

Working Paper

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Ross School of Business Working Paper Series
Working Paper No. 902
September 2002

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Abstract

Recent research lends credence to the belief that supply chain structure and product variety are closely linked. We explore the effect of supply chain configuration on product variety, in a competitive setting. We treat both *non-functional* as well as *functional* variety, and for various supply chain configurations we examine equilibrium variety outcomes, assuming that the structure of the market can be represented by a static one-period model belonging to the class of *Market Share Attraction Models*.

1 Introduction

Recent research lends credence to the belief that supply chain structure and product variety are closely linked (Ramdas (2002); Randall et al. (2002); Randall and Ulrich (2001)). Ramdas (2002) provides a framework for understanding and explaining the relationship between variety and supply chain structure. Under this framework, variety decisions include *variety-creation* decisions that determine the amount, type, and timing of end-product variety, and *variety-implementation* decisions, which focus on the design of internal processes as well as a supply chain to support a firm's variety-creation strategy. Supply chain structure would therefore be categorized as a variety-implementation decision under this framework. Using business examples, Randall et al. (2002) explain that firms making the decision to offer high levels of product variety strategically choose specific inventory structures. Randall and Ulrich (2001) characterize supply chain structure by the degree to which production facilities are scale-efficient and by the distance of the production facility from the target market. Using data from the U.S. bicycle industry they hypothesize that firms with scale-efficient production will offer *types* of variety associated with high production costs, and firms with local production will offer types of variety associated with high market mediation costs¹. This hypothesis implies that there is a coherent way to match product variety with supply chain

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¹Market mediation costs include variety-related inventory holding costs, product mark-down costs occurring when supply exceeds demand, and the costs of lost sales occurring when demand exceeds supply.

structure. Their empirical results suggest that firms which match supply chain structure to the type of product variety they offer outperform firms which fail to match such choices.

Existing literature treats product variety as either of two types - *non-functional variety* or *functional variety*. Examples of non-functional variety include videos in a video rental store and colors of fabric lengths in a cloth shop (Schaffir (1963); Lancaster (1980); Hohenbalken and West (1991)). Characteristics of such variety include similar demand and identical production and marketing costs (Schaffir (1963); Kelvin Lancaster (1980)). Baumol and Ide (1956) consider variety simply as the number of the different items available in a store and through a simple yet insightful model, consider the effect of this aggregate variety on store profits. Schaffir (1963) conducts a marginal analysis to examine profitability of adding an item to a set of existing items in a product line, that are available for sale. Kekre and Srinivasan (1990) empirically assess the impact of breadth of the product line on profitability and market share and conclude that significant market share and profitability benefits arise from having a broader product line. However, none of these papers consider variety in a competitive setting, nor do they incorporate the effect of supply chain structure on variety.

Functional variety has been dealt with in economics literature through models similar in spirit to that in Hotelling (1929). Rosen (1974) studies a spatial equilibrium in which the set of implicit prices² guides both consumer and producer location decisions in the characteristics space. Prescott and Visscher (1977) consider a modification of Hotelling's model in which location decisions are made by firms sequentially, and once-and-for-all. Each firm takes into consideration the effect of its location decision upon the ultimate configuration of the industry, and the equilibrium number and location of firms are identified in specific examples. Stiglitz (1979) provides a descriptive account of the nature of equilibria in markets with quality dispersion and product variety, under imperfect information. Anderson et al. (1995) apply discrete choice models³ to oligopolistic competition and deduce the equilibrium number of firms. Shaked and Sutton (1982) consider a game-theoretic model of monopolistic competition in which potential entrants compete in three stages - entry, quality choice and price. They conclude that the only perfect equilibrium is one in which only two firms enter. None of the above papers study the effect of supply chain structure, though competition is modelled in some of them.

Additional examples observed in practice reinforce the belief that variety and supply chain structure are indeed related. Radio Shack® manufactures electronics and communications products and offers limited variety within each family in its broad product line. Radio Shack stores operate on a franchise basis and exclusively offer Radio Shack products in addition to few other brands. Best Buy™ stores, on the other hand, operate as retail outlets for various manufacturers and compete with Radio Shack franchises. Best Buy has certain product families in common with Radio Shack, but offers greater product variety in each of these families. For instance, in the *portable CD player* family, as on August 30, 2002, Best Buy offered 36 brands of portable CD players while Radio Shack offered only 10.⁴ Similarly, Do it Best® and Home Depot® have different supply chain

²Customers have utilities for characteristics of a product; associated with these utilities are implicit prices.

³Discrete choice models are used to describe heterogeneous customer tastes.

⁴Information obtained from www.bestbuy.com and www.radioshack.com.

arrangements and offer different levels of variety in their common and hence competing product families.

However, it is not clear whether existing supply chain configuration and firms' desire for profitability result in the observed product variety, or whether customer demand for variety and firms' desire for profitability result in the observed supply chain configuration. In this paper we treat both non-functional and functional variety and examine the impact of supply chain configuration and the competition within, on equilibrium variety.

We consider two competing retailers catering to the same geographic market. These retailers have to decide on variety, i.e., the brands to offer and volume in each brand, and the price for each brand. We focus on a particular product family and define an *offering*⁵ as the triple (i, j, p_{ij}) where p_{ij} is the price for brand j at retailer i . These offerings are partially substitutable. The demand for an offering decreases with an increase in its own price, and increases with an increase in a competing offering's price. We assume that all customer demand must be satisfied.⁶ The retailers' objective is to maximize total product family profit; they source their products from profit-maximizing manufacturers. The aforementioned decisions and resulting equilibrium outcomes are influenced by the structure of the supply chain (e.g. whether decentralized or centralized) and the retailers' and manufacturers' cost structures (e.g. dependence of cost on variety). We assume complete information. For various supply chain configurations, we analyze the resulting games and their equilibrium outcomes, assuming that the structure of the market can be represented by a static one-period model which belongs to the class of *Market Share Attraction Models*.

In general, a market share attraction model specifies that the market share of a firm, in either quantity or revenue terms, is equal to its "attraction" divided by the total attraction of all firms in the market, where a firm's attraction is a function of the values of its marketing instruments. Such models are theoretically appealing because they are logically consistent; they yield market share values that are between zero and one, and these values sum to one across all firms in the market (Karnani, (1985)). Karnani (1985) investigates the conduct of firms and the performance of the market when the structure of the market can be represented by a static one-period market share attraction model. The solution concept of a Nash equilibrium is used, and strategic implications of such an equilibrium are deduced. It is shown that, at equilibrium, each active competitor must have market share above a certain threshold value, and that this threshold value is a function of both - the cost structures of the firms as well as the demand structure of the market. The model also predicts a positive relationship between market share and profitability. It is shown that these implications are consistent with previous empirical research in marketing and business policy. Bell et al. (1975) provide examples of empirical studies that have used the attraction specification to model market shares. Cooper and Nakanishi (1988) provide a thorough treatment of market-share models. They demonstrate how model extensions can accommodate differential and cross-competitive effects (Lilien et al. (1992)). To further justify our choice of a market share attraction model, we quote

⁵Where unambiguous, the terms *brand* and *offering* are used interchangeably.

⁶In practice, a customer can place an order with the store if the demanded item is not available in stock.

Cooper and Nakanishi (1988) - “We report that many studies on predictive accuracy of market-share models found the logical-consistency property of the (market share attraction) models to produce only marginally better predictions than the linear and multiplicative models. Why then all this fuss about the (market share attraction) models? ...we do not believe that predictive accuracy is the only important criterion for judging the value of a model. We would rather find the answer in the *construct validity* (i.e., intrinsic meaningfulness) of market share attraction models.”

To the best of our knowledge, there is no research to date which explores the effect of supply chain configuration on product variety in a competitive setting. In particular, we consider the supply chain configurations as in Figure 1. M stands for manufacturer and R for retailer. The supply chain structures we consider are similar to those in McGuire and Staelin (1983). The dotted lines indicate flow of product from manufacturer to retailer and the solid lines represent competition for final demand.

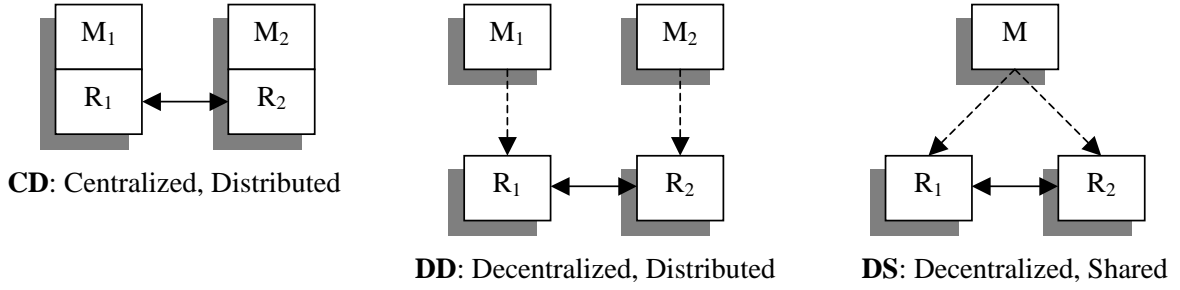


Figure 1: Supply Chain Structures Considered

In the CD structure, M_i and R_i are one integrated firm F_i . The sequence of actions in the game is as follows. The F_i s simultaneously decide their sets of offerings $\{(i, j, p_{ij})\}$, demand for each offering is realized, and finally costs are spent and revenues are received. Note that in all cases $p_{ij} = \infty$ implies that brand j is, in effect, not offered by retailer i . In the DD structure, M_i s and R_i s are separate entities. The sequence of actions in this game is as follows. Simultaneously, the M_i s decide on the linear wholesale prices w_{ij} s to offer to the respective R_i s, then the R_i s simultaneously decide their sets of offerings $\{(i, j, p_{ij})\}$, demand for each offering is realized, and finally costs are spent and revenues are received. In the DS structure, M and R_i s are separate entities. The sequence of actions in this game is as follows. M decides on the linear wholesale prices w_{ij} s to offer to the R_i s, then the R_i s simultaneously decide their sets of offerings $\{(i, j, p_{ij})\}$, demand for each offering is realized, and finally costs are spent and revenues are received.

As a logical buildup, the sections progress from the CD structure to the DD structure and finally to the DS structure. For each structure, we begin with the single brand⁷ case and then extend the analysis to the multiple brand setting. In the single brand case, retailers have access

⁷“In No Choice, Klein indicates how the promise of a huge array of consumer choice has been betrayed by mergers, franchising and corporate censorship. Single-brand stores now predominate in many urban centres - Nike

to one brand each, whereas in the multiple brand case, each retailer can offer a subset of several available brands. Our notation and specifications of market share and profit functions are similar to those in Karnani (1985). However, we assume that marketing expenses are sunk and that the retailers compete on product price alone.

2 Centralized Distributed (CD) Configuration

In the *CD* configuration, the integrated firms F_i s compete for final demand. We begin by analyzing the single brand case and then move on to the multiple brand setting. As mentioned above, our notation and specifications of market share and profit functions are similar to those in Karnani (1985).

2.1 Single Brand Case

Notation:

p_i = product price at retailer i

c_i = cost parameter for retailer i

y_i = sales volume in physical units for retailer i

β_i = factor representing economies of scale for retailer i ; $0 < \beta_i \leq 1$

K_i = factor capturing ‘attractiveness’ of retailer i ’s offering

s_i = market share in terms of revenue for retailer i

Q_i = factor representing the contribution of relative market share to profitability of retailer i

$-i$ = index for i ’s competing retailer

R = total market size in terms of revenue

V_i = profit function for retailer i

α, θ are industry-specific parameters.

$\alpha > 0$ captures the price sensitivity of demand in the industry. As in Karnani (1985) we assume that consumers in the industry have decreasing marginal utility and that the total market size is a non-decreasing, concave function of the total attraction. Since we assume that the retailers compete on price alone, we can simplify the notation used in Karanani (1985). Thus,

$$s_i = \frac{K_i p_i^{-\alpha}}{\sum_{j=1}^n K_j p_j^{-\alpha}}; \quad R = \left[\sum_{j=1}^n K_j p_j^{-\alpha} \right]^\theta, \quad 0 \leq \theta < 1 \quad (1)$$

Since retailers competing in the same market are concerned with profitability as well as relative market share, we specify the objective⁸ for retailer i as:

Town, Roots, The Body Shop - and single-brand pioneer Disney has gone far beyond Disney World to create the Disney Magic cruise ship, and Celebration, Fla., where you can live the complete Disney life.” - Bronwyn Drainie, “Brand names? Hate ’em,” The Globe and Mail, January 15, 2000.

⁸Since we consider a competitive one-period model, relative market share can be thought of as a surrogate for

$$\text{Maximize } V_i = p_i y_i - c_i y_i^{\beta_i} - E_i + Q_i \frac{s_i}{s_{-i}}; \quad y_i = \frac{s_i R}{p_i}. \quad (2)$$

Note that in this specification, the quantity sold by retailer i is a function of prices p_i and p_{-i} .

2.1.1 Parameter Restrictions

The results of this subsection apply to the single brand cases of all the supply chain structures considered. In order for the model to be a reasonable representation of reality, we need to specify ranges of admissible values for its parameters. We therefore state the following.

Proposition 1 *For self-price elasticities to be negative and cross-price elasticities to be positive, the parameter θ should satisfy $0 \leq \theta < 1$.*

Proof: The self-price elasticity of retailer i 's sales revenue with respect to its price p_i , is:

$$\begin{aligned} \nu_{i,i} &:= \frac{p_i}{(p_i y_i)} \cdot \frac{\partial(p_i y_i)}{\partial p_i} \\ &= \frac{p_i}{[(K_i p_i^{-\alpha})(K_i p_i^{-\alpha} + K_{-i} p_{-i}^{-\alpha})^{\theta-1}]} \cdot \frac{\partial[(K_i p_i^{-\alpha})(K_i p_i^{-\alpha} + K_{-i} p_{-i}^{-\alpha})^{\theta-1}]}{\partial p_i} \\ &= \frac{-\alpha(K_i p_i^{-\alpha})(K_i p_i^{-\alpha} + K_{-i} p_{-i}^{-\alpha})^{\theta-1} + \alpha(1-\theta)(K_i p_i^{-\alpha})^2(K_i p_i^{-\alpha} + K_{-i} p_{-i}^{-\alpha})^{\theta-2}}{(K_i p_i^{-\alpha})(K_i p_i^{-\alpha} + K_{-i} p_{-i}^{-\alpha})^{\theta-1}} \\ &= \frac{-\alpha(K_i p_i^{-\alpha} + K_{-i} p_{-i}^{-\alpha}) + \alpha(1-\theta)(K_i p_i^{-\alpha})}{(K_i p_i^{-\alpha})(K_i p_i^{-\alpha} + K_{-i} p_{-i}^{-\alpha})} \end{aligned}$$

Since $\alpha, p_i, p_{-i}, K_i, K_{-i} \geq 0, 0 \leq \theta \leq 1$ is a sufficient condition for $\nu_{i,i}$ to be negative.

The cross-price elasticity of retailer i 's sales revenue with respect to the competitor's price p_{-i} , is:

$$\begin{aligned} \eta_{i,-i} &:= \frac{p_{-i}}{(p_i y_i)} \cdot \frac{\partial(p_i y_i)}{\partial p_{-i}} \\ &= \frac{p_{-i}}{[(K_i p_i^{-\alpha})(K_i p_i^{-\alpha} + K_{-i} p_{-i}^{-\alpha})^{\theta-1}]} \cdot \frac{\partial[(K_i p_i^{-\alpha})(K_i p_i^{-\alpha} + K_{-i} p_{-i}^{-\alpha})^{\theta-1}]}{\partial p_{-i}} \\ &= \alpha(1-\theta)(K_{-i} p_{-i}^{-\alpha})(K_i p_i^{-\alpha} + K_{-i} p_{-i}^{-\alpha})^{-1} \end{aligned}$$

$\theta < 1$ is a necessary condition for $\eta_{i,-i}$ to be positive. ■

future earnings. The profit function in Karnani (1985) does not include this additional term. Hiam and Schewe (1992) note that a striking trend in the Profit Impact of Marketing Strategy (PIMS) database is that market share and return on investment (ROI) vary together. A smaller share is associated with a lower ROI and vice versa. In fact, the relationship is virtually a straight line, varying from an average ROI of 11 percent for businesses with market shares of 10 percent or less, up to an ROI of 40 percent for businesses with shares of 50 percent or more.

2.1.2 Analysis

Retailer i 's profit function is not concave in price p_i - but a simple reformulation elegantly permits us to establish properties relating to the retailers' profit functions. Defining⁹ $x_i := p_i^{-\alpha}$, $m_i := \frac{\beta_i}{\alpha}(\alpha + 1)$, $n_i := (\theta - 1)\beta_i$, we have $V_i = (K_i x_i)(K_i x_i + K_{-i} x_{-i})^{\theta-1} - c_i K_i^{\beta_i} x_i^{m_i} (K_i x_i + K_{-i} x_{-i})^{-n_i} - E_i + Q_i \frac{x_i}{x_{-i}}$.

Lemma 1 *Profit function V_i is strictly concave in x_i if $m_i - n_i > 1$. Thus, if $m_i - n_i > 1$, there exists $p_i^* = x_i^{*-1/\alpha}$ that uniquely maximizes V_i .*¹⁰

Proof: For notational convenience, denote

$$I_i := (K_i x_i)(K_i x_i + K_{-i} x_{-i})^{\theta-1} + Q_i \frac{x_i}{x_{-i}};$$

$$C_i = x_i^{m_i} (K_i x_i + K_{-i} x_{-i})^{-n_i} - \frac{E_i}{c_i K_i^{\beta_i}};$$

$$A := K_i x_i + K_{-i} x_{-i}$$

$$\begin{aligned} \frac{dI_i}{dx_i} &= K_i A^{\theta-2} [K_i x_i (\theta - 1) + A] + \frac{Q_i}{x_{-i}} \\ \frac{d^2 I_i}{d^2 x_i} &= (1 - \theta)(2 - \theta) K_i^3 x_i A^{\theta-3} - 2(1 - \theta) K_i^2 A^{\theta-2} \\ &= (1 - \theta) K_i^2 A^{\theta-3} [(2 - \theta) K_i x_i - 2A] \\ &< 0 \end{aligned} \tag{3}$$

$$\begin{aligned} \frac{dC_i}{dx_i} &= x_i^{m_i-1} A^{-n_i-1} [m_i A - n_i K_i x_i] \\ \frac{d^2 C_i}{d^2 x_i} &= x_i^{m_i-2} A^{-n_i-2} [m_i(m_i - 1)A^2 - 2m_i n_i x_i A + n_i(n_i + 1)K_i^2 x_i^2] \\ &= x_i^{m_i-2} A^{-n_i-2} [K_i^2 x_i^2 (m_i(m_i - 1) - 2m_i n_i + n_i(n_i + 1)) \\ &\quad + 2K_i K_{-i} x_i x_{-i} (m_i(m_i - 1) - m_i n_i) + K_{-i}^2 x_{-i}^2 m_i(m_i - 1)] \end{aligned} \tag{4}$$

If $m_i - n_i > 1$, then

$$m_i(m_i - 1) - 2m_i n_i + n_i(n_i + 1) = (m_i - n_i)^2 - (m_i - n_i) > 0,$$

$$m_i(m_i - 1) - m_i n_i = m_i(m_i - n_i - 1) > 0, \text{ and therefore}$$

$$\frac{d^2 C_i}{d^2 x_i} > 0 \tag{5}$$

⁹Note that the function $x = p^{-\alpha}$ is a monotonic one-one correspondence. Also note that a concave function of a monotonic function is pseudo-concave.

¹⁰Karnani (1985) differentiates a similar profit function with respect to price and equates the derivative to 0, in order to arrive at the optimal price. Numerical counterexamples demonstrate that without restrictions as in Lemma 1 the profit function in Karnani (1985) is, in some situations, quasi-convex and therefore that the use of first order conditions can yield a non-optimal price.

Since $V_i = I_i - c_i K_i^{\beta_i} C_i$, from (3) and (5) we have $\frac{d^2 V_i}{dx_i^2} < 0$, i.e., the profit function V_i is strictly concave in x_i if $m_i - n_i > 1$. Thus, if $m_i - n_i > 1$ there exists an x_i^* that uniquely maximizes V_i . Since the function $x_i = p_i^{-\alpha}$ is a one-one correspondence, there exists $p_i^* = x_i^*{}^{-\frac{1}{\alpha}}$ that uniquely maximizes V_i . ■

From this point forward, we will assume that $m_i - n_i > 1, \forall i$. This translates into the following condition on the model's parameters.

$$\frac{\alpha}{1 + \alpha\theta} < \beta \leq 1 \quad (6)$$

In other words, for the profit function, V_i , to be concave in the transform, x_i , of price, economies of scale cannot be arbitrarily large.¹¹

We will also assume that the retailers will not sell at zero price. This means that x_i is bounded above for all i . Let B_i denote an arbitrarily large upper bound on x_i . We denote the strategy space for retailer i as $S_i = \{x_i : x_i \in \mathbb{R}^1, 0 \leq x_i \leq B_i\}$ and the set of strategy profiles as $X = \{x := (x_i, x_{-i}) : x_i \in S_i, x_{-i} \in S_{-i}\}$. Denote $x_i^*(x_{-i})$ as retailer i 's profit maximizing (best response) to x_{-i} . We state the following fact by inspection of the profit function V_i .

Fact 1 (a) $Q_i = 0 \Rightarrow x_i^*(0) < \infty$; (b) $Q_i = 0 \Rightarrow x_i^*(\infty) < \infty$; (c) $Q_i > 0 \Rightarrow x_i^*(0) = \infty$; (d) $Q_i > 0 \Rightarrow x_i^*(\infty) < \infty$.

Proof: (a) By contradiction. If $x_i^*(0) = \infty$, since $m_i - n_i > 1$, the term $c_i K_i^{\beta_i} x_i^{m_i} (K_i x_i + K_{-i} x_{-i})^{-n_i}$ in the expression for V_i dominates so that $V_i = -\infty$. We are better off choosing $x_i(0) = 0$.

(b) By contradiction. If $x_i^*(\infty) = \infty$, since $m_i - n_i > 1$, the term $c_i K_i^{\beta_i} x_i^{m_i} (K_i x_i + K_{-i} x_{-i})^{-n_i}$ in the expression for V_i dominates so that $V_i = -\infty$. We are better off choosing $x_i(\infty) = 0$.

(c) If $x_i^*(0) = \infty$, the term $Q_i \frac{x_i}{x_{-i}}$ in the expression for V_i dominates so that $V_i = +\infty$.

(d) By contradiction. If $x_i^*(\infty) = \infty$, since $m_i - n_i > 1$, the term $c_i K_i^{\beta_i} x_i^{m_i} (K_i x_i + K_{-i} x_{-i})^{-n_i}$ in the expression for V_i dominates so that $V_i = -\infty$. We are better off choosing $x_i(\infty) = 0$. ■

Lemma 2 *The reaction function $x_i^*(x_{-i})$ is downward sloping when $Q_i \rightarrow \infty$.*

Proof: Since the profit function V_i is concave in x_i , the first order condition, $\frac{dV_i}{dx_i} = 0$, yields the reaction function $x_i^*(x_{-i})$.

$$\frac{dV_i}{dx_i} = 0 \Rightarrow f(x_i, x_{-i}) := K_i A^{\theta-2} [K_i x_i (\theta - 1) + A] - c_i K_i^{\beta_i} x_i^{m_i-1} A^{-n_i-1} [m_i A - n_i K_i x_i] + \frac{Q_i}{x_{-i}} = 0$$

¹¹We believe that this is not a restrictive assumption. Note that lower β implies greater economies of scale. Several numerical examples indicate that if economies of scale are larger than that specified by the bound in (6), the profit function behaves in a non-concave manner.

Using the implicit function theorem, the slope of the reaction function is:

$$\frac{dx_i}{dx_{-i}} = -\frac{\left(\frac{\partial f}{\partial x_{-i}}\right)}{\left(\frac{\partial f}{\partial x_i}\right)}$$

Where,

$$\begin{aligned} \frac{\partial f}{\partial x_{-i}} &= (1 - \theta)K_i K_{-i} A^{\theta-3} [(2 - \theta)K_i x_i - A] - nc_i K_i^\beta K_{-i} x_i^{m-1} A^{-n-2} [(n + 1)K_i x_i - mA] \\ &\quad - \frac{Q_i}{x_{-i}^2} \\ \frac{\partial f}{\partial x_i} &= (1 - \theta)(2 - \theta)K_i^3 x_i A^{\theta-3} - 2(1 - \theta)K_i^2 A^{\theta-2} - c_i K_i^\beta x_i^{m-2} A^{-n-2} [K_i^2 x_i^2 (m_i(m_i - 1) \\ &\quad - 2m_i n_i + n_i(n_i + 1)) + 2K_i K_{-i} x_i x_{-i} (m_i(m_i - 1) - m_i n_i) + K_{-i}^2 x_{-i}^2 m_i(m_i - 1)] \end{aligned}$$

From Lemma 1, we know that $\frac{\partial f}{\partial x_i} < 0$. When $Q_i \rightarrow \infty$, $\frac{\partial f}{\partial x_{-i}} < 0$, and $\frac{dx_i}{dx_{-i}} < 0$. Thus, when $Q_i \rightarrow \infty$, the reaction function $x_i^*(x_{-i})$ is downward sloping. ■

The reason for stating Lemma 2 in the above manner is because a closed form expression for the threshold value of Q_i , above which the reaction function is downward sloping, does not exist. The Lemma helps in supporting the fact there is a threshold value of Q_i , perhaps large, beyond which the optimal response to a price decrease, is a price increase. Figure 2 shows a typical response curve for large Q .

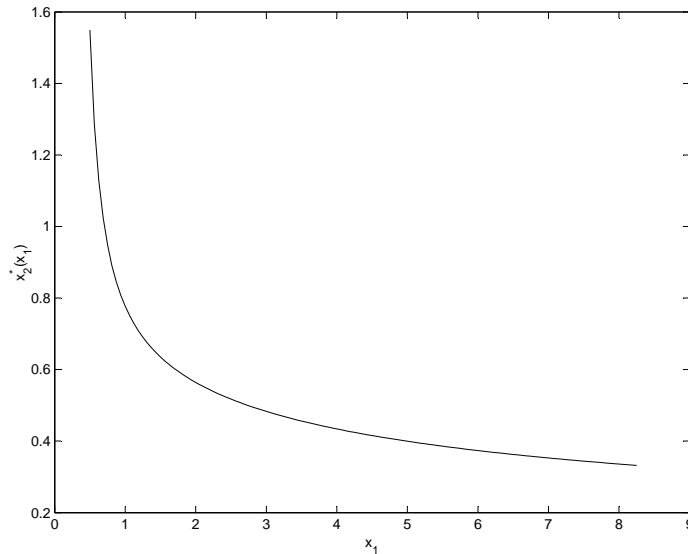


Figure 2: Downward Sloping Reaction Function (Q large)

The reason for the observed shape can be explained as follows. For a low value of x_1 there are large revenue and market share benefits to R_2 in responding with a high x_2 . As x_1 increases, some

market share benefits still remain with R_2 but since market size grows, it becomes costly for R_2 to serve the market and therefore R_2 responds by decreasing x_2 . As x_1 is increased further, it becomes increasingly costly to serve the growing market and R_2 's optimal response is to decrease x_2 . Facts 1(c) and 1(d) corroborate this explanation.

The following observation is stated without mathematical proof because of absence of closed form expressions.

Observation 1 *The reaction function $x_i^*(x_{-i})$ is unimodal when $Q_i = 0$.*

Observation basis: The graph in Figure 3 from a numerical example depicts the typical form of the response function $x_i^*(x_{-i})$. Reaction functions in all numerical studies conducted behaved in an identical manner.

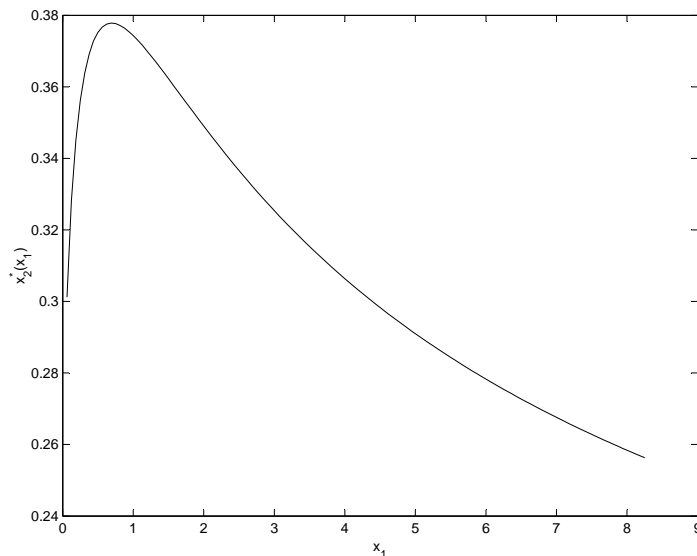


Figure 3: Unimodal Reaction Function ($Q = 0$)

An explanation for the above behavior is that as x_1 increases from 0, R_2 responds by increasing x_2 with the aim of getting a larger fraction of the market revenue. This continues but only up to a particular point. At this threshold, both R_1 and R_2 have a low price, and the market size is relatively large and costly to serve. If R_1 increases x_1 beyond this threshold, R_2 has to respond by now decreasing x_2 in order to keep the market size under control and to still profitably serve the market. Facts 1(a) and 1(b) corroborate to this explanation.

Lemma 3 *There exists at least one pure strategy Nash equilibrium in the game.*

Proof: The strategy space $S_i = \{x_i : x_i \in \mathbb{R}^1, 0 \leq x_i \leq B_i\}$ of retailer i , is a non-empty compact convex subset of the Euclidean space \mathbb{R}^1 . The profit function V_i is continuous in $x := (x_i, x_{-i})$ and

concave in x_i . Therefore, using Theorem 1.2 in Fudenberg and Tirole (1998), there exists at least one pure strategy Nash equilibrium in the game. ■

Define the symmetric case (SC) as one in which $K_i = K_{-i} = K$, $c_i = c_{-i} = c$, $\beta_i = \beta_{-i} = \beta$, $Q_i = Q_{-i} = Q$.

Proposition 2 *An asymmetric equilibrium is possible even in the symmetric case, i.e., there are SC situations in which, at equilibrium, $x_i^* \neq x_{-i}^*$, and therefore $p_i^* \neq p_{-i}^*$.*

Proof: By example. Consider the numerical example with the following parameter values: $K = 100$, $Q = 2$, $\alpha = 1$, $\theta = 0.4$, $\beta = 0.75$, $c = 1.5$. There are three Nash equilibria, i.e., three points at which the reaction curves $x_i^*(x_{-i})$, and $x_{-i}^*(x_i)$ intersect. One is a symmetric equilibrium and the other two are asymmetric. Figure 4 shows *one* asymmetric equilibrium and *the* symmetric equilibrium. ■

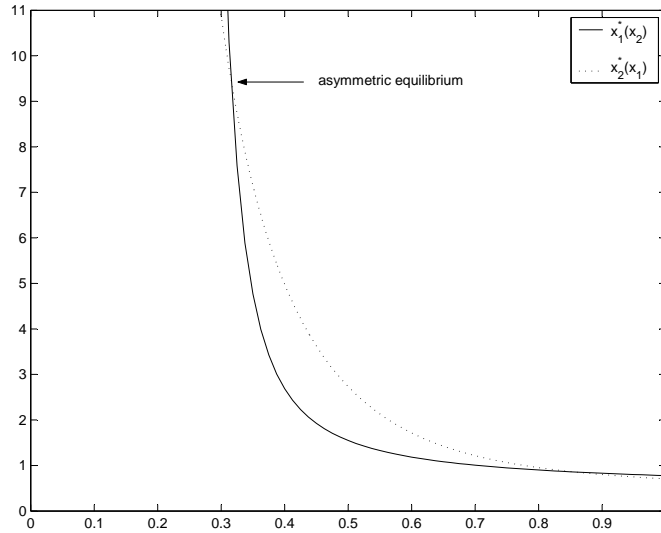


Figure 4: Example of Asymmetric Equilibrium

In the numerical example, the asymmetric equilibria in terms of transform of price are $x^1 := (0.3157, 9.4707)$ and $x^2 := (9.4707, 0.3157)$. In terms of price these equilibria are $p^1 := (3.1676, 0.1056)$ and $p^2 := (0.1056, 3.1676)$ respectively. Asymmetric equilibria exist because of the shape of the reaction curve. As explained before, for large enough Q , the reaction curve is downward sloping. For a low x_i , the optimal response of R_{-i} is a high x_{-i} because of large revenue and relative market share benefits. With an increase in x_i , R_{-i} 's relative market share benefit decreases at a faster rate than cost, with the result that R_{-i} best responds by decreasing x_{-i} convexly in order to keep the market size under control. As Figures 5 and 6 point out, R_{-i} 's best response to $x_i = 0.3157$, is $x_{-i} =$

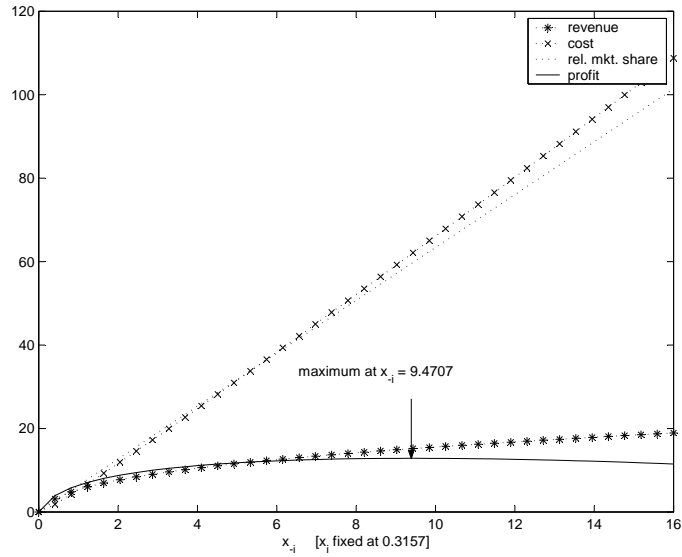


Figure 5: $x_{-i} = 9.4707$ is the best response to $x_i = 0.3157$

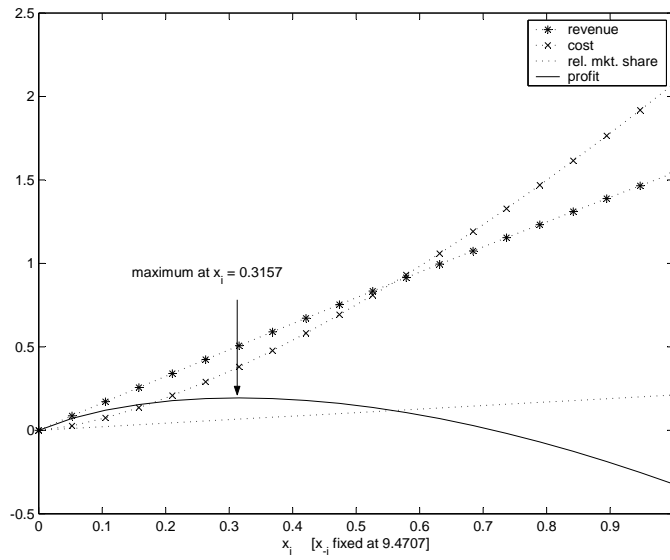


Figure 6: $x_i = 0.3157$ is the best response to $x_{-i} = 9.4707$

9.4707; and R_i 's best response to $x_{-i} = 9.4707$, is $x_i = 0.3157$.

2.2 Multiple Brand Case

For the multiple brand case, we use an additional index which represents the index of a particular brand at a particular retailer. As mentioned earlier, p_{ij} is the price for brand j at retailer i . Notation for other parameters and variables are to be similarly interpreted. Assume that the number of brands at retailer i is ξ_i . The profit function for retailer i is:

$$\begin{aligned}
V_i &= \sum_j (p_{ij}y_{ij}) - \sum_j (c_{ij}y_{ij}^{\beta_{ij}}) - E_i + Q_i \frac{\sum_j s_{ij}}{\sum_k s_{-ik}} \\
&= \left[\sum_j K_{ij}x_{ij} \right] \left[\sum_j K_{ij}x_{ij} + \sum_k K_{-ik}x_{-ik} \right]^{\theta-1} - \sum_j c_{ij} K_{ij}^{\beta_{ij}} x_{ij}^{m_{ij}} \left[\sum_j K_{ij}x_{ij} + \sum_k K_{-ik}x_{-ik} \right]^{-n_{ij}} - E_i \\
&\quad + Q_i \frac{\sum_j x_{ij}}{\sum_k x_{-ik}}, \text{ where } 1 \leq j \leq \xi_1, \text{ and } 1 \leq k \leq \xi_2.
\end{aligned}$$

2.2.1 Parameter Restrictions

The results of this subsection apply to the multiple brand cases of all supply chain structures considered.

Proposition 3 *For self-price elasticities to be negative and cross-price elasticities to be positive, the parameter θ should satisfy $0 \leq \theta < 1$.*

Proof: The elasticity of retailer i 's revenue from sales of brand j , with respect to its price p_{ij} , is:

$$\nu_{ij,ij} := \frac{p_{ij}}{(p_{ij}y_{ij})} \cdot \frac{\partial(p_{ij}y_{ij})}{\partial p_{ij}}$$

The proof that $0 \leq \theta < 1$ is a sufficient condition for $\nu_{ij,ij} < 0$ is identical to the corresponding proof in Proposition 1.

The cross-price elasticity of retailer i 's sales revenue with respect to the competitor's price p_{-ir} of brand r , is:

$$\begin{aligned}
\eta_{i,-ir} &:= \frac{p_{-ir}}{(\sum_j p_{ij}y_{ij})} \cdot \frac{\partial(\sum_j p_{ij}y_{ij})}{\partial p_{-ir}} \\
&= \frac{p_{-ir}}{\left([\sum_j K_{ij}p_{ij}^{-\alpha}] [\sum_j K_{ij}p_{ij}^{-\alpha} + \sum_k K_{-ik}p_{-ik}^{-\alpha}]^{\theta-1} \right)} \cdot \frac{\partial \left([\sum_j K_{ij}p_{ij}^{-\alpha}] [\sum_j K_{ij}p_{ij}^{-\alpha} + \sum_k K_{-ik}p_{-ik}^{-\alpha}]^{\theta-1} \right)}{\partial p_{-ir}} \\
&= \alpha(1-\theta)(K_{-ir}p_{-ir}^{-\alpha}) \left(\sum_j K_{ij}p_{ij}^{-\alpha} + \sum_k K_{-ik}p_{-ik}^{-\alpha} \right)^{-1}
\end{aligned}$$

Since $\alpha, p_{ij}, p_{-ik}, K_{ij}, K_{-ik} \geq 0, 0 \leq \theta < 1$ is a required condition for $\eta_{i,-ir}$ to be positive. ■

2.2.2 Analysis

Lemma 4 *Profit function V_i defined above is strictly jointly concave¹² in $\mathbf{x}_i := \{x_{ij}, 1 \leq j \leq \xi_i\}$ if $m_{ij} - n_{ij} > 1 \forall j$. Thus, if $m_{ij} - n_{ij} > 1 \forall j$, there exists a vector of prices $\mathbf{p}_i^* = \{p_{ij}^*, 1 \leq j \leq \xi_i\}$ (where $p_{ij}^* = x_{ij}^{*\frac{1}{\alpha}}$), that uniquely maximizes V_i .*

¹²Briefly stated as "jointly concave in x_i 's"

Proof: We begin by proving that $[\sum_j K_{ij}x_{ij}][\sum_j K_{ij}x_{ij} + \sum_k K_{-ik}x_{-ik}]^{\theta-1}$ is strictly jointly concave in $x_{ij}s$. Because sums of jointly concave functions are again jointly concave, it is sufficient to prove that $K_{i1}x_{i1}[\sum_j K_{ij}x_{ij} + \sum_k K_{-ik}x_{-ik}]^{\theta-1}$ is strictly jointly concave in $x_{ij}s$, or, that $f(\{x_{ij}\}) := x_{i1}[1 + \sum_j A_{ij}x_{ij}]^{\theta-1}$ is strictly jointly concave in $x_{ij}s$, where $A_{ij}s$ are positive constants. We are done if we can prove that $f(\frac{\{x_{ij}^{(1)} + x_{ij}^{(2)}\}}{2}) > \frac{f(\{x_{ij}^{(1)}\}) + f(\{x_{ij}^{(2)}\})}{2}$.

$$\begin{aligned}
f\left(\frac{\{x_{ij}^{(1)} + x_{ij}^{(2)}\}}{2}\right) &= \left(\frac{x_{i1}^{(1)} + x_{i1}^{(2)}}{2}\right) \left[1 + \sum_j A_{ij} \left(\frac{x_{ij}^{(1)} + x_{ij}^{(2)}}{2}\right)\right]^{\theta-1} \\
&> \left(\frac{x_{i1}^{(1)} + x_{i1}^{(2)}}{2}\right) \left[\left[\frac{1}{2} + \sum_j \frac{A_{ij}x_{ij}^{(1)}}{2}\right]^{\theta-1} + \left[\frac{1}{2} + \sum_j \frac{A_{ij}x_{ij}^{(2)}}{2}\right]^{\theta-1}\right] \\
&= \left(\frac{1}{2}\right)^\theta x_{i1}^{(1)} \left[1 + \sum_j A_{ij}x_{ij}^{(1)}\right] + \left(\frac{1}{2}\right)^\theta x_{i1}^{(2)} \left[1 + \sum_j A_{ij}x_{ij}^{(2)}\right] \\
&> \left(\frac{1}{2}\right)x_{i1}^{(1)} \left[1 + \sum_j A_{ij}x_{ij}^{(1)}\right] + \left(\frac{1}{2}\right)x_{i1}^{(2)} \left[1 + \sum_j A_{ij}x_{ij}^{(2)}\right] \\
&= \frac{f(\{x_{ij}^{(1)}\}) + f(\{x_{ij}^{(2)}\})}{2}
\end{aligned}$$

Thus we have proved the (strict) joint concavity of $[\sum_j K_{ij}x_{ij}][\sum_j K_{ij}x_{ij} + \sum_k K_{-ik}x_{-ik}]^{\theta-1}$. We now proceed to prove that $\sum_j c_{ij}K_{ij}^{\beta_{ij}}x_{ij}^{m_{ij}}[\sum_j K_{ij}x_{ij} + \sum_k K_{-ik}x_{-ik}]^{-n_{ij}}$ is strictly jointly convex in the $x_{ij}s$ if $m_{ij} - n_{ij} > 1$. We are done if we can prove the strict joint convexity of $x_{i1}^{m_{i1}}[1 + \sum_j A_{ij}x_{ij}]^{-n_{i1}}$ (where $A_{ij}s$ are positive constants), assuming that $m_{i1} - n_{i1} > 1$. We first consider the case when $\xi_i = 2$. Denote $f(x, y) := x^m(1 + A_1x + A_2y)^{-n}$. We use the Hessian matrix to show that $f(x, y)$ is jointly convex in (x, y) .

$$\begin{aligned}
f_x &= mx^{m-1}(1 + A_1x + A_2y)^{-n} - nx^m A_1(1 + A_1x + A_2y)^{-n-1} \\
f_{xx} &= m(m-1)x^{m-2}(1 + A_1x + A_2y)^{-n} - 2mnx^{m-1}A_1(1 + A_1x + A_2y)^{-n-1} \\
&\quad + n(n+1)x^m A_1^2(1 + A_1x + A_2y)^{-n-2} \\
f_y &= -nx^m A_2(1 + A_1x + A_2y)^{-n-1} \\
f_{yy} &= n(n+1)x^m A_2^2(1 + A_1x + A_2y)^{-n-2} \\
f_{xy} &= -mnA_2x^{m-1}(1 + A_1x + A_2y)^{-n-1} + n(n+1)x^m A_1A_2(1 + A_1x + A_2y)^{-n-2}
\end{aligned}$$

For $f_{xx} > 0$, require that $m(m-1)(1 + A_1x + A_2y)^2 - 2mnxA_1(1 + A_1x + A_2y) + n(n+1)x^2A_1^2 > 0$ i.e.,

$$m(m-1)(1 + A_1^2x^2 + A_2^2y^2 + 2A_1x + 2A_2y + 2A_1A_2xy) - 2mn(A_1x + A_1^2x^2 + A_1A_2xy) + n(n+1)A_1^2x^2 > 0$$

The above will hold if $m(m-1) - 2mn + n(n+1) > 0$; i.e., if $(m-n)^2 - (m-n) > 0$; i.e., if $(m-n) > 1$.

The determinant of the Hessian matrix is $f_{xx}f_{yy} - f_{xy}^2$.

$$f_{xx}f_{yy} - f_{xy}^2 = mn(m-1)(n+1)A_2^2x^{2m-2}(1 + A_1x + A_2y)^{-2n-2} - m^2n^2x^{2m-2}A_2^2(1 + A_1x + A_2y)^{-2n-2}$$

$f_{xx}f_{yy} - f_{xy}^2 > 0$, if $mn(m-1)(n+1) - m^2n^2 > 0$; i.e., if $(m-n) > 1$.

Since $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$, the Hessian matrix is positive definite. Thus, when $(m-n) > 1$, $f(x, y) := x^m(1 + A_1x + A_2y)^{-n}$ is strictly jointly convex in (x, y) . We now prove the strict joint convexity of $x_{i1}^{m_{i1}}[1 + \sum_j A_{ij}x_{ij}]^{-n_{i1}}$. When $(m_{i1} - n_{i1}) > 1$, we know that $\forall \{(x_{i1}^{(1)}, y_1), (x_{i1}^{(2)}, y_2)\}$,

$$\begin{aligned} \left(\frac{x_{i1}^{(1)} + x_{i1}^{(2)}}{2}\right)^{m_{i1}} \left[1 + A_{i1}\left(\frac{x_{i1}^{(1)} + x_{i1}^{(2)}}{2}\right) + A_2\left(\frac{y_1 + y_2}{2}\right)\right]^{-n_{i1}} &< \frac{1}{2}[x_{i1}^{(1)}]^{m_{i1}}(1 + A_{i1}x_{i1}^{(1)} + A_2y_1)^{-n_{i1}} \\ &+ \frac{1}{2}[x_{i1}^{(2)}]^{m_{i1}}(1 + A_{i1}x_{i1}^{(2)} + A_2y_2)^{-n_{i1}} \end{aligned} \quad (7)$$

In particular, if $y_1 = \frac{\sum_{j:j \neq 1} A_{ij}x_{ij}^{(1)}}{A_2}$, and $y_2 = \frac{\sum_{j:j \neq 1} A_{ij}x_{ij}^{(2)}}{A_2}$, then inequality (7) becomes,

$$\begin{aligned} \left(\frac{x_{i1}^{(1)} + x_{i1}^{(2)}}{2}\right)^{m_{i1}} \left[1 + \sum_j A_{ij}\left(\frac{x_{ij}^{(1)} + x_{ij}^{(2)}}{2}\right)\right]^{-n_{i1}} &< \frac{1}{2}[x_{i1}^{(1)}]^{m_{i1}}(1 + \sum_j A_{ij}x_{ij}^{(1)})^{-n_{i1}} \\ &+ \frac{1}{2}[x_{i1}^{(2)}]^{m_{i1}}(1 + \sum_j A_{ij}x_{ij}^{(2)})^{-n_{i1}} \end{aligned} \quad (8)$$

Which proves the strict joint convexity of $x_{i1}^{m_{i1}}[1 + \sum_j A_{ij}x_{ij}]^{-n_{i1}}$, and therefore the strict joint convexity of $\sum_j c_{ij}K_{ij}^{\beta_{ij}}x_{ij}^{m_{ij}}[\sum_j K_{ij}x_{ij} + \sum_k K_{-ik}x_{ik}]^{-n_{ij}}$.

Finally, since $-E_i + Q_i \frac{\sum_j x_{ij}}{\sum_k x_{-ik}}$ is clearly jointly concave in $x_{ij}s$, we conclude that the profit function V_i for retailer i is strictly jointly concave in $x_{ij}s$ if $m_{ij} - n_{ij} > 1 \forall j$. The function $x = p^{-\alpha}$ is a one-one correspondence. Hence, if $m_{ij} - n_{ij} > 1$, there exists a vector of prices, $\mathbf{p}_i^* = \{p_{ij}^*, 1 \leq j \leq \xi_i\}$ (where $p_{ij}^* = x_{ij}^*{}^{-\frac{1}{\alpha}}$), that uniquely maximizes V_i . ■

Denote the strategy space for retailer i as the ξ_i -dimensional space $\mathbf{S}_i = \{x_{ij} : x_{ij} \in \mathfrak{R}^1, 0 \leq x_{ij} \leq B_{ij}\}^{\xi_i}$ and the set of strategy profiles as $\mathbf{X} = \{\mathbf{x} := (\mathbf{x}_i, \mathbf{x}_{-i}) : \mathbf{x}_i \in \mathbf{S}_i, \mathbf{x}_{-i} \in \mathbf{S}_{-i}\}$.

Lemma 5 *There exists at least one pure strategy Nash equilibrium in the game.*

Proof: The strategy space $\mathbf{S}_i = \{x_{ij} : x_{ij} \in \mathfrak{R}^1, 0 \leq x_{ij} \leq B_{ij}\}^{\xi_i}$ of retailer i , is a convex, non-empty, closed and bounded and therefore compact subset of the Euclidean space \mathfrak{R}^{ξ_i} . The profit function V_i is continuous in $\mathbf{x} := (\mathbf{x}_i, \mathbf{x}_{-i})$ and concave in \mathbf{x}_i . Therefore, using Theorem 1.2 in Fudenberg and Tirole (1998), we conclude that there exists at least one pure strategy Nash equilibrium in the game. ■

3 Decentralized Distributed (DD) Configuration

In this configuration, the sequence of actions is as follows. Simultaneously, the M_i s decide on the linear wholesale prices $w_{ij}s$ to offer to the respective R_i s, then the R_i s simultaneously decide their

sets of offerings $\{(i, j, p_{ij})\}$, demand for each offering is realized, and finally costs are spent and revenues are received.

3.1 Single Brand Case

The profit function for manufacturer M_i , is $V_{M_i} = w_i y_i - c_i y_i^{\beta_i}$, and that for retailer R_i is $V_i = p_i y_i - w_i y_i - E_i + Q_i \frac{s_i}{s_{-i}} = (K_i x_i)(K_i x_i + K_{-i} x_{-i})^{\theta-1} - w_i K_i x_i^{\frac{\alpha+1}{\alpha}} (K_i x_i + K_{-i} x_{-i})^{\theta-1} - E_i + Q_i \frac{x_i}{x_{-i}}$.

Fact 2 *Manufacturer M_i 's profit function V_{M_i} is strictly convex and increasing in y_i for $y_i > (\frac{c_i \beta_i}{w_i})^{\frac{1}{1-\beta_i}}$.*

Proof:

$$\begin{aligned} \frac{dV_{M_i}}{dy_i} &= w_i - c_i \beta_i y_i^{\beta_i-1} \\ \frac{d^2 V_{M_i}}{dy_i^2} &= \beta_i (1 - \beta_i) c_i y_i^{\beta_i-2} \\ &> 0, \text{ since } 0 < \beta_i < 1. \end{aligned}$$

Therefore, V_{M_i} is convex in y_i . V_{M_i} is increasing in y_i when $w_i - c_i \beta_i y_i^{\beta_i-1} > 0$, i.e., when $y_i > (\frac{c_i \beta_i}{w_i})^{\frac{1}{1-\beta_i}}$. ■

From this point forward, we will assume that profit function V_{M_i} is increasing in y_i .¹³

Fact 3 *Given wholesale prices w_i s, retailer R_i 's profit function V_i is strictly concave in x_i if $m_i - n_i > 1$. Thus, if $m_i - n_i > 1$, there exists $p_i^* = x_i^{*-1/\alpha}$ that uniquely maximizes V_i .*

Proof: From Lemma 1, we know that $(K_i x_i)(K_i x_i + K_{-i} x_{-i})^{\theta-1} - E_i + Q_i \frac{x_i}{x_{-i}}$ is strictly concave in x_i . It suffices to show that $\Psi_i := x_i^a (K_i x_i + K_{-i} x_{-i})^{\theta-1}$ is convex in x_i , where $a = \frac{\alpha+1}{\alpha} > 1$.

$$\begin{aligned} \frac{d\Psi_i}{dx_i} &= a x_i^{a-1} (K_i x_i + K_{-i} x_{-i})^{\theta-1} - (1 - \theta) K_i x_i^a (K_i x_i + K_{-i} x_{-i})^{\theta-2} \\ \frac{d^2 \Psi_i}{dx_i^2} &= a(a-1) x_i^{a-2} (K_i x_i + K_{-i} x_{-i})^{\theta-1} - 2a(1-\theta) K_i x_i^{a-1} (K_i x_i + K_{-i} x_{-i})^{\theta-2} \\ &\quad + (1-\theta)(2-\theta) x_i^a K_i^2 (K_i x_i + K_{-i} x_{-i})^{\theta-3} \\ &= K_i^2 x_i^2 [a(a-1) - 2a(1-\theta) + (1-\theta)(2-\theta)] + K_{-i}^2 x_{-i}^2 [a(a-1)] \\ &\quad + 2a K_i K_{-i} x_i x_{-i} [(a-1) - (1-\theta)] \end{aligned} \tag{9}$$

¹³ $y_i > (\frac{c_i \beta_i}{w_i})^{\frac{1}{1-\beta_i}}$ may not hold when $y_i \approx 0^+$. However, for most practical situations, it is safe to make the said assumption

Consider the terms within square braces in (9) separately.

$$a(a-1) - 2a(1-\theta) + (1-\theta)(2-\theta) = (a+\theta)^2 - 3(a+\theta) + 2 > 0, \text{ since } a+\theta > 1.$$

$$a(a-1) > 0, \text{ since } a > 1.$$

$$(a-1) - (1-\theta) = a+\theta-2; (m_i-n_i) > 1 \Rightarrow a-(1-\theta) > \frac{1}{\beta} \Rightarrow a+\theta-1 > 1.$$

Since $K_i, K_{-i} > 0$, the right hand side of (9) is greater than 0, implying that $\Psi_i := x_i^a(K_i x_i + K_{-i} x_{-i})^{\theta-1}$ is strictly convex in x_i . ■

Proposition 4 *There exists at least one pure strategy Nash equilibrium in the subgame between the retailers.*

Proof: Given wholesale prices w_i s, the strategy space $S_i = \{x_i : x_i \in \mathbb{R}^1, 0 \leq x_i \leq B_i\}$ of retailer i , is a non-empty compact convex subset of the Euclidean space \mathbb{R}^1 . The profit function V_i is continuous in $x := (x_i, x_{-i})$ and concave in x_i . Therefore, using Theorem 1.2 in Fudenberg and Tirole (1998), there exists at least one pure strategy Nash equilibrium in the subgame between the retailers. ■

Lemma 6 *There exists at least one Nash equilibrium in the overall game.*

Proof: Let W_i^* be an arbitrarily large upper bound on wholesale price w_i such that if M_i prices beyond this bound then R_i would stay out of the market and make zero profit rather than participate in the market and make a loss. Such an upper bound exists because as $w_i \rightarrow \infty$, $V_i \rightarrow -\infty$. Thus the strategy space of M_i is constrained to be a closed and bounded convex, compact set $[0, W_i^*]$. Since the extensive form game is finite and strategy spaces of M_i s and R_i s are closed and bounded, using the technical properties in section 8.3.3 of Fudenberg and Tirole (1998) we conclude that there exists at least one Nash equilibrium in the overall game. ■

Corollary 1 *There exists at least one subgame perfect pure strategy Nash equilibrium in the overall game.*

Proof: Since for any specification of wholesale prices there exists at least one pure strategy Nash equilibrium in the subgame between the retailers, it is possible to deduce equilibrium y_i^* s (derived from equilibrium x_i^* s), for all combinations of w_i between 0 and W_i^* and w_{-i} between 0 and W_{-i}^* . Since M_i 's profit is assumed to be increasing in the quantity y_i sold to R_i , we can pick w_i^* which results in the largest equilibrium y_i^* , given w_{-i} . The continuous reaction functions $w_i^*(w_{-i})$ and $w_{-i}^*(w_i)$ yield a subgame perfect pure strategy Nash equilibrium. ■

Lemma 7 *For any specification of wholesale prices, the symmetric equilibrium quantity in the retailer subgame of the DD configuration is smaller than the symmetric equilibrium quantity in the*

*CD configuration game. Hence, the symmetric equilibrium quantity in the overall DD configuration game is smaller than that in the CD configuration game.*¹⁴

Proof: For clarity, denote the integrated firm i 's profit in the *CD* configuration, as V_i^{CD} . We know that $V_i^{CD} = (K_i x_i)(K_i x_i + K_{-i} x_{-i})^{\theta-1} - c_i K_i^{\beta_i} x_i^{m_i} (K_i x_i + K_{-i} x_{-i})^{-n_i} - E_i + Q_i \frac{x_i}{x_{-i}}$. Denote R_i 's profit in the *DD* configuration as V_i^{DD} .

$$V_i^{DD} = V_i^{CD} + c_i K_i^{\beta_i} x_i^{m_i} (K_i x_i + K_{-i} x_{-i})^{-n_i} - w_i K_i x_i^{\frac{\alpha+1}{\alpha}} (K_i x_i + K_{-i} x_{-i})^{\theta-1}$$

Denoting $a = \frac{\alpha+1}{\alpha}$ and $A = K_i x_i + K_{-i} x_{-i}$,

$$\frac{dV_i^{DD}}{dx_i} = \frac{dV_i^{CD}}{dx_i} + c_i K_i^{\beta_i} (m_i x_i^{m_i-1} A^{-n_i} - n_i K_i x_i^{m_i} A^{-n_i-1}) - w_i K_i (a x_i^{a-1} A^{\theta-1} - (1-\theta) K_i x_i^a A^{\theta-2}) \quad (10)$$

We consider the last two terms on the right hand side of (10) separately.

$$\begin{aligned} & c_i K_i^{\beta_i} (m_i x_i^{m_i-1} A^{-n_i} - n_i K_i x_i^{m_i} A^{-n_i-1}) - w_i K_i (a x_i^{a-1} A^{\theta-1} - (1-\theta) K_i x_i^a A^{\theta-2}) \\ &= x_i^{-1} a (c_i \beta_i y_i^{\beta_i} - w_i y_i) - (1-\theta) A^{-1} K_i (c_i \beta_i y_i^{\beta_i} - w_i y_i) \\ &= \frac{1}{K_i x_i} (aA - (1-\theta)K_i) (c_i \beta_i y_i^{\beta_i} - w_i y_i) \end{aligned}$$

$(aA - (1-\theta)K_i) > 0$ because $a > 1$ and $0 \leq \theta < 1$. If M_i 's profit is increasing in the quantity y_i sold to R_i , then from Fact 2, $c_i \beta_i y_i^{\beta_i} - w_i y_i < 0$. Therefore $\frac{dV_i^{DD}}{dx_i} < \frac{dV_i^{CD}}{dx_i}$, and $x_i^{DD*} < x_i^{CD*}$, because V_i^{DD} and V_i^{CD} are concave in x_i . Since $y_i = K_i x_i^a A^{\theta-1}$, $\frac{dy_i}{dx_i} = K_i x_i^{a-1} A^{\theta-2} (aA - (1-\theta)K_i x_i) > 0$; implying that $y_i^{DD*} < y_i^{CD*}$. Thus, for any given x_{-i} , the best response x_i^* and hence y_i^* is smaller in the *DD* configuration than in the *CD* configuration. The reaction functions $x_i^*(x_{-i})$ and $x_{-i}^*(x_i)$ are shifted closer to the origin in the *DD* configuration, and so is the symmetric equilibrium. ■

3.2 Multiple Brand Case

The notation we use is identical to that used in section 2.2. The profit function for manufacturer M_i , is $V_{M_i} = \sum_j w_{ij} y_{ij} - \sum_j c_{ij} y_{ij}^{\beta_{ij}}$, and that for retailer R_i , is:

$$V_i = \sum_j p_{ij} y_{ij} - \sum_j w_{ij} y_{ij} - E_i + Q_i \frac{\sum_j s_{ij}}{\sum_k s_{-ik}}$$

¹⁴Through numerical studies, we observe the following effect of decentralization on asymmetric equilibria. The retailer that prices low in the *CD* configuration asymmetric equilibrium, prices still lower in the asymmetric equilibrium of the retailer subgame in the *DD* configuration (if such an equilibrium exists). Vice-versa for the retailer that prices high. The absence of scale economies at the retailers' in the *DD* configuration diminishes the effect of relative market share significantly. Therefore, asymmetric equilibria rarely exist in the retailer subgame of the *DD* configuration.

$$\begin{aligned}
&= \left[\sum_j K_{ij} x_{ij} \right] \left[\sum_j K_{ij} x_{ij} + \sum_k K_{-ik} x_{-ik} \right]^{\theta-1} - \sum_j w_{ij} K_{ij} x_{ij}^a \left[\sum_j K_{ij} x_{ij} + \sum_k K_{-ik} x_{-ik} \right]^{\theta-1} - E_i \\
&\quad + Q_i \frac{\sum_j x_{ij}}{\sum_k x_{-ik}}, \text{ where } a = \frac{\alpha+1}{\alpha}, 1 \leq j \leq \xi_1, \text{ and } 1 \leq k \leq \xi_2.
\end{aligned}$$

Fact 4 *Manufacturer M_i 's profit function V_{M_i} is strictly jointly convex and increasing in $\mathbf{y}_i = \{y_{ij}, 1 \leq j \leq \xi_i\}$ if $y_{ij} > \left(\frac{c_{ij}\beta_{ij}}{w_{ij}}\right)^{\frac{1}{1-\beta_{ij}}} \forall j$.*

Proof: $V_{M_i}(\mathbf{y}_i)$ is separable in the components of \mathbf{y}_i ; i.e., we can write $V_{M_i}(\mathbf{y}_i) = \sum_j V_{M_{ij}}(y_{ij})$, where $V_{M_{ij}}(y_{ij}) = w_{ij}y_{ij} - c_{ij}y_{ij}^{\beta_{ij}}$. From Fact 2, we know that $V_{M_{ij}}$ is strictly convex and increasing in y_{ij} if $y_{ij} > \left(\frac{c_{ij}\beta_{ij}}{w_{ij}}\right)^{\frac{1}{1-\beta_{ij}}}$. Since $V_{M_i}(\mathbf{y}_i)$ is separable and its separate parts are strictly convex and increasing, we have that V_{M_i} is strictly jointly convex and increasing in $\mathbf{y}_i = \{y_{ij}, 1 \leq j \leq \xi_i\}$ if $y_{ij} > \left(\frac{c_{ij}\beta_{ij}}{w_{ij}}\right)^{\frac{1}{1-\beta_{ij}}} \forall j$. ■

Fact 5 *Given wholesale prices w_{ij} s, retailer R_i 's profit function V_i is strictly jointly concave in x_{ij} s if $m_{ij} - n_{ij} > 1 \forall j$. Thus, if $m_{ij} - n_{ij} > 1 \forall j$, there exists there exists a vector of prices $\mathbf{p}_i^* = \{p_{ij}^*, 1 \leq j \leq \xi_i\}$ (where $p_{ij}^* = x_{ij}^{*-\frac{1}{\alpha}}$), that uniquely maximizes V_i .*

Proof: From Lemma 4, we know that $[\sum_j K_{ij} x_{ij}] [\sum_j K_{ij} x_{ij} + \sum_k K_{-ik} x_{-ik}]^{\theta-1} - E_i + Q_i \frac{\sum_j x_{ij}}{\sum_k x_{-ik}}$ is strictly jointly concave in x_{ij} s. What remains to be shown is that $\sum_j w_{ij} K_{ij} x_{ij}^a [\sum_j K_{ij} x_{ij} + \sum_k K_{-ik} x_{-ik}]^{\theta-1}$ is strictly jointly convex in x_{ij} s. Since the sum of jointly convex functions is again jointly convex, it suffices to show the joint convexity of $\Psi_{i1} := x_{i1}^a [1 + \sum_j A_{ij} x_{ij}]^{\theta-1}$ (where A_{ij} s are positive constants), assuming that $m_{i1} - n_{i1} > 1$.

$m_{i1} - n_{i1} > 1 \Rightarrow a\beta_{i1} - (1-\theta)\beta_{i1} > 1 \Rightarrow a - (1-\theta) > 1$, since $0 < \beta_{i1} < 1$. The proof of Lemma 4 can be replicated by replacing m with a , and n with $(1-\theta)$. ■

Proposition 5 *There exists at least one pure strategy Nash equilibrium in the subgame between the retailers.*

Proof: Given wholesale prices w_{ij} s, the strategy space $\mathbf{S}_i = \{x_{ij} : x_{ij} \in \mathfrak{R}^1, 0 \leq x_{ij} \leq B_{ij}\}^{\xi_i}$ of retailer i is a non-empty compact convex subset of the Euclidean space \mathfrak{R}^{ξ_i} . The profit function V_i is continuous in $\mathbf{x} := (\mathbf{x}_i, \mathbf{x}_{-i})$ and concave in \mathbf{x}_i , where $\mathbf{x}_i \in \mathbf{S}_i, \mathbf{x}_{-i} \in \mathbf{S}_{-i}$. Therefore, using Theorem 1.2 in Fudenberg and Tirole (1998), there exists at least one pure strategy Nash equilibrium in the subgame between the retailers. ■

The proof of the existence of at least one Nash equilibrium in the overall DD game is similar to the proof of Lemma 6 and is therefore omitted. The proof of the existence of a subgame perfect pure strategy Nash equilibrium in the overall game is similar to the proof of Corollary 1 and is also omitted.

Corollary 2 *For any specification of wholesale prices, the symmetric equilibrium quantities in the retailer subgame of the DD configuration are respectively smaller than the symmetric equilibrium quantities in the CD configuration game. Hence, the symmetric equilibrium quantities in the overall DD configuration game are respectively smaller than those in the CD configuration game.*

Proof: Since the profit function of each retailer R_i is jointly concave in the x_{ij} s, the first order condition with respect to each x_{ij} must hold at the maximum of the respective retailer's profit function. The remainder of the proof is similar to the proof of Lemma 7 *mutatis mutandis*. ■

4 Decentralized Shared (DS) Configuration

In the *DS* structure, M and R_i s are separate entities. The sequence of actions in this game is as follows. M decides on the linear wholesale prices w_{ij} s to offer to the R_i s, then the R_i s simultaneously decide their sets of offerings $\{(i, j, p_{ij})\}$, demand for each offering is realized, and finally costs are spent and revenues are received.

4.1 Single Brand Case

The profit function for the manufacturer M , is $V_M = \sum_i w_i y_i - \sum_i c_i y_i^{\beta_i}$, and that for retailer R_i is $V_i = p_i y_i - w_i y_i - E_i + Q_i \frac{s_i}{s-i} = (K_i x_i)(K_i x_i + K_{-i} x_{-i})^{\theta-1} - w_i K_i x_i^{\frac{\alpha+1}{\alpha}} (K_i x_i + K_{-i} x_{-i})^{\theta-1} - E_i + Q_i \frac{x_i}{x_{-i}}$.

Fact 6 *Manufacturer M 's profit function V_M is strictly jointly convex and increasing in $\mathbf{y} = \{y_i, i = 1, 2\}$ if $y_i > (\frac{c_i \beta_i}{w_i})^{\frac{1}{1-\beta_i}} \forall i$.*

Proof: $V_M(\mathbf{y})$ is separable in the components of \mathbf{y} ; i.e., we can write $V_M(\mathbf{y}) = \sum_i V_{M_i}(y_i)$, where $V_{M_i}(y_i) = w_i y_i - c_i y_i^{\beta_i}$. From Fact 2, we know that V_{M_i} is strictly convex and increasing in y_i if $y_i > (\frac{c_i \beta_i}{w_i})^{\frac{1}{1-\beta_i}}$. Since $V_M(\mathbf{y})$ is separable and its separate parts are strictly convex and increasing, we have that V_M is strictly jointly convex and increasing in $\mathbf{y} = \{y_i, i = 1, 2\}$ if $y_i > (\frac{c_i \beta_i}{w_i})^{\frac{1}{1-\beta_i}} \forall i$. ■

Fact 7 *Given wholesale prices w_i s, retailer R_i 's profit function V_i is strictly concave in x_i if $m_i - n_i > 1$. Thus, if $m_i - n_i > 1$, there exists $p_i^* = x_i^{*- \frac{1}{\alpha}}$ that uniquely maximizes V_i .*

Proof: Identical to the proof of Fact 3. Hence omitted. ■

Proposition 6 *There exists at least one pure strategy Nash equilibrium in the subgame between the retailers.*

Proof: Given wholesale prices w_i s, the strategy space $S_i = \{x_i : x_i \in \mathbb{R}^1, 0 \leq x_i \leq B_i\}$ of retailer i , is a non-empty compact convex subset of the Euclidean space \mathbb{R}^1 . The profit function V_i is continuous in $x := (x_i, x_{-i})$ and concave in x_i . Therefore, using Theorem 1.2 in Fudenberg and Tirole (1998), there exists at least one pure strategy Nash equilibrium in the subgame between the retailers. ■

For both the single brand as well as the multiple brand cases of the *DS* configuration, the proof of the existence of at least one Nash equilibrium in the overall game is similar to the proof of Lemma 6 and is therefore omitted. Also, the proof of the existence of a subgame perfect pure strategy Nash equilibrium in the overall game is similar to the proof of Corollary 1 and is omitted.

Lemma 8 *The symmetric equilibrium quantity in the overall DS configuration game is no less than that in the overall DD configuration game.*

Proof: We prove the lemma by argument. In the *DD* game, manufacturer M_i chooses w_i conditional on the competing manufacturer's w_{-i} in order to achieve the largest equilibrium y_i . However, in the *DS* game, since the same manufacturer M supplies to both retailers, the choice of both w_i as well as w_{-i} is within the control of M . Manufacturer M thus optimizes over a set larger than that available to each of the M_i s in the *DD* game. Hence M cannot be worse off than the M_i s. Since, by construction, a manufacturer's profit is increasing in quantity sold, we can conclude that manufacturer M sells no less than manufacturer M_i , to retailer R_i . ■

4.2 Multiple Brand Case

The notation we use is identical to that used in section 3.2. The profit function for manufacturer M , is $V_M = \sum_i \sum_j w_{ij} y_{ij} - \sum_i \sum_j c_{ij} y_{ij}^{\beta_{ij}}$, and that for retailer R_i , is:

$$\begin{aligned} V_i &= \sum_j p_{ij} y_{ij} - \sum_j w_{ij} y_{ij} - E_i + Q_i \frac{\sum_j s_{ij}}{\sum_k s_{-ik}} \\ &= \left[\sum_j K_{ij} x_{ij} \right] \left[\sum_j K_{ij} x_{ij} + \sum_k K_{-ik} x_{-ik} \right]^{\theta-1} - \sum_j w_{ij} K_{ij} x_{ij}^a \left[\sum_j K_{ij} x_{ij} + \sum_k K_{-ik} x_{-ik} \right]^{\theta-1} - E_i \\ &\quad + Q_i \frac{\sum_j x_{ij}}{\sum_k x_{-ik}}, \text{ where } a = \frac{\alpha+1}{\alpha}, 1 \leq j \leq \xi_1, \text{ and } 1 \leq k \leq \xi_2. \end{aligned}$$

Fact 8 *Manufacturer M 's profit function V_M is strictly jointly convex and increasing in $\mathbf{y} = \{y_{ij}, i = 1, 2; 1 \leq j \leq \xi_i\}$ if $y_{ij} > \left(\frac{c_{ij}\beta_{ij}}{w_{ij}}\right)^{\frac{1}{1-\beta_{ij}}} \forall i, j$.*

Proof: $V_M(\mathbf{y})$ is separable in the components of \mathbf{y} ; i.e., we can write $V_M(\mathbf{y}) = \sum_i \sum_j V_{M_{ij}}(y_{ij})$, where $V_{M_{ij}}(y_{ij}) = w_{ij} y_{ij} - c_{ij} y_{ij}^{\beta_{ij}}$. From Fact 2, we know that $V_{M_{ij}}$ is strictly convex and increasing in y_{ij} if $y_{ij} > \left(\frac{c_{ij}\beta_{ij}}{w_{ij}}\right)^{\frac{1}{1-\beta_{ij}}}$. Since $V_M(\mathbf{y})$ is separable and its separate parts are strictly convex and increasing, we have that V_M is strictly jointly convex and increasing in $\mathbf{y} = \{y_{ij}, 1 = 1, 2; 1 \leq j \leq \xi_i\}$ if

$$y_{ij} > \left(\frac{c_{ij}\beta_{ij}}{w_{ij}}\right)^{\frac{1}{1-\beta_{ij}}} \quad \forall i, j. \quad \blacksquare$$

Fact 9 *Given wholesale prices w_{ij} s, retailer R_i 's profit function V_i is strictly jointly concave in x_{ij} s if $m_{ij} - n_{ij} > 1 \quad \forall j$. Thus, if $m_{ij} - n_{ij} > 1 \quad \forall j$, there exists there exists a vector of prices $\mathbf{p}_i^* = \{p_{ij}^*, 1 \leq j \leq \xi_i\}$ (where $p_{ij}^* = x_{ij}^{*\frac{1}{\alpha}}$), that uniquely maximizes V_i .*

Proof: Identical to the proof of Fact 5. Hence omitted. \blacksquare

Proposition 7 *There exists at least one pure strategy Nash equilibrium in the subgame between the retailers.*

Proof: Given wholesale prices w_{ij} s, the strategy space $\mathbf{S}_i = \{x_{ij} : x_{ij} \in \mathbb{R}^1, 0 \leq x_{ij} \leq B_{ij}\}^{\xi_i}$ of retailer i is a non-empty compact convex subset of the Euclidean space \mathbb{R}^{ξ_i} . The profit function V_i is continuous in $\mathbf{x} := (\mathbf{x}_i, \mathbf{x}_{-i})$ and concave in \mathbf{x}_i , where $\mathbf{x}_i \in \mathbf{S}_i, \mathbf{x}_{-i} \in \mathbf{S}_{-i}$. Therefore, using Theorem 1.2 in Fudenberg and Tirole (1998), there exists at least one pure strategy Nash equilibrium in the subgame between the retailers. \blacksquare

Corollary 3 *The symmetric equilibrium quantities in the overall DS configuration game are respectively no less than those in the overall DD configuration game.*

Proof: Similar to the proof of Lemma 8. \blacksquare

5 Conclusion and Future Work

Recent research supports the belief that supply chain configuration and equilibrium product variety are closely linked. In this paper we have explored the effects of supply chain configuration on equilibrium product variety, in a competitive setting. Thus far we have been able to prove important existence results pertaining to equilibrium variety, and specific properties of such equilibria, under three distinct supply chain configurations - *CD*, *DD*, and *DS* - depicted in Figure 1. In the *CD* configuration, when relative market share is a concern for the vertically integrated retailers, we find that asymmetric equilibria exist even in symmetric situations. The propensity for asymmetric equilibria in symmetric situations is diminished when, in a decentralized situation, retailers no longer have manufacturing economies of scale. While we have dealt with the *CD* configuration in detail, further analysis with regard to the *DD* and *DS* configurations is needed to gain deeper insights into the relationship between supply chain structure and product variety. In addition, a detailed comparative analysis of equilibrium variety outcomes of the three configurations, is required. As a further direction, we foresee that an examination of the impact of cost structure on equilibrium variety will help us gain a better understanding of equilibria associated with functional variety.

References

- Anderson, Simon P., Andre de Palma, and Yurii Nesterov, "Oligopolistic Competition and the Optimal Provision of Products", *Econometrica*, Vol. 63, No. 6 (Nov., 1995), 1281-1301.
- Baumol, William J. and Edward A. Ide, "Variety in Retailing", *Management Science*, Vol. 3, No. 1 (Oct., 1956), 93-101.
- Bell, David E., Ralph L. Keeney, and John D. C. Little, "A Market Share Theorem", *Journal of Marketing Research*, Vol. 12 (May, 1975), 136-141.
- Betancourt, Roger and David Gautschi, "The Economics of Retail Firms", *Managerial and Decision Economics*, Vol. 9, No. 2 (Jun., 1988), 133-144.
- Cooper, Lee G., Masao Nakanishi, *Market-Share Analysis*, Kluwer Academic Publishers (1998).
- Fudenberg, Drew and Jean Tirole, *Game Theory*, MIT Press (1998).
- Hiam, Alexander and Charles D. Schewe, *The Portable MBA in Marketing*, John Wiley & Sons (1992).
- Hotelling, Harold, "Stability in Competition", *Economic Journal*, Vol. 39 (Mar., 1929), 41-57.
- Karnani, Aneel, "Strategic Implications of Market Share Attraction Models", *Management Science*, Vol. 31, No. 5 (May, 1985), 536-547.
- Kekre, Sunder and Kannan Srinivasan, "Broader Product Line: A Necessity to Achieve Success?", *Management Science*, Vol. 36, No. 10 (Oct., 1990), 1216-1231.
- Lancaster, Kelvin, "Competition and Product Variety", *The Journal of Business*, Vol. 53, No. 3 (Jul., 1980), S79-S103.
- Lilien, Gary L., Philip Kotler, and K. Sridhar Moorthy, *Marketing Models*, Prentice Hall (1992).
- Lindsey, Robin, Balder Von Hohenbalken, and Douglas S. West, "Spatial Price Equilibrium with Product Variety, Chain Stores, and Integer Pricing: An Empirical Analysis", *The Canadian Journal of Economics*, Vol. 24, No. 4 (Nov., 1991), 900-922.
- McGuire, Timothy W. and Richard Staelin, "An Industry Equilibrium Analysis of Downstream Vertical Integration", *Marketing Science*, Vol. 2, No. 2 (Spring, 1983), 161-191.

Prescott, Edward C. and Michael Visscher, “Sequential Location among Firms with Foresight”, *The Bell Journal of Economics*, Vol. 8, No. 2 (Autumn 1977), 378-393.

Ramdas, Kamalini, “Managing Product Variety: An Integrative Review and Research Directions”, *Working Paper*, Darden Graduate School of Business, University of Virginia (Mar., 2002).

Randall, Taylor and Karl Ulrich, “Product Variety, Supply Chain Structure, and Firm Performance: Analysis of the U.S. Bicycle Industry”, *Management Science*, Vol. 47, No. 12 (Dec., 2001), 1588-1604.

Randall, Taylor (David Eccles School of Business, University of Utah), Serguei Netessine (The Wharton School of Business, University of Pennsylvania), and Nils Rudi (W. E. Simon School of Business, University of Rochester), “Virtual Order Fulfilment: Key to Success or Path to Failure?”, *Working Paper* (Mar., 2002).

Rosen, Sherwin, “Hedonic Prices and Implicit Markets: Product Differentiation in Pure Competition”, *The Journal of Political Economy*, Vol. 82, No. 1 (Jan.-Feb., 1974), 34-55.

Schaffir, Kurt H., “The Economics of Nonfunctional Variety”, *Operations Research*, Vol. 11, No. 5 (Sep.-Oct., 1963), 702-720.

Shaked, Avner and John Sutton, “Relaxing Price Competition through Product Differentiation”, *The Review of Economic Studies*, Vol. 49, No. 1 (Jan., 1982), 3-13.

Stiglitz, J. E., “Equilibrium in Product Markets with Imperfect Information”, *The American Economic Review*, Vol. 69, No. 2 (May, 1979), 339-345.