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FUNCTIONAL ANALYSIS AND BOUNDARY VALUE PROBLEMS

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FUNCTIONAL ANALYSIS AND BOUNDARY VALUE PROBLEMS[(*)]

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1. A Review of Research

1.1 EXISTENCE AND STABILITY OF PERIODIC SOLUTIONS OF PERTURBATION PROBLEMS

In the years 1952-60 a great deal of work was done by a group of us (Cesari [5a,b, 1, 6a,b], R. A. Gambill [2, 9a,b,c], J. K. Hale [10, 12a,b,c,d,e,f,g,h,i,j], and others) in relation to periodic solutions of perturbation problems for ordinary differential equations. This work was based on functional analysis, and particularly on fixed point theorems, projector operators, and bifurcation equations in terms of functional analysis.

We obtained criteria for the existence of periodic solutions (harmonic, subharmonics, ultraharmonic) of systems of periodic ordinary differential equations of the perturbation type

$$du/dt = Au + \varepsilon f(t,u,\varepsilon) \, ,$$

$$u = (u_1,\ldots,u_n), \quad f(t + T,u,\varepsilon) = f(t,u,\varepsilon) \, ,$$

criteria for the existence of cycles of systems of autonomous equations

$$du/dt = Au + \varepsilon f(u,\varepsilon), \quad u = (u_1,\ldots,u_n) \, ,$$

and methods for the analysis of the asymptotic stability of periodic solutions, and of asymptotic orbital stability of cycles. For the particular purpose of a new and convergent method of successive approximations was devised.

In particular, J. K. Hale proved that nonlinear autonomous perturbation type systems of ordinary differential equations may easily present families of periodic solutions, or cycles, under suitable conditions of symmetry. J. K. Hale gave also simple criteria for this occurrence, and for the determination of the dimension of the family of cycles [12c, 5b].

I presented aspects of this first phase of the research in [5c] and in my book ([5d], pp. 66-78, and pp. 123-135). Halanay reported the method of successive approximations in his book ([11], pp. 308-317) together with the proof of its convergence and applications. J. K. Hale gave a detailed account of aspects of this first phase of the research in his books ([12k], pp. 27-94).

This work on periodic solutions of perturbation type problems in ordinary differential equations was then continued by C. Imaz [17], and by J. Mawhin [21a,b,c,d]. C. Imaz discussed particularly the case in which the underlying linear problem has multiple characteristic roots. J. Mawhin, in an extensive and systematic investigation still underway, has improved some of the results of J. K. Hale and R. A. Gambill.

In view of numerical analysis C. Banfi [3a,b] and C. Banfi and G. Casadei [4] considered a variant of the method of successive approximations, suited for high speed computers, and in which the actual successive approximations and the analysis and solution of the bifurcation equation are combined. They successfully experimented with the method.

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1.2 BOUNDARY VALUE PROBLEMS FOR GENERAL NONLINEAR PROBLEMS

In my paper [5e] I presented the underlying ideas of the method, in an abstract form, for general boundary value problems for ordinary and partial differential equations, even, strongly nonlinear. Also, I showed connections of the method with Galerkin's approach. In [5e] I essentially considered problems whose underlying linear formulation was selfadjoint. J. Locker [20a,b], and J. K. Hale, S. Bancroft, and D. Sweet [13] gave suitable extensions of the method to "nonselfadjoint" problems.

In [5f] I considered the problem of periodic solutions of systems of periodic ordinary differential equations nonnecessarily of the perturbation type, from the view point of the method as presented in [5c]. Also I proved in [5f] that the method contains as a particular case the much simpler approach used earlier for perturbation problems only. My work in [5f] was extended to problems of periodic solutions of ordinary differential equations with time lags by A. M. Rodionov [25].

S. A. Williams [27] showed that the method under consideration, as described in [5c], has a strict theoretical link with the Leray-Schauder approach [29]. Hale, Bancroft, and D. Sweet in their paper [13] emphasized other connections with previous or current work of D. C. Lewis, H. A. Antosiewicz, Jane Cronin, R. G. Bartle, L. Nirewberg, and Y. Sibuya.

I presented aspects of this second phase of the research in [5ℓ]. Jane Cronin reported the main idea of the method, as presented in [5f] for periodic solutions only, in her book ([8], pp. 180-185). J. K. Hale gave a detailed account of aspects of this second phase of research in his books ([12k], pp. 96-99, and [12m], pp. 252-290).

For the particular problem of periodic solutions of ordinary differential equations, nonnecessarily of the perturbation type further theoretical work was done by H. W. Knobloch [18a] in connection with the use of the uniform norm, suitable estimates of the Fourier approximations, and a discussion of the bifurcation equations for any order of approximation by topological considerations based on C. Miranda's version of Bro wer's fixed joints theorem, version which had been already used in [5b]. H. W. Knobloch proved then simple existence theorems [18b] for periodic solutions of nonlinear periodic equations $y'' + g(t,y,y') = o$, based only on qualitative properties of g, and this analysis led him to comparison and oscillation theorems for the same equations [18c].

In [5g,i,j] I studied periodic solutions in x and y of the partial differential equation

$$u_{xy} = g(x,y,u,u_x,u_y)$$

where u is an n-vector, in the frame of the same method as presented in [5e]. By assuming g continuous in all its argument and Lipschitzian in u_x, u_y, or in u, u_x, u_y, I obtained criteria for the existence of periodic solutions $u(x,y)$ continuous in E_2 with u_x, u_y and u_{xy}. In [5h] I applied the same approach to the problem of solutions $u(x,y)$ to the same partial differential equation above, which are periodic with respect to x only and continuous with u_x, u_y, u_{xy} in a thin strip $[- \infty < x < + \infty, |y| \leq a]$. The results of [5g,i,j] yield analogous results for the nonlinear wave equation

$$u_{tt} - u_{xx} = g(t,x,u,u_t,u_x)$$

By the use of the same method, based on [5f], and actually the same projection

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operators, more stringent quantitative estimates have been successively obtained

by A. K. Aziz [Proc. Amer. Math. Soc. 17, 1966]. J. K. Hale [12ℓ], by the same

method applied directly to the nonlinear wave equation, and the use of a dif-

ferent choice of projector operators, obtained perspicuous criteria for doubly

periodic solutions in E_2. Recently D. Petrovanu [24a] has used the same method

in the study of periodic solutions of the equation

$$u_{xyz} = g(x,y,z,u,u_x,u_y,u_z)$$

and in the study of the periodic solutions [24b] of the Tricomi system of equa-

tions

$$u_x = g(x,y,u,v), \quad v_y = h(x,y,u,v)$$

where u and v are m- and n-vectors respectively.

A. Naparstek [22] has also used the same method for periodic distributional

solutions of the nonlinear wave equation

$$u_{tt} - \sigma^2 u_{xx} = \varepsilon\, g(t,x,u)$$

where g is periodic (of period 2π) in x and y, and either σ is a rational number,

or an irrational one of a known class of real numbers badly approximated by

rationals and everywhere dense on the positive real line. The result extend

to the problem of periodic solutions of equations of the form

$$u_{tt} - \sigma^2 u_{xx} = \varepsilon g(t,x,u,u_t,u_x)$$

and required a great deal of mathematical investigation. The bifurcation equa-

tion is studied in this work in terms of the theory of monotone operators.

In the same line W. S. Hall [14] has recently proved the existence of doubly periodic solutions of partial differential equations of the form

$$u_{tt} + (-1)^p D_x^{2p} u = \varepsilon f(t,x,u,u_t,u_x,\ldots,D_x^p u)$$

in suitable Banach spaces of periodic functions and distributions. The deep analysis which was needed for this work includes a discussion of the bifurcation in the lines of Minty and Browder's theory on monotone operators, a discussion of suitable boundary conditions for the problem under consideration, and smoothness properties of the solutions.

E. M. Landesman and A. C. Lazer [19] have recently proved a general existence theorem for the Dirichlet problem for nonlinear partial differential equations of the form

$$Lu + au + g(u) = h(x) ,$$

where L is a general second order selfadjoint uniformly elliptic operator in a domain G, with bounded measurable coefficients, where a is a positive constant, h is a given function in $L_2(G)$, and z is a monotone continuous real valued bounded functions on E_1. The very subtle proof makes use of a technique which has points of contact with the one of the method under consideration.

Cesari [5k] by the use of the process described in [5f], has discussed the existence of solutions $u(x,y)$ for the boundary value problem

$$\Delta u = g(x,y,u) \quad \text{for} \quad (x,y) \in A, \quad u = o \quad \text{for} \quad (x,y) \in \partial A ,$$

where A is the unit circle in the xy-plane, ∂A its boundary, g is measurable in

in x,y for every u, and Lipschitzian in u for every x,y, and where the solutions

u(x,y) are required to be continuous in the closed circle $A \cup \partial A$, with continuous

u_x, u_y in the open circle A, and $\Delta u = \partial^2 u/\partial x^2 + \partial^2 u/\partial y^2$————in the sense of the

theory of distributions—is a measurable function in A.

Recently, W. A. Harris, Y. Sibuya, and L. Weinberg [15] have used the same

approach under consideration here to obtain new and extremely simple proofs of

the classical theorems of Cauchy, Frobenius, Perron, and Lettenmeyer on linear

ordinary differential equations and systems in the complex field. These authors

are actually concerned with systems of the form

$$z^{s_i} \, du_i/dz = \sum_{j=1}^{n} a_{ij}(z)u_j \, , \quad i = 1,\ldots,n,$$

where z = x + iy, and the coefficients a_{ij} are holomorphic functions of z in a

complex neighborhood $V = [|z| < \delta]$ of the origin z = o. In the regular case

(all s_i = o) the bifurcation equation is trivial, and the system has a funda-

mental system of holomorphic solutions in V. In the regular singular case (all

s_i = 1) the bifurcation equation reduces essentially to the indicial equation,

and the system has the expected number of solutions of the forms $z^\rho P(z)$, ρ

complex, P(z) a convergent power series of z. For $d = \max_i s_i < n$, the given

system has at least n - d solutions holomorphic in V.

2.   A Direct Proof of Cauchy-Kovalevsky's Theorem

2.1   FORMULATION OF THE PROBLEM

The Cauchy-Kovalevsky theorem for partial differential equations in the complex field is usually proved by the method of majorants, which, for the problem under investigation, leads to a first order ordinary differential equation with separable variables, whose solutions are then proved to be majorants of the solutions of the given system of partial differential equations. All this is avoided in the following simple proof (No. (2.5) below), based on functional analysis.

Let $\mathcal{S}$ denote the class of all functions u of t and $z = (z_1,\ldots,z_\nu)$, which are holomorphic in some complex neighborhood of the origin (t = o, z = o), and thus possess power series expansion $u(t,z) = \sum_\ell \sum_m u_{\ell m} t^\ell z_1^{m_1}\ldots z_\nu^{m_\nu}$, $m = (m_1,\ldots,m_\nu)$, which is convergent in some neighborhood of the origin.

For the sake of simplicity we limit ourselves to the linear Cauchy problem, and we know that it is not restrictive (see I. G. Petrovsky [32], pp. 16-17) to formulate the linear Cauchy problem as follows: Given elements a_{ijh}, b_{ij}, $c_i \in \mathcal{S}$, determine elements $u_i \in \mathcal{S}$ such that

$$u_{it} = \sum_{j=1}^{n} \sum_{h=1}^{\nu} a_{ijh} u_{jz_h} + \sum_{j=1}^{n} b_{ij} u_j + c_i , \qquad (1)$$

$$u_i(o,z_1,\ldots,z_\nu) = o \quad , \quad i = 1,\ldots,n ,$$

in some complex neighborhood of the origin, and where u_{it}, u_{jz_h} denote partial derivatives with respect to t and z_h, h = 1,...,ν. If $U(t,z) = (u_1,\ldots,u_\nu)$,

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and $U_z = [u_{iz_h}, \ i = 1,\ldots,n, \ h = 1,\ldots,\nu]$, then (1) takes the form

$$U_t = F(t,z,U,U_z) \ , \quad U(o,z) = o \ .$$

The solution $U = (u_1,\ldots,u_n) \in \mathscr{S}^n$ of problem (1) is uniquely determined by the usual argument: If $\sum_\ell \sum_m u_{i\ell m} \ t^\ell z_1^{m_1}\ldots z_\nu^{m_\nu}$, $i = 1,\ldots,n$, are power series satisfying formally $U_t = F(t,z,U,U_z)$, and the coefficients u_{iom} are known, then all remaining coefficients can be determined by induction. Indeed, each power $t^\ell z_1^{m_1}\ldots z_\nu^{m_\nu}$ appears in the first member of (1) with coefficients $(\ell+1)u_{i,\ell+1,m}$ and in the second member with coefficients $E_{i\ell m}$ which are finite linear combinations of coefficients $u_{j\lambda s}$, $s = (s_1,\ldots,s_\nu)$, $j = 1,\ldots,n$, with $o \leq \lambda \leq \ell$. For a solution $U \in \mathscr{S}^n$ of problem (1) all u_{iom} are zero.

A proof of the following theorem will be given in (2.5).

(2.i) (Cauchy-Kovalevsky). If all a_{ijh}, b_{ij}, c_i are elements of $\mathscr{S}$, then there is a unique element $u \in \mathscr{S}$ satisfying (1) in a complex neighborhood of the origin $t = o$, $z = o$.

2.2 BANACH SPACES OF HOLOMORPHIC FUNCTIONS

For $\delta > 0$, $0 < \alpha \leq 1$, and $k \geq 0$ integer, let $S_k = S_{\delta\,k}$ be the class of all elements $u \in \mathscr{S}$ with coefficients $u_{\ell m}$ for which

$$\|u\|_{\delta\alpha k} = \mathrm{Sup}_{\ell,m}\left[|u_{\ell m}| \ \ell! m_1!\ldots m_\nu! k! ((\ell+m_1+\ldots+m_\nu+k)!)^{-1}\alpha^\ell \ \delta^{\ell+m_1+\ldots+m_\nu} \right] < \infty \ .$$

Then S_k is a Banach space with norm $\|u\|_k = \|u\|_{\delta\alpha k}$. Actually, we shall need below only S_o and S_1 .

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If $u \in S_k$, then u_t, $u_{z_h} \in S_{k+1}$ with $\|u_t\|_{k+1} \leq (k+1)(\alpha\delta)^{-1}\|u\|_k$,

$\|u_{z_h}\|_{k+1} \leq (k+1)\delta^{-1}\|u\|_k$, $h = 1,\ldots,\nu$. Conversely, if $v \in S_{k+1}$ and $u(t,z) = \int_0^t v(\tau,z)d\tau$ (formal integration), then $u \in S_k$ and $\|u\|_k = (k+1)^{-1}(\alpha\delta)\|v\|_{k+1}$.

If $0 < \delta < \sigma$, $u \in S_{\delta\alpha k}$, $v \in S_{\sigma\alpha k}$, then $uv \in S_{\delta\alpha k}$ and $\|uv\|_{\delta\alpha k} \leq (1-\delta/\sigma)^{-k-1}$

$\|u\|_{\delta\alpha k}\|v\|_{\sigma\alpha k}$.

Note that $S_{\delta\alpha,k+1} \supset S_{\delta\alpha k}$, and that $u \in S_{\delta\alpha k}$ implies $\|u\|_{\delta\alpha k} \geq \|u\|_{\delta\alpha,k+1}$. Also, for $0 < \alpha < \beta \leq 1$, we have $S_{\delta\alpha k} \supset S_{\delta\beta k}$, and $\|u\|_{\delta\alpha k} \leq \|u\|_{\delta\beta k}$.

For $U(t,z) = (u_1,\ldots,u_n) \in (S_{\delta\alpha k})^n$, that is, each $u_i \in S_{\delta\alpha k}$, we take

$\|U\|_{\delta\alpha k} = (\sum_{i=1}^n \|u_i\|^2_{\delta\alpha k})^{1/2}$. Note that if $u \in S_k$ then u has the majorant

$\|u\|_k(1-\delta^{-1}(t/\alpha + z_1+\ldots+z_\nu))^{-k-1}$, the same kind of majorant used in the clas-

sical proof of Cauchy-Kovalevsky theorem (see I. G. Petrovsky [32], pp. 18-24).

2.3 A PRIORI ESTIMATE

Let $\sigma > 0$ be such that a_{ijh}, b_{ij}, $c_i \in S_{\sigma 10}$, and take $A = \text{Max}[\|a_{ijh}\|_{\sigma 10}$,

$\|b_{ij}\|_{\sigma 10}]$. Then, for every $0 < \delta \leq \sigma$, $0 < \alpha \leq 1$, we have $\|a_{ijh}\|_{\delta\alpha 0}$, $\|b_{ij}\|_{\delta\alpha 0} \leq A$.

If C denotes the n-vector $C = [c_i, i = 1,\ldots,n]$, then we have also $\|C\|_{\delta\alpha 0} \leq \|C\|_{\sigma\alpha 0}$.

Let us choose δ and α so that $0 < \delta \leq 2^{-1}\sigma$, $0 < \alpha \leq 1$, $4An^2(\delta + \nu)\alpha < 1$. Let

us prove that for any solution $U \in S_0^n = (S_{\delta\alpha 0})^n$ of (1), if any exists, we have

$$\|U\|_0 \leq \alpha\delta(1 - 4n^2A(\nu + \delta)\alpha)^{-1}\|C\|_0 = \mu\|C\|_0 . \qquad (2)$$

Indeed, if $U \in S_0^n$, then U_t, $U_{z_h} \in S_1^n = (S_{\delta\alpha 1})^n$, $h = 1,\ldots,\nu$. On the other

hand, $U(t,z) = \int_0^t U_\tau(\tau,z)d\tau$, hence $\|U\|_0 = \alpha\delta\|U_t\|_1$. Thus,

$$\|U\|_0 = \alpha\delta\|U_t\|_1 = \alpha\delta\|F(t,z,U,U_z)\|_1$$

$$\leq \alpha\delta[n^2\nu(1-\delta/\sigma)^{-2}A (\text{Max}\|U_{z_h}\|_1 + n^2(1-\delta/\sigma)^{-2}A\|U\|_1 + \|C\|_1] ,$$

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where $1 - \delta/\sigma \geq 2^{-1}$, $\|U_{z_h}\|_1 \leq \delta^{-1}\|U\|_0$, $\|U\|_1 \leq \|U\|_0$, $\|C\|_1 \leq \|C\|_0$. We obtain

$$\|U\|_0 \leq \alpha\delta[4n^2\nu A\delta^{-1}\|U\|_0 + 4n^2 A\|U\|_0 + \|C\|_0] \, ,$$

which yields (2).

Note that relation (2) yields another proof that there can be at most one

solution $U \in \mathcal{S}$ to problem (1). Indeed, any two solutions U_1, $U_2 \in \mathcal{S}$ must belong

to the same space $S_{\delta\alpha_0}$ for δ, α chosen as above, and then $U_1 - U_2$ is a solution

of (1) with $C = 0$. By (2) we have then $\|U_1 - U_2\|_0 = 0$, or $U_1 = U_2$.

2.4 PROOF OF THE EXISTENCE OF A SOLUTION TO PROBLEM (1)

Let us denote by H the operator of integration with respect to t from 0 to

t already considered in No. 3. Then, problem (1) is equivalent to

$$U(t,z) = H \, F(t,z,U,U_z) \, .$$

Thus, if $T: \mathcal{S}^n \to \mathcal{S}^n$ denotes the operator defined by,

$$V = TU = H \, F(t,z,U,U_z)$$

all we have to do is to find the fixed points of T.

Let σ, A, δ, α be chosen as in No. 4 so that $k = 4n^2 A(\nu+\delta)\alpha < 1$. Note

that, if $U \in S_0^n$, then $U_z \in S_1^n$, and $F(t,z,U,U_z) \in S_1^n$. Then, $H \, F \in S_0^n$, and T

is actually a map $T: S_0^n \to S_0^n$. If U_1, $U_2 \in S_0^n$, then

$$\|TU_1 - TU_2\|_0 = \alpha\delta\|TU_1 - TU_2\|_1 =$$

$$= \alpha\delta\|F(t,z,U_1,U_{1z}) - F(t,z,U_2,U_{2z})\|_1$$

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$$\leq \alpha\delta[n^2\nu(1-\delta/\sigma)^{-2}A(\max\|U_{1z_h} - U_{2z_h}\|_1)$$

$$+ n^2(1-\delta/\sigma)^{-2} A\|U_1 - U_2\|_1]$$

$$\leq 4n^2A\ (\nu+\delta)\ \alpha\|U_1 - U_2\|_0 = k\|U_1 - U_2\|_0\ .$$

Since $k < 1$, $T : S_o^n \to S_o^n$ is a contraction in the norm $\|\ \|_0$ of S_o^n, and possesses, therefore, a unique fixed point $U \in S_o^n \subset \mathcal{S}^n$.

2.5 PROJECTION OPERATORS

For fixed integers $N \geq 1$, $M \geq 1$, let us denote by P_N^t, $P_M^z : \mathcal{S} \to \mathcal{S}$, the projection operators defined by

$$P_N^t(\sum_\ell \sum_m u_{\ell m}\ t^\ell\ z_1^{m_1}\ldots z_\nu^{m_\nu}) = \sum_{\ell=o}^N \sum_m u_{\ell m}\ t^\ell\ z_1^{m_1}\ldots z_\nu^{m_\nu}$$

$$P_M^z(\sum_\ell \sum_m u_{\ell m}\ t^\ell\ z_1^{m_1}\ldots z_\nu^{m_\nu}) = \sum_\ell \sum_{m_1=o}^M\ldots\sum_{m_\nu=o}^M u_{\ell m}\ t^\ell\ z_1^{m_1}\ldots z_\nu^{m_\nu}\ ,$$

so that we have also $P_N^t, P_M^z : S_k \to S_k$. For fixed N and M, both $P_N^t S_k$ and $P_M^z S_k$ are subspaces of S_k, actually Banach spaces in the same norm $\|u\|_k$ of S_k.

For any fixed $M \geq 1$ we may consider the Cauchy problem analogous to (1):

$$U_t = P_M^z\ F(t,z,U,U_z)\ ,\ U(0,z) = 0, \tag{3}$$

for which we seek solutions $U \in (P_M^z S_o)^n$. The argument of No. 2 leading to the uniqueness of a solution U, if any exists, repeats in the present situation. The same holds for the argument of No. 4 leading to an a priori estimate. Thus, and solution $U \in (P_m^z S_o)^o$ of problem (3), if any exists, is uniquely determined and satisfies relation (2).

Analogously, for fixed N, $M \geq 1$ we may consider the Cauchy problem analogous to (1) and (3):

$$U_t = P_{N-1}^t \ P_M^z \ F(t,z,U,U_z) \ , \quad U(0,z) = 0 \ , \tag{4}$$

for which we seek solutions $U \in (P_N^t \ P_M^z \ S_o)^n$ (polynomials in t and z). Both the argument leading to the uniqueness of a solution, and the argument leading to the a priori estimate hold in the present situation. But now U is a polynomial, and thus we conclude that a polynomial solution $U \in (P_N^t \ P_M^z \ S_o)^n$ of problem (4) exists and U satisfies (2).

2.6 AN ALTERNATE PROOF OF THE EXISTENCE OF A SOLUTION TO PROBLEM (1)

For every $M = 1,2,\ldots$, let us take $N = M$ in problem (4), and let $U^M(t,z)$ be the polynomials so obtained, $U^M \in (P_N^t \ P_M^z \ S_o)^n$. Let $u_{i\ell m}^M$ be the coefficient of the polynomials U^M. All these U^M are elements of the unique Banach space S_o^n, and satisfy the a priori estimate $\|U^M\|_o \leq \mu$, or (2), as proved. Let $u_{i\ell m}$ be the coefficients of the formal power series expansions corresponding to problem (1), as mentioned in No. 2. Then for every fixed ℓ and $m = (m_1,\ldots,m_\nu)$ and all M large enough, we have $u_{i\ell m}^M = u_{i\ell m}$. Hence, also $u_{i\ell m}$ are the coefficients of an element $U \in S_o^n$ and $\|U\|_o \leq \mu$. Thus $U \in S_o^n \subset \mathcal{S}$, and U is a solution to problem (1).

2.7 ANOTHER PROOF

Let $g(t,z) = (g_1,\ldots,g_n)$ be any n-vector polynomial, or element $g \in (P_N^t \ P_M^z \ S_o)^n$, with $g(0,z) = 0$, and let $g_{i\ell m}$, $0 \leq \ell \leq N$, $0 \leq m_h \leq M$, denote the coefficients of g_i (all g_{iom}, and all $g_{i\ell m}$ with $\ell > N$ or $m_h > M$, being zero).

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For every $U(t,z) = (u_1,\ldots,u_n) \in S_o^n$, let $u_{i\ell m}$ denote the coefficients of the power series of u_i, and let $H: S_o \to S_o$ denote the operation of integration with respect to t as in No. 3, or $HU = V(t,z) = (V_1,\ldots,V_n)$ with

$$V_i(t,z) = \sum_{\ell=N}^{\infty} \sum_m u_{\ell m} (\ell+1)^{-1} t^{\ell+1} z_1^{m_1} \ldots z_\nu^{m_\nu} \quad, \quad i = 1,\ldots,n \;.$$

Then, $(\partial/\partial t)HU = U - P_{N-1}^t U$, $\|HU\|_o \leq (N+1)^{-1} \|U\|_o$, and also $H: (P_M^z S_o)^n \to (P_M^z S_o)^n$. Finally, let $T: (P_M^z S_o)^n \to (P_M^z S_o)^n$ be the operator defined by taking

$$V = TU = g + (I - P_M^z)\, F(t,z,U,U_z) \;.$$

Note that, for $U \in (P_M^z S_o)^n$ we have $U_{z_h} \in (P_M^z S_o)$, and $\|U_{z_h}\|_o \leq M \|U\|_o$, $h = 1,\ldots,\nu$. Then, for $U, V \in (P_M^z S_o)^n$ we have also, as in No. 4,

$$\|TU - TV\|_o \leq (N+1)^{-1}(2AMn^2\nu + 2An^2) \|U - V\|_o \;.$$

By choosing $N = N(M)$ sufficiently large, T is a contraction. The fixed point $U = TU \in (P_M^z S_o)^n$ is an element $U = TU = \mathcal{C}\, g$ which depends on g, and which satisfies $U = g + H\, P_M^z\, F(t,z,U,U_z)$ with $u_{i\ell m} = g_{i\ell m}$ for $o \leq \ell \leq N$. By differentiation we obtain

$$U_t = P_M^z\, F(t,z,U,U_z) + \Delta \;, \tag{5}$$

where $\Delta = g_t - P_{N-1}^t\, P_M^z\, F$. Thus, $U = TU = \mathcal{C}\, g$ is a solution of (4) if and only if the determining (or bifurcation) equation

$$g_t = P_{N-1}^t\, F(t,z,U,U_z) \tag{6}$$

is satisfied. Here $u_{i\ell m} = g_{i\ell m}$ for all $\ell \leq N$ and $g_{iom} = 0$, and the determination

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of a polynomial g satisfying (6) is a problem similar to (4). Thus g can be uniquely determined so as to satisfy (6), and correspondingly $U = TU = \mathcal{T}g$ satisfies problem (4). If we denote by $U^M \in (P_M^z S_o)^n$ the element so determined, and we take M = 1,2,..., we have a sequence $[U^M]$ as in No. (2.6) and a solution U of the original problem (1) can be derived.

3. A Direct Proof of an Hormander's Theorem

3.1 STATEMENT OF THE PROBLEM

The proof and the remaining considerations of No. 2 extend to a more general statement (Hormander [28], Th. 5.1., p. 116), from which Hormander deduces as corollaries the Cauchy-Kovalevsky theorem, a theorem by Darboux, Goursat, and Bendon, and a number of other statements ([28], pp. 118-126). For the sake of simplicity we limit ourselves to partial differential equations with one unknown function. The extension to systems is easy.

Let $\mathcal{S}$ denote the class of all holomorphic functions $u(z)$, $z = (z_1,\ldots,z_\nu)$, which are holomorphic in some complex neighborhood of the origin $z = o$, and thus possesses power series expansion $\sum_m u_m z^m$, or $\sum_m u_m z_1^{m_1}\ldots z_\nu^{m_\nu}$, $m = (m_1,\ldots,m_\nu)$, $|m| = m_1+\ldots+m_\nu$. Hormander's problem is as follows:

Given any multiindex $\beta = (\beta_1,\ldots,\beta_\nu)$, $|\beta| > o$, and elements $f, a^\alpha, \varphi \in \mathcal{S}$, $o \leq |\alpha| \leq |\beta|$, determine an element $u \in \mathcal{S}$ such that in a neighborhood of $z = o$ we have

$$D^\beta u = \sum_{|\alpha|\leq|\beta|} a^\alpha(z)\, D^\alpha u + f(z), \text{ and} \qquad (1)$$

$$D_j^k(u-\varphi) = o \quad \text{when} \quad z_j = o \quad \text{if} \quad o \leq k < \beta_j, \quad j = 1,\ldots,\nu. \qquad (2)$$

A proof of the following theorem will be given in No. 3.5 below. As usual $D^\beta u$ denotes $\partial^{|\beta|} u/\partial z_1^{\beta_1}\ldots\partial z_\nu^{\beta_\nu}$.

(3.1.i) (Hormander, [28] Th.5.1.1, p. 116). If $\sum\limits_{|\alpha|=|\beta|} |a^{\alpha}(o)|$ is smaller than a positive number depending only on $|\beta|$, then problem (1), (2) has one and only one solution $u \in \mathcal{S}$.

3.2 THE CLASSICAL UNIQUENESS ARGUMENT

The substitution $u = v + \varphi$ reduces the proof to the case $\varphi = o$ so we may assume that $\varphi = o$ from the beginning. Thus, we replace (2) by

$$D_j^k u = o \quad \text{when} \quad z_j = o \quad \text{if} \quad o \leq k < \beta_j, \quad j = 1,\ldots,\nu. \tag{3}$$

If α,β are any two multiindices $\alpha = (\alpha_1,\ldots,\alpha_\nu)$, $\beta = (\beta_1,\ldots,\beta_\nu)$ we shall say that $\alpha < \beta$ [or $\alpha \leq \beta$] provided $\alpha_i < \beta_i$ [or $\alpha_i \leq \beta_i$] for all $i = 1,\ldots,\nu$. If $\alpha \leq \beta$ we shall write as usual

$$\binom{\beta}{\alpha} = \binom{\beta_1}{\alpha_1}\ldots\binom{\beta_\nu}{\alpha_\nu} \quad \text{with} \quad \binom{\beta_i}{\alpha_i} = \frac{\beta_i!}{\alpha_i!(\beta_i-\alpha_i)!} \; .$$

Given $\beta = (\beta_1,\ldots,\beta_\nu)$ and any integer $\lambda \geq |\beta|$, we shall denote by $N_{\lambda\beta}$ the number of integral solutions of the equations $|\ell| = \lambda$, $\ell \geq \beta$, or $\ell_1 +\ldots+ \ell_\nu = \lambda$, $\ell_i \geq \beta_i$, $i = 1,\ldots,\nu$. Also, we shall denote by μ_β the number of solutions of the equations $|\alpha| = |\beta|$, or $\alpha_1 +\ldots+ \alpha_\nu = \beta_1 +\ldots+ \beta_\nu$.

The solution $u \in \mathcal{S}$ of problem (1-3) is uniquely determined. This can be proved by a simple modification of the usual argument for the Cauchy-Kovalevsky theorem (see [5], p. 19). Indeed, if $u \in \mathcal{S}$ satisfies (1-3), and d_ℓ, $A_{\alpha\ell}$, $\ell = (\ell_1,\ldots,\ell_\nu)$ denote the derivatives $d_\ell = D^\ell u\big|_{z=o}$, $A_{d\ell} = D^\ell a^\alpha\big|_{z=o}$, then all derivatives d_ℓ with at least one $\ell_i < \beta_i$ are zero. In other words, $d_\ell = o$ for every $\ell \not\geq \beta$. For $\ell \geq \beta$, the derivatives d_ℓ can be obtained from Leibniz rule by differentiating (1) exactly $\ell - \beta$ lines, that is, applying the operator

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$D^{\ell-\beta}$ on both sides, and taking z = o. We obtain

$$d_\ell = \sum_{|\alpha| \le |\beta|} \sum_{\ell' \le \ell} \binom{\ell}{\ell'} A_{\alpha, \ell-\ell'} d_{\ell'+\alpha-\beta}; \qquad (4)$$

and in this sum we may well restrict to only those terms with $\ell' + \alpha - \beta \ge \alpha$, or $\ell' \ge \beta$. Note that, for any given $\ell \ge \beta$ we have in (4) $\ell' \le \ell$, $\ell \ge \beta$, $\ell' \ge \beta$, $|\alpha| \le |\beta|$, and finally

$$|\ell'+\alpha-\beta| = |(\ell'-\beta)+\alpha| = |\ell'-\beta|+|\alpha| = |\ell'|-|\beta|+|\alpha| \le |\ell'| \le |\ell| .$$

Thus, for any given $\lambda \ge 0$, $\lambda \ge |\beta|$, $|\ell| = \lambda$, we may have in (4) $|\ell'+\alpha-\beta| = \lambda$ if and only if $|\ell'| = |\ell|$, $|\beta| = |\alpha|$. In other words, the N_λ derivatives d_ℓ with $|\ell| = \lambda$, $\ell \ge \beta$, satisfy the system of N_λ equations

$$d_\ell - \sum_{|\alpha|=|\beta|} A_{\alpha o} d_{\ell+\alpha-\beta} = E_\ell , \quad |\ell| = \lambda, \quad \ell \ge \beta, \qquad (5)$$

where E_ℓ denote finite linear combinations of derivatives d_s with $|s| < \lambda$, and where $A_{\alpha o} = a^\alpha(o)$. In each equation (5) there appear, therefore, besides the unknown d_ℓ with coefficient one, at most μ_β other unknowns $d_{\ell+\alpha-\beta}$ with $\ell+\alpha-\beta = |\lambda| = |\ell|$, with coefficients $A_{\alpha o}$ whose sum of the absolute values is $\le \sum_{|\alpha|=|\beta|} |a^\alpha(o)|$. Thus, if this sum is < 1, the N_λ equations (5) have a unique solution ([22] p. 29).

3.3 BANACH SPACES OF HOLOMORPHIC FUNCTIONS

For $\delta > 0$ and $k \ge 0$ integer, let $S_{\delta k}$ be the class of all elements $u \in \mathcal{S}$, $u = \sum_m u_m z_1^{m_1} \ldots z_\nu^{m_\nu}$, $m = (m_1, \ldots, m_\nu)$, for which

$$\|u\|_{\delta k} = \mathrm{Sup}_m [|u_m| m_1! \ldots m_\nu! \, k! \, ((m_1+\ldots+m_\nu+k)!)^{-1} \delta^{m_1+\ldots+m_\nu}] < \infty. \qquad (6)$$

Then $S_{\delta k}$ is a Banach space with norm $\|u\|_{\delta k}$. Note that $S_{\delta k} \subset S_{\delta,k+1}$ and that

for $u \in S_{\delta k}$ we have $\|u\|_{\delta k} \geq \|u\|_{\delta,k+1}$. Also, $S_{\delta 0} \subset S_{\delta 1} \subset \ldots \subset S_{\delta k} \subset \ldots \subset \mathcal{S}$,

for $u \in S_{\delta 0}$ we have $\|u\|_{\delta 0} \geq \|u\|_{\delta 1} \geq \ldots \geq |u_0|$. For any given element $u \in S_{\sigma 0}$

we can take δ, $0 < \delta \leq \sigma$, so small that $\|u\|_{\delta 0}$ is a close as we want to $|u_0|$.

If $u \in S_{\delta k}$ then each derivative $D_{x_i} u \in S_{\delta,k+1}$ with $\|D_{x_i}u\|_{\delta,k+1} \leq (k+1)\delta^{-1}\|u\|_{\delta k}$.

Conversely, if $v \in S_{\delta,k+1}$ and $u(z) = D_{x_i}^{-1} v$ (formal integration), then $u \in S_{\delta k}$

and $\|u\|_{\delta k} = (k+1)^{-1}\delta \|v\|_{\delta,k+1}$. These are immediate consequences of the defini-

tions. As a consequence, for every multiindex $\alpha = (\alpha_1,\ldots,\alpha_\nu)$ and $u \in S_{\delta k}$, we

have $D^\alpha u \in S_{\delta,k+|\alpha|}$ and $\|D^\alpha u\|_{\delta_1 k+|\alpha|} \leq (k+1)\ldots(k+|\alpha|)\delta^{-|\alpha|}\|u\|_{\delta k}$. Conversely,

if $v \in S_{\delta,k+|\alpha|}$ and $u = D^{-\alpha}v$ (formal integration), then $u \in S_{\delta k}$ and $\|u\|_{\delta k} =$

$(k+1)^{-1}\ldots(k+|\alpha|)^{-1}\delta^{|\alpha|}\|u\|_{\delta,k+|\alpha|}$.

Note that, if $0 < \delta < \sigma$, $u \in S_{\delta k}$, $v \in S_{\sigma k}$, then $uv \in S_{\delta k}$ and $\|uv\|_{\delta k}$

$\leq (1 - \delta/\sigma)^{-k-1}\|u\|_{\delta k} \|v\|_{\sigma k}$. Also note that, if $u \in S_{\delta k}$, then

$$|u_m| \leq \|u\|_{\delta k} (m_1+\ldots+m_\nu + k)! \, (m_1!\ldots m_\nu!k!)^{-1} \delta^{-m_1-\ldots-m_\nu}$$

that is, $u(z)$ has the majorant

$$\|u\|_{\delta k} \sum_m (m_1+\ldots+m_\nu + k)!(m_1!\ldots m_\nu! \, k!)^{-1} z_1^{m_1}\ldots z_\nu^{m_1} \delta^{-m_1-\ldots-m_\nu}, \qquad (7)$$

or

$$\|u\|_{\delta k} (1 - (z_1+\ldots+z_\nu)\delta^{-1})^{-k-1}.$$

Conversely, if $u(z)$ has a majorant $\mu(1-(z_1+\ldots+z_\nu)\delta^{-1})^{-k-1}$, then $u \in S_{\delta k}$ and

$\|u\|_{\delta k} \leq \mu$.

3.4 A PRIORI ESTIMATE

We prove here an a priori estimate for the solutions $u \in \mathcal{S}$ of problem (1-3) above (that is, for problem (5.1.1) of [28], p. 116).

(3.4.i) Given β, there is a number $\tau = \tau(\beta) > 0$ and, for all $a^\alpha \in \mathcal{S}$, $|\alpha| \leq |\beta|$ with $\sum_{|\alpha| = |\beta|} |a^\alpha(0)| \leq \tau$, there is another number $\delta_0 > 0$ such that, for any $f \in S_{\delta_0}$, $0 < \delta \leq \delta_0$, and solution $u \in \mathcal{S}$ of (1-3) (if any) with $u \in S_{\delta_0}$, we have also

$$\|u\|_{\delta_0} \leq 2(|\beta|!)^{-1} \delta^{|\beta|} \|f\|_{\delta_0} . \tag{8}$$

Proof. Let $\tau = 2^{-4-|\beta|}$. Given $a^\alpha \in \mathcal{S}$, $|\alpha| \leq |\beta|$, let $\sigma > 0$ be such that $a^\alpha \in S_{\sigma 0}$, $|\alpha| \leq |\beta|$, and let $A = \text{Max}\, [\|a^\alpha\|_{\sigma 0}, |\alpha| < |\beta|]$. Then, for any r integer we have $\|a^\alpha\|_{\sigma r} \leq \|a^\alpha\|_{\sigma 0} \leq A$ for all α with $|\alpha| < |\beta|$. If $\sum_{|\alpha| = |\beta|} |a^\alpha(0)| \leq \tau$, then we certainly have

$$2^{|\beta|+1} \sum_{|\alpha| = |\beta|} |a^\alpha(0)| < 2^{-3} .$$

Let δ_0 be so chosen that $0 < \delta_0 \leq 2^{-1}\sigma$,

$$2^{|\beta|+1} A \sum_{|\alpha| < |\beta|} \delta_0^{|\beta|-|\alpha|} < 2^{-3} ,$$

and such that

$$\sum_{|\alpha| = |\beta|} \|a^\alpha\|_{\delta 0} < \sum_{|\alpha| = |\beta|} |a^\alpha(0)| + 2^{-p-4}$$

for every δ, $0 < \delta \leq \delta_0$. Then

$$k_0 = 2^{|\beta|+1} A \sum_{|\alpha| < |\beta|} \delta_0^{|\beta|-|\alpha|} + 2^{|\beta|+1}\left(\sum_{|\alpha| = |\beta|} |a^\alpha(0)| + 2^{-|\beta|-4}\right) \tag{9}$$

$$< 2^{-3} + 2^{-3} + 2^{-3} < 2^{-1} .$$

Thus, for $o < \delta \leq \delta_o$ we have $1-\delta/\sigma \geq 2^{-1}$, and if $k(\delta)$ is the number defined as in (9) with δ replacing δ_o and $\mu(\delta) = 1 - k(\delta)$, we have $o < k(\delta) \leq k_o < 2^{-1}$, $1 > \mu(\delta) > 1 - 2^{-1} = 2^{-1}$.

If $f \in S_{\delta o}$ and $u \in \mathcal{S}$, $u \in S_{\delta o}$, is any solution of (1-3), then $D^{\beta}u \in S_{\delta,|\beta|}$, and $u = D^{-\beta}(D^{\beta}u)$ (formal integration as in No. (3.3)). Then from (1) and the remarks made in No. (3.3),

$$\|u\|_{\delta o} = (|\beta|!)^{-1} \delta^{|\beta|} \|D^{\beta}u\|_{\delta,|\beta|} =$$

$$= (|\beta|!)^{-1} \delta^{|\beta|} \| \sum_{|\alpha| \leq |\beta|} a^{\alpha} D^{\alpha}u\|_{\delta,|\beta|}$$

$$\leq (|\beta|!)^{-1} \delta^{|\beta|} \left[(1-\delta/\sigma)^{-|\beta|-1} \sum_{|\alpha| \leq |\beta|} \|a^{\alpha}\|_{\sigma,|\beta|} \|D^{\alpha}u\|_{\delta,|\beta|} + \|f\|_{\delta,|\beta|} \right]$$

$$\leq (|\beta|!)^{-1} \delta^{|\beta|} \left[2^{|\beta|+1} A \sum_{|\alpha| < |\beta|} (|\beta|-|\alpha|+1)\ldots(|\beta|)\delta^{-|\alpha|} \|u\|_{\delta,|\beta|-|\alpha|} \right.$$

$$\left. + 2^{|\beta|+1} (|\beta|!) \delta^{-|\beta|} (\sum_{|\alpha|=|\beta|} |a^{\alpha}(o)| + 2^{-\beta-4}) \|u\|_{\delta o} + \|f\|_{\delta o} \right]$$

$$\leq 2^{|\beta|+1} A \sum_{|\alpha| < |\beta|} \delta^{|\beta|-|\alpha|} + 2^{|\beta|+1} (\sum_{|\alpha|=|\beta|} |a^{\alpha}(o)| + 2^{-\beta-4})\|u\|_{\delta o} + (|\beta|!)^{-1}$$

$$\delta^{|\beta|} \|f\|_{\delta o} ,$$

$$\leq k(\delta) \|u\|_{\delta o} + (|\beta|!)^{-1} \delta^{|\beta|} \|f\|_{\delta o} ,$$

or

$$\|u\|_{\delta o} \leq (\mu(\delta))^{-1} (|\beta|!)^{-1} \delta^{|\beta|} \|f\|_{\delta o} ,$$

where $\mu(\delta) = 1 - k(\delta) \geq \mu_o > 2^{-1}$, and (8) is thereby proved.

3.5 PROOF OF THE EXISTENCE OF A SOLUTION TO PROBLEM (1-3)

Let us denote by H the operation of integration $H = D^{-\beta}$ already considered in Nos. (3.3) and (3.4). Then, problem (1-3) is equivalent to

$$u(z) = H\left(\sum_{|\alpha| \leq |\beta|} a^{\alpha} D^{\alpha} u + f(z) \right).$$

If $T: \mathscr{S} \to \mathscr{S}$ denotes the operator defined by

$$v = Tu = H\left(\sum_{|\alpha| \leq |\beta|} a^{\alpha} D^{\alpha} u + f(z) \right),$$

all we have to do is to find fixed points of T.

Let σ, A, δ be chosen as in No. (3.4), so that $0 < k(\delta) \leq k_0 < 2^{-1}$.

Note that, if $u \in S_{\delta o}$, then $D^{\alpha} u \in S_{\delta, |\alpha|}$, $0 \leq |\alpha| \leq |\beta|$, and $\sum_{|\alpha| \leq |\beta|} a^{\alpha} D^{\alpha} u \in S_{\delta, |\beta|}$. For $f \in S_{\delta, o}$ we certainly have $H(\sum_{|\alpha| \leq |\beta|} a^{\alpha} D^{\alpha} u + f) \in S_{\delta o}$, and T is, therefore, a map $T: S_{\delta o} \to S_{\delta o}$. If $u_1, u_2 \in S_{\delta o}$, then

$$\|Tu_1 - Tu_2\|_{\delta o} = \left\| H\left(\sum_{|\alpha| \leq |\beta|} a^{\alpha} D^{\alpha} (u_1 - u_2) \right) \right\|_{\delta o}$$

$$= (|\beta|!)^{-1} \delta^{|\beta|} \left\| \sum_{|\alpha| \leq |\beta|} a^{\alpha} D^{\alpha} (u_1 - u_2) \right\|_{\delta, |\beta|}$$

$$\leq (|\beta|!)^{-1} \delta^{|\beta|} (1 - \delta/\sigma)^{-|\beta|-1} \sum_{|\alpha| \leq |\beta|} \|a^{\alpha}\|_{\sigma, |\beta|} \|D^{\alpha} u\|_{\delta, |\beta|}$$

$$\leq (|\beta|!)^{-1} \delta^{|\beta|} \left[2^{|\beta|+1} A \sum_{|\alpha| < |\beta|} (|\beta| - |\alpha| - 1) \cdots (|\beta|) \delta^{-|\alpha|} \|u_1 - u_2\|_{\delta, |\beta| - |\alpha|} + \right.$$

$$\left. + 2^{|\beta|+1} (|\beta|!) \delta^{-|\beta|} \left(\sum_{|\alpha| = |\beta|} |a^{\alpha}(o)| + 2^{-|\beta|-4} \right) \right] \|u_1 - u_2\|_{\delta o}$$

$$\leq \left[2^{|\beta|+1} \sum_{|\alpha| < |\beta|} \delta^{|\beta| - |\alpha|} + 2^{|\beta|+1} \left(\sum_{|\alpha| = |\beta|} |a^{\alpha}(o)| + 2^{-|\beta|-4} \right) \right] \|u_1 - u_2\|_{\delta o}$$

$$= k(\delta) \|u_1 - u_2\|_{\delta o},$$

where $k(\delta) < 2^{-1}$. Thus, $T: S_{\delta o} \to S_{\delta o}$ is a contraction in the norm $\|u\|_{\delta o}$ of $S_{\delta o}$, and possesses, therefore, a unique fixed point $u \in S_{\delta o}$. Theorem (3.1.i) is thereby proved.

Considerations analogous to the ones in Nos. (2.6-8) could be repeated here, and are omitted for the sake of brevity.

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