Transient and Steady State, Free and Natural Convection, Numerical Solutions: Part I. The Isothermal, Vertical Plate

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Numerical values are presented for the transient velocity field, temperature field, and local heat transfer coefficient. These results were obtained by solving the partial differential equations describing the conservation of mass, momentum, and energy on an IBM-704 computer with finite difference methods in time-dependent form. The computed values for short times agree very well with the analytical solution for conduction only, and the limiting values for long time agree well with previous solutions for the steady state. The existence of a temporal minimum in the heat transfer coefficient is confirmed. The time required for the heat transfer coefficient to approach its steady state is shown to be less than previously predicted.

This paper presents a complete transient solution for free convection in an unconfined fluid initially at rest and at uniform temperature, adjacent to a semi-infinite vertical plate at a different uniform temperature (Figure 1). The solution was obtained by numerical integration of the partial differential equations describing the conservation of mass, momentum, and energy in the fluid. The integrations were carried out on the time-dependent form of the equations by finite difference methods with an IBM-704 computer. The method of solution is described in detail in references *1* and *2*.

PRIOR WORK

Pohlhausen (*3*) developed an analytical solution and Ostrach (*4*) a numerical solution for the isothermal, vertical plate at steady state. Their results agree well with the experimental values of Schmidt and Beckmann (*3*) for a uniform heat flux density at the plate. For very short times the fluid velocities are very small, and the analytical solution for conduction alone is presumed to be applicable. Siegel (*5*) studied the transient case by an integral method and obtained an estimate of the time required to attain steady state. Gebhart (*6*) developed an approximate solution for the transient behavior with a constant heat flux density at the plate, but no comparable results have been obtained for an isothermal plate.

MATHEMATICAL DESCRIPTION

The temperature of a plate immersed in a fluid at temperature T_i is postulated to be raised suddenly and then maintained at a higher, uniform, and constant value T_w. Fluid motion develops slowly following the development of nonuniformity in the temperature field. During this initial period heat transfer is almost entirely by conduction, and the classical solution for conduction into a half space is applicable. The temperature field is thus given by Equation (1) and the heat transfer coefficient by Equation (2):

$$\frac{T - T_i}{T_w - T_i} = 1 - \operatorname{erf}\left(\frac{y}{2\sqrt{\alpha t}}\right) \quad (1)$$

$$\frac{h\sqrt{\pi \alpha t}}{k} = 1 \quad (2)$$

The conservation of momentum, energy, and mass in the fluid at all times is described approximately by the following dimensionless equations:

$$\frac{\partial U}{\partial \tau} + U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = \phi + \frac{\partial^2 U}{\partial Y^2} \quad (3)$$

$$\frac{\partial \phi}{\partial \tau} + U\frac{\partial \phi}{\partial X} + V\frac{\partial \phi}{\partial Y} = \frac{1}{N_{PR}}\frac{\partial^2 \phi}{\partial Y^2} \quad (4)$$

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0 \quad (5)$$

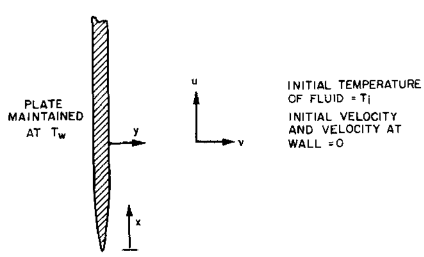

Fig. 1. Coordinate system for flat plate problem.

The boundary and initial conditions corresponding to the stated problem are

$U = \phi = 0$
at $X = 0$ and $Y = \infty$
$U = V = 0, \phi = 1$
at $Y = 0$
$U = V = \phi = 0$
at $\tau = 0$

The idealizations in this representation were discussed in an earlier paper (*7*) in which it was also shown that the number of independent variables could be reduced so that $U/X^{1/2}$, $VX^{1/4}$, and ϕ depend only on $Y/X^{1/4}$, $\tau/X^{1/2}$, and N_{PR}. It follows that the composite heat transfer group $(h/k)(\nu^2 x/g\beta\Delta T)^{1/4}$ depends only on $\tau/X^{1/2}$ and N_{PR}.

Pohlhausen (*3*) utilized equivalent composite variables to reduce Equations (3), (4), and (5) to ordinary differential equations for the steady state case. A series solution was then developed, and the coefficients in the series were evaluated by iteration. The more recent steady state solution obtained by Ostrach (*4*) by numerical integration of the same ordinary differential equations will be compared with the results of this investigation.

RESULTS

Equations (3), (4), and (5) were solved for the unsteady state. The computations were carried out for N_{PR} = 0.733, since Pohlhausen and Ostrach gave results for this value. The computed values are plotted in Figures 2, 3, 4, 5, and 6 in terms of composite variables.

Figures 2 and 3 present the transient velocity field in terms of the composite dimensionless variables. Figure 2 can be interpreted as a plot of velocity at any chosen elevation x vs. distance y normal to the plate for a

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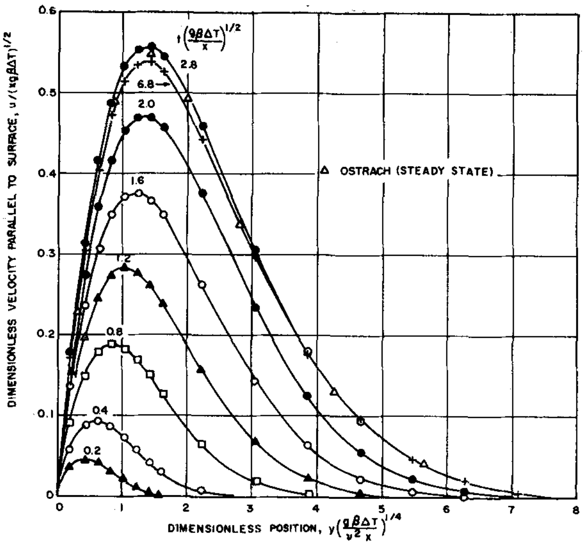

Fig. 2. Transient velocity profiles.

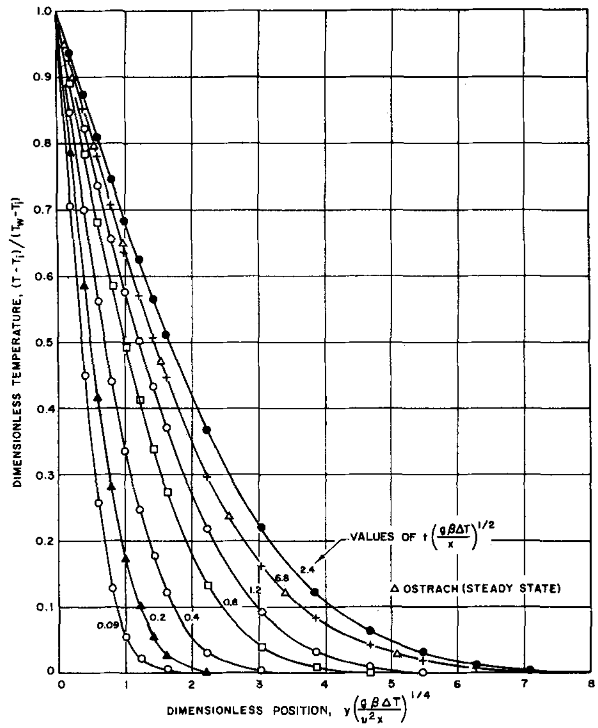

Fig. 4. Transient temperature profiles.

series of values of time. Figure 3 can similarly be interpreted as the history of the velocity at any elevation x for a series of distances y normal to the plate. The velocity at a given point increases with time to a maximum and then decreases slightly to a steady value. The calculations were carried out to a dimensionless time of 6.8, but the abscissa of Figure 3 is terminated at a 4.0 since essentially no change occurs beyond this value. The open triangles in Figure 2 represent the values computed by Ostrach for the steady state. The maximum difference between the two computations is about 2%.

Figures 4 and 5 show the dimensionless temperature profiles and histories. The temperature at a given point also increases with time to a maximum and then decreases slightly to a steady value. Again excellent agreement is noted with the open triangles representing Ostrach's steady state solution.

The variation of the heat transfer coefficient with position and time is shown in Figure 6. The dimensionless heat transfer group is initially infinite since the wall temperature is presumed to change discontinuously at time zero. It decreases to a minimum and finally increases to a steady value of 0.365. Ostrach computed a value of 0.359 for the steady state.

Equation (1) may be rearranged in terms of the dimensionless variables of Figure 6 to give

$$\frac{h}{k}\left(\frac{\nu^2 x}{g\beta\Delta T}\right)^{1/4} = \left[\frac{N_{PR}}{\pi}\right]^{1/2} \left[t\left(\frac{g\beta\Delta T}{x}\right)^{1/2}\right]^{-1/2} \quad (6)$$

Equation (6), which would be expected to hold for short times, is represented in Figure 6 by the dashed curve. For abscissa values of 2.4 or

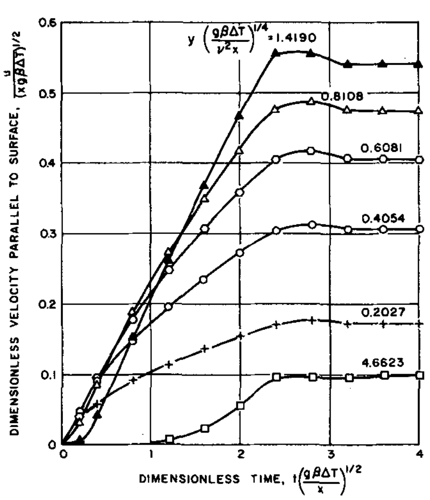

Fig. 3. Transient velocity at various positions.

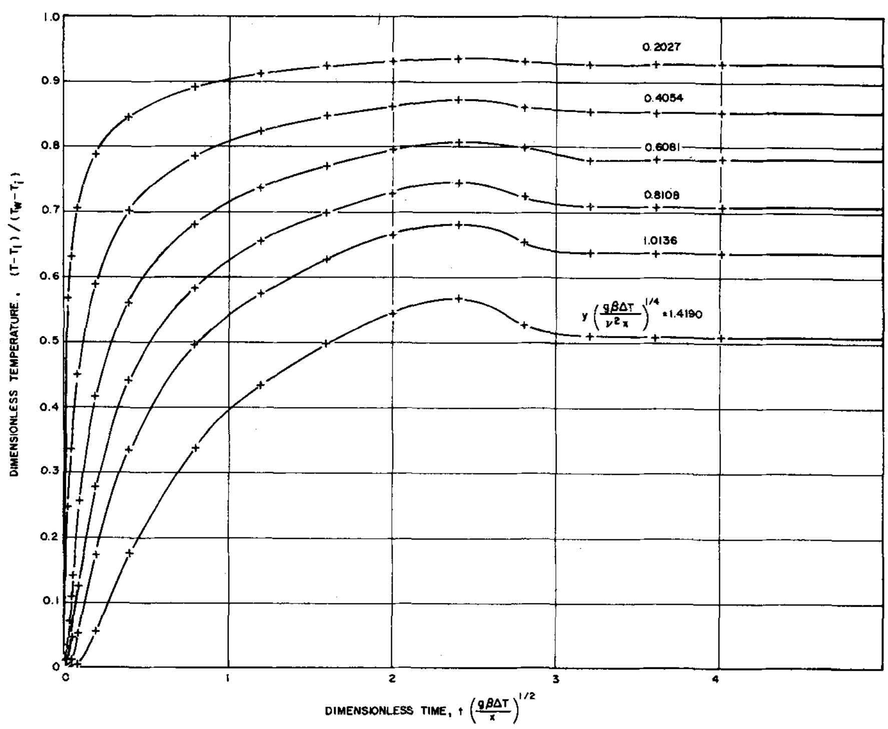

Fig. 5. Temperature histories.

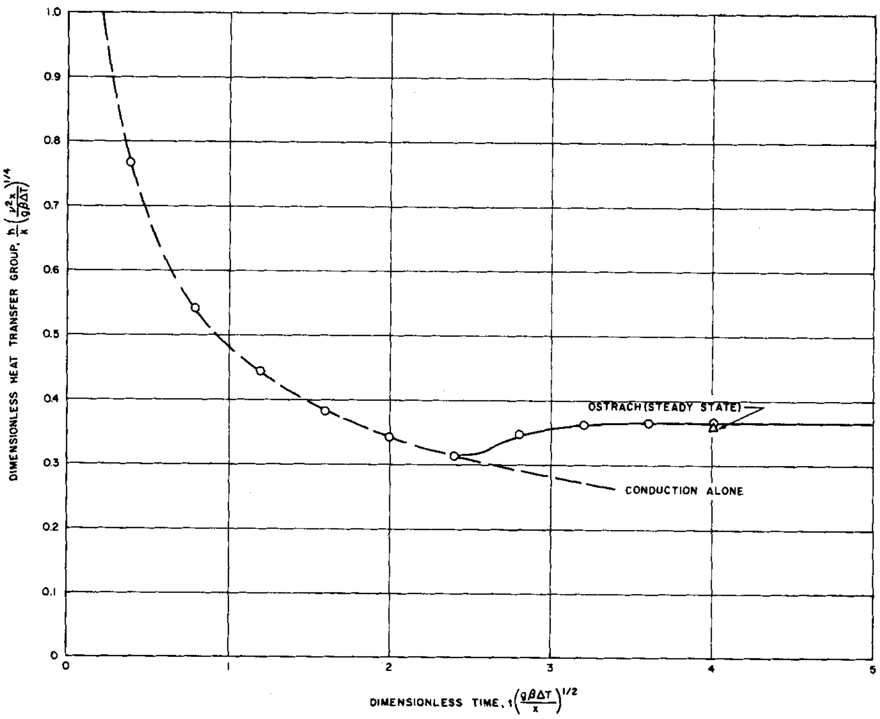

Fig. 6. Transient heat flux.

less the analytical solution for conduction alone and the numerical solution are indistinguishable; that is the solution for conduction alone is valid for most of the time required to approach the steady state value.

Siegel (2) estimated that the solution for conduction alone would be valid for $t(g\beta\Delta T/x)^{1/2} \leq 2.7$, which is in good agreement with the computed limit of 2.4. However his estimate of 7.1 for the dimensionless time required to reach steady state is about twice the value indicated in Figure 6. This latter value of course depends on the arbitrary choice of the fraction of the steady state value which is attained. When the dimensionless time equals 4.0, the heat transfer group is within 0.3% of the steady value. A dimensionless time of 7.0 is required to obtain agreement with the steady state to five significant figures.

The existence of a temporal minimum in the heat transfer coefficient has been noted previously. Klei (8) observed a minimum in his constant heat flux experiments, and both Siegel (5) and Gebhart (6) predicted one by analysis.

CONCLUSIONS

The results herein are apparently the first complete solution for transient free convection in any geometry.

The excellent agreement between the numerical solution and previous solutions for short times and the steady state gives credence to the results for intermediate times.

The existence of a temporal minimum in the heat transfer coefficient is confirmed. The velocity and temperature at all points go through corresponding maximum values.

The time required for the heat transfer coefficient to approach its steady state value is somewhat less than the time predicted by Siegel (5).

Dimensional analysis indicates that the numerical solution presented herein is reliable only for $N_{Pr} = 0.733$ but is applicable for any fluid properties or temperatures within this restriction, insofar as the idealizations in the theoretical model are valid.

NOTATION

g = acceleration due to gravity
h = local heat transfer coefficient
k = thermal conductivity of fluid
N_{Pr} = ν/α = Prandtl number of fluid
t = time
T = temperature
T_i = initial temperature of fluid
T_w = temperature of plate
u = component of velocity in vertical direction
U = $u/(\nu g\beta\Delta T)^{1/3}$
v = component of velocity in horizontal direction
V = $v/(\nu g\beta\Delta T)^{1/3}$
x = vertical distance above bottom of plate
X = $x(g\beta\Delta T/\nu^2)^{1/3}$
y = horizontal distance from plate
Y = $y(g\beta\Delta T/\nu^2)^{1/3}$

Greek Letters

α = thermal diffusivity of fluid
β = coefficient in equation of state of fluid (see reference 7)
ΔT = $T_w - T_i$
ν = kinematic viscosity of fluid
ϕ = $(T - T_i)/\Delta T$
ρ = density of fluid
τ = $t(g\beta\Delta T)^{2/3}/\nu^{1/3}$

Part II. The Region Inside a Horizontal Cylinder

Numerical values are presented for the transient and steady state temperature field, velocity field, and local heat transfer coefficient within an infinitely long, horizontal cylinder with vertical halves of the wall at different uniform temperatures. Despite the many idealizations in the theoretical model the solution agrees reasonably well with previous experimental data for the steady state. The steady state solutions encompass a greater range of N_{Gr} and N_{Pr} than the experimental data, and the asymptotic solution for large N_{Gr} is found to be a reasonable approximation for a wide range of N_{Pr} and N_{Gr}, permitting generalization of the numerical results. The transient motion is found to be quite complex, and the local heat transfer coefficients are found to decrease to a minimum and then go through a maximum before attaining a steady value.

A numerical solution for transient, free convection in the unconfined region adjacent to an isothermal, vertical plate was presented in Part I of this paper. The region confined by an infinitely long, horizontal cylinder with vertical halves at different uniforr