b}$$

$$S_n = \frac{-[2v_n + K(\tanh v_n + \tan v_n)] \cos v_n}{(v_n + K \tanh v_n)} \tag{A.6c}$$

$$\begin{aligned}A_n = \Bigg[&1 + \frac{\sin 2v_n}{2v_n} + R_n^2\left(\frac{\sin 2v_n}{2v_n} + 1\right) + 2S_n^2 \\ &+ \frac{2R_n}{v_n}(\sinh v_n \cos v_n + \cosh v_n \sin v_n) \\ &+ 4S_n \frac{\sin v_n}{v_n} + 4R_n S_n \frac{\sinh v_n}{v_n}\Bigg]^{-\frac{1}{2}}.\end{aligned} \tag{A.6d}$$

The mode shapes satisfy the orthogonality condition

$$\int_{-1}^{1} Z_m Z_n \, d\Gamma = \delta_{mn},$$

where δ_{mn} is the Kronecker delta and Z_m represents either a symmetric or an asymmetric mode.

Résumé On utilise la théorie des coques peu profondes pour étudier la déformation plane non linéaire d'un panneau cylindrique circulaire élastiquement retenu à ses supports contre la rotation. On détermine les configurations critiques ou à l'équilibre qui peuvent exister sous une charge nulle. En examinant la stabilité locale des diverses configurations, on obtient la rigidité critique en rotation au dessus de laquelle la coque ne peut pas donner lieu à une rupture dynamique sous une charge par impulsions. On donne finalement une condition suffisante de stabilité pour la gamme de géométries et de rigidité de rotation pour lesquelles la rupture peut exister.

Zusammenfassung—Die Theorie flacher Schalen wird benützt, um die nichtlineare ebene Verformung einer kreisförmigen zylindrischen Platte zu untersuchen, die elastisch an den Unterstützungen gehalten wird, um eine Rotation zu verhindern. Die kritischen oder Gleichgewichtskonfigurationen, die bei Nullbelastung bestehen können, werden bestimmt. Durch Untersuchung der lokalen Stabilität verschiedener Konfigurationen wird die kritische Rotationssteifheit erhalten. oberhalb derer die Schale kein dynamisches Durchknicken unter stossartiger Beanspruchung zeigen kann. Schliesslich wird für den Bereich von Geometrien und Rotationssteifheiten. für die ein Durchknicken möglich ist. eine hinreichende Bedingung für Stabilität angegeben.

Аннотация—Применяется теория пологих оболочек с целью исследования нелинейной плоской деформации круглой цилиндрической панели на опорах упруго препятствующих поворотом. Определяются критические или равновесные конфигурации, которые могут существовать при нулевой нагрузке, Изучая местную устойчивость разных конфигураций получается критическую жесткость (устойчивость) по отношению поворота, выше которой оболочка под влиянием импульсной нагрузки не можеь проявлять динамического хлопка. Наконéц приводится достаточное условие устойчивости для области геометрических конфигураций и жесткости против поворота, для которых хлопок может происходить.